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UV^k–mappings on homology manifolds

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We prove a strong controlled generalization of a theorem of Bestvina and Walsh, which states that a $(k+1)$–connected map from a topological n–manifold to a polyhedron, $2k+3 \leq n$, is homotopic to a UV^k–map, that is, a surjection whose point preimages are, in some sense, k–connected. One consequence of our main result is that a compact *ENR* homology n–manifold, $n \geq 5$, having the disjoint disks property satisfies the linear $UV^{\lfloor (n-3)/2 \rfloor}$–approximation property for maps to compact *ANR*s. The method of proof is general enough to show that *any* compact *ENR* satisfying the disjoint $(k+1)$–disks property has the linear UV^k–approximation property.

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1 Introduction

A deformation theorem of Bestvina and Walsh [2] states that, below middle and adjacent dimensions, a $(k+1)$–connected mapping of a compact topological manifold to compact polyhedron can be deformed to a UV^k–mapping; that is, a surjection whose fibers are in some sense k–connected. For example, if one has a map f from the n–sphere to the m–sphere, where $n \leq m$, one generally expects a typical point inverse image to be a finite set (usually empty, if $n < m$), but the truth, however, may be rather the opposite: if $n > 4$, then f is homotopic to a surjection with simply connected point inverses. This is predicted by the high connectivity of the homotopy fiber of the map. It is sometimes more useful to consider approximations by maps that behave like these "space-filling curves", which are closer models of the underlying abstract homotopy theory, rather than adopt the usual strategy of approximating by smooth or piecewise linear maps. Controlled versions of this phenomenon were essential in the construction of nonresolvable homology manifolds (Bryant, Ferry, Mio and Weinberger [5]; see also Pedersen, Quinn and Ranicki [24] and Ferry [13]) and in the "desingularization" of higher-dimensional homology manifolds (Bryant, Ferry, Mio and Weinberger [6]).

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In this paper we establish a strong controlled version of the Bestvina–Walsh Theorem for maps from an *ENR* homology n–manifold (with the disjoint disks property, or *DDP*, if $n \geq 5$) to an *ANR*. To our knowledge this is the first "controlled homotopy improvement" result for maps from spaces having no local linear structure. Among the main motivations for the paper are questions related to the cell-like approximation problem for homology manifolds explained below. The key observation is found in Theorem 6, a "UV–expansion theorem" for compact *ENR*s. One consequence of our main theorem is that a (controlled) homotopy equivalence $f\colon X \to Y$ between compact *ENR* homology n–manifolds is (controlled) homotopic to a $UV^{\lfloor (n-3)/2 \rfloor}$–map, provided X has the disjoint disks property (*DDP*) if $n \geq 5$. If the Quinn [26] resolution obstruction for X, $\iota(X) \in 1 + 8\mathbb{Z}$, is different from that of Y, then f is *not* homotopic to a $UV^{\lfloor (n-1)/2 \rfloor}$–map. This follows from arguments of Lacher (see remark below), which can be used to show that a $UV^{\lfloor (n-1)/2 \rfloor}$–homotopy equivalence between *ENR* homology n–manifolds, $n \geq 5$, is cell-like. If, however, f is a sufficiently fine homotopy equivalence, depending on the metric on Y, then the results of Quinn [26] show that $\iota(X) = \iota(Y)$. Thus it is natural to suggest the following:

CE-approximation conjecture *If Y is a compact ENR homology n–manifold and $\epsilon > 0$ then there exists $\delta > 0$ such that if X is a compact ENR homology n–manifold, $n \geq 5$, with the DDP and $f\colon X \to Y$ is a δ–homotopy equivalence, then f is ϵ–homotopic to a cell-like map.*

The CE-approximation conjecture can be used to establish a version of the Chapman–Ferry [9] α–approximation theorem for homology manifolds.

α–approximation conjecture for homology manifolds *Given a compact ENR homology n–manifold Y, $n \geq 5$, with the DDP and $\epsilon > 0$ there is a $\delta > 0$ such that if X is a compact ENR homology n–manifold (with the DDP if $n \geq 5$) and $f\colon X \to Y$ is a δ–equivalence, the f is ϵ–homotopic to a homeomorphism.*

It is not difficult to show that the α–approximation conjecture implies that an *ENR* homology n–manifold, $n \geq 5$, with the disjoint disks property is topologically homogeneous, as conjectured in Bryant et al [5].

The methods we develop here are new, even in the case of *PL* or smooth manifolds, and provide an alternative proof of the Bestvina–Walsh Theorem referred to above. Other results of this type are due to Keldyš [16], Anderson [1], Frum-Ketkov [14], Wilson [31; 32], Walsh [30], Černavskii [10] and Ferry [12]. In fact, they apply to *any* *ENR* having sufficient general position properties, and the essential propositions and lemmas will be presented in this setting. These methods allow us to take a map that, in

Quinn's terminology [25], is $(\epsilon, k+1)$–connected, which we call a $UV^k(\epsilon)$–map, and "squeeze" it in a controlled fashion to be $(\mu, k+1)$–connected, for arbitrarily small μ. The desired UV^k–map is obtained by taking a limit. The controls on the homotopies have sufficient uniformity to show that a compact *ENR* with the disjoint $(k+1)$–disks property (DDP^{k+1}) has the linear UV^k–approximation property introduced in Bryant et al [6]. In the same work, Bryant et al constructed "resolutions", which have the linear UV^1–approximation property, and suggested that this latter property is stronger than the *DDP*. But a consequence of our main result is that *every ENR* homology n–manifold, $n \geq 5$, with the *DDP* has the linear $UV^{\lfloor (n-3)/2 \rfloor}$–approximation property. This is a considerable strengthening of the disjoint disks property and indicates yet another way in which the exotic homology manifolds constructed in Bryant et al [5] resemble topological manifolds. Our techniques are strong enough to yield a relative theorem, which asserts that the homotopies of a given map to a UV^k–map may be kept fixed on a sufficiently nicely embedded compact set. As a result we obtain a strong relative theorem for maps from a homology manifold with boundary.

Here is our main result. (LCC^k subsets are defined in the next section. Informally, they are subsets that can be avoided by maps of a $(k+1)$–dimensional polyhedron into the ambient space.)

Theorem 1 *Suppose X is a compact, connected ENR satisfying the disjoint $(k+1)$–disks property, B is a compact ANR, Y is a metric space and $p\colon B \to Y$ is a map. If $f\colon X \to B$ is $UV^k(\epsilon)$ over Y, then f is $(C(k)\cdot\epsilon)$–homotopic (over Y) to a UV^k–map, where $C(k)$ is a positive constant depending only on k.*

Moreover, if Z is a compact, LCC^k subset of X, then the homotopy of f to a UV^k–map can be chosen to be fixed on Z.

As Theorem 1 essentially defines the relative linear UV^k–approximation property, we get as a corollary one of the results that motivated this paper.

Theorem 2 *Suppose X is a compact ENR homology n–manifold, $n \geq 3$, with boundary ∂X. If $n \geq 5$ assume that X has the DDP and that ∂X is LCC^1 in X. Then $(X, \partial X)$ has the relative linear $UV^{\lfloor (n-3)/2 \rfloor}$–approximation property.*

Proof It is well-known that a connected *ANR* of dimension ≥ 1 is arc-wise connected and locally arc-wise connected. In particular, any continuous map of [0, 1] into X can be approximated by one whose image has dimension ≤ 1. If $n = 3$ or 4, the DDP^1 property of X follows from this fact together with Alexander duality: if U is any connected open subset of X and A is a closed, 1–dimensional subset of U, then

$H_1(U, U-A) \cong \check{H}^{n-1}(A) = 0$. (Integer coefficients are understood throughout this paper.) This, in turn, implies that the reduced homology group $\widetilde{H}_0(U-A) = 0$. The LCC^0 property of ∂X in X follows immediately from the homology conditions given in the definition below.

If $n \geq 5$, then the results of Walsh [29] and Bryant [4] show that a homology n–manifold with the disjoint disks property also has the disjoint $\lfloor (n-1)/2 \rfloor$–disks property. (See the discussion in the next section.) If U is an open subset of X, then, by definition (below), the inclusion $U - \partial X \subseteq U$ induces an isomorphism on homology. If $X - \partial X$ is locally simply connected at points of ∂X, then, by the eventual Hurewicz Theorem [11], $U - \partial X \subseteq U$ also induces an isomorphism on homotopy groups, hence, is a homotopy equivalence. (A subset of a space X with this property is called a *Z–set*.) □

A similar argument establishes a hybrid version.

Theorem 3 *Suppose X is a compact ENR homology n–manifold, $n \geq 3$, possibly with boundary, ∂X and Z is a compact, LCC^0 subset of X containing ∂X. If $n \geq 5$ assume further that X has the DDP and that Z is $LCC^{\lfloor (n-3)/2 \rfloor}$ in X. Then (X, Z) has the relative linear $UV^{\lfloor (n-3)/2 \rfloor}$–approximation property.*

As a special case (Y = a point) we recover the analogue of the theorem of Bestvina and Walsh for "nice" homology manifolds.

Theorem 4 *Suppose X is a compact, connected, ENR homology n–manifold, with boundary ∂X, and suppose B is a compacted, connected ANR. Suppose $f\colon X \to B$ is a $(k+1)$–connected map for some $k \geq 0$, $2k+3 \leq n$. If $k \geq 1$, we assume further X has the DDP and ∂X is LCC^1 in X. Then f is homotopic, rel $f|\partial X$, to a UV^k–map.*

Remark Lacher [20, Sections 5 and 7] (see also Frum-Ketkov [14]), has shown that a $UV^{\lfloor (n-1)/2 \rfloor}$–map between compact n–manifolds must be cell-like if n is odd and, if n is even, it must be a *spine map*, in which spines of connected summands are collapsed to points. Thus, the result in Theorem 1 is best possible for maps from the n–sphere S^n to itself of degree $d \neq \pm 1$.

As a separate application we invoke a theorem of Krupski [17], who has shown that a homogeneous *ENR* of dimension ≥ 3 has the DDP^1. Recall that a space X is *homogeneous* if, given points $x, y \in X$, there is a homeomorphism of X onto itself taking x to y. Combining Krupski's result with Theorem 1 we obtain:

Theorem 5 *If X is a compact, connected, homogeneous ENR of dimension ≥ 3, then X has the linear UV^0–approximation property. In particular any map of X to a compact, simply-connected ANR is homotopic to a monotone map, that is, a surjection with connected point-inverses.*

2 Definitions and preliminary results

A *homology n–manifold* is a space X having the property that for each $x \in X$,

$$H_k(X, X - x; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = n, \\ 0 & k \neq n. \end{cases}$$

$H_*(*, *; \mathbb{Z})$ is understood to mean singular homology with integer coefficients. We say that X is an *homology n–manifold with boundary* if the condition $H_n(X, X-x; \mathbb{Z}) \cong \mathbb{Z}$ is replaced by $H_n(X, X-x; \mathbb{Z}) \cong \mathbb{Z}$ or 0 and, if $\partial X = \{x \in X : H_n(X, X-x; \mathbb{Z}) \cong 0\}$, then ∂X is a homology $(n-1)$–manifold and $H_*(U, U - \partial X; \mathbb{Z}) = 0$ for every open subset U of X. (Mitchell [23] shows that, if X is an *ENR*, ∂X is a homology $(n-1)$–manifold.)

