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Restricting the topology of 1–cusped arithmetic 3–manifolds

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This paper makes progress on classifying those closed orientable 3–manifolds M that contain knots K so that $M \setminus K$ is arithmetic.

57M50, 57M25; 57M10

1 Introduction

Let d be a square-free positive integer, let O_d denote the ring of integers in $\mathbb{Q}(\sqrt{-d})$ and let Q_d denote the Bianchi orbifold $\mathbb{H}^3/\mathrm{PSL}(2, O_d)$. A finite volume, noncompact hyperbolic 3–manifold X is called *arithmetic* if X and Q_d are commensurable, that is to say they share a common finite sheeted cover (see Maclachlan and the second author [15, Chapter 8] for more on this). This paper is concerned with those 1–cusped arithmetic 3–manifolds $X = \mathbb{H}^3/\Gamma$ that actually cover a Bianchi orbifold Q_d. If M is a closed orientable 3–manifold, a knot K is called *arithmetic*, if $M \setminus K$ is arithmetic. Therefore, we are concerned with those arithmetic knots $K \subset M$ for which there is a finite cover $M \setminus K \to Q_d$.

It is known that there are only finitely many commensurability classes of 1–cusped arithmetic hyperbolic 3–orbifolds; see Chinburg, Long and the second author [6]. On the other hand, 1–cusped arithmetic hyperbolic 3–manifolds have provided many interesting examples in the study of the geometry and topology of hyperbolic 3–manifolds; eg the smallest volume orientable cusped hyperbolic 3–manifolds (see Cao and Meyerhoff [5]) and examples arising in connection with exceptional Dehn fillings Gordon [12]. In addition, as discussed by the authors in [2], the existence of arithmetic knots in such M was connected to an approach to the Poincaré Conjecture, and to that end, in [2] we exhibited closed orientable 3–manifolds M that do not contain any arithmetic knots. One such family of manifolds is obtained by taking lens spaces $L(p,q)$ where $p \neq 5$ is odd. However, when $p = 5$ there are well-known arithmetic examples arising from the double cover of the figure-eight knot complement and the sister manifold of the figure-eight knot complement. Indeed, these two manifolds both cover the Bianchi orbifold Q_3. In addition, upon analyzing manifolds from

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the SnapPea census ([8] and Coulson, Goodman, Hodgson and Neumann [7]), in [2, Table 1], we showed that there is at least one more example of a manifold with finite fundamental group containing a knot K determining a 1–cusped manifold that also covers Q_3 ([2, Table 1] gives several examples of arithmetic 1–cusped manifolds).

In this paper we obtain further restrictions on 1–cusped arithmetic 3–manifolds and thereby further illuminate the structure of arithmetic knots in certain 3–manifolds in a way that extends the results of [2], and, in addition, also generalizes the fact that the figure-eight knot is the only arithmetic knot in S^3; see the second author [17].

To explain these classes further, note that given Perelman's solution to the Geometrization Conjecture, a 3–manifold is called *spherical* if it has finite fundamental group. As noted above, there are several spherical 3–manifolds that contain arithmetic knots, and one of the aims of this paper is to make a start towards classifying all such examples. Another natural class of examples that generalize the case of the figure-eight knot in S^3 is that of arithmetic knots in *integral homology* 3*–spheres*. Clearly $1/n$–Dehn surgery on the figure-eight knot produces integral homology 3–spheres that contain an arithmetic knot. However, somewhat mysteriously no other examples appear to be known.

The main result of this paper makes progress on the existence of arithmetic knots in spherical 3–manifolds and integral homology 3–spheres and which Bianchi orbifolds these knot complements cover. In particular we prove the following.

Theorem 1.1 *Let M be a closed orientable 3–manifold and $K \subset M$ an arithmetic knot for which $M \setminus K$ is a finite cover of Q_d.*

(1) *If M is a spherical 3–manifold, then $d = 3$.*
(2) *If M is an integral homology 3–sphere, then $d = 1, 3$.*

Remark As we point out in Section 2, the hypothesis that $M \setminus K$ covers Q_d is automatically satisfied in (2) of Theorem 1.1.

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2 Preliminaries and outline of the proof

We start with a discussion of the plan of the proof of Theorem 1.1. The proofs of both (1) and (2) of Theorem 1.1 begin with a (by now) standard analysis to reduce consideration to a small list of Bianchi orbifolds that a candidate knot complement (as in Theorem 1.1) can cover. Part (1) now proceeds by restricting the possible parabolic elements that correspond to surgery curves that can produce a spherical 3–manifold (using the 6-Theorem). This is done in Section 2. In Section 3 we complete the analysis that eliminates all possible candidates. This part relies on Dunbar's lists of spherical orbifolds [9] as well as group theory packages in Magma [3].

The proof of (2) of Theorem 1.1 is given in Section 4. This has some similarities to the structure of the proof of (1), but requires some different methods. For example, we exploit the infinite cyclic first homology group in both geometric and algebraic ways.

For convenience, we begin by recalling some of [2], as well as a discussion about Dehn filling on cusped orbifolds that will be used in what follows.

2.1

We record the following well-known proposition (see [2, Proposition 2.3]).

Proposition 2.1 *Let M be a closed orientable 3–manifold and $K \subset M$ arithmetic with $M \setminus K \to Q_d$. Then $\mathbb{Q}(\sqrt{-d})$ has class number 1, and furthermore, if M is a rational homology 3–sphere, then*

$$d \in \{1, 2, 3, 7, 11, 19\}.$$

Since spherical 3–manifolds are rational homology 3–spheres, an obvious corollary of this that is useful for us to state is the following.

Corollary 2.2 *Let M be a spherical 3–manifold and $K \subset M$ with $M \setminus K \to Q_d$. If $d \neq 3$, then $d \in \{1, 2, 7, 11, 19\}$.*

In the case of an arithmetic knot K in an integral homology 3–sphere M, it follows from standard techniques (see [2; 17] for example) that $M \setminus K$ is a finite cover of some Q_d. Hence, in reference to Theorem 1.1(2), the assumption that $M \setminus K$ covers Q_d given in the first sentence of Theorem 1.1 is automatically satisfied.

2.2

Let M be a spherical 3–manifold and assume we have a finite cover

$$M \setminus K = \mathbb{H}^3 / \Gamma \to Q_d, \quad d \neq 3.$$

Since $\mathrm{PSL}(2, O_d)$ obviously contains parabolic elements fixing ∞, there is a parabolic element μ in Γ fixing ∞ which is a "meridian" of K, in the sense that trivially filling $M \setminus K$ along μ gives back M. Let

$$\mu = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

for some $x \in O_d$. This notation will be fixed throughout. The following proposition is a consequence of Lemma 4.2 and Case 1 of the proof of Theorem 1.2 of [2].

Proposition 2.3 *In the notation above, $|x| \leq 6$ and x is not a unit in O_d.*

Proof That $|x| \leq 6$ follows from the fact that since M is spherical, the length of μ on any horospherical cusp cross-section is at most 6 (see Agol [1] and Lackenby [14]). That x is not a unit holds, since as shown in Case 1 of the proof of Theorem 1.2 of [2], x is a unit only when $M \setminus K$ is homeomorphic to the complement of the figure-eight knot. However, we are assuming $d \neq 3$. □

2.3

Maintaining the hypotheses in the first sentence of the previous subsection, since M is spherical and x is not a unit (by Proposition 2.3), we have the following diagram of finite covers. In the diagram, I denotes the principal O_d–ideal generated by x and $\Gamma(I)$ is the principal congruence subgroup of level I in $\mathrm{PSL}(2, O_d)$. A key point, derived from Lemma 2.4 below, is that $S^3 \setminus J$ is a regular cover of Q_d.

$$\begin{tikzcd}
 & S^3 \setminus J \arrow[dl, "f_4"'] \arrow[dr, "f_3"] & \\
\mathbb{H}^3/\Gamma(I) \arrow[dr, "f_2"'] & & M \setminus K \arrow[dl, "f_1"] \\
 & Q_d &
\end{tikzcd}$$

Figure 1

Let $M \setminus K = \mathbb{H}^3/\Gamma$. Now, the covering f_1 is given by assumption, and f_3 is a finite regular cover with covering group $\pi_1(M)$ which comes from the universal cover $S^3 \to M$. Let $S^3 \setminus J$ denote this link complement cover and assume that $S^3 \setminus J = \mathbb{H}^3/\Gamma_J$. Note that $\mu \in \Gamma(I)$ by hypothesis and $\mu \in \Gamma_J$ by definition of the cover f_3. Furthermore, since Γ_J is generated by Γ–conjugates of μ and these also lie in $\Gamma(I)$, we deduce that $\Gamma_J < \Gamma(I)$ which gives the covering f_4.

