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Algebraic & Geometric Topology 13 (2013) 2335–2346

Idempotent functors that preserve cofiber sequences and split suspensions

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We show that an f–localization functor L_f commutes with cofiber sequences of $(N-1)$–connected finite complexes if and only if its restriction to the collection of $(N-1)$–connected finite complexes is R–localization for some unital subring $R \subseteq \mathbb{Q}$. This leads to a homotopy theoretical characterization of the rationalization functor: the restriction of L_f to simply connected spaces (not just the finite complexes) is rationalization if and only if $L_f(S^2)$ is nontrivial and simply connected, L_f preserves cofiber sequences of simply connected finite complexes and for each simply connected finite complex K, there is a k such that $\Sigma^k L_f(K)$ splits as a wedge of copies of $L_f(S^n)$ for various values of n.

55P60, 55P62; 55P35, 55P40

Introduction

Let $f\colon P \to Q$ be a continuous function from one CW complex to another. The f–localization functor L_f is the universal example of a homotopically idempotent functor from spaces to spaces which carries f to a weak equivalence. These functors are the primary—conceivably the only[1]—examples of homotopically idempotent functors.

The R–localization functors are among the earliest defined and best behaved examples of localization functors, defined for (unital) subrings $R \subseteq \mathbb{Q}$; we denote them by L_R. There are various constructions of these, but all of them agree on simply connected spaces. They were constructed with the intention of lifting to spaces the algebraic operation of R–localization applied to homotopy and homology groups. This prescribed algebraic effect guarantees two nice homotopy theoretical properties of R–localization: L_R carries cofiber sequences of simply connected spaces to cofiber sequences and similarly for fiber sequences.

[1] Casacuberta, Scevenels and Smith have shown [2] that it is impossible to prove in ZFC that all homotopically idempotent functors take the form L_f for some f; but in the presence of Vopěnka's Principle (which is thought, but not known, to be consistent with ZFC), every homotopically idempotent functor *is* of the form L_f for some well-chosen map f.

Published: 30 June 2013 DOI: 10.2140/agt.2013.13.2335

Rationalization is the special case $R = \mathbb{Q}$, and here the theory is most powerful. The central theorems of Quillen [6] and Sullivan [8] show that the homotopy theory of simply connected rational spaces (ie, simply connected spaces for which $X \to L_{\mathbb{Q}}(X)$ is a weak equivalence) is perfectly modeled by the algebraic homotopy theories of DGLAs and of CDGAs, respectively. Rational spaces have two additional homotopy theoretic properties beyond those enjoyed by all simply connected R–local spaces: their suspensions split as wedges of copies of $L_{\mathbb{Q}}(S^n)$ for various values of n and, dually, their loop spaces split as products of copies of $K(\mathbb{Q}, n)$.

Many noncomputational theorems about rationalizations which were first proved via the algebraic machinery provided by Quillen and Sullivan can also be proved by appealing to the basic homotopy theoretical properties noted above. For example, the celebrated Mapping Theorem for Lusternik–Schnirelmann category was originally proved using Sullivan models by Félix and Halperin [4], but later a much simpler proof was found that was based on the splitting of loop spaces; see Félix and Lemaire [5]. Examples like this led the author to wonder if every localization functor L_f that satisfies the four homotopy theoretical properties listed above must be related to rationalization in some way.

In fact, rationalization is determined—overdetermined!—by these properties: it is the only f–localization functor of simply connected spaces that preserves cofiber sequences and splits suspensions. Furthermore, the restriction of R–localization to simply connected finite complexes is characterized simply by the fact that it commutes with cofiber sequences. These are the main results of this paper.

We write $\mathcal{K}(N)$ for the collection of all $(N-1)$–connected finite complexes.