A *Euclidean neighborhood retract (ENR)* is a space homeomorphic to a closed subset of Euclidean space that is a retract of some neighborhood of itself, that is, a locally compact, finite-dimensional *ANR*. Topological manifolds and locally compact, finite-dimensional polyhedra are the most well-known examples of *ENR*s, but there are many other interesting types of examples, such as the exotic homology manifolds constructed in Bryant et al [5]. Perhaps the most important property of a topological manifold or locally compact polyhedra that generalizes to an arbitrary *ENR* X is the existence of *mapping cylinder neighborhoods*, which we have already mentioned above: If X is LCC^1 embedded in a topological manifold M, $\dim M - \dim X \geq 3$, then X has a topological manifold neighborhood W with boundary in M, which admits a retraction $p\colon W \to X$, such that W is the mapping cylinder of $p|\partial W$ and p is the mapping cylinder retraction (Miller [22] and Quinn [25]). This generalizes the notion of normal bundle neighborhoods for topological manifolds and regular neighborhoods for polyhedra. In fact, there are stable classification theorems for mapping cylinder neighborhoods of *ENR* homology manifolds analogous to those for normal bundle neighborhoods of topological manifolds (Bryant and Mio [7]).

A space X satisfies the *disjoint disks property* (*DDP*) if for every $\epsilon > 0$ and maps $f, g\colon D^2 \to X$, there are maps $f', g'\colon D^2 \to X$ so that $d(f, f') < \epsilon$, $d(g, g') < \epsilon$ and $f'(D^2) \cap g'(D^2) = \varnothing$. More generally, we say that a space X has the *disjoint k–disks property*, or DDP^k, if any two maps of a k–cell into X can be approximated by maps with disjoint images. The DDP^k implies that maps $f\colon D^i \to X$ and $g\colon D^j \to X$ can be approximated by maps with disjoint images whenever $i, j \leq k$.

Given $\epsilon > 0$ and a map $p\colon B \to C$, a map $f\colon A \to B$ is $UV^k(\epsilon)$ *over* C, if it has the ϵ–homotopy lifting property over C for $(k+1)$–dimensional polyhedra. That is, if (P, Q) is a polyhedral pair with $\dim P \leq k+1$, $\alpha_0\colon Q \to A$ and $\alpha\colon P \to B$, with $f \circ \alpha_0 = \alpha|Q$, then there is a map $\overline{\alpha}\colon P \to A$ extending α_0 such that $f \circ \overline{\alpha}$ is

ϵ–homotopic over C to α in B, rel $\alpha|Q$. The lift $\overline{\alpha}$ of α will be called an *ϵ–lift of α, rel α_0* (or, sometimes, rel Q), over C, identical to the notion of Quinn [25, Definition 5.1] of a relatively $(\epsilon, k+1)$–connected map over C.

There are two important special cases of this definition representing the two extremes on the degree of control. If p is a constant map or, equivalently, C is a point, then we have the usual notion of a k–connected map $f\colon A \to B$. This is equivalent to f inducing isomorphisms on homotopy groups through dimension k and an epimorphism in dimension $k+1$. At the other extreme we have $C = B$ and $p = \mathrm{id}_B$. In this case we will often omit reference to B as the control space and just say $f\colon A \to B$ is $UV^k(\epsilon)$.

A compact metric space C *has property* UV^k, $k \geq 0$, if for some (and, hence, any) embedding of C in an *ANR* X and every neighborhood U of C in X, there is a neighborhood V of C lying in U such that the inclusion induced map $\pi_i(V) \to \pi_i(U)$ is 0 for $0 \leq i \leq k$. In the language of shape theory, the shape homotopy groups of C vanish through dimension k (see eg Mardešić [21]). A surjection $f\colon A \to B$ between compact *ANR*s is UV^k, $k \geq 0$, if its point inverses have property UV^k. A UV^{-1}–map is a surjection.

Remark For *ANR*s, property UV^k is equivalent to k–connectedness. For non-*ANR*s, especially nonlocally connected spaces, the situation is quite different. For example, the fundamental group of the dyadic solenoid, $\Sigma = \operatorname{proj\,lim}\{S^1, z \to z^2\}$, is trivial, but it fails to have property UV^1. By contrast, the "topologists' sine curve" in the plane and the Whitehead continuum, $Wh \subseteq S^3$, are not locally connected, but satisfy property UV^k for all $k \geq 0$.

The following basic result is due to Lacher [18; 20].

Proposition 1 *A surjection $f\colon A \to B$ between compact ANRs is UV^k if and only if it is $UV^k(\epsilon)$ for every $\epsilon > 0$.*

A compact metric pair (X, Z) has the *relative linear UV^k–approximation property* if, for a given compact (metric) *ANR* B and map $p\colon B \to Y$ of B to a metric space Y, every map $f\colon X \to B$ that is $UV^k(\epsilon)$ over Y, for some $\epsilon > 0$, is $(C \cdot \epsilon)$–homotopic over Y, keeping $f|Z$ fixed, to a UV^k–map, where C is a constant depending only on k.

As the following proposition indicates, it is sufficient to prove Theorem 1, as well as the propositions and lemmas leading up to its proof, for mappings to compact polyhedra.

Proposition 2 *The (relative) UV^k–approximation property is equivalent to the (relative) UV^k–approximation property for maps to compact polyhedra.*

Proof Suppose a compact metric pair (X, Z) has the relative linear UV^k–approximation property for maps to compact polyhedra. Suppose B is a compact *ANR*, $p\colon B \to Y$ is a map to a metric space, Y, and suppose a map $f\colon X \to B$ is $UV^k(\epsilon)$ over Y, for some $\epsilon > 0$.

If B is finite-dimensional, then B has a mapping cylinder neighborhood N in $\mathbb{R}^n$ with mapping cylinder projection $\varphi\colon N \to B$. The composition $f' = \iota \circ f\colon X \to N$ is $UV^k(\epsilon)$ over Y with respect to the control map $p' = p \circ \varphi\colon N \to B$. Since N is triangulable as a finite polyhedron, f' is $(C \cdot \epsilon)$–homotopic over Y, keeping $f|Z$ fixed, to a UV^k–map, $g'\colon X \to N$, where C is a constant depending only on k. Since φ is cell-like, $g = \varphi \circ g'\colon X \to B$ is UV^k and g is $(C \cdot \epsilon)$–homotopic to f over Y, keeping $f|Z$ fixed.

If B is infinite-dimensional, we invoke a famous result of Chapman [8] which states that, if I^∞ is the Hilbert cube, then $B \times I^\infty \cong K \times I^\infty$ for some finite polyhedron K. The results of Chernavskiĭ [10] and Ferry [12] show that, for $n \geq 2k+3$ and $\delta_n > 0$, there is a UV^k map $g_n\colon I^n \to I^{n+1}$ such that $\pi g_n\colon I^n \to I^n$ is δ_n–homotopic to the identity, where $\pi\colon I^{n+1} \to I^n$ is the natural projection. For fixed $j \geq 0$ and $n \geq 2k+3$ let $g_{n,j} = g_{n+j} \circ \dots \circ g_n\colon I^n \to I^{n+j+1}$ and let $G_{n,j}\colon I^\infty \to I^\infty$ be defined by

$$G_{n,j}(t_1, t_2, \dots) = (g_{n,j}(t_1, \dots t_n), t_{n+j+2}, \dots).$$

Given $\delta > 0$, we can choose the sequence δ_{n+j} so that the sequence $\{G_{n,j}\}$ converges to a map

$$G_n\colon I^n \to I^\infty$$

such that, if $\pi_n\colon I^\infty \to I^n$ is the natural projection, $d(\pi_n G_n, \mathrm{id}_{I^n}) < \delta$. By a theorem of Lacher [20, Page 505], G_n is UV^k, since each $G_{n,j}$ is UV^k, being the composition of a UV^k–map and the cell-like map $K \times I^\infty \to K \times I^\infty$ that omits the $(n+j+1)$–coordinate of I^∞.

Assume the product metric on $B \times I^\infty$, and provide $K \times I^\infty$ with the metric induced by a homeomorphism

$$B \times I^\infty \xrightarrow{\cong} K \times I^\infty.$$

For any given $\mu > 0$ we may choose $n \geq 2k+3$ and $G_n\colon I^n \to I^\infty$ such that $\mathrm{id}_K \times G_n\colon K \times I^n \to K \times I^\infty$ is a UV^k–map and $\mathrm{id}_K \times (\pi_n \circ G_n)\colon K \times I^n \to K \times I^n$ is μ–homotopic to the identity over $B \times I^\infty$ (with respect to inclusion).

Let $\overline{p}\colon B \times I^\infty \to Y$ be the composition of the projection $\pi_B\colon B \times I^\infty \to B$ with p. Given $\delta > 0$ we can choose $n \geq 2k+3$ and $G_n\colon I^n \to I^\infty$ as above so that $\mathrm{id}_K \times \pi_n\colon K \times I^\infty \to K \times I^n$ and $\mathrm{id}_K \times (\pi_n \circ G_n)\colon K \times I^n \to K \times I^n$ are δ–equivalences over Y. If $f\colon X \to B$ is $UV^k(\epsilon)$ over Y, as above, then, for appropriate δ and n, the map

$$\bar{f} = (\mathrm{id}_K \times \pi_n) \circ f\colon X \to K \times I^n$$

is $UV^k(2\epsilon)$ over Y. By hypothesis, $\bar{f}$ is $(2C\epsilon)$–homotopic, rel Z, over Y to a UV^k–map

$$\bar{g}\colon X \to K \times I^n.$$

The map

$$g = \pi_B \circ (\mathrm{id}_K \times G_n) \circ \bar{g}\colon X \to B$$

is UV^k, since $\bar{g}$ and $(\mathrm{id}_K \times G_n)$ are UV^k and π_B is cell-like. For suitably chosen n and δ we can arrange it so that g is $(3C\epsilon)$–homotopic to f, and this will complete the nonrelative version.

To get a relative version we observe that, for any $\mu > 0$, we can arrange it so that the map g constructed above will have the property that $g|Z$ is μ–homotopic *over* B to $f|Z$. Applications of the estimated homotopy extension will produce a map $g_\mu\colon X \to B$ such that g_μ is μ–homotopic to g over B and f is $(4C\epsilon)$–homotopic to g_μ, rel Z, over Y. By Lemma 2, below, g_μ is $UV^k(2\mu)$ over B. Applying this argument inductively, with control lifted to B, we can construct a sequence of $UV^k(\delta_n)$–maps $g_n\colon X \to B$ *over* B that converge to a map $g\colon X \to B$, which is $(5C\epsilon)$–homotopic to f over Y. As a consequence of Proposition 1, g is UV^k. □

A subset A of an *ANR* X *is locally k–coconnected, or LCC^k, in* X if, for every connected open set $U \subseteq X$, $\pi_i(U, U - A) = 0$ for $0 \le i \le k+1$. This is equivalent to the condition that the inclusion map $\iota\colon (X - A) \to X$ is $UV^k(\epsilon)$ for every $\epsilon > 0$. If X is a topological n–manifold and A is a closed subset of dimension r, $n - r \ge 3$, then A is LCC^{n-r-2} if and only if A is LCC^1. This is essentially a consequence of Alexander duality and the Hurewicz Isomorphism Theorem (see Bryant [3] and Štan'ko [28]). This remains true if X is an *ENR* homology n–manifold, $n \ge 5$, with the *DDP* (Bryant [4]; Walsh [29]).

Proposition 3 *If an ENR X has the DDP^k and A is an LCC^{k-1} subset of X, then any map f of a k–dimensional polyhedron K into X can be approximated by an LCC^{k-1} embedding that misses A. Moreover, if f is already an LCC^{k-1} embedding on a subpolyhedron L (into the complement of A), then the approximation can be made to agree with f on L.*

Outline of proof This proposition is proved using techniques similar to those used to prove the main results of Bryant [4] and Walsh [30]. Since there are some differences, we outline a proof here.