Now, as in [2, Section 3.1], we can say more. We include the argument for completeness, but will need some notation. If G is a group, we let $\langle b\rangle_G$ denote the normal closure in G of the element $b \in G$. In the case when $G = \mathrm{PSL}(2, O_d)$, we simply use the notation $\langle b\rangle$.

Lemma 2.4 $\langle\mu\rangle_\Gamma = \langle\mu\rangle$.

Proof As above, let P_d denote the peripheral subgroup of $\mathrm{PSL}(2, O_d)$ fixing ∞. Since $M \setminus K$ and Q_d both have one cusp, it follows as in [17; 2] that $\mathrm{PSL}(2, O_d) = P_d.\Gamma$. If $d \neq 1, 3$, the only elements fixing ∞ are translations and so μ commutes with these. In the case of $d = 1$, the additional element

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

also fixes ∞ and conjugates μ to μ^{-1}. Thus we see that

$$\langle\mu\rangle = \langle\mu\rangle_{P_d.\Gamma} = \langle\mu\rangle_\Gamma,$$

as required. □

Hence we deduce from Lemma 2.4 the statement made prior to Figure 1.

Corollary 2.5 *The covering $S^3 \setminus J \to Q_d$ is a regular cover.*

2.4

We briefly recall the notion of Dehn surgery on an orbifold (for more details see Dunbar and Meyerhoff [10]). Let Q be a finite volume orientable 1–cusped hyperbolic 3–orbifold. The cusp end of the orbifold has the form $T \times [0, \infty)$, where T is an orientable Euclidean 2–orbifold. In the case of a torus, one proceeds to define Dehn surgery in exactly the same way as in the manifold case. In the case when the horospherical cusp cross-section is a pillowcase P (ie the quotient of the torus by the hyperelliptic involution $\tau\colon T^2 \to T^2$ acting as -1 on $H_1(T^2; \mathbb{Z})$) one proceeds as follows.

The map τ defines an orbifold covering map $\pi : T^2 \to P$. The involution τ extends to a self-map of the solid torus, and so π extends to a map between the solid torus and the solid pillowcase. By choosing a homology basis for the 2–fold cover of P we can define p/q–surgery on the end $P \times [0, \infty)$ to mean cutting off the end and regluing it in a way that induces p/q–surgery on the 2–fold cover of the end. This corresponds to attaching a disc to a p/q–curve γ say, in the 2–fold cover of the end so that under the map π, γ projects to a loop in P. The other cusp cross-sections do not admit Dehn surgeries.

In the context of Bianchi orbifolds, as is well-known, apart from the case of $d = 1, 3$, the 1–cusped Bianchi orbifolds, Q_d, have a torus as a horospherical cusp cross-section, whilst in the case of $d = 1$ it is a pillowcase. Thus Dehn surgery can be performed in all cases except $d = 3$, ie for those d stated in Corollary 2.2.

Moreover, from the discussion in Section 2.3, note that the (finite) coverings

$$S^3 \setminus J \to M \setminus K \to Q_d$$

can be extended over μ–Dehn surgeries to give (finite) coverings,

$$S^3 \to M \to Q_d(\mu),$$

where the orbifold $Q_d(\mu)$ will have finite orbifold fundamental group since it is finitely covered by S^3. Note that $\pi_1^{\mathrm{orb}}(Q_d(\mu)) = \mathrm{PSL}(2, O_d)/\langle\mu\rangle$. Thus, to prove Theorem 1.1 we must rule out such finite covers. This is done in Section 3.

2.5

The proofs of both (1) and (2) of Theorem 1.1 use presentations for the Bianchi groups as well as explicit pictures of the orbifolds Q_d for $d \in \{1, 2, 7, 11, 19\}$ (see Swan [18] for the presentations and Frohman and Fine [11] for the orbifolds).

$$\begin{aligned}
&\mathrm{PSL}(2, O_1)\\
&\quad = \langle a, \ell, t, u \mid \ell^2 = (t\ell)^2 = (u\ell)^2 = (a\ell)^2 = a^2 = (ta)^3 = (ua\ell)^3 = 1, [t, u] = 1\rangle,\\
&\mathrm{PSL}(2, O_2) = \langle a, t, u \mid a^2 = (ta)^3 = (au^{-1}au)^2 = 1, [t, u] = 1\rangle,\\
&\mathrm{PSL}(2, O_7) = \langle a, t, u \mid a^2 = (ta)^3 = (atu^{-1}au)^2 = 1, [t, u] = 1\rangle,\\
&\mathrm{PSL}(2, O_{11}) = \langle a, t, u \mid a^2 = (ta)^3 = (atu^{-1}au)^3 = 1, [t, u] = 1\rangle,\\
&\mathrm{PSL}(2, O_{19})\\
&\quad = \langle a, b, t, u \mid a^2 = (ta)^3 = b^3 = (bt^{-1})^3 = (ab)^2 = (at^{-1}ubu^{-1})^2 = 1, [t, u] = 1\rangle.
\end{aligned}$$

In addition in all cases

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} 1 & \omega_d \\ 0 & 1 \end{pmatrix},$$

(with the obvious abuse of notation between SL and PSL) and

$$\omega_d = i, \sqrt{-2}, \frac{1+\sqrt{-7}}{2}, \frac{1+\sqrt{-11}}{2}, \frac{1+\sqrt{-19}}{2}.$$

The orbifolds Q_d are drawn in Figure 2.

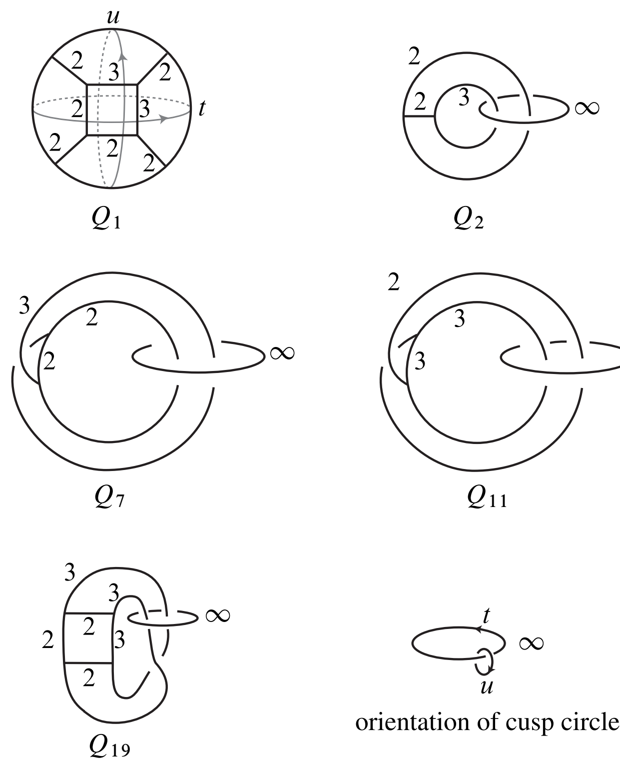


Figure 2

As remarked above, for $d \neq 1$, Q_d has a torus cusp cross-section and the underlying space is an open solid torus which we view as the complement of an unknotted circle in S^3. This circle is labelled ∞ while segments of the singular locus of cone angle π (resp. $2\pi/3$) are labelled 2 (resp. 3). The meridian (resp. longitude) of the cusp torus corresponds to the parabolic element u (resp. t) above. Our convention for orientation is also shown in Figure 2.

The orbifold Q_1 has underlying space the 3–ball with cusp cross-section being a pillowcase. As above, the segments of the singular locus are labelled 2 and 3, whilst in this case u and t correspond to the curves shown in Figure 2.

We also note for future reference, that, by inspecting the singular locus of the orbifolds in Figure 2, one sees that for $d = 1, 2, 11$ and 19, $\mathrm{PSL}(2, O_d)$ contains a copy of A_4 whereas $\mathrm{PSL}(2, O_7)$ does not, but it does contain a copy of S_3. Hence, a manifold cover of Q_d has degree $12k$ (when $d = 1, 2, 11$) and $6k$ when $d = 7$ (see also Grunewald and Schwermer [13]).