Theorem 1 *Let $f\colon P \to Q$ be a map of CW complexes. Let $N \geq 2$ and suppose $L_f(S^N)$ is simply connected and not contractible. Then the following are equivalent:*

(1) *there is a subring $R \subseteq \mathbb{Q}$ and a natural transformation $\xi\colon L_R \to L_f$ such that ξ_K is a weak equivalence for every $K \in \mathcal{K}(N)$;*

(2) *L_f commutes with cofiber sequences[2] of spaces in $\mathcal{K}(N)$.*

Taking $N = 2$ in Theorem 1, we obtain a characterization of the rationalization of simply connected spaces in terms of the elementary notions of homotopy theory.

[2] There are natural comparison maps $\xi_\alpha\colon C_{L_f(\alpha)} \to L_f(C_\alpha)$; L_f commutes with the cofiber sequence $X \xrightarrow{\alpha} Y \to C_\alpha$ if ξ_α is a weak equivalence.

Theorem 2 *The restriction of a localization functor L_f to simply connected spaces is rationalization if and only if the following three conditions hold:*

- *$L_f(S^2)$ is simply connected and not weakly contractible;*
- *L_f commutes with cofiber sequences of simply connected finite complexes;*
- *if K is a simply connected finite complex, then for some $k \in \mathbb{N}$, the suspension $\Sigma^k L_f(K)$ splits as a wedge of copies of $L_f(S^n)$ for various values of n.*

Interestingly, the hypotheses of Theorem 2 only concern finite complexes, but the conclusion applies to *all* simply connected spaces. One of the referees has pointed out that we do not need the full power of the third property: all that is required is that the f–localization of certain Moore spaces $M(\mathbb{Z}/p, n)$ should split as wedges of localized spheres after repeated suspension.

We conclude this introduction with a bit of speculation. Our proof of Theorem 1 comes very close to showing that L_f commutes with cofiber sequences (and hence restricts to R–localization on $\mathcal{K}(N)$) if and only if L_f 'respects the smash and suspension structure of spheres'—that is, if and only if $\Sigma L_f(S^n)$ and $L_f(S^n) \wedge L_f(S^m)$ are f–local for all $m, n \geq N$. Using a relative version of the theory of resolving classes,[3] this can be proved under the assumption that f factors up to homotopy through a finite dimensional complex (we have not included that proof here). Is it true for all maps f?

Acknowledgements The author gratefully acknowledges the assistance of Steven Landsburg (via MathOverflow) concerning the algebraic implications of the vanishing of Tor. Javier Gutiérrez provided valuable feedback on an earlier version of this paper. Two (not necessarily distinct) anonymous referees provided valuable suggestions that led to substantial improvements in both the structure and the content of this paper.

1 Preliminaries

We will work in a fixed convenient category $\mathcal{T}_*$ of pointed topological spaces; for example $\mathcal{T}_*$ could be the category of compactly generated weak Hausdorff spaces. If so inclined, the reader may pretend that this paper was written simplicially.

We write $\operatorname{conn}(X) = n$ if X is n–connected but not $(n+1)$–connected.

[3] See the author [7] for an account of the absolute theory.

1.1 Localization functors

Let $f\colon P \to Q$ be a map between pointed CW complexes. A pointed space X is said to be *f–local* if the induced map

$$f^*\colon \mathrm{map}_*(Q, X) \longrightarrow \mathrm{map}_*(P, X)$$

of pointed mapping spaces is a weak homotopy equivalence; a map $q\colon X \to Y$ is called an *f–equivalence* if for every f–local space Z the induced map

$$q^*\colon \mathrm{map}_*(Y, Z) \longrightarrow \mathrm{map}_*(X, Z)$$

is a weak homotopy equivalence. A map $i\colon X \to L$ is said to be an *f–localization* of X if L is f–local and i is an f–equivalence.

The following important properties follow easily from the definitions.

Lemma 3 (a) *An f–equivalence between f–local spaces is a weak equivalence.*

(b) *Let $i\colon X \to L$ be f–localization and let $g\colon X \to Z$ with Z f–local. Then in the diagram*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "i"'] & Z \\ L \arrow[ur, dashed, "\gamma"'] & \end{tikzcd}$$

there is a map γ, unique up to homotopy, making the triangle commute up to homotopy.