Suppose K is a k–dimensional polyhedron and $f\colon K \to X$ is a map. Let $K_1, K_2, \dots$ be a sequence of triangulations of K with mesh tending to 0. Use the DDP^k property of X to get a sequence f_j, $j = 1, 2, \dots$ of maps, where f_j is an approximation of f_{j-1}, $j \geq 1$, ($f_0 = f$), such that $f_j(\sigma) \cap f_j(\tau) = \varnothing$ whenever σ and τ are disjoint k–simplexes of K_j. By taking extra care in choosing the sizes of subsequent approximations, we can guarantee that the limit map $\overline{f}\colon K \to X$ satisfies this property for every j and, hence, is an embedding. Likewise, we can assume that the first and all subsequent approximations are chosen so that their images, as well as the image of $\overline{f}$, misses A. Arguments such as these may be found in Hurewicz and Wallman [15].

In order to get an LCC^{k-1} embedding we need an extra ingredient. Let N be a mapping cylinder neighborhood of X in some Euclidean space of dimension $\geq 2k+1$ with mapping cylinder projection $p\colon N \to X$. Let $T_1 \subseteq T_2 \subseteq \cdots$ be the k–skeletons of a sequence of triangulations of N with mesh tending to 0. Given a map $f\colon K \to X$ as above, we combine the process above with a sequence $p_j\colon N \to X$, where p_j is an approximation of p_{j-1}, $j \geq 1$, ($p_0 = p$) so that $p_j(T_j) \cap f_j(K) = \varnothing$ and the limit maps $\overline{p} = \lim p_j$ and $\overline{f} = \lim f_j$ satisfy $\overline{p}(\bigcup T_j) \cap \overline{f}(K) = \varnothing$. If $\alpha\colon (P, Q) \to (X, X - \overline{f}(K))$ is a map of a k–dimensional polyhedral pair, then there is a small homotopy of α to a map of P into T_j for some j. We can choose j large enough and the homotopy small enough so that the image of the composition of the homotopy restricted to Q with p does not meet $\overline{f}(K)$. After composing this map with p and using the estimated homotopy extension theorem (Bryant et al [5]), we can get a small homotopy of α, rel $\alpha|Q$, to a map into $X - \overline{f}(K)$.

This argument can easily be adapted to get the relative result. □

The following property of UV^k–maps between compact *ANR*s is similar to results that may be found in Lacher [18; 19].

Proposition 4 *If a map $f\colon X \to Y$ between compact ANRs is UV^k, then, for each point y and each pair of open sets U and V in Y such that $y \in \overline{V} \subseteq U$ and V is contractible to a point in U, the induced homomorphisms $\pi_i(f^{-1}(V)) \to \pi_i(f^{-1}(U))$ are zero for $0 \leq i \leq k$.*

We can use Proposition 4 inductively to prove:

Proposition 5 *Suppose $f\colon X \to Y$ is a UV^k–map between compact ANRs and $y \in Y$. Then for any neighborhood U of y, there is a connected neighborhood V of y such that if $g\colon (P, p_0) \to (f^{-1}(V), g(p_0))$ is a map of a polyhedron P of dimension $\leq k$ into $f^{-1}(V)$, then g is homotopic, rel p_0, in $f^{-1}(U)$ to a constant map.*

The following proposition illustrates a "Seifert–van Kampen"-type property of UV^k–maps.

Proposition 6 *Suppose $X = A \cup B$, $A \cap B = C$, where A, B and C are compact ANRs, and suppose a mapping $f\colon X \to Y$ of X to a compact ANR Y has the property that $f|A$ and $f|B$ are UV^k and $f|C$ is UV^{k-1}, $k \geq 0$. Then f is UV^k.*

Proof Suppose $X = A \cup B$ and $f\colon X \to Y$ are given as above. For any $Z \subseteq Y$, set $Z^* = f^{-1}(Z)$. If $k = 0$, then, for each $y \in Y$, $y^* = (y^* \cap A) \cup (y^* \cap B)$ and $(y^* \cap A) \cap (y^* \cap B) = y^* \cap C \neq \varnothing$. Thus, y^* is connected, and f is UV^0 on X.

Assume that $k \geq 1$. Suppose $y \in Y$ and $W \subseteq V \subseteq U$ are neighborhoods of y in Y such that $W \subseteq V$ satisfy the conclusion of Proposition 5 for $f|C$ (for UV^{k-1}–maps) and $V \subseteq U$ satisfy the conclusion of Proposition 5 for $f|A$ and $f|B$ (for UV^k–maps). Suppose $g\colon S^i \to W^*$ is a mapping of the i–sphere, $1 \leq i \leq k$. Assume that $g^{-1}(A - B)$ and $g^{-1}(B - A)$ are nonempty. (Otherwise, g is homotopic to a constant map in either A or B by Proposition 5.) Let Q be a polyhedral neighborhood of $g^{-1}(C)$ in S^i, let $P = \mathcal{C}\ell 1(S^i - Q)$ and let P^A and P^B be the union of the closures of the components of $S^i - Q$ lying in $g^{-1}(A - B)$ and $g^{-1}(B - A)$, respectively.

Using the *ANR* properties of A, B and C, we can choose a fine enough neighborhood Q of $g^{-1}(C)$ so that there is a small homotopy of g, rel $g|g^{-1}(C)$, in W^* to a map we will still call $g\colon S^i \to W^*$ such that $g(Q) \subseteq C$ and the homotopy keeps P^A and P^B mapped into A and B, respectively. As a separating polyhedron in a *PL* i–manifold, Q collapses to polyhedron $Q_0 \subseteq Q$ of dimension $i - 1$ using only collapses across i–dimensional simplexes. The associated deformation retraction of Q onto Q_0 extends to a homotopy $R\colon S^i \times [0,1] \to S^i$ of the identity map of S^i to a map $r\colon S^i \to S^i$ such that $R((Q \cup P^A) \times [0,1]) \subseteq Q \cup P^A$ and $R((Q \cup P^B) \times [0,1]) \subseteq Q \cup P^B$. Let P_0^A and P_0^B be the union of the closures of the components of $S^i - Q_0$ that contain P^A and P^B, respectively. Precomposing g with R gives a homotopy of g to a map $h\colon S^i \to W^*$ such that $h(Q_0) \subseteq C$, $h(P_0^A) \subseteq A$ and $h(P_0^B) \subseteq B$.

By Proposition 5 $h|Q_0\colon Q_0 \to W^* \cap C$ is homotopic to a constant map c in $V^* \cap C$. Use regular neighborhoods of Q_0 in P_0^A and P_0^B to extend this homotopy to a homotopy of h in $W^* \cup (V^* \cap C)$ to a map $h_0\colon S^i \to W^* \cup (V^* \cap C) \subseteq V^*$ such that $h_0(P_0^A) \subseteq A$ and $h_0(P_0^B) \subseteq B$. Let L be a quotient polyhedron of S^i topologically homeomorphic to S^i/Q_0, containing subpolyhedra L^A homeomorphic to P_0^A/Q_0 and L^B homeomorphic to P_0^B/Q_0 so that $L^A \cap L^B$ is the point $[Q_0] \in S^i/Q_0$. Then h_0 induces a map $\overline{h}\colon L \to V^*$ such that $\overline{h}(L^A) \subseteq (V^* \cap A)$, $\overline{h}(L^B) \subseteq (V^* \cap B)$ and $\overline{h}([Q_0]) = c(Q_0)$ is a point of C. Proposition 5 then applies to each of $\overline{h}|L^A$ and $\overline{h}|L^B$ to provide a homotopy of $\overline{h}$, rel $[Q_0]$, in U^* to a constant. Thus, the inclusion homomorphism $\pi_i(W^*) \to \pi_i(U^*)$ is zero proving that y^* is UV^k for each $y \in Y$. □

Proposition 7 *Suppose A and B are compact, connected ANRs of dimension ≥ 1, $\epsilon > 0$, Z is an LCC^{-1} subset of A, and $p\colon B \to C$ is a map, where C is a metric space. If $f\colon A \to B$ is $UV^{-1}(\epsilon)$ over C, then f is 2ϵ–homotopic (over C), rel $f|Z$, to a surjection.*

Proof Assume all measurements are made in C. Let P be a finite subset of B such that every point of B can be joined to a point of P by an arc of diameter $\leq \epsilon/2$ in both B and C. By hypothesis, there is a map $\alpha\colon P \to A$ whose composition with f is ϵ–homotopic to the inclusion. Since $\dim A \geq 1$, we may assume α is one-to-one and, since Z is LCC^{-1}, we may assume $\alpha(P) \cap Z = \varnothing$. Let $P' = \alpha(P)$. Using the homotopy extension theorem on a small neighborhood of P' in A, which is disjoint from Z, we can get an extension of the ϵ–homotopy of $f|P'$ to α^{-1} to an ϵ–homotopy of f to a map that sends P' to P. Thus there is an ϵ–homotopy of f, rel $f|Z$, to a map that is $UV^{-1}(\epsilon/2)$ over both B and C. A sequence of such maps can be constructed so as to converge to a surjection that is 2ϵ–homotopic to f. □

The next lemma gives a criterion for determining when an extension of a $UV^k(\epsilon)$–map is (almost) $UV^k(\epsilon)$.

Lemma 1 *Suppose $X_1 \subseteq X_2$ and B are compact ANRs, $\delta > 0$ and $\epsilon > 0$, suppose $p\colon B \to Y$ is a map of B to a metric space Y, and suppose that for some integer $k \geq 0$, $f\colon X_2 \to B$ is a map such that*

(i) *$f|X_1$ is $UV^k(\epsilon)$ over Y, and*

(ii) *if g is a map of a k–dimensional polyhedron R into X_2, then g is δ–homotopic over Y to a map of R into X_1.*

Then f is $UV^k(2\delta + \epsilon)$ over Y.

Proof Suppose (P, Q) is a polyhedral pair, $\dim P \leq k+1$, and suppose $\alpha\colon P \to B$ and $\alpha_0\colon Q \to X_2$ satisfy $f \circ \alpha_0 = \alpha|Q$. For any $\mu > 0$ there is a μ–homotopy over B of the identity on P to a map $r\colon P \to P$, which is fixed on Q and outside a neighborhood of Q, that deformation retracts a small regular neighborhood N of Q onto Q. By precomposing α with such a map, we can get a μ–homotopy (over B) of α to a map $\alpha_1\colon P \to B$, whose restriction to N can be lifted by $\alpha_0 \circ r|N$. Thus, if μ is sufficiently small, $\alpha\colon P \to B$ is δ–homotopic to $\alpha_1\colon P \to B$, rel $\alpha|Q$, such that $\alpha_1|N$ can be lifted to an extension of $\alpha_0\colon Q \to X_2$ to a map we will still call $\alpha_0\colon N \to X_2$ such the $f \circ \alpha_0 = \alpha_1|N$.