Notation Let m and n be integers (not necessarily coprime). We denote the orbifold obtained by (m,n)–Dehn surgery on Q_d (using the framings described above) by $Q_d(m,n)$.

3 Proof of Theorem 1.1(1)

Part (1) of Theorem 1.1 will be proved once the following two propositions are established. Recalling that

$$\mu = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

for some $x \in O_d$, we declare x to be *allowable* if $\mathrm{PSL}(2, O_d)/\langle\mu\rangle$ is finite. The set of allowable values of x is a subset of those for which $|x| \le 6$. Since the rings of integers in quadratic imaginary number fields are discrete subsets of $\mathbb{C}$, the set of allowable values for x is finite. By our previous discussion, the case of x a unit can be excluded from consideration.

Proposition 3.1 *The only values of x that are allowable, together with the corresponding elements (modulo inverses) of $\mathrm{PSL}(2, O_d)$ and orders of the finite groups are those stated in Table 1 below.*

Table 1

d	x	**element of** $\mathrm{PSL}(2, O_d)$	**Order of** $\mathrm{PSL}(2, O_d)/\langle\mu\rangle$
1	$\pm(1 \pm i)$	$tu, t^{-1}u$	24
1	$\pm(2 \pm i)$	$t^2u, t^{-2}u$	120
1	$\pm(1 \pm 2i)$	tu^2, tu^{-2}	120
1	$\pm(3 \pm i)$	$t^3u, t^{-3}u$	2880
1	$\pm(1 \pm 3i)$	tu^3, tu^{-3}	2880
2	$\pm(1 \pm \sqrt{-2})$	$tu, t^{-1}u$	24
2	$\pm(2 \pm \sqrt{-2})$	$t^2u, t^{-2}u$	576
7	$\pm(1 \pm \sqrt{-7})/2$	$t^{-1}u, u$	6
7	$\pm(3 \pm \sqrt{-7})/2$	$tu, t^{-2}u$	48
7	$\pm(1 \pm \sqrt{-7})$	$t^{-2}u^2, u^2$	288
11	$\pm(1 \pm \sqrt{-11})/2$	$t^{-1}u, u$	12
11	$\pm(3 \pm \sqrt{-11})/2$	$tu, t^{-2}u$	1440
19	$\pm(1 \pm \sqrt{-19}/2$	$t^{-1}u, u$	60

Proposition 3.2 *No allowable x determines a 1–cusped cover of Q_d.*

These propositions will be proved in the following subsections.

3.1 Proof of Proposition 3.1

3.1.1 We begin with some general discussion of the process to eliminate a given μ. So, let $\mu = u^m t^n \in \mathrm{PSL}(2, O_d)$ be a candidate as described prior to Proposition 3.1 (where the convention is that of Section 2.5). Since $\mathrm{PSL}(2, O_d)/\langle u^m t^n \rangle = {\pi_1}^{\mathrm{orb}}(Q_d(m,n))$, this group is finite if and only if $Q_d(m,n)$ is a spherical 3–orbifold, in which case the covering group has order equal to the degree of the cover $S^3 \to Q_d(m,n)$.

Our approach is to describe a sequence of operations on Q_d (that we describe below for $d \neq 1$ and in Section 3.1.7 for $d = 1$) so that an orbifold $X_d \to Q_d$ is produced for which there is an extension of this finite cover to $X'_d \to Q_d(m,n)$. That $Q_d(m,n)$ is spherical is equivalent to X'_d is spherical, and at this point we invoke the tables of [9] which give (amongst other things) a complete list of spherical 3–orbifolds, allowing us to decide if X'_d is spherical or not.

For $d \neq 1$, the covering is produced by performing some or all of the following three operations.

(1) Take an m–fold cyclic cover Q'_d of Q_d branched over the circle labelled ∞ in Figure 2. Now Q'_d is again homeomorphic to the complement of an unknotted circle in S^3 with meridian (resp. longitude) corresponding to u^m (resp. t). Hence this cover extends to a cover $Q'_d(1,n) \to Q_d(m,n)$.

(2) Modify Q'_d by θ^n, where θ is a left twist homeomorphism in the disk D bounded by the circle labelled ∞ in Figure 2 (see Figure 3).

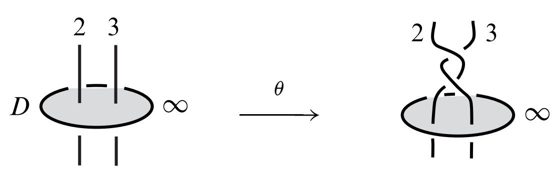


Figure 3

Note that θ sends the $(1,n)$–Dehn surgery curve to the $(1,n-1)$ curve and that θ^n corresponds to n left twists (resp. n right twists) if $n > 0$ (resp. $n < 0$). Now letting Q''_d denote $\theta^n(Q'_d)$ we obtain the cover $Q''_d(1,0) \to Q_d(m,n)$ where $Q''_d(1,0)$ has underlying space S^3.

Note that $Q_d(m,n)$ is spherical if and only if $Q''_d(1,0)$ is spherical.

(3) Take a k–fold cyclic cover (for $k = 2,3$), $Y \to Q''_d(1,0)$, branched over a circle labelled k in its singular locus.

Remarks **(1)** Since $Q_d(m,n) = Q_d(-m,-n)$ we can assume $m \geq 0$. In fact, we can assume $m > 0$ since $Q_d(0,n)$ is an orbifold with underlying space $S^2 \times S^1$ and so is not spherical.

(2) Complex conjugation induces an automorphism of PSL(2, O_d) which fixes t and sends u to u^{-1} (resp. $u^{-1}t$) if $d = 1, 2 \pmod 4$ (resp. $d = 3 \pmod 4$). Hence we can assume that $n \geq 0$ in the case of $d = 1, 2 \pmod 4$ and that $Q_d(-m,n) = Q_d(m, n-m)$ if $d = 3 \pmod 4$.

3.1.2 We list, in Figure 4 below, the ten spherical orbifolds (modulo diffeomorphism of S^3) from [9] that we obtained by applying the operations described in Section 3.1.1 to $Q_d(m,n)$. Here the order refers to the size of the fundamental group of the orbifold or, equivalently, to the degree of an S^3 cover.

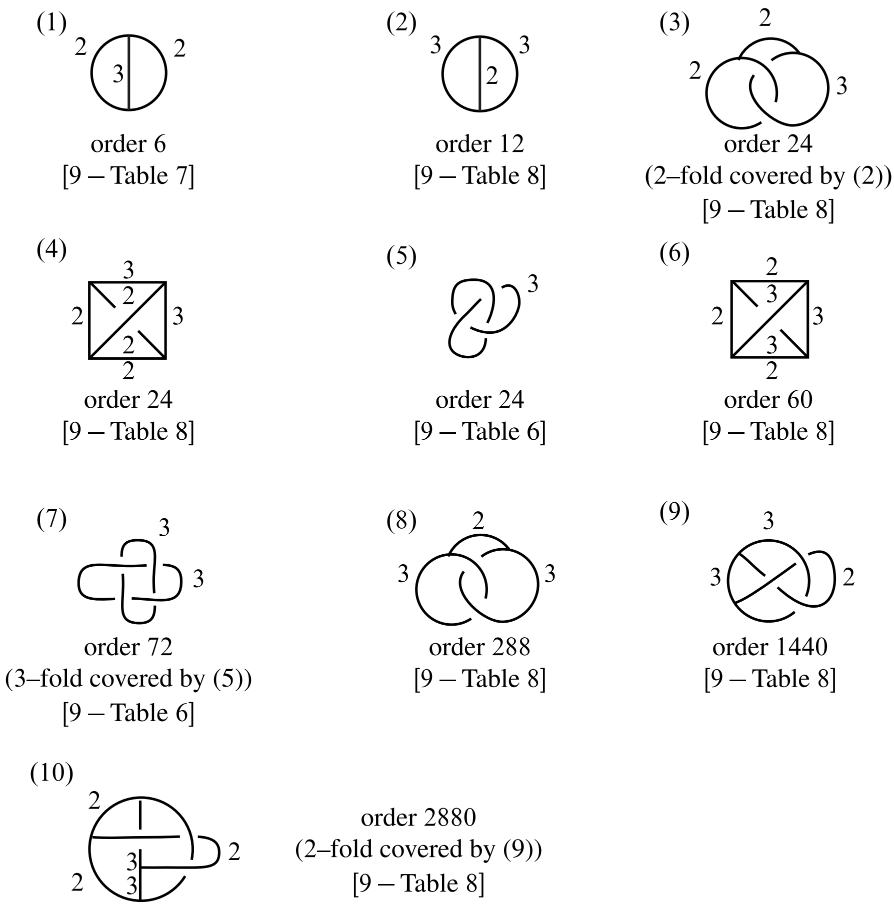


Figure 4

3.1.3 The case $d = 7$ First note that $Q_7(m,n)$ is non-spherical for $m \geq 3$, since the cover $Q_7'' \to Q_7$ constructed by taking an m–fold cyclic cover of Q_7 and applying θ^n extends to a cover $Q_7''(1,0) \to Q_7(m,n)$ where $Q_7''(1,0)$ is an orbifold with base S^3 and singular locus as shown in Figure 5(a).