A *coaugmented functor* is a functor $F\colon \mathcal{T}_* \to \mathcal{T}_*$ equipped with a natural transformation (its coaugmentation) $\iota\colon \mathrm{id} \to F$. A coaugmented functor F is *homotopically idempotent* if for every X the maps

$$\iota_{F(X)}, F(\iota_X)\colon F(X) \longrightarrow F(F(X))$$

are homotopic to one another and both are weak equivalences. An *f–localization functor* is a homotopically idempotent functor L_f such that, for every space X, the coaugmentation $\iota_X\colon X \to L_f(X)$ is an f–localization of X.

The main existence theorem is as follows [3, Theorem 1.A.3].

Theorem 4 (Bousfield, Farjoun) *For any map $f\colon P \to Q$ between CW complexes, there exists an f–localization functor L_f.*

Here are some basic properties of f–localization.

Proposition 5 *Let $f\colon P \to Q$ be a map of CW complexes.*

(a) *If $q\colon X \to Y$ is an f–equivalence, then so is $q \wedge \mathrm{id}_Z$ for any space Z.*

(b) *$L_f(q)$ is a weak equivalence if and only if q is an f–equivalence.*

(c) *For any X and Y, the natural map $L_f(X \times Y) \to L_f(X) \times L_f(Y)$ is a weak equivalence.*

Proof Parts (a) and (b) are [3, Example 1.D.5 and Proposition 1.C.5], respectively. Part (c) is [3, 1.A.8(e.4)]. □

If $q\colon X \to Y$ is an f–equivalence, then every f–local space is also q–local, and so the transformation j_{L_f} in the square

$$\begin{tikzcd} \mathrm{id} \arrow[r, "j"] \arrow[d, "\iota"'] & L_q \arrow[d, "L_q(\iota)"] \\ L_f \arrow[r, "j_{L_f}"] & L_q \circ L_f \end{tikzcd}$$

evaluates to a weak equivalence $j_{L_f(X)}\colon L_f(X) \to L_q(L_f(X))$ for every space X. Thus we say that $\iota\colon \mathrm{id} \to L_f$ factors through $j\colon \mathrm{id} \to L_q$ 'up to weak homotopy equivalence.' Since $L_q \circ L_f(X) \sim L_f(X)$ for all X, the functor $L_q \circ L_f$ is a perfectly good choice of f–localization functor. In this situation we will abuse notation by silently redefining L_f and referring to $L_q(\iota)$ as a *comparison transformation* $\xi\colon L_q \to L_f$.

Proposition 6 *Suppose q is an f–equivalence. If $L_q(X)$ is f–local, then the comparison $\xi_X\colon L_q(X) \to L_f(X)$ is a weak equivalence.*

Proof Any q–equivalence is *a fortiori* an f–equivalence. Since $X \to L_q(X)$ is a q–equivalence by definition and $L_q(X)$ is f–local by hypothesis, we have weak equivalences $L_f(X) \to L_f(L_q(X)) \leftarrow L_q(X)$. □

1.2 *R*–localization

We record here the definition and basic properties of localization with respect to a subring $R \subseteq \mathbb{Q}$. Write $p_n\colon S^n \to S^n$ for the degree p map from the n–sphere to itself (p may be any integer, but we usually take p to be a prime). Now for a subring $R \subseteq \mathbb{Q}$, let $\mathcal{P}(R)$ denote the set of all primes that are invertible in R, and define

$$q_R = \bigvee_{p \in \mathcal{P}(R)} p_1\colon \bigvee_{p \in \mathcal{P}(R)} S^1 \longrightarrow \bigvee_{p \in \mathcal{P}(R)} S^1.$$

The localization L_{q_R} is called *R–localization*; we abbreviate it L_R and refer to a q_R–equivalence as an *R–equivalence*. There are other (different) definitions for R–localization, but they all agree up to weak homotopy equivalence on simply connected spaces [3, 6.1.E.2–3].