Let $P_0 = \mathcal{C}\ell(P - N)$ and let $Q_0 = N \cap P_0 = \mathrm{bd}(N)$. Since $\dim Q_0 \leq k$, there is a δ–homotopy (over Y) of $\alpha_0|Q_0$ that takes Q_0 into X_1. Since Q_0 is collared in N, this homotopy can be extended to a δ–homotopy of α_0 on N (over Y) that is fixed on Q. Call the resulting map $\overline{\alpha}_0\colon N \to X_2$. Composing with f gives a δ–homotopy of $\alpha_1|N$ in B (over Y), rel $\alpha_1|Q = \alpha|Q$, which can be extended to a δ–homotopy of α_1 (over Y), rel $f \circ \overline{\alpha}_0|Q = \alpha|Q$, on P to $\alpha_2\colon P \to B$, since Q_0 is collared in P_0. By hypothesis, $f|X_1$ is $UV^{k-1}(\epsilon)$ over Y, and so $\alpha_2|P_0$ can be ϵ–lifted to X_1 (over Y), rel $\overline{\alpha}_0|Q_0$, to $\overline{\alpha} : P_0 \to X_2$. That is, $f \circ \overline{\alpha}\colon P_0 \to B$ is ϵ–homotopic (over Y) to $\alpha_2\colon P_0 \to B$, rel $\alpha_2|Q_0$. The map $\overline{\alpha}\colon P_0 \to X_1$ extends to a map we shall still call $\overline{\alpha}\colon P \to X_2$ such that $\overline{\alpha}|N = \overline{\alpha}_0$. Thus, $f \circ \overline{\alpha}_1\colon P \to X_2$ is ϵ–homotopic to $\alpha_2\colon P \to B$ (over Y), rel $\alpha_2|N = \alpha|N$ (hence rel $\alpha|Q$). Stacking these homotopies gives a $(2\delta+\epsilon)$–homotopy of $f \circ \overline{\alpha}$ to α rel $\alpha|Q$. $\alpha_0\colon Q \to X_2$, undergoes two δ–homotopies over Y to a map $\alpha_2\colon P \to B$ with the following properties: there is a regular neighborhood N of Q in P, with $Q_0 = \operatorname{bd} N$ and $P_0 = \mathcal{C}\ell(P-N)$, and an extension of α_0 to a map $\overline{\alpha}_0\colon N \to X_2$ lifting $\alpha_2|N$ such that $\overline{\alpha}_0(Q_0) \subseteq X_1$. The hypothesis gives an ϵ–lift $\overline{\alpha}\colon P_0 \to X_1$, of $\alpha_2|P_0$, rel $\alpha_2|Q_0$ (over Y). Extending $\overline{\alpha}$ over N *via* $\overline{\alpha}_0$ produces an ϵ–lift (over Y) of α_2 we still call $\overline{\alpha}\colon P \to X_2$, rel $\alpha_2|N$; hence, $\overline{\alpha}$ is a $(2\delta+\epsilon)$–lift of α over Y, rel $\alpha|Q$. □

An argument virtually identical to the one just given also proves the following lemma.

Lemma 2 *Suppose X and B are compact ANRs, $\delta > 0$ and $\epsilon > 0$. If $f\colon X \to B$ is $UV^k(\epsilon)$ over a metric space Y and g is δ–homotopic to f over Y, then g is $UV^k(2\delta+\epsilon)$ over Y.*

The proof of the next lemma is an easy application of the definition.

Lemma 3 *Suppose A, B and C are compact metric spaces and $f\colon B \to C$ is a $UV^k(\epsilon)$–map for some $\epsilon > 0$. Then there exists $\delta > 0$ such that if $g\colon A \to B$ is $UV^k(\delta)$ over B, then $f \circ g\colon A \to C$ is $UV^k(2\epsilon)$ (over C).*

3 Simple homotopies and *ENR*s

The proof of our main result involves a double induction argument in which the principal inductive step is established in two parts. Suppose X is a compact *ENR* satisfying the *DDP*$^{k+1}$, B is a finite complex and $f\colon X \to B$, is a map such that f is UV^{k-1} and $UV^k(\epsilon)$ over B, for some $\epsilon > 0$. We will want to prove that, for arbitrary $\mu > 0$, there is a $(D \cdot \epsilon)$–homotopy of f to a $UV^k(\mu)$–map for some constant D, depending only on k.

One part is to show that there is a "simple homotopy solution" to this shrinking problem. That is, we will show:

(1) *X "ϵ–expands" to an ENR $\overline{X}$ and f extends to a $UV^k(\mu)$–map $\bar{f}\colon \overline{X} \to B$.*

The second part is to show that this is sufficient:

(2) *For every $\eta > 0$, the inclusion $X \subseteq \overline{X}$ is η–homotopic over X to a $UV^k(\eta)$ map $g\colon X \to \overline{X}$.*

We are generalizing here the notion of *collapse* and *expansion* from *PL* topology to *ENR*s in the obvious manner. That is, an elementary k–collapse (k–expansion) $Y \,{}_{e}\!\searrow X$ ($X \,{}^{e}\!\nearrow Y$) means Y is obtained from X by affixing a k–cell D to X by a map $a\colon C \to X$, where C is a (nice) $(k-1)$–cell in ∂C. Alternatively, we can think of $Y = X \cup_a D$ as the mapping cylinder of a, rel ∂C. (We do not require, in general, that the attaching map be an embedding.) This provides a natural deformation retraction of Y to X. A collapse (expansion) $Y \searrow X$ ($X \nearrow Y$) is a sequence of elementary collapses (expansions).

The main goal of this section is to prove Theorem 6, below, which is a special case of part (2) for elementary expansions in which the attaching map is a "nice" embedding. But first we make some elementary observations about maps defined on balls and spheres.

Let $S^m \subseteq \mathbb{R}^{m+1}$ denote the *PL* m–sphere, defined as the join of $(m+1)$ copies of $S^0 = \{-1, 1\} \subseteq \mathbb{R}$, and let $B^m = S^{m-1} * 0 \subseteq \mathbb{R}^m$, with natural inclusions $S^{m-1} = S^{m-1} \times 0 \subseteq S^m$ and $B^m = B^m \times 0 \subseteq B^{m+1}$. In $\mathbb{R}^{n+1} = \mathbb{R}^k \times \mathbb{R}^{n-k} \times \mathbb{R}$, consider the $(n,k), (n+1, k+1)$ ball pairs

$$(B^n, B^k) \subseteq (B^n, B^k) * \{1\} = (E, D).$$

If $p\colon \mathbb{R}^n \to \mathbb{R}^k$ is the natural projection, then $p^{-1}(x) \cap \partial B^k$ is an $(n-k-1)$–sphere, a point or the empty set, accordingly as $x \in \operatorname{int} B^k$, $x \in \partial B^k$ or $x \notin B^k$. Similarly, if $p_+\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^k \times 0 \times \mathbb{R}$ is the natural projection, $p_+^{-1}(x) \cap (\partial E - \operatorname{int} B^n)$ is an $(n-k-1)$–sphere, a point or the empty set, accordingly as $x \in \operatorname{int} D$, $x \in \partial D$ or $x \notin D$. The projection $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ maps $\partial_+ E = (\partial E - \operatorname{int} B^n)$ homeomorphically onto B^n. The inverse of this homeomorphism composed with the projection p_+ give a UV^{n-k-2}–map

$$q\colon B^n \longrightarrow D.$$

In terms of join structures, q is the composition

$$B^n = S^{n-k-1} * S^{k-1} * 0 \xrightarrow{\cong} S^{n-k-1} * S^{k-1} * 1 \xrightarrow{p_+} 0 * S^{k-1} * 1 = D.$$

Let

$$B^n_+ = B^k * 2S^{n-k-1} \supseteq B^n.$$

Using the product structure on $(B^n_+ - \operatorname{int} B^n)$ (pinched on ∂B^k) we can extend q to a UV^{n-k-2}–map we will still call

$$q\colon B^n_+ \longrightarrow B^n_+ \cup D,$$

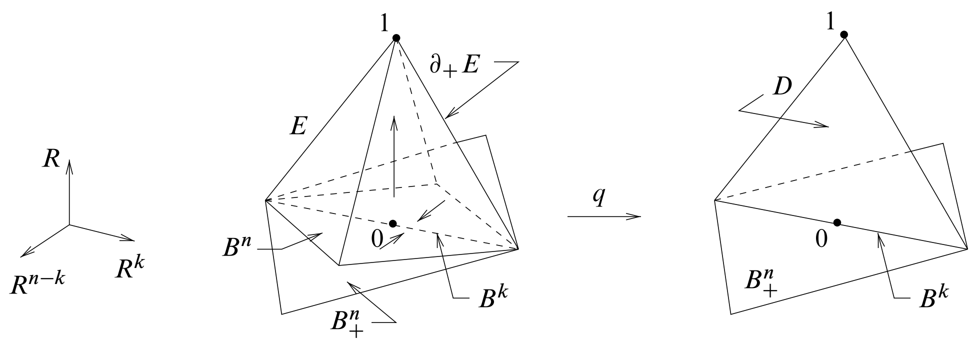
which is the identity on ∂B^n_+ and one-to-one on $(B^n_+ - B^n)$. Using join structures, one can construct a homotopy

$$h\colon B^n_+ \times [0,1] \longrightarrow B^n_+ \cup D$$

such that

$$h_0 = \iota\colon B^n_+ \hookrightarrow B^n_+ \cup D, \quad h_1 = q,$$

and, for each $t \in [0,1]$ and each $y \in B^n_+ \cup D$, $h_t^{-1}(y)$ is either empty, a point or an $(n-k-1)$–sphere.



If M is a topological n–manifold and D is a $(k+1)$–cell attached to M along a k–cell A that is nice in both M and ∂D, then we can use the model above to construct homotopy from the inclusion map $M \hookrightarrow M \cup_A D$ to a map $q\colon M \to M \cup_A D$, which is the identity outside a relative regular neighborhood of A, rel ∂A, whose point inverses are either points or $(n-k-1)$–spheres. This implies q is UV^{n-k-2}. If $2k+1 \le n$, then M has the DDP^k and q will be UV^{k-1}. The main theorem of this section asserts that arbitrarily close approximate versions of this result hold for an *ENR* X with the DDP^k. (Ultimately, the main results will apply to show that the inclusion $X \subseteq X \cup_A D$ is homotopic to a UV^{k-1}–map.)

Given a closed pair (A, B) in a space X, a *relative neighborhood* of A, rel B, in X is a subset U of X containing A such that $U - B$ is a neighborhood of $A - B$ in $X - B$.

Theorem 6 *Suppose X is an ENR with the DDP^k, $k \geq 0$ and suppose $\gamma\colon C \to X$ is an embedding of a k–cell C onto an LCC^{k-1} k–cell $A \subseteq X$, and $\overline{X} = X \cup_A D$ is the relative mapping cylinder of γ, rel ∂C, with mapping cylinder retraction $d\colon \overline{X} \to X$. Assume a metric on $\overline{X}$ extending a given one on X. Then for every relative neighborhood U of A, rel A, in X and every $\eta > 0$, there is an η–homotopy $h\colon X \times I \to \overline{X}$ over X of the inclusion $\iota\colon X \to \overline{X}$ such that*

(i) *each h_t is the identity outside U,*

(ii) *$d \circ h\colon X \times I \to X$ is an η–homotopy that deformation retracts a relative neighborhood of A, rel ∂A, onto A inside U,*

(iii) *$h_1\colon X \to \overline{X}$ is $UV^{k-1}(\eta)$ over $\overline{X}$.*

Proof Assume that X is tamely embedded in $\mathbb{R}^m$, $m > 2\dim X$, so as to have a mapping cylinder neighborhood N with mapping cylinder projection $\pi\colon N \to X$ (Miller [22]). Given any triangulation of N, π restricted to its k–skeleton can be approximated arbitrarily closely by an LCC^{k-1} embedding whose image misses A. For any $\epsilon > 0$, there is a triangulation of N, with k–skeleton T such that any map of a k–dimensional polyhedral pair (P, Q) into (N, T) can be ϵ–homotoped, rel Q, into T. Thus, for a given sequence $\epsilon_0, \epsilon_1, \dots$ of positive numbers, there is a sequence $T_0 \subseteq T_1 \subseteq \cdots$ of k–dimensional polyhedra, LCC^{k-1} embedded in $X - A$, such that any map of a k–dimensional polyhedral pair (P, Q) into (X, T_j), $j < i$, can be ϵ_i–homotoped, rel Q, into T_i.