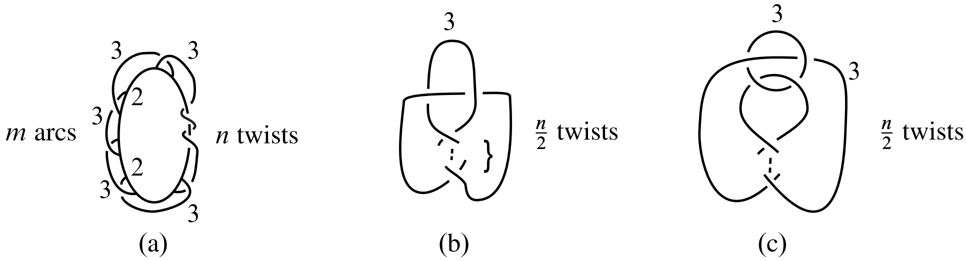


Figure 5

This singular locus consists of one circle of cone angle π together with m disjoint arcs of cone angle $2\pi/3$ attached to it. This does not appear in Dunbar's list of spherical orbifolds if $m \geq 3$.

Next note that $Q_7(1,n)$ is spherical if and only if $n = -2, -1, 0, 1$. To see this first apply θ^n to $Q_7(1,n)$ to obtain $Q_7'(1,0)$ and let $Y \to Q_7'(1,0)$ be the 2–fold cover branched over the circle of cone angle π. This orbifold has base S^3 with singular locus as shown in Figure 5(b). It is easily checked from Dunbar's lists that Y is spherical only for $n = -2, -1, 0, 1$. The values $n = -1, 0$ both yield the unknot, and so Y is the 2–fold cover of the orbifold 1 in Figure 4. When $n = -2, 1$, we get singular locus the trefoil, and Y is the orbifold 5 in Figure 4.

Finally $Q_7(2,n)$ is spherical if and only if $n = -2, 0$. Indeed, taking a 2–fold cover of Q_7 followed by an n–fold twist θ^n yields a 2–fold cover $Q_7''(1,0) \to Q_7(2,n)$. Now let Y be be the 2–fold cover of $Q_7''(1,0)$ branched over its circle of cone angle π. This has singular locus as pictured in Figure 5(c). This orbifold is spherical only for $n = -2, 0$ both corresponding to orbifold 7 in Figure 4.

3.1.4 The case $d = 11$ First note that $Q_{11}(m,n)$ is non-spherical if $m \geq 2$. This can be seen exactly as in Section 3.1.3. Briefly, let $Q_{11}''(1,0) \to Q_{11}(m,n)$ be the cover obtained by taking the m–fold cyclic cover of Q_{11} and applying the twist homeomorphism θ^n as before. Then $Q_{11}''(1,0)$ has underlying space S^3 with singular locus as shown in Figure 6. Now we have that the only spherical orbifolds with this singular locus are the orbifolds 2 and 9 of Figure 4. It is straightforward to check that orbifold 2 corresponds to $Q_{11}(1,0) = Q_{11}(1,-1)$ while orbifold 9 corresponds to $Q_{11}(1,1) = Q_{11}(1,-2)$.

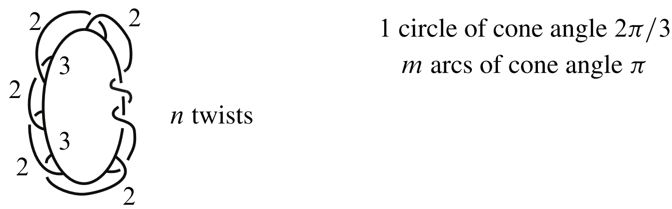


Figure 6

3.1.5 The case $d = 19$ The singular locus of Q_{19} (see Figure 2) is sufficiently complicated so as to have only two spherical surgeries, $Q_{19}(1,0)$ and $Q_{19}(1,-1)$, which both correspond to orbifold 6 in Figure 4.

3.1.6 The case $d = 2$ In this case we need only consider $m > 0$ and $n \geq 0$ (see Remarks 1, 2 in Section 3.1.1).

To begin with, note that $Q_2(m,n)$ is non-spherical if $m > 1$ since $Q_2''(1,0)$ is an orbifold with underlying space S^3 and singular locus as shown in Figure 7(a). However, spherical orbifolds with this configuration can have at most one such arc of cone angle π.

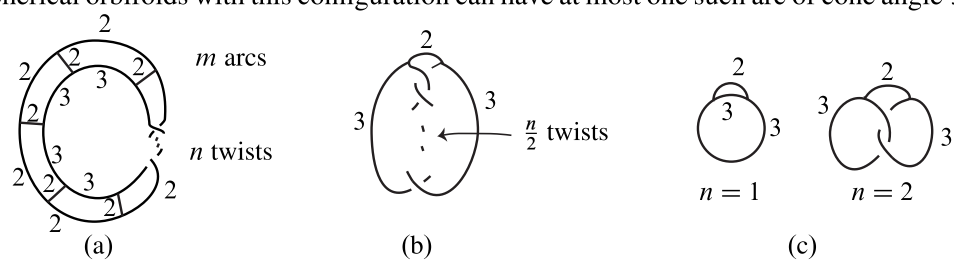


Figure 7

Next, note that $Q_2(1,n)$ is spherical if and only if $n = 1, 2$. Indeed, applying θ^n to $Q_2(1,n)$ yields $Q_2'(1,0)$. Now take Y to be the 2–fold cover of $Q_2'(1,0)$ branched over the circle of cone angle π pictured in Figure 7(b). From Dunbar's lists Y is a spherical orbifold only when $n = 1$ and $n = 2$ as pictured in Figure 7(c). These orbifolds are numbers 2 and 8 in Figure 4.

3.1.7 The case $d = 1$ For $d = 1$ we have an additional automorphism of Q_1 induced by the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & ib \\ -ic & d \end{pmatrix}.$$

This sends $u \to t^{-1}$ and $t \to u$. Thus we can assume that $\mu = u^m t^n$ with $m \geq n \geq 0$. Also recall from discussion at the start of Section 3 that $m^2 + n^2 \leq 36$ (and the case $m = 1$, $n = 0$ is also excluded since we need not consider units).

As explained in Section 2.4, (m,n)–Dehn surgery on the cusp of Q_1 means attaching a solid ball to the boundary sphere of Q_1 so as to induce (m,n)–Dehn surgery on the 2–fold torus cover. The cases of $Q_1(1,0)$ and $Q_1(0,1)$ are the S^3 orbifolds shown in Figure 8(a).