We will need the following simple result.

Lemma 7 *If X is $(N-1)$–connected, then $X \to L_{\Sigma^{N-1} q_R}(X)$ is R–localization.*

It is easy to detect simply connected R–local spaces using their standard algebraic invariants. An abelian group G is said to be *R–local* if the map $G \otimes_{\mathbb{Z}} \mathbb{Z} \to G \otimes_{\mathbb{Z}} R$ induced by the inclusion $\mathbb{Z} \hookrightarrow R$ is an isomorphism.

Proposition 8 (Sullivan [8]) *Let $R \subseteq \mathbb{Q}$ be a unital subring. If X is simply connected, then the following are equivalent:*

(1) *X is R–local;*

(2) *$\pi_n(X)$ is R–local for all $n \geq 2$;*

(3) *$\widetilde{H}_n(X; \mathbb{Z})$ is R–local for all $n \geq 2$.*

Together with the exactness of the R–localization of abelian groups, Proposition 8 implies the two basic homotopy theoretical properties of R–localization of spaces.

Theorem 9 (Sullivan [8]) *If $R \subseteq \mathbb{Q}$, then L_R commutes with both fiber sequences and cofiber sequences of simply connected spaces.*

2 Proof of Theorem 1

Since it is well known that (1) implies (2), we suppose (2) and prove (1). The plan is to prove that $S^n \to L_f(S^n)$ is R–localization for $n \geq N$ and then work by induction on the cells of a finite CW complex.

2.1 The connectivity of $L_f(S^n)$

We begin by showing that $L_f(S^n)$ is simply connected for $n \geq N$. Applying (2) to the cofiber sequences

$$S^n \to * \to S^{n+1}, \quad S^n \xrightarrow{*} S^m \to S^m \vee S^{n+1},$$

reveals that $L_f(S^{n+1}) \sim \Sigma L_f(S^n)$ and $L_f(S^m \vee S^{n+1}) \sim L_f(S^m) \vee L_f(S^{n+1})$, as long as $m, n \geq N$. The first of these provides, by induction, weak equivalences $S^m \wedge L_f(S^n) \xrightarrow{\sim} L_f(S^m \wedge S^n)$ for all $m \geq 1$ and $n \geq N$. Next (with both $m, n \geq N$) apply L_f to the cofiber sequence

$$S^m \vee S^{n+1} \to S^m \times S^{n+1} \to S^m \wedge S^{n+1}$$

to obtain

$$\begin{aligned} L_f(S^m \wedge S^{n+1}) &\sim L_f(S^m \times S^{n+1})/L_f(S^m \vee S^{n+1}) \\ &\sim (L_f(S^m) \times L_f(S^{n+1}))/(L_f(S^m) \vee L_f(S^{n+1})) \\ &\sim L_f(S^m) \wedge L_f(S^{n+1}), \end{aligned}$$

using Proposition 5(c). This shows, in particular, that $L_f(S^n)$ is simply connected for $n \geq N$ and that $L_f(S^m) \wedge L_f(S^{n+1}) \sim L_f(S^{n+m+1})$ is f–local for $n, m \geq N$.

We use these preliminary results to construct the dotted arrows in the diagram

$$\begin{tikzcd} S^m \wedge S^{n+1} \arrow[r, "\mathrm{id} \wedge \iota_{S^{n+1}}"] \arrow[d, "\iota_{S^m \wedge S^{n+1}}"'] & S^m \wedge L_f(S^{n+1}) \arrow[d, "\iota_{S^m} \wedge \mathrm{id}"] \arrow[dl, dotted, "\iota"'] \\ L_f(S^m \wedge S^{n+1}) & L_f(S^m) \wedge L_f(S^{n+1}) \arrow[l, dotted, "\epsilon_{m,n+1}"'] \end{tikzcd}$$