Suppose we are given $\overline{X} = X \cup_A D$. Let $X' = (X \times 0) \cup (A \times I) \subseteq X \times I$, $I = [0,1]$ and let $p\colon X' \to X$ be projection to the first factor. Let $g\colon X' \to \overline{X}$ be the map that sends each of the vertical intervals in $\partial A \times I$ to a point, but is otherwise one-to-one. We may assume $d\colon \overline{X} \to X$ is the map induced by p. Equip X' with the metric ρ inherited from the embedding into $\mathbb{R}^m \times [0,1]$ with the product metric, where $X \subseteq \mathbb{R}^m \times 0$ as above and $(x,t) \mapsto (x,t)$ if $x \in A$. Since the quotient map $g\colon X' \to \overline{X}$ is cell-like, it is sufficient, by Lemma 3, to prove the theorem with $\overline{X}$ replaced by X' and $d\colon \overline{X} \to X$ replaced by $p\colon X' \to X$.

Suppose then that we are given $\eta > 0$. Let $\{0 = t_0 < t_1 < \cdots < t_\ell = 1\}$ be a subdivision of I of mesh $< \eta/3$. Given a relative neighborhood U of A, rel A, positive numbers $\epsilon_0, \dots, \epsilon_\ell$ and k–dimensional polyhedra $T_0 \subseteq T_1 \subseteq \cdots \subseteq T_\ell$ in $X - A$, as above, construct a sequence of relative neighborhoods

$$V_\ell \subseteq \cdots \subseteq V_1 \subseteq V_0 \subseteq U$$

of A, rel A and an ϵ_0–homotopy $R\colon X \times I \to X$ as follows:

(1) $R_0 = \mathrm{id}_X$.

(2) $R_t|[(X-U)\cup A] = \mathrm{id}$ for all $t \in I$.

(3) $R(U \times I) \subseteq U$.

(4) $R(\mathcal{C}\ell(V_0) \times I) \subseteq U$.

(5) $R_1^{-1}(A) = \mathcal{C}\ell(V_0)$.

(6) $R(\mathcal{C}\ell(V_i) \times I) \subseteq (V_{i-1} - T_{i-1})$ for $1 \le i \le \ell$.

(7) $R|V_i \times I$ is an ϵ_i–homotopy, $0 \le i \le \ell$.

That is, R is an ϵ_0–deformation retraction of a neighborhood V_0 of A onto A inside a neighborhood U of A, which has been extended to X by the estimated homotopy extension theorem of Bryant et al [5]. (For the remainder of this proof, all neighborhoods of A are relative neighborhoods of A, rel ∂A.) Having constructed R satisfying (1)–(5), the neighborhoods $V_1, \dots, V_\ell$ satisfying properties (6) and (7) are obtained from continuity of R. The positive number ϵ_0 will be chosen so that subsets of X of diameter $< \epsilon_0$ will have diameter $< \eta/2$ throughout the homotopy R. The numbers ϵ_i (and the polyhedra T_i), $i \ge 1$, will be chosen inductively so that, for any polyhedral pair (P, Q) of dimension $\le k+1$ and map $\alpha\colon (P, Q) \to (V_{i-1}, T_{i-1})$, there is an ϵ_i–homotopy of α, rel $\alpha|Q$, in V_{i-2} to a map of P into T_i (where $U = V_{-1}$). We assume, furthermore, that $\epsilon_i < \min\{\epsilon_0/3, \mathrm{dist}(A, X - V_i)\}$, for $i > 0$.

For each $i = 1, \dots, \ell$, let $\lambda_i\colon (\mathcal{C}\ell(V_{i-1}) - V_i) \to [t_{i-1}, t_i]$ be an Urysohn function that takes $\mathrm{bd}(V_{i-1})$ to t_{i-1} and $\mathrm{bd}(V_i)$ to t_i. Combine these maps to get a map $\lambda\colon X \to I$ that takes $X - V_0$ to 0 and V_ℓ to 1.

A map $q\colon X \to X'$ can then be defined by setting

(a) $q(x) = (R_1(x), 0)$, if $x \in (X - V_0)$,

(b) $q(x) = (R_1(x), \lambda_i(x))$, if $x \in (V_{i-1} - V_i)$, $i = 1, \dots, \ell$, and

(c) $q(x) = (R_1(x), 1)$, if $x \in V_\ell$.

Then $R_1 = p \circ q$, and the homotopy $\mathrm{id}_{X'} \simeq p$ composed with q gives a homotopy of q to $p \circ q = R_1$. Piecing this homotopy together with R gives a homotopy $h'\colon X \times I \to X'$ from the inclusion $X \subseteq X'$ to q.

The claim now is that $q\colon X \to X'$ is $UV^{k-1}(\eta)$.

To this end, suppose we are given a polyhedral pair (P, Q) of dimension $\le k$ and maps $\alpha\colon P \to X'$ and $\alpha_0\colon Q \to X$ with $q \circ \alpha_0 = \alpha|Q$. As in the proof of Lemma 1 we may assume that, after a small perturbation of α, rel $\alpha|Q$, there is a small regular neighborhood W of Q in P and an extension of α_0 to W lifting $\alpha|W$. This perturbation is obtained by precomposing α with a perturbation of the identity on P that deformation retracts W to Q. Let $Q_0 = \mathrm{bd}(W)$ and let $P_0 = P - \mathrm{int}(W)$. After a second small perturbation of α we may assume each $S_i = \alpha^{-1}(A \times [t_{i-1}, t_i]) \cap P_0$ $(i \geq 1)$ and each $B_i = \alpha^{-1}(A \times t_i) \cap P_0$ $(i \geq 0)$ are subpolyhedra of P_0. Set $S_0 = \alpha^{-1}(X) \cap P_0$. Thus, we have

$$P = W \cup P_0 = W \cup S_0 \cup S_1 \cup \cdots \cup S_\ell,$$

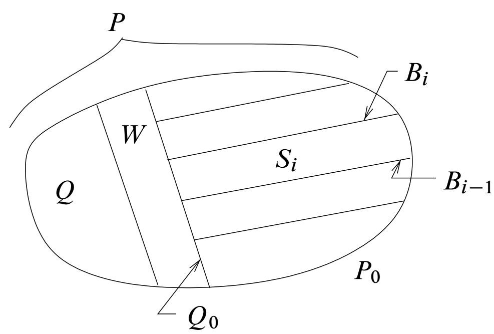
where $W \cap P_0 = \mathrm{bd}(W)$ and $S_{i-1} \cap S_i = B_{i-1}$ for $1 \leq i \leq \ell$.

Observe that $h'\colon X \times I \to X'$ provides a homotopy from $\alpha_0\colon W \to X$ to $\alpha|W\colon W \to X'$. Set $\alpha' = p \circ \alpha$ and observe that $\alpha'(P_0 - S_0) \subseteq A$.

Proceed inductively to move $\alpha'(P_0 - \mathrm{int}(S_\ell))$ off of A using the moves below:

- An ϵ_0–homotopy of α' to a map α'_0, that takes B_0 into T_0 and is constant outside a small neighborhood of B_0 in P_0 that misses S_i, $i \geq 2$.
- An ϵ_1–homotopy of α'_0 to a map α'_1 that takes S_1 into T_1 and is constant on S_0 and outside a small neighborhood of S_1 that misses S_i, $i \geq 3$. Since $\alpha'_0(S_2) \subseteq A$, our choice of ϵ_1 ensures that $\alpha'_1(B_1) \subseteq \alpha'_1(S_2) \subseteq V_1$.
- An ϵ_2–homotopy of α'_1 to a map α'_2 that takes S_2 into T_2 and is constant on $S_0 \cup S_1$ and outside a small neighborhood of S_2 that misses S_i, $i \geq 4$. Since $\alpha'_1(S_3) \subseteq A$, our choice of ϵ_2 ensures that $\alpha'_2(B_2) \subseteq \alpha'_2(S_3) \subseteq V_2$.

Continuing this process produces a homotopy of α' to $\alpha'_{\ell-1}\colon P_0 \to X$, which moves no point of P_0 more than twice, such that $\alpha'_{\ell-1}(S_i) \subseteq V_{i-2} - V_{i+1}$ for $1 \leq i \leq \ell$ (where $V_{\ell+1} = \varnothing$). Since W is a (small) regular neighborhood of Q in P, this homotopy, restricted to $\mathrm{bd}(W)$, can be extended over W to a $2\epsilon_0$–homotopy of $\alpha'|W$ that is constant on Q by the estimated homotopy extension theorem. The resulting map $\overline{\alpha}\colon P \to X$ satisfies $\overline{\alpha}|Q = \alpha_0$.



Our choice of ϵ_0 ensures that $p \circ \overline{\alpha}$ is $\eta/2$–homotopic to $p \circ \alpha$. Since $\overline{\alpha}(S_i) \subseteq V_{i-2} - V_{i+1}$, $q \circ \overline{\alpha}$ is η–homotopic to α. □

Addendum *If $Z \subseteq X - A$ is a closed subset of X, then the homotopy $h\colon X \times I \to \overline{X}$ can be chosen to be fixed on Z.*

4 Special case

We will establish Theorem 1 by first proving the special case in which $C = B$ and $p = \mathrm{id}_B$.

Theorem 7 *Suppose X is a compact, connected ENR satisfying the disjoint $(k+1)$–disks property, B is a compact, connected ANR and $f\colon X \to B$ is $UV^k(\epsilon)$ for some $\epsilon > 0$. Then f is $(C(k)\cdot\epsilon)$–homotopic to a UV^k–map, where $C(k)$ is a positive constant depending only on k.*

Moreover, if Z is an LCC^k subset of X, then the homotopy of f to a UV^k–map can be chosen to be fixed on Z.

In Section 5 we indicate how the proof of Theorem 7 can be modified to obtain our main result. We shall separate the proof of Theorem 7 into two cases: $k = 0$ and $k \geq 1$. The intent is to present the main ideas first in a somewhat less cluttered setting, so that they may be a bit more transparent. This approach has, of course, introduced redundancies into the exposition, but we hope they prove to be of value to the reader.

4.1 UV^0

In this section we assume only that X is a compact *ENR* satisfying the DDP^1, also known as the *disjoint arcs property*, and that Z is a compact, LCC^0 subset of X. Recall from Proposition 2 that we may also assume throughout that B is a compact polyhedron. We start by proving a simple homotopy analogue of our main result in the base case $k = 0$. Keep in mind that all measurements are made in B unless specifically indicated otherwise.