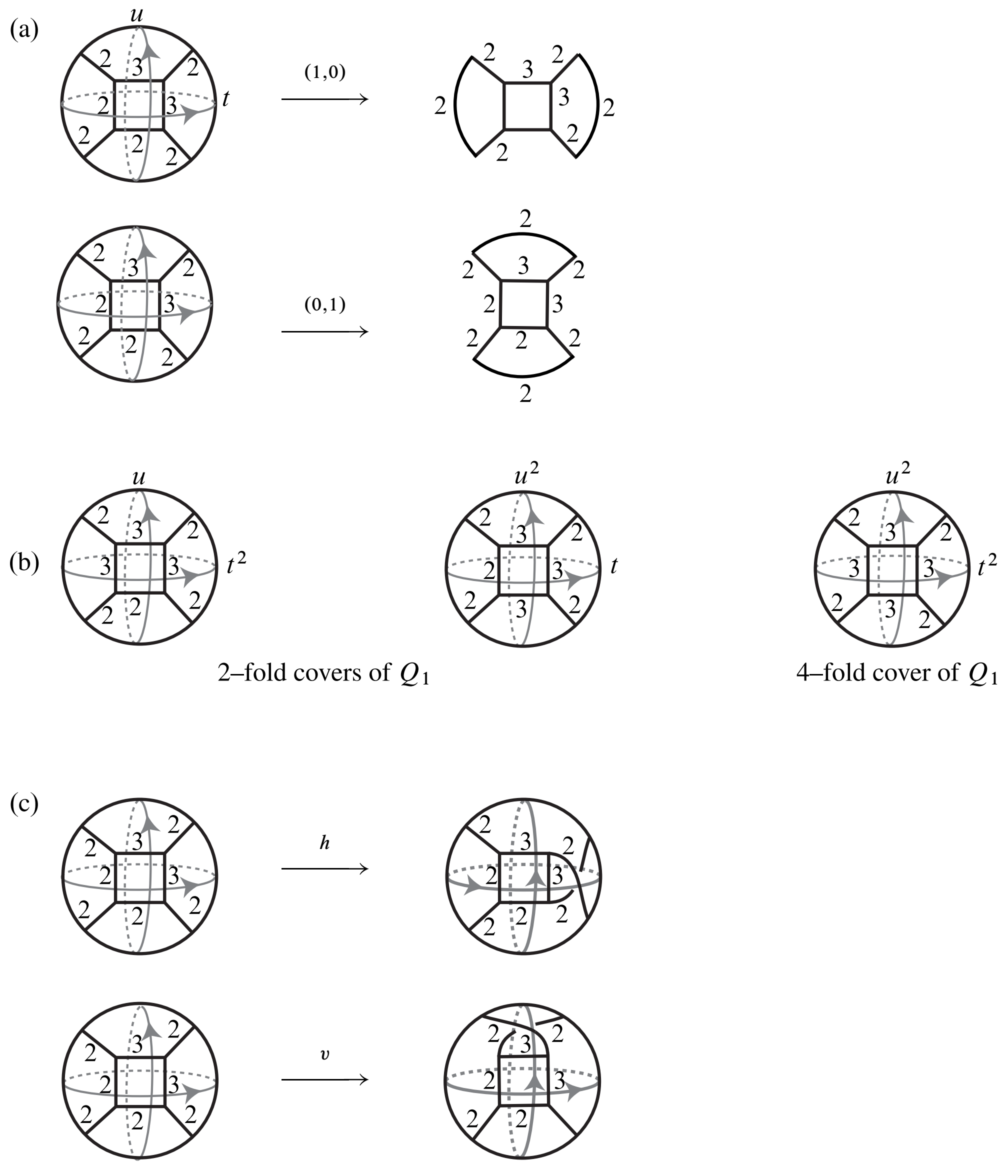


Figure 8

The cover $X_1' \to Q_1(m,n)$ described in Section 3.1.1 is produced using the following operations on Q_1.

(1) Taking 2– and 4–fold covers. Since $\mathrm{PSL}(2, O_1)$ has abelianization $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, then Q_1 has the pair of 2–fold covers and the 4–fold cover shown in Figure 8(b). These are obtained by branching over arcs of cone angle π.

(2) Performing a series of horizontal (denoted h) and vertical (denoted v) half twists in order to modify the filling parameters. These are pictured in Figure 8(c).

Now up to the equivalences on (m,n) mentioned above, $Q_1(m,n)$ is spherical if and only if $(m,n) = (1,1), (2,1), (3,1)$. This is proved in cases (i)–(vi) below.

(i) If $n = 0$, then $Q_1(m,0)$ has singular locus as in Figure 9.

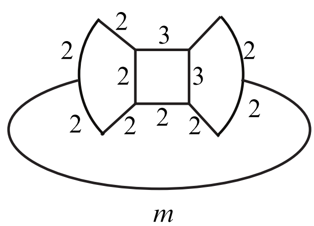


Figure 9

Hence $Q_1(m,0)$ is nongeometric if $m = 1$ and non-spherical if $m \geq 2$.

(ii) If $m = 2m'$ and $n = 2n'$ are even and nonzero, then $Q_1(m,n)$ is non-spherical, since it is 4–fold covered by $Q_1'(m',n')$ where Q_1' is the 4–fold cover of Q_1 pictured in Figure 8(b). The singular locus of $Q_1'(m',n')$ consists of a circle of cone angle $2\pi/3$, 2 arcs of cone angle π and, if $\gcd(m',n') > 1$, an arc of cone angle $2\pi/\gcd(m',n')$. Thus $Q_1'(m',n')$ cannot be a spherical orbifold on comparing with Dunbar's lists.

(iii) If $m = 2m'$ is even and n is odd, then $Q_1(m,n)$ is 2–fold covered by the orbifold $Q_1'(m',n)$ where Q_1' is the second 2–fold cover of Q_1 pictured in Figure 8(b). The singular locus of Q_1' contains three arcs of cone angle $2\pi/3$ in addition to five arcs of cone angle π. This configuration places strong constraints on the filling parameters (m',n). Namely, $Q_1'(m',n)$ is spherical only for $m' = n = 1$ in which case $Q_1'(1,1)$ is equivalent, after applying the half twist v, to $Q_1''(1,0)$ with singular locus pictured in Figure 10, which is orbifold 6 in Figure 4.
Note that this orbifold 2–fold covers $Q_1(2,1)$ which therefore has fundamental group of order 120.
Note that this argument applies verbatim to the case of m odd and n even.

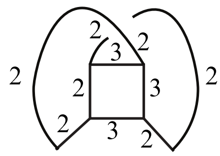


Figure 10

(iv) If $m = n$ is odd, then $Q_1(m,n)$ is equivalent to $Q'_1(m,0)$, where Q'_1 is obtained from Q_1 by performing a half twist v. This gives an orbifold with singular locus pictured in Figure 11.

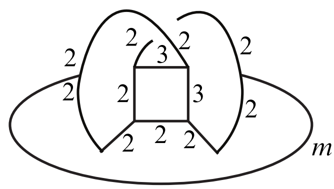


Figure 11

This is a non-spherical orbifold unless $m = 1$, in which case $Q_1(1,1) = Q'_1(1,0)$ has singular locus pictured in Figure 12, which is orbifold 4 in Figure 4.

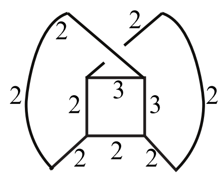


Figure 12

(v) If m, n are odd with $m > 1$ and $n = 1$, then $Q_1(m,1)$ is equivalent to $Q'_1(0,1)$, where Q'_1 is the orbifold obtained from Q_1 by applying by h^m. This is spherical if $m = 3$ and non-spherical if $m = 5$.
Indeed, $Q_1(3,1)$ is given in Figure 13; this is orbifold 10 in Figure 4.

(vi) The last case not covered is $Q_1(5,3)$ which is non-spherical since it is equivalent to $Q'_1(0,1)$, where $Q'_1 = h^2 \circ v \circ h(Q_1)$. The singular locus is shown in Figure 14, which is equivalent to the singular locus in Figure 15.