for $m, n \geq N$. Proposition 5(a) implies that the solid arrows in the diagram are f–equivalences. Thus the composite $S^m \wedge S^{n+1} \to L_f(S^m) \wedge L_f(S^{n+1})$ is an f–equivalence to an f–local space, and therefore is f–localization. It follows from Lemma 3(b) that (up to homotopy) there is a unique map $\epsilon_{m,n+1}$ making the diagram commute up to homotopy. We define ι to be the composite in the diagram; but since $\mathrm{id} \wedge \iota_{S^{n+1}}$ is f–localization, ι is also uniquely determined up to homotopy by the commutativity of the upper left triangle. The (homotopy) commutativity of the diagram implies that $\epsilon_{m,n+1}$ and ι are f–equivalences. Since the three spaces in the lower triangle are f–local, the maps in that triangle are weak equivalences by Lemma 3(a).

Now we determine the connectivity of $L_f(S^n)$. Write $\mathrm{conn}(L_f(S^N)) = c$; since $L_f(S^N)$ is simply connected by hypothesis, we have

$$\mathrm{conn}(L_f(S^n)) = \mathrm{conn}(\Sigma^{n-N} L_f(S^N)) = c + (n - N)$$

for all $n \geq N$, so we may work with $n > N$, and we have the equation

$$\mathrm{conn}(L_f(S^n) \wedge L_f(S^n)) = 2(c + (n - N)) + 1.$$

Since $L_f(S^n)$ is simply connected, the homotopy commutative diagram of weak equivalences

$$\begin{tikzcd}
S^n \wedge L_f(S^n) \arrow[r, "\sim"] \arrow[dr, "\sim"'] & L_f(S^n) \wedge L_f(S^n) \arrow[d, "\epsilon_{n,n}", "\sim"'] & L_f(S^n) \wedge S^n \arrow[l, "\sim"'] \arrow[dl, "\sim"] \\
& L_f(S^{2n}) &
\end{tikzcd}$$

shows that

$$n+(c+(n-N)) = \operatorname{conn}(\Sigma^n L_f(S^n)) = \operatorname{conn}(L_f(S^n) \wedge L_f(S^n)) = 2(c+(n-N))+1,$$

so $c = N-1$ and consequently

$$\operatorname{conn}(L_f(S^n)) = n-1, \quad \text{for all } n \geq N.$$

2.2 $L_f(S^n)$ Is a Moore space $M(R,n)$

Next we show that there is an abelian group R such that $L_f(S^n)$ is the Moore space $M(R,n)$ for every n. This is done by determining $A_* = \Sigma^{-n}\widetilde{H}_*(L_f(S^n);\mathbb{Z})$, which is canonically identified with $\Sigma^{-m}\widetilde{H}_*(L_f(S^m);\mathbb{Z})$ for $m \geq N$ using the weak equivalences established in Section 2.1.

We expand the previous diagram to a three-dimensional diagram, writing $L = L_f(S^n)$, with $n > N$:

$$\begin{tikzcd}
S^n \wedge S^n \arrow[rr, equal] \arrow[dd, equal] \arrow[dr] & & S^n \wedge S^n \arrow[rr, equal] \arrow[dd, equal] \arrow[dr] & & S^n \wedge S^n \arrow[dd, equal] \arrow[dr] & \\
& S^n \wedge L \arrow[rr, "\iota\wedge\mathrm{id}"] \arrow[dd] & & L \wedge L \arrow[dd] & & L \wedge S^n \arrow[ll, "\mathrm{id}\wedge\iota"'] \arrow[dd] \\
S^n \wedge S^n \arrow[rr, equal] \arrow[dr] & & S^n \wedge S^n \arrow[rr, equal] \arrow[dr] & & S^n \wedge S^n \arrow[dr] & \\
& L_f(S^n \wedge L) \arrow[rr, "\sim"] & & L_f(S^{2n}) & & L_f(L \wedge S^n), \arrow[ll, "\sim"']
\end{tikzcd}$$