Proposition 8 *Suppose a surjection $f\colon X \to B$ is a $UV^0(\delta)$–map and $\mu > 0$. Then there is an ENR $\overline{X}$ obtained by adding 1– and 2–cells to $X - Z$ and an extension $\overline{f}\colon \overline{X} \to B$ such that $\overline{f}$ is $UV^0(\mu)$ and $\overline{X}$ 2δ–collapses to X.*

Proof Triangulate B so that the diameter of the star of each simplex is less than $\mu' < \mu/3$, where μ' is chosen so that maps into B that are μ'–close are $\mu/3$–homotopic. The inverse image under f of each simplex $\sigma \in B$ is compact. If U_σ is a small path-connected open neighborhood of σ in B, then $f^{-1}(\sigma)$ is contained in finitely many components of $f^{-1}(U_\sigma)$. Attach finitely many 1–cells to $X - Z$ connecting the components of $f^{-1}(U_\sigma)$ that contain points of $f^{-1}(\sigma)$ so that their boundaries are mutually exclusive, and extend the map f over each of these 1–cells so that their images lie in U_σ. Doing this for each $\sigma \in B$ produces a space X_1 and an extension $f_1\colon X_1 \to B$ of f. If the neighborhood U_σ of each $\sigma \in B$ is sufficiently small, f_1 is $UV^0(\mu/3)$: For each simplex σ in B, choose a neighborhood V_σ of σ lying in U so that $f^{-1}(V_\sigma)$ meets only components of $f^{-1}(U_\sigma)$ which meet $f^{-1}(\sigma)$. A path in B can be broken into finitely many segments, each lying in one of these sets V_σ. It suffices to μ'–lift one such segment relative to given lifts on the ends. But this is easily accomplished using the 1–cells of X_1.

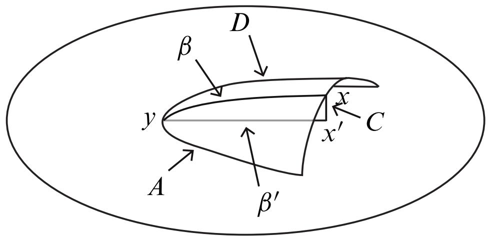
Let C be a 1–cell in $\mathcal{C}\ell(X_1 - X)$. Since $f\colon X \to B$ is $UV^0(\delta)$, $f_1|C$ has a δ–lift to X, rel ∂C, which we may assume is an embedding into $X - Z$. Call the image arc A. Attach a 2–cell D to X_1 by identifying its boundary with $A \cup C$. Call the result X_2, and use the δ–homotopy from $f_1(C)$ to A to extend f_1 to $f_2\colon X_2 \to B$. Unfortunately, the map f_2 is no longer $UV^0(\mu/3)$, since all we know about the image of D is that it has size δ in B.

We remedy this as follows. Parameterize D as the quotient of $A \times I$ with the intervals over ∂A identified to points, and identify A with $A \times 0$ and C with $A \times 1$. Let A_0 be a finite subset of A such that every point of D is within $\mu/3$ (measured in B) of a point of $A_0 \times I \subseteq D$. Let y be a point of A_0, let $\beta = y \times I \subseteq D$ and let $x = y \times 1 \in C$. Since f is surjective, there is a point x' in X such that $f_2(x) = f(x')$. By changing f by a small homotopy, if necessary, we can assume $x' \notin Z$. Since f_1 is $UV^0(\mu/3)$, there is a path β' in $X_1 - Z$ connecting y to x' such that $f_2 \circ \beta$ is $(\mu/3)$–homotopic to $f_1 \circ \beta'$ (rel $\{x, y\}$). We have a map from β to β' sending x to x' and y to y, so we can attach its mapping cylinder (rel y) to X_2. We can extend the map f_2 to this mapping cylinder, using the $(\mu/3)$–homotopy above, so that mapping cylinder fibers have size $< \mu/3$ in B. Thus, all points on the new 2–cell are $(\mu/3)$–close to X, as well. Performing this construction for all $y \in A_0$ produces a relative 2–complex X_3, and a map $f_3\colon X_3 \to B$, which, by Lemma 1, is $UV^0(\mu)$. X_3 δ–collapses to X_2, which, in turn, δ–collapses to $X_1 - \operatorname{int} C$.

Repeat this construction for each 1–cell, $C \subseteq \mathcal{C}\ell(X_1 - X)$, making sure that the corresponding family of attaching arcs is mutually exclusive in X. The resulting space $\overline{X}$ 2δ–collapses to X and admits a $UV^0(\mu)$–map $\overline{f}\colon \overline{X} \to B$. □

The figure below illustrates a single 2–cell attached to X_2 and a single point $y \in A_0$. The placement of the path β' is misleading, however, since it can wind about the other 1–cells we attached to X when we formed X_1.

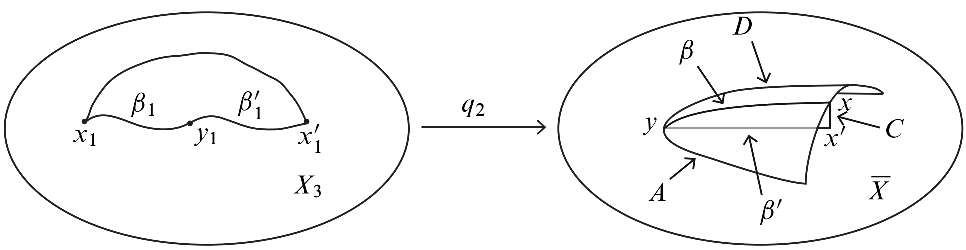
The following proposition provides the key to proving Theorem 7 for the case $k = 0$.



Proposition 9 *Suppose $f\colon X \to B$ is $UV^0(\epsilon)$, and $\mu > 0$. Then f is 10ϵ–homotopic, rel $f|Z$, to a $UV^0(\mu)$–map.*

Proof Suppose X and B are given as in the hypothesis, and suppose $\mu > 0$. By Proposition 7, we can get a 2ϵ–homotopy of f to a surjection. By Lemma 2 the resulting map, which we shall still call f, is $UV^0(5\epsilon)$. Set $\delta = 5\epsilon$.

Proceed as in the proof of Proposition 8. Obtain $X_1 \subseteq X_2$ from X by attaching 1–cells to $X - Z$ to get X_1 and 2–cells to $X_1 - Z$ to get X_2, together with extensions $f_1 \subset f_2$ of $f\colon X \to B$ to X_1 and X_2, respectively. These were constructed so that f_1 is $UV^0(\mu')$ and f_2 is $UV^0(\delta)$, where $\mu' > 0$ will be determined later. We may assume that the arcs in X along which the 2–cells are attached to form X_2 are mutually exclusive. Enclose the attaching arcs in neighborhoods whose closures are mutually exclusive and miss Z . Let D be a 2–cell of $X_2 - X_1$ attached to X along an arc A. (The complementary arc $C \subseteq \partial D$ was added when X_1 was constructed.) The arc $\beta \subseteq D$ and path $\beta' \subseteq X_1$ from points $x \in C$ and $x' \in X$, respectively, to a point y in A, were chosen so that $f_2(x) = f(x')$ and $f_2|\beta$ and $f_1|\beta'$ are μ'–homotopic in B.



For a given $\eta_2 > 0$, Theorem 6 provides us with a homotopy $h\colon X \times I \to X_2$ of the inclusion $\iota\colon X \to X_2$ to a $UV^0(\eta_2)$–map $q_2 : X \to X_2$ *over* X_2 such that h is fixed at the identity on the complement of the union of the neighborhoods of the attaching arcs and h composed with the collapse $X_2 \searrow X$ is an η_2–homotopy on X. In particular, h is fixed on Z. Let y_1, x_1, x_1' be points of X that map to y, x, x', respectively. Then there are arcs β_1 and β_1' in $X - Z$ joining y_1 to x_1 and y_1 to x_1', respectively, such that $q_2(\beta_1)$ and $q_2(\beta_1')$ are η_2–homotopic to β and β', respectively. We may assume that β_1 and β_1' are embedded and that $\beta_1 \cap \beta_1' = y_1$. We may also assume that the collection of all the arcs $\beta_1 \cup \beta_1'$ is mutually exclusive. It is possible to arrange it so that $q_2(\beta_1) = \beta$ and $q_2(\beta_1') = \beta'$ at the expense of ending up with a map q_2 that is $UV^0(6\eta_2)$ over X_2: Given $\beta_1 \cup \beta_1'$ in X, let X' be the space obtained by attaching $(\beta_1 \cup \beta_1') \times I$ to X so that $(\beta_1 \cup \beta_1') \times 0$ is identified with $(\beta_1 \cup \beta_1')$ and the intervals over the endpoints of β_1 and β_1' are identified to points. Construct a map $X' \to X_2$ extending q_2 using the η_2–homotopy from $q_2(\beta_1 \cup \beta_1')$ to $\beta \cup \beta'$, rel the endpoints of β and β'. Then, by Lemma 1, this map is $UV^0(3\eta_2)$ over X_2. By Lemma 3 and Theorem 6, we can find a map from X to X' so that the composition $X \to X' \to X_2$ is $UV^0(6\eta_2)$ over X_2. Thus, after rescaling, we may assume that q_2 is $UV^0(\eta_2)$ over X_2, $q_2(\beta_1) = \beta$ and $q_2(\beta_1') = \beta'$. In Proposition 8 this construction is performed a finite number of times for each of the 2–cells added to X to form X_2. Since the collection of arcs $\beta_1 \cup \beta_1'$ is mutually exclusive, we can perform this construction for all of the arcs simultaneously; hence, we can assume that we have a $UV^0(\eta_2)$–map $q_2\colon X \to X_2$ over X_2 that works as above for all of the (β, β') arc-path pairs. The next step in the proof of Proposition 8 was to add mapping cylinders of the maps $\beta \to \beta'$ (rel y) to X_2. The *ENR* $\overline{X}$ is obtained from X_2 by attaching 2–cells (the mapping cylinders) along the family of arcs $\beta \cup \beta'$. We also obtain an extension $\bar{f}\colon \overline{X} \to B$ of f_2 that is $UV^0(\mu')$ and δ–homotopic to the collapse from $\overline{X}$ to X_2 composed with f_2. Form the space X_3 by attaching 2–cells to X along the arcs $\beta_1 \cup \beta_1'$, and get a $UV^0(\eta_2)$–map $q'\colon X_3 \to \overline{X}$ over $\overline{X}$ by combining $q_2\colon X \to X_2$ with a map between corresponding attaching 2–cells that realizes the mapping cylinder identification. That is, the 2–cell attached along $\beta_1 \cup \beta_1'$ should be thought of as the product $\beta_1 \times I$, with $\beta_1 \times 0$ identified with β_1, $\beta_1 \times 1$ identified with β_1' and $y_1 \times I$ identified to the point y_1. Given an $\eta_3 > 0$ apply Theorem 6 to get a $UV^0(\eta_3)$–map $q_3\colon X \to X_3$ over X_3, along with accompanying homotopies. Lemma 3 tells us that we can choose μ', η_2 and η_3 sequentially so that, after performing the constructions above, the composition

$$X \xrightarrow{q_3} X_3 \xrightarrow{q'} \overline{X} \xrightarrow{\bar{f}} B$$

is $UV^0(\mu)$. During this process, f has undergone two δ– or one 10ϵ–homotopy, and each of these homotopies can be chosen to fixed on Z. □

Proof of Theorem 7 in the case $k = 0$ To get a UV^0–map from a $UV^0(\epsilon)$–map, simply apply Proposition 9 inductively to get a sequence of homotopies of maps from X to B, which starts with f and converges to a homotopy of f to a map that is $UV^0(\delta)$ for every $\delta > 0$ and is fixed on Z. We may make the positive number μ in Proposition 9 small enough so that the homotopy from the $UV^0(\mu)$–map to a UV^0–map has size $< \epsilon$; hence, f is 11ϵ–homotopic to a UV^0–map, rel $f|Z$. □

4.2 UV^k, $k \geq 1$

Throughout this section we will assume that X is a compact *ENR* with the DDP^{k+1}, $k \geq 1$, Z is a compact, LCC^k subset of X and B is a compact polyhedron. To proceed, we need the following finite generation lemma.