This completes the proof of the list of the allowable x. □

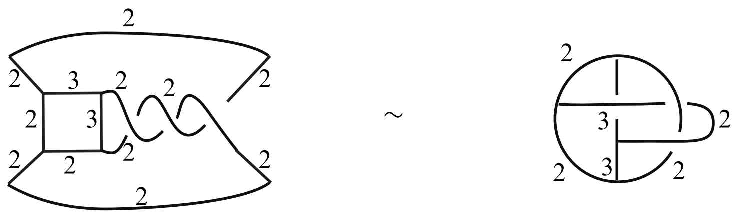


Figure 13

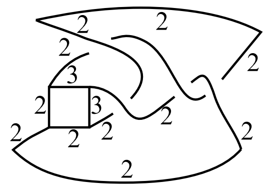


Figure 14

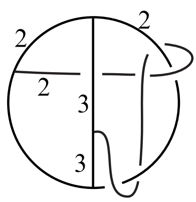


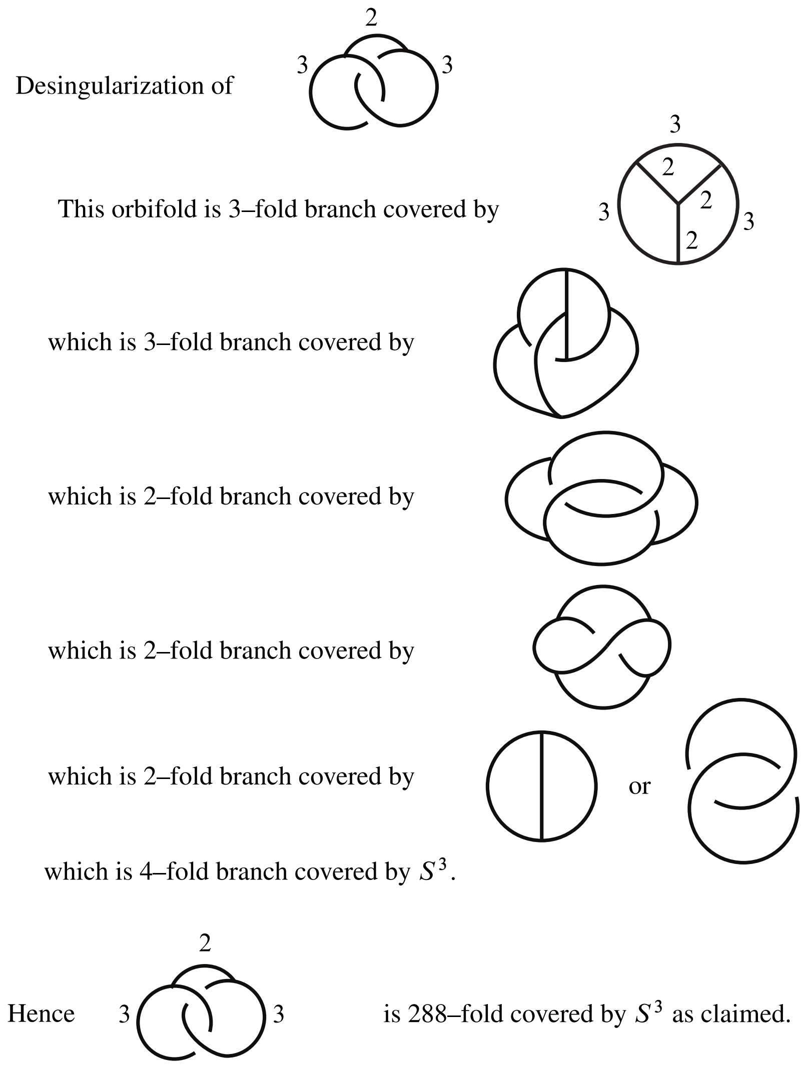
Figure 15

The orders of $\mathrm{PSL}(2, O_d)/\langle\mu\rangle$ given in Table 1 can be checked by Magma or found directly using the above operations to construct an S^3 cover of the orbifold $Q_d(\mu)$. We illustrate this construction for orbifold 8 in Figure 4, showing that it is indeed 288–fold covered by S^3 (see Figure 16).

3.2 Proof of Proposition 3.2

3.2.1 This is handled by a case-by-case analysis. Consider Table 1 and Figure 1. The order of $\mathrm{PSL}(2, O_d)/\langle\mu\rangle$ equals the degree of the cover $S^3 \setminus J \to Q_d$, which must equal the product, $\deg(f_1) \times \deg(f_3)$, of the degrees of the covering maps f_1 and f_3.

Now the index of a torsion-free subgroup in $\mathrm{PSL}(2, O_d)$ is a multiple of 12 when $d = 1, 2, 11, 19$ and a multiple of 6 when $d = 7$ (recall Section 2.5), hence the same holds for $\deg(f_1)$. On the other hand, $\deg(f_3)$ must be a multiple of the number of cusps of $S^3 \setminus J$ which is itself a multiple of the number of cusps of $\mathbb{H}^3/\,\Gamma(I)$ by Lemma 3.4 (proved below). The number of cusps of the relevant $\mathbb{H}^3/\,\Gamma(I)$ are given in Lemma 3.3 below.



NB Unlabelled components are of cone angle π.

Figure 16

For all cases in Table 1 except ($d = 2$; order 576) and ($d = 11$; order 1440), it is clear that $\deg(S^3 \setminus J \to Q_d)$ can not equal $\deg(f_1) \times \deg(f_3)$. Indeed, for ($d = 1$; order 120) we have $\deg(f_1) = 12k$ and $\deg(f_3) = 6l$ since $\mathbb{H}^3 / \Gamma(\langle 2+i \rangle)$ has 6 cusps. However $(12) \times (6) = 72$ doesn't divide 120. As for the case ($d = 1$; order 2880), we have $\deg(f_1) = 12k$ and $\deg(f_3) = 18l$ since $\mathbb{H}^3 / \Gamma(\langle 3+i \rangle)$ has 18 cusps. Here again 2880 is not a multiple of $(12) \times (18)$.

All of the remaining cases except ($d = 2$; order 576) and ($d = 11$; order 1440) are eliminated in this manner.

Lemma 3.3 *Let I denote one of the following ideals:*

(1) $\langle 1 \pm i \rangle$, $\langle 2 \pm i \rangle$, *or* $\langle 3 \pm i \rangle$ *when* $d = 1$;

(2) $\langle 1 \pm \sqrt{-2} \rangle$ *or* $\langle 2 \pm \sqrt{-2} \rangle$ *when* $d = 2$;

(3) $\langle (1 \pm \sqrt{-7})/2 \rangle$, $\langle (3 \pm \sqrt{-7})/2 \rangle$, *or* $\langle 1 \pm \sqrt{-7} \rangle$ *when* $d = 7$;

(4) $\langle (1 \pm \sqrt{-11})/2 \rangle$ *or* $\langle (3 \pm \sqrt{-11})/2 \rangle$ *when* $d = 11$;

(5) $\langle (1 \pm \sqrt{-19})/2 \rangle$ *when* $d = 19$.

Then $\mathbb{H}^3/\Gamma(I)$ *has correspondingly* 3, 6 *or* 18 *cusps when* $d = 1$; 4 *or* 12 *cusps when* $d = 2$; 3, 6 *or* 18 *cusps when* $d = 7$; 4 *or* 12 *cusps when* $d = 11$; *and* 12 *cusps when* $d = 19$.

Proof Denote by P_d the subgroup of $\mathrm{PSL}(2, O_d)$ fixing ∞. The number of cusps of $\mathbb{H}^3/\Gamma(I)$ is obtained by dividing the order of the quotient group $\mathrm{PSL}(2, O_d)/\Gamma(I)$ by the order of the image of P_d. As noted above, $P_d = \langle t, u \rangle$ in the cases when $d \neq 1$. A formula for the order of $\mathrm{PSL}(2, O_d)/\Gamma(I)$ can be found in Newman [16, Chapter 7]. In the case $d = 2$, the groups $\mathrm{PSL}(2, O_2)/\Gamma(I)$ are of order 12 and 72, while the corresponding images of P_2 are cyclic of order 3 and 6, giving 4 or 12 cusps in 2. The cases $d = 7, 11, 19$ are handled the same way.

In the case $d = 1$, we have $P_1 = \langle t, u, \ell \rangle$ where

$$\ell = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The image of P_1 has order 2, 10, and 20 in $\mathrm{PSL}(2, O_1)/\Gamma(I)$ which is of order 6, 60 and 360, thus giving 3, 6 or 18 cusps as stated. □

Lemma 3.4 *In the notation of Figure 1, the number of cusps of $S^3 \setminus J$ is an integer multiple of the number of cusps of $\mathbb{H}^3/\Gamma(I)$.*

Proof Let y be a nontorsion point on the cusp of Q_d. Since the cover $S^3 \setminus J \to Q_d$ is regular (Corollary 2.5) it follows that all cusps of $S^3 \setminus J$ contain an equal number of preimages of the point y. Now f_2 is also a regular cover, so that the cusps of $\mathbb{H}^3/\Gamma(I)$ all contain the same number of preimages of y. Thus the preimage of each cusp of $\mathbb{H}^3/\Gamma(I)$ with respect to f_4 contains the same number of preimages of y and hence the same number of cusps of $S^3 \setminus J$. □

3.2.2 The case $d = 11$; order 1440 Here it is not sufficient to know that $\deg(f_3)$ is a multiple of 12. A more detailed analysis, in Lemma 3.5 below, shows that $S^3 \setminus J$ has 48 cusps, hence $\deg(f_3) = 48l$. But, as before, $(12) \times (48)$ does not divide 1440 hence this case is eliminated.