Lemma 4 *Suppose $f\colon X \to B$ is UV^{k-1}, where $k \geq 1$. Given $\mu > 0$, we can attach finitely many $(k+1)$–cells to $X - Z$ along their boundaries to obtain an ENR X_1 and an extension of f to an $UV^k(\mu)$–map $f_1 : X_1 \to B$.*

Proof First observe that, if A and B are connected open subsets of X with $\mathcal{C}\ell(A) \subseteq B$, then there is a compact polyhedron P and maps $\iota\colon A \to P$ and $\rho\colon P \to B$ such that $\rho\iota\colon A \to B$ is the inclusion map. This follows from the fact that X is compact and is a retract of an open set in some Euclidean space. Thus, the images of the inclusion-induced maps $\pi_1(A) \to \pi_1(B)$ and $H_*(A) \to H_*(B)$ are finitely generated.

Triangulate B so that each open vertex star U has diameter $\ll \mu$, and its closure lies in a contractible open set V. Given $\alpha\colon I^{k+1} \to B$ with a lift $\alpha_0\colon \partial I^{k+1} \to X$, choose a subdivision of I^{k+1} so that the image of each simplex lies in a vertex star of the triangulation of B. Since f is UV^{k-1}, we can lift the k–skeleton of this subdivision and assume the lifts to be embeddings into $X - Z$. Attaching $(k+1)$–cells to X allows us to extend the lift over the $(k+1)$–skeleton of I^{k+1}. We could then produce the desired $UV^k(\mu)$–map, provided the image of $\pi_k(f^{-1}(U))$ in $\pi_k(f^{-1}(V))$ is finitely generated. This is certainly true if $k = 1$, as observed above.

If $k \geq 2$, then $\pi_i(f^{-1}(U)) = 0$ for $0 \leq i < k$, since f is UV^{k-1} (see Lacher [18, Theorem 3.2]). Thus, $\pi_k(f^{-1}(U)) = H_k(f^{-1}(U); \mathbb{Z})$, by the Hurewicz Isomorphism Theorem; hence, the image of $\pi_k(f^{-1}(U))$ in $\pi_k(f^{-1}(V))$ is finitely generated. Choosing a finite set of generators for each such image and attaching $(k+1)$–cells to kill the images completes the construction. □

The next result is the analogue of Proposition 8 for $k \geq 1$.

Proposition 10 *Suppose $f\colon X \to B$ is UV^{k-1} and $UV^k(\delta)$. For every $\mu > 0$ there exists an ENR $\overline{X}$ obtained by adding cells of dimension $\leq k+2$ to $X - Z$ and an extension $\overline{f}\colon \overline{X} \to B$ so that $\overline{f}$ is $UV^k(\mu)$ and $\overline{X}$ 2δ–collapses to X.*

Proof Since f is UV^{k-1}, Lemma 4 ensures that we can attach finitely many $(k+1)$–cells to X along their boundaries, forming X_1, and extend f to $f_1\colon X_1 \to B$ so that f_1 is $UV^k(\mu')$, where $0 < \mu' \ll \mu$. By Proposition 3, we may assume the attaching spheres are LCC^k embedded and mutually exclusive in $X - Z$. Let C be one such

$(k+1)$–cell. Since f is $UV^k(\delta)$, $f_1|C$ has a δ–lift to X, rel ∂, which we may assume to be an LCC^k embedding into $X-Z$. Call the image A. Attach a $(k+2)$–cell D to X_1 along $A\cup C$, obtaining X_2. The δ–homotopy of $f|A$ to $f_1|C$, rel $f|\partial A(=\partial C)$, gives us an extension of f_1 to $f_2\colon X_2\to B$ so that $f_2(D)$ has size δ in B.

Unfortunately, f_2 is only $UV^k(\delta)$. We modify the proof of Proposition 8 so that we can recover property $UV^k(\mu')$.

Use the δ–homotopy of $f|A$ to $f_1|C$, rel $f|\partial A$, to parameterize D as the quotient of $A\times I$ with the intervals in $\partial A\times I$ identified to points. Here, A is identified with $A\times 0$ and C is identified with $A\times 1$. Suppose $0<\eta\ll\mu'$. Introduce the following notation:

- J is the k–skeleton of a fine triangulation of A.
- $K\subseteq J$ is the $(k-1)$–skeleton of A.
- $R=J\times[0,1]\subseteq D$.
- $S=K\times[0,1]\subseteq R\subseteq D$.
- $L=S\cup(J\times\{0,1\})\subseteq R\subseteq D$.

Choose the triangulation of A fine enough so that if P is an i–dimensional polyhedron, $0\le i\le k$, then any map of P into D can be η–homotoped into R (over B).

By the inductive hypothesis we can η'–lift the map $f_2|L$ to X (rel $f_2|L\cap A$), for any preassigned $\eta'>0$. This gives a map $\alpha_0\colon L\to X$, which is the identity on $L\cap A$, and which we may assume results in an LCC^k embedding of $L\cup A$ into $X-Z$ (Proposition 3). Let L' be the image of this map. Since η' can be made arbitrarily small, we may use the estimated homotopy extension theorem to deform $f_2|D$ (rel $f_2|A$) slightly so that this lift is exact. Thus, we also have a map of the mapping cylinder M of the map $\alpha_0\colon L\to L'$ (rel $J\times 0$) into B with mapping cylinder fibers projecting to points in B. Attach M to X_2 along $L\cup L'$ to get X_2' and an extension $f_2'\colon X_2'\to B$ that sends mapping cylinder fibers of M to points. Observe that if M' is the portion of this mapping cylinder under S, then $X_2\cup M'$ δ–collapses to X_2.

We now have a map $\alpha=f_2|R\colon (R,L)\to B$ and a lift α_0 of $\alpha|L$ to $X-Z$. Thus, there is a μ'–lift $\overline{\alpha}\colon R\to X_1-Z$, and the μ'–homotopy between $f_1\circ\overline{\alpha}$ and α is fixed on L. This μ'–homotopy provides an extension of f_2' to the mapping cylinder $M_1\supseteq M$ (rel $R\cap A$) of $\overline{\alpha}$ so that mapping cylinder fibers have size μ' in B. Attach this mapping cylinder to X_2' along $M\cup R\cup\overline{\alpha}(R)$ to get $\overline{X}$, which δ–collapses to X_2, and extend f_2' to $\overline{f}\colon\overline{X}\to B$.

The result of this construction is to produce a relative $(k+2)$–complex $(\overline{X},X)$, which 2δ–collapses to X, such that every map of a k–dimensional polyhedron into $\overline{X}$ can be $(\eta + \mu')$–homotoped into X (over B). Lemma 1 guarantees that, if η and μ' are sufficiently small, then $\bar{f}$ is $UV^k(\mu)$.

One should observe that, although $\bar{f}$ is UV^{k-1} on X, it is *not* UV^{k-1} on $\bar{X}$. $\square$

Here is the key proposition for the proof of Theorem 7 when $k \geq 1$.

Proposition 11 *Suppose $f\colon X \to B$ is $UV^k(\epsilon)$. Then there is a constant $D(k)$, depending only on k, such that, for every $\mu > 0$, f is $(D(k)\cdot\epsilon)$–homotopic, rel $f|Z$, to a $UV^k(\mu)$–map. Moreover, the constants $D(k)$, $k \geq 0$, are related to the constants $C(k)$, $k \geq 0$ of Theorem 7 by the formula $D(k) = 2(2C(k-1)+1)$.*

Proof of Theorem 7 for $k \geq 1$ assuming Proposition 11 Suppose $f\colon X \to B$ is $UV^k(\epsilon)$. Given arbitrary $\mu > 0$, Proposition 11 assures us of the existence of a $(2(2C(k-1)+1))$–homotopy of f, rel $f|Z$, to a $UV^k(\mu)$–map. If μ is sufficiently small, we can repeat this process to get an ϵ–homotopy, rel Z, of the resulting map to one that is $UV^k(\eta)$ for every $\eta > 0$, hence, UV^k by Proposition 1. Thus, $C(k) = 4C(k-1)+3$. Since $C(-1) = 2$ (Proposition 7), we get the explicit formula $C(k) = 3\cdot 4^{k+1} - 1$. $\square$

Proof of Proposition 11 We use induction on k, the case $k = 0$ having already been established. The proof of the inductive step follows closely the proof for the case $k = 0$. We will assume Theorem 7 in dimensions $< k$. Keep in mind throughout that, unless otherwise indicated, all measurements are made in B.

Assume that $k \geq 1$ and $f\colon X \to B$ is $UV^k(\epsilon)$ for some $\epsilon > 0$. Assume, inductively, that f is $(C(k-1)\cdot\epsilon)$–homotopic, rel $f|Z$, to a UV^{k-1}–map, which we shall still call f. Then, by Lemma 3 the "new" f is now $UV^k((2C(k-1)+1)\epsilon)$. Set $\delta = (2C(k-1)+1)\epsilon$.

As in the proof of Proposition 10 build a relative $(k+2)$–complex $(\bar{X}, X)$, which 2δ–collapses to X and on which the map f extends to a $UV^k(\mu)$–map $\bar{f}\colon \bar{X} \to B$ for a given $\mu > 0$. As in the proof for $k = 0$ we need to retrace the steps in the construction of $\bar{X}$.

We start by constructing $X_1 \subseteq X_2$ from X by attaching $(k+1)$–cells to $X - Z$ to get X_1 and $(k+2)$–cells to $X_1 - Z$ to get X_2. These relative complexes come with extensions $f_1 \subset f_2$ of $f\colon X \to B$ to X_1 and X_2, respectively, such that f_1 is $UV^k(\mu')$ and f_2 is $UV^k(\delta)$, where $0 < \mu' \ll \mu$, and X_2 δ–collapses to X. Each $(k+2)$–cell D is attached to X_1 along $\partial D = A \cup C$, where C is a $(k+1)$–cell attached to X while forming X_1, and $A \subseteq X$ is the complementary $(k+1)$–cell in ∂D. We may assume, by Proposition 3, that the collection of cells A is mutually exclusive and lies in $X - Z$.

In each $(k+2)$–cell D attached to X (along a $(k+1)$–cell A in its boundary) we identify a $(k+1)$–complex $R = J \times [0,1]$, where J is the k–skeleton of a fine triangulation of A. The next step is to attach the mapping cylinder M of a map $R \to R' \subseteq X_1 \subseteq X_2$ (rel $R \cap A$) to X_2, and, after doing this for each $(k+2)$–cell D, we obtain the space $\overline{X} \supseteq X_2$ and an extension of f_2 to $\overline{f}\colon \overline{X} \to B$ that is $UV^k(\mu')$. The space $\overline{X}$ 2δ–collapses to X: the first δ–collapse comes from the collapses $M \searrow (R \cup R')$ of the relative mapping cylinders, and the second comes from the collapses $D \searrow A$.

For a given $\eta_2 > 0$, apply Theorem 6 to get a map $q_2\colon X \to X_2$ that is $UV^k(\eta_2)$ over X_2 and equal to the identity on Z. We can η_2–lift each of the complexes $R \cup R'$ to $R_1 \cup R'_1 \subseteq X - Z$ and assume by Proposition 3 that each of R_1 and R'_1 is homeomorphic to R, that each $R_1 \cup R'_1$ is embedded, and that the collection of all such lifts is mutually exclusive. By an argument similar to the one in the proof for $k = 0$, we may assume that the lifts are exact. Thus, for each complex $R_1 \cup R'_1$, there is a homeomorphism $r\colon R_1 \to R'_1$, which is the identity on $R_1 \cap R'_1$, that commutes with q_2. For each (R_1, R'_1)–pair attach the mapping cylinder M_1 of r to X forming X_3, and extend the map $q_2\colon X \to X_2$ to a map $q'\colon X_3 \to \overline{X}$, which is $UV^k(\eta_2)$ over $\overline{X}$ and the identity on Z, using the mapping cylinder identifications $M_1 \to M$.

For a given η_3, apply Theorem 6 again to get an $UV^k(\eta_3)$–map $q_3\colon X \to X_3$ over X_3, which is the identity on Z. Lemma 3 tells us that we can choose μ', η_2 and η_3 sequentially so that, after performing the constructions above, the composition

$$X \xrightarrow{q_3} X_3 \xrightarrow{q'} \overline{X} \xrightarrow{\overline{f}} B$$

is $UV^k(\mu)$ over B.

During this process f has undergone two δ–homotopies (each of which fixed Z) so that $D(k) = 2(2C(k-1)+1)$. Although the resulting map is $UV^k(\mu)$, it may no longer be UV^i for any $i = 0, \dots, k$. □

5 Proof of Theorem 1

We now show how to alter the proof of Theorem 7 to prove Theorem 1. The key is in establishing an analogous simple homotopy version corresponding to Propositions 8 and 10. We maintain our basic assumption that X is a compact *ENR* with the DDP^{k+1} and B is a compact polyhedron.

Proposition 12 *Suppose Y is a metric space, $p\colon B \to Y$ is a map, $k \geq 0$ and $f\colon X \to B$ is a UV^{k-1}– and a $UV^k(\delta)$–map over Y for some $\delta > 0$ and Z is a compact, LCC^k subset of X. Then for every $\mu > 0$, there is an ENR $\overline{X}$ obtained by adding cells of dimension $\leq k+2$ to $X - Z$ and an extension $\overline{f}\colon \overline{X} \to B$ so that $\overline{f}$ is $UV^k(\mu)$ over B and $\overline{X}$ 2δ–collapses to X over Y.*

Proof Since f is UV^{k-1}, we can attach finitely many $(k+1)$–cells to $X - Z$ along their boundaries, forming X_1, and extend f to $f_1\colon X_1 \to B$ so that f_1 is $UV^k(\mu')$ (over B), where $0 < \mu' \ll \mu$ (Lemma 4). Let C be one such $(k+1)$–cell. Since f is $UV^k(\delta)$ over Y, the map $f_1|C\colon C \to B$ has a δ–lift $g\colon C \to X$ (over Y), rel $g|\partial C$. Let $A = g(C)$ and assume, by Proposition 3, that A is LCC^{k-1} embedded in $X - Z$. Using the δ–homotopy of $f_1|C$ to $f \circ g$ (over Y), we may attach the mapping cylinder D of g, rel ∂C, to X_1 and extend f_1 to $X_1 \cup D$. Then $X_1 \cup D$ δ–collapses to X over Y.

The rest of the proof now follows as in the proofs of Propositions 8 and 10. As in the proofs of these two propositions, the map f_2 is no longer $UV^k(\mu')$ over B. The construction that remedies this defect, however, is exactly the same. □

Proof of Theorem 1 After constructing $\overline{X}$ using Proposition 12, we can apply Theorem 6 to get a homotopy of f, fixing Z and controlled over Y, to map that is $UV^k(\mu)$–map over B *and* over Y, for some preassigned $\mu > 0$. The resulting map satisfies the hypotheses of Theorem 7, which takes over to complete the proof. We need only ensure that subsequent homotopies are small enough in B so that their sizes add up to $< \epsilon$ in Y. □

6 Pseudoisotoping codimension 1 submanifolds to UV^k–maps

In this section we establish a theorem in the spirit of the early results of Keldyš [16], Cernavskii [10] and Ferry [12]. We start with the following observation.

Proposition 13 *Suppose M is a compact topological $(n+1)$–manifold, N is a locally flat n–dimensional closed submanifold, separating M into submanifolds M_1 and M_2, such that the inclusion $N \subseteq M_i$, $i = 1, 2$, is $UV^k(\mu)$, for some $\mu > 0$. Then the inclusion $N \subseteq M$ is $UV^k(\mu)$.*

Proof The proof of the proposition is fairly straightforward. Given a map $\alpha\colon (P, Q) \to (M, N)$, where P is a polyhedron of dimension $\leq k+1$, deform α slightly, keeping $\alpha|Q$ fixed, so that α sends a small regular neighborhood W of Q in P into N

as in the proof of Lemma 1. Set $P_0 = P - \operatorname{int} W$ and $Q_0 = \operatorname{bd} W$, and assume $\alpha|P_0\colon (P_0, Q_0) \to (M, N)$ are tame embeddings. By a further adjustment of α, keeping $\alpha|Q_0$ fixed, we may assume that $P_i = \alpha^{-1}(M_i)$, $i = 1, 2$, is a subpolyhedron of P_0. Now apply the $UV^k(\mu)$ assumptions on the inclusions $N \subseteq M_i$, $i = 1, 2$, to the separate pieces of P_0. □

Theorem 8 *Suppose M is a compact topological $(n+1)$–manifold and N is a locally flat, closed n–dimensional submanifold separating M into submanifolds M_1 and M_2 such that each inclusion $N \subseteq M_i$, $i = 1, 2$, is $UV^k(\epsilon)$, for some $\epsilon > 0$ and k, $2k + 3 \leq n$. Then there is constant $D(k) > 0$, depending only on k, such that, for every $\mu > 0$, there is an ambient $(D(k) \cdot \epsilon)$–isotopy on M, supported in an arbitrarily preassigned neighborhood of a $(k+2)$–dimensional polyhedron, to a homeomorphism $h\colon M \to M$ such that each of the inclusions $h(N) \subseteq M_i$, $i = 1, 2$, is $UV^k(\mu)$. Thus, there is a constant $C(k)$, depending only on k, such that the inclusion $N \subseteq M$ is ambient $(C(k) \cdot \epsilon)$–pseudoisotopic to a UV^k–map.*

Moreover, if $W \subseteq N$ is a compact, $(n-1)$–dimensional, locally flat submanifold of N, separating N into submanifolds N_1 and N_2, such that each of the inclusions $N_j \subseteq M_i$, $i, j = 1, 2$, is $UV^k(\epsilon)$, for some $\epsilon > 0$ and k, $2k + 3 \leq n$, then the isotopies can be chosen to be fixed on ∂W.

By an *ambient pseudoisotopy* on M we mean a level-preserving map $H\colon M \times I \to M \times I$ such that $H_0 = \mathrm{id}_M$ and $H|M \times [0,1)\colon M \times [0,1) \to M \times [0,1)$ is a homeomorphism.

Suppose N is a locally flat n–dimensional submanifold of a topological $(n+1)$–manifold M. Suppose $g\colon I^k \to M_1$ is a locally flat embedding of the k–cell into M such that $A = g(I^{k-1} \times 0) = g(I^k) \cap N$ is a locally flat $(k-1)$–cell in N. Let $E \subseteq N$ be a locally flat n–cell in N containing A as a properly embedded $(k-1)$–cell. The embedding g extends to a locally flat embedding, which we shall still call $g\colon E \times I \to M$, such that $g(E \times 0) = E$. Using a local collar structure of N in M in a neighborhood of E, one can find an ambient isotopy H on M, fixed outside any preassigned neighborhood of $g(E \times I)$ and on $N - \operatorname{int} E$ taking N to $(N - \operatorname{int} E) \cup g(E)$. Moreover, $H|N$ can be made arbitrarily small with respect to the projection $N \cup E \times I \to N$. We will refer to the isotopy H as a *k–shelling*. The discussion following Lemma 3 shows that there is a UV^{n-k-2}–map $h\colon (N - \operatorname{int} E) \cup g(E) \to N \cup g(I^k)$.

If $g_j\colon I^k \to M$, $j = 1, \dots, r$, is a finite collection of mutually exclusive such embeddings, and the associated shellings have mutually exclusive supports, then they can be done simultaneously, and the resulting ambient isotopy H on M will be called a *multi-k–shelling*.

Proof of Theorem 8 Proceed inductively following the proof of Theorem 1. Suppose $N \subseteq M$, M_1 and M_2, are given as in the statement of the theorem with the inclusion $N \subseteq M_i$ $UV^k(\epsilon)$, $i = 1, 2$, for some $\epsilon > 0$, $2k + 3 \le n$. Assume $G\colon M \times I \to M \times I$ is an ambient $(C(k-1)\cdot\epsilon)$–pseudoisotopy on M such that $g = G_1|N\colon N \to M$ is UV^{k-1} and, for $0 \le t < 1$, each of the inclusions $G_t(N) \subseteq M_i$, $i = 1, 2$, is $UV^{k-1}(\epsilon_t)$, where $\epsilon_t \to 0$ as $t \to 1$.

Given $\mu > 0$, apply Proposition 10 (or Proposition 8) to get an *ENR* N_i, $i = 1, 2$, obtained by adding cells of dimension $\le k + 2$ to N and an extension $g_i\colon N_i \to M_i$ of g so that g_i is $UV^k(\mu)$ and N_i 2ϵ–collapses to N. Since $2k + 3 \le n$, we may assume the cells added to N to get N_1 are disjoint from those added to get N_2.

By the estimated homotopy extension theorem there is a t, $0 \le t < 1$, such that, if $g' = G_t|N$, then g' can be extended to a $UV^k(\mu)$–map $g'_i\colon N_i \to M_i$, $i = 1, 2$, as well. If $2k + 3 < n$, we can assume the maps $g'_i\colon N_i \to M_i$ are embeddings. If $2k + 3 = n$, then we can use a standard "piping" construction to make each g'_i an embedding at the possible expense of doubling the size of the collapses $N_i \searrow N$ over M. We can now appeal to Rourke and Sanderson [27, Theorem 3.26] to get multi-$(k + 2)$–shelling G' on M such that $G'_1 G_t|N\colon N \to M$ is $UV^k(\mu)$. Controls needed to accomplish this as we expand along the cells in the collapse are the same as in the proof of Theorem 1. The only difference is that the constructions are performed inside of M.

Proposition 13 shows that the limit map $N \to M$ is UV^k.

The "moreover" part of the theorem is easily established using the methods above. □

The following is a corollary to the proof of Theorem 8.

Theorem 9 *Suppose X is a compact ENR and $Y \subseteq X$ is a closed subset such that $X - Y$ is an open topological $(n + 1)$–manifold and Y is LCC^{k+1} in X. Suppose $N \subseteq X - Y$ is a locally flat, closed n–dimensional submanifold separating X into closed components X_1 and X_2 such that each inclusion $N \subseteq X_i$, $i = 1, 2$, is $UV^k(\epsilon)$, for some $\epsilon > 0$ and $2k + 3 \le n$. Then there is constant $D(k) > 0$, depending only on k, such that, for every $\mu > 0$, there is an ambient $(D(k)\cdot\epsilon)$–isotopy on X, supported in an arbitrarily preassigned neighborhood of a $(k + 2)$–dimensional polyhedron lying in $X - Y$, to a homeomorphism $h\colon X \to X$ such that each of the inclusions $h(N) \subseteq X_i$, $i = 1, 2$, is $UV^k(\mu)$. Thus, there is a constant $C(k)$, depending only on k, such that the inclusion $N \subseteq X$ is ambient $(C(k)\cdot\epsilon)$–pseudoisotopic to a UV^k–map.*

Moreover, if $W \subseteq N$ is a compact, $(n-1)$–dimensional, locally flat submanifold of N, separating N into submanifolds N_1 and N_2, such that each of the inclusions $N_j \subseteq X_i$, $i, j = 1, 2$, is $UV^k(\epsilon)$, for some $\epsilon > 0$ and k, $2k + 3 \le n$, then the isotopies can be chosen to be fixed on ∂W.

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