Lemma 3.5 *Let $d \in \{2, 11\}$, and P_d be a peripheral subgroup of* $\mathrm{PSL}(2, O_d)$*. The order of the image of P_d in the finite groups of order 576 and 1440 of Proposition 3.1 is 24 and 30 respectively. Hence $S^3 \setminus J$ has 24 cusps in the first case and 48 cusps in the second.*

Proof We exploit the group theory package Magma [3] using the presentations of the groups $\mathrm{PSL}(2, O_d)$ which are given above in the cases of $d = 2, 11$. We indicate the Magma computations in the case of $d = 2$ and the case of the normal closure of the element t^2u.

$g\langle a, t, u\rangle := \mathrm{Group}\langle a, t, u \mid a^2, (t * a)^3, (a * u^{-1} * a * u)^2, (t, u), t^2 * u\rangle$;

$>$ print Order(g);

576

$> h :=$ sub$\langle g \mid t, u\rangle$;

$>$ print Order(h);

24.

The other case is handled in a similar way. □

3.2.3 The case $d = 2$; order 576 The above analysis shows that the only two possibilities in this case are $\deg(f_1) = 12$ and $\deg(f_3) = 48$ or $\deg(f_1) = 24$ and $\deg(f_3) = 24$. We can rule out the first possibility since, by [13], there are no 1–cusped manifolds that 12–fold cover Q_2. Thus we are left with $d = 2$ and $\pi_1(M) = 24$, which we consider below.

Using Boyer and Zhang [4], we can identify the only possibilities for $\pi_1(M)$ as the binary tetrahedral group T or the even D–type group $\langle x, y \mid x^2 = y^2 = (xy)^2\rangle \times \mathbb{Z}/3\mathbb{Z}$ denoted in what follows by D (note that the cyclic case is excluded by [2]). We will give the details in the case when $x = 2 + \sqrt{-2}$ (so $I = \langle 2 + \sqrt{-2}\rangle$ has norm 6) and the corresponding element of $\mathrm{PSL}(2, O_2)$ is t^2u; ie $\mathrm{PSL}(2, O_2)/\langle t^2u\rangle$ is a group of order 576. The other ideal of norm 6 in the table is handled similarly.

We shall analyze the possibilities forced on the group $\Gamma\Gamma(I)$. Since $\Gamma \cap \Gamma(I) > \Gamma_J$, elementary group theory shows that $[\mathrm{PSL}(2, O_2) : \Gamma\Gamma(I)] = 3, 6, 12, 24$.

The cases of index $[\mathrm{PSL}(2, O_2) : \Gamma\Gamma(I)] = 12$ or 24 can be easily dealt with. In the former case it follows that $\Gamma \cap \Gamma(I)$ has index 6 in Γ. Hence $\mathbb{H}^3/(\Gamma \cap \Gamma(I))$ has at most 6 cusps. However, by Lemma 3.3, $\mathbb{H}^3/\Gamma(I)$ has 12 cusps, a contradiction.

Now in the index 24 case, it follows that $\Gamma(I) \subset \Gamma$ is of index 3. However, $\mathbb{H}^3/\Gamma$ has only 1 cusp, and a 3–fold cover cannot therefore have 12 cusps, once again contradicting Lemma 3.3.

For the cases of index 3 and 6 we will use Magma once again. For the index 3 case, we can argue as follows. When the index is 3, it follows that $\Gamma\Gamma(I)/\Gamma(I) \cong \Gamma/\Gamma \cap \Gamma(I)$ has order 24, and so must be isomorphic to $\pi_1(M)$.

Now the quotient group $\mathrm{PSL}(2, O_2)/\Gamma(I)$ has order 72, and it is isomorphic to $S_3 \times A_4 \cong \mathrm{PSL}(2, \mathbb{F}_2) \times \mathrm{PSL}(2, \mathbb{F}_3)$. It can be easily checked (by Magma for example) that $S_3 \times A_4$ has two subgroups of index 3 and their abelianizations are $\mathbb{Z}/6\mathbb{Z}$ and $(\mathbb{Z}/2\mathbb{Z})^3$. From the previous paragraph, one of these groups must be isomorphic to T or D. However, the abelianizations of T and D are not consistent with those just stated. This contradiction completes the proof in this case.

For the case of index 6, a more detailed Magma analysis is required. We describe the calculations. Let H denote the cyclic subgroup of $\mathrm{PSL}(2, O_2)$ generated by t^2u. The first step in the calculation has Magma list index 6 subgroups containing the subgroup H. The Magma computations use g and h to denote $\mathrm{PSL}(2, O_2)$ and H. This is done by using

$$l := \mathrm{LowIndexSubgroups}(g, \langle 6, 6\rangle : \mathrm{Subgroup} := h).$$

This produces a list of 16 groups. Since these candidate subgroups are to contain Γ and since $H_1(M \setminus K; \mathbb{Q}) \cong \mathbb{Q}$, these candidate subgroups must have abelianization with free part of rank 1. A check of abelianizations using Magma, shows that only three of these groups satisfy this condition.

Of these, one contains elements of order 3 and these would persist in Γ (having index 4), a contradiction. For the other two, we can obtain presentations for the groups using the "Rewrite" package in Magma. We can now run the low index subgroup routine used previously (this time restricting to index 4 subgroups containing H). Checking abelianizations rules out the existence of any satisfying the condition on the first Betti number. □

4 Proof of Theorem 1.1(2)

Henceforth, let M be an integral homology 3–sphere and that we have a finite cover $M \setminus K = \mathbb{H}^3/\Gamma \to Q_d$. Then $d \in \{1, 2, 3, 7, 11, 19\}$ by Proposition 2.1, and so the result will follow once we eliminate the cases $d = 2, 7, 11, 19$.

We begin with some discussion of how the proof is structured. Since it is no longer true that M is spherical, bounds on the length of the meridian μ and Dunbar's lists used in Section 3 no longer apply. Instead, note that M an integral homology sphere implies that $H_1(M \setminus K; \mathbb{Z}) \cong \mathbb{Z}$, generated by a meridian μ for K. We deal with the cases $d = 2, 11, 19$ by using the infinite cyclic first homology of $M \setminus K$, coupled with the geometry of Q_d (in particular we exploit certain totally geodesic embedded surfaces) to show that $M \setminus K$ cannot cover Q_d. Finally we eliminate $d = 7$ by constructing an intermediate cover $M \setminus K \to X \to Q_d$ with the property that X has three cusps, an obvious contradiction.

4.1

Given a group G, we denote its abelianization by G^{ab}. The presentations for the Bianchi groups in Section 2.5 (where $d = 2, 7, 11, 19$) yield $\mathrm{PSL}(2, O_d)^{\mathrm{ab}} = \langle u, t \rangle$ where u and t are as defined in Section 2.5, with u having infinite order in the abelianization, and t having order $6, 2, 3, 1$, which correspond to the cases $d = 2, 7, 11, 19$, respectively.

Now let $\langle \mu, \lambda \rangle$ be a meridian–longitude pair for $M \setminus K$. In particular, λ is chosen to be null homologous, and so in terms of u and t it follows that $\lambda = t^r$, which forces $\mu = u^m t^n$, with $m \neq 0$. We will assume, as before, that $m > 0$.

Our arguments will make repeated use of the following simple lemma.

Lemma 4.1 *Let $M \setminus K = \mathbb{H}^3/\Gamma \to \mathbb{H}^3/G_1$, $G_1 < \mathrm{PSL}(2, O_d)$, be a finite cover and let $\mathbb{H}^3/G_2 \to \mathbb{H}^3/G_1$ be an s–fold cyclic cover, where $G_2 = \ker(j\colon G_1 \to \mathbb{Z}/s\mathbb{Z})$. Then*

(i) $M \setminus K = \mathbb{H}^3/\Gamma \to \mathbb{H}^3/G_2$, *if* $j(\mu) = \mathrm{id}$ *(ie if* $\mu \in G_2$*);*

(ii) *if all parabolic elements of G_1 are in the kernel of j, then $\mathbb{H}^3/G_2$ has s cusps.*

Remarks **(1)** Since $M \setminus K = \mathbb{H}^3/\Gamma \to \mathbb{H}^3/G_1$ by hypothesis, we can assume that $\mathbb{H}^3/G_1$ has only one cusp.

(2) Note that the hypothesis of (ii) implies (i) since we have that the parabolic $\mu \in \Gamma$ is also in G_1.

Proof of Lemma 4.1 The proof of (ii) is a standard covering space argument. As for (i), that $j(\mu) = \mathrm{id}$ implies $\Gamma < \ker(j) = G_2$ follows easily from the commutative diagram below and the hypothesis that $\Gamma^{\mathrm{ab}} = \mathbb{Z}$.

$$\begin{tikzcd}
\Gamma \arrow[r, "i"] \arrow[d] & G_1 \arrow[d] \arrow[dr, "j"] & \\
 & & \mathbb{Z}/s\mathbb{Z} \\
\Gamma^{ab} \arrow[r, "i_*"'] & G_1^{ab} \arrow[ur, "j_*"'] &
\end{tikzcd}$$

4.2 The cases of $d = 2, 11, 19$

For convenience, we recall from Section 3.2 that the index of a torsion-free subgroup of finite index in $\mathrm{PSL}(2, O_d)$ for $d = 2, 11, 19$ is a multiple of 12. Hence, in the present context, it follows that the cover $M \setminus K \to Q_d$ is a multiple of 12.

4.2.1 As described in Section 4.1, $\mu = u^m t^n$, $\lambda = t^r$ is a meridian–longitude pair for $M \setminus K$. Let Q'_d denote the m–fold cyclic cover of Q_d branched over the cusp circle labelled ∞ in Figure 2. Then, $Q'_d = \mathbb{H}^3/G$, where $G = \ker(j\colon \mathrm{PSL}(2, O_d) \to \mathbb{Z}/m\mathbb{Z})$ is defined by the composition

$$\mathrm{PSL}(2, O_d) \to \mathrm{PSL}(2, O_d)^{\mathrm{ab}} \to \mathbb{Z}/m\mathbb{Z} = \langle \epsilon \rangle,$$

where the last homomorphism (denoted j_*) is projection on the infinite cyclic factor and then reduction modulo m. In particular, $j_*(u) = \epsilon$ and $j_*(t) = \mathrm{id}$, so $j(\mu) = \mathrm{id}$. Thus, $M \setminus K \to Q'_d$ by Lemma 4.1.

Now Q'_d is again homeomorphic to the complement of an unknotted circle in S^3 with meridian (resp. longitude) corresponding to u^m (resp. t) as in Section 3.1.1. It is also easy to see using Figure 2 that the group G contains A_4, so that the cover $M \setminus K \to Q'_d$ must also be a multiple of 12, say $12k$.

Since $M \setminus K$ and Q'_d are both 1–cusped, the peripheral subgroup $\langle u^m t^n, t^r \rangle$ of Γ must therefore also be of index $12k$ in the peripheral subgroup $\langle u^m, t \rangle$ of G. Given that both these groups are free abelian, it follows that $r = 12k$ and that our meridian–longitude pair for $M \setminus K$ is $\mu = u^m t^n$, $\lambda = t^{12k}$.

4.2.2 Abusing notation slightly, we will let $D \subset Q'_d$ be the disk with two marked points bounded by the cusp circle (as in Figure 3), and let $S \subset M \setminus K$ be the embedded surface covering D. Now from Section 4.2.1, $S \to D$ is a $12k$–fold cover, and since D is orientable, S is also orientable. However, we claim that

(i) S has Euler characteristic $\chi(S) = -2k$;

(ii) S has one boundary component in the cusp torus of $M \setminus K$.

Given this, we deduce that S is nonorientable, and this is the desired contradiction.

Indeed, since D has a cone point of order 2 and a cone point of order 3, the Riemann–Hurwitz formula gives

$$\chi(S) = 12k - 6k - 4k(3-1) = -2k,$$

which proves (i). Finally, since the boundary of D in the cusp torus of Q'_d corresponds to the longitude t, it follows that the boundary of S in the cusp torus of $M \setminus K$ must correspond to the longitude $\lambda = t^{12k}$. But S is a $12k$–fold cover of D, so that S can only have one boundary component, which proves (ii).

4.3 The case of $d = 7$

The argument in Section 4.2 does not work for $d = 7$ because torsion-free subgroups of finite index in $\mathrm{PSL}(2, O_7)$ have index of the form $6k$ (since there is an S_3 subgroup but no A_4). Instead, we eliminate $d = 7$ by constructing a sequence of covers

$$M \setminus K \to X \to Y \to Q'_d \to Q_d,$$

for which X has three cusps.

4.3.1 Letting $Q'_7 \to Q_7$ be the m–fold cyclic cover branched over the circle labelled ∞ in Figure 2, we have $M \setminus K \to Q'_7 \to Q_7$ as in Section 4.2. The singular locus of Q'_7 consists of an unknotted circle of cone angle π with m arcs of cone angle $2\pi/3$ attached to it. The case of $m = 2$ is pictured in Figure 17.

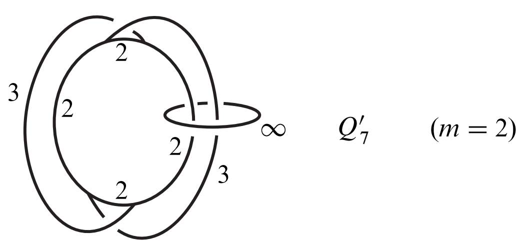


Figure 17

4.3.2 Denoting $Q_7' = \mathbb{H}^3/G$, then it can be checked that $G^{\mathrm{ab}} = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ generated by u^m and t. Now let $Y = \mathbb{H}^3/\ker(j) \to Q_7'$ where $j\colon G \to \mathbb{Z}/2\mathbb{Z}$ is the composition of the abelianization homomorphism $G \to G^{\mathrm{ab}}$ with $j_*\colon G^{\mathrm{ab}} \to \mathbb{Z}/2\mathbb{Z}$ where j_* is defined as follows.

Note G^{ab} has three homomorphisms to $\mathbb{Z}/2\mathbb{Z}$ determined by what happens to u^m and t. The choice of which one of these homomorphisms to take for j_* (and hence j) will be determined by $\mu = u^m t^n$. In all cases we take $j_*(t) = 1$ and then we make a choice as follows. Take the homomorphism that additionally sets $j_*(u^m) = 0$ (resp. 1) if n is even (resp. n is odd). Then by construction, the induced homomorphism j satisfies $j(\mu) = j(u^m t^n) = 0$.

Geometrically, $Y \to Q_7'$ is obtained by taking the 2–fold cover of Q_7' branched over the circle of cone angle π (when n is even). If n is odd, we modify Y as follows.

(i) Let D' (resp. D) be the 2–orbifolds with underlying space the disc which are bounded by the circle labelled ∞ (see Figures 17 and 18). Note that D' 2–fold covers D, branched over the cone point of order 2 (see Figure 18).

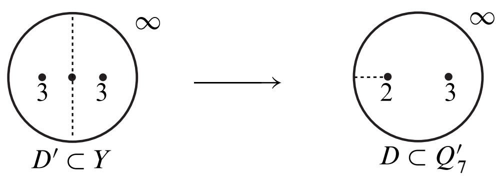


Figure 18

(ii) Now cut Y along D', twist by π, and reglue as in Figure 19.

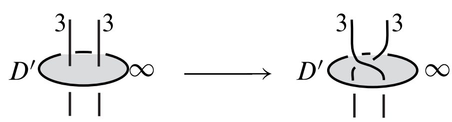
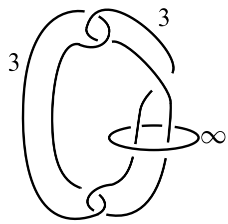


Figure 19

The resulting Y for $m = 2$ and n odd is pictured in Figure 20 (see also Figure 17). Note that the singular arcs labelled 3 in Q_7' lift to unknotted circles labelled 3 in Y, and that each of these circles has linking number zero with cusp circle of Y.

4.3.3 Let $\Sigma_3 \subset Y$ be a singular circle of cone angle $2\pi/3$, and $X \to Y$ be the 3–fold cyclic cover branched over Σ_3. Since Σ_3 has zero linking number with the cusp circle of Y, it follows from Lemma 4.1 that X has three cusps and is covered by $M \setminus K$, giving the desired contradiction. $\square$



$Y \qquad (m = 2, n \text{ odd})$

Figure 20

Remark The cases of $d = 2$ and 11 can also be handled by constructing a multi-cusped X such that $M \setminus K \to X$ as above.

References

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