in which the unmarked arrows are f–localizations, and the vertical arrows are weak equivalences. Applying integral homology to the front face of this diagram results in the diagram

$$\begin{tikzcd}
\mathbb{Z}\otimes_{\mathbb{Z}} A_* \arrow[r] \arrow[d, "\kappa", "\cong"'] & A_*\otimes_{\mathbb{Z}} A_* \arrow[d, "\kappa"] & A_*\otimes_{\mathbb{Z}}\mathbb{Z} \arrow[l] \arrow[d, "\kappa", "\cong"'] \\
A_* \arrow[r, "\cong"] \arrow[d, "\cong"'] & \Sigma^{-2n}\widetilde{H}_*(L\wedge L;\mathbb{Z}) \arrow[d, "\cong"] & A_* \arrow[l, "\cong"'] \arrow[d, "\cong"] \\
A_* \arrow[r, equal] & A_* \arrow[r, equal] & A_*,
\end{tikzcd}$$

in which we use κ to denote the Künneth exterior product map. The commutativity of the diagram implies that $\kappa\colon A_*\otimes A_*\to\Sigma^{-2n}\widetilde{H}_*(L\wedge L;\mathbb{Z})$ is an isomorphism (κ is injective by the Künneth theorem). Write A_+ for the positive degree part of A_*. Since $(\mathbb{Z}\otimes_{\mathbb{Z}} A_+)\cap(A_+\otimes_{\mathbb{Z}}\mathbb{Z})=0$ in $A_*\otimes_{\mathbb{Z}} A_*$, it must be that $A_+\cong\mathbb{Z}\otimes_{\mathbb{Z}} A_+=0$ or else the vertical composite could not be injective in positive degrees. Thus A_* is some abelian group R concentrated in degree zero.

2.3 The structure of R

We complete the determination of $S^n\to L_f(S^n)$ by showing that R is isomorphic to a unital subring of $\mathbb{Q}$ and the induced map on homology is simply the inclusion $\mathbb{Z}\hookrightarrow R$.

The vertical map $\mu\colon A_*\otimes A_*\to A_*$ in the diagram of Section 2.2 endows the graded abelian group A_* with a multiplication having a two-sided unit $\mathbb{Z}\to A_*$ induced by $S^n\to L_f(S^n)$. If $n>N$ is even, this multiplication is commutative and associative. To see this, write $T\colon S^n\wedge S^n\to S^n\wedge S^n$ for the twist map that switches smash factors. Working through the proof in Section 2.1 that $L_f(S^n\wedge S^n)\sim L_f(S^n)\wedge L_f(S^n)$ reveals that $L_f(T)$ may be identified with the twist map $T\colon L_f(S^n)\wedge L_f(S^n)\to L_f(S^n)\wedge L_f(S^n)$. Since we take n to be even, $T\simeq\mathrm{id}$, and we have the homotopy commutative diagram

$$\begin{tikzcd}
S^n\wedge S^n \arrow[r, "T"] \arrow[d, "\iota\wedge\iota"'] & S^n\wedge S^n \arrow[d, "\iota\wedge\iota"] \\
L_f(S^n)\wedge L_f(S^n) \arrow[r, "T"] \arrow[d, "\epsilon_{n,n}"'] & L_f(S^n)\wedge L_f(S^n) \arrow[d, "\epsilon_{n,n}"] \\
L_f(S^{2n}) \arrow[r, "\mathrm{id}"] & L_f(S^{2n}),
\end{tikzcd}$$

which implies that the multiplication $\mu\colon A_* \otimes_{\mathbb{Z}} A_* \to A_*$ is commutative. The proof of associativity is similar, where the map T is replaced with the associator map $A\colon S^n \wedge (S^n \wedge S^n) \xrightarrow{\cong} (S^n \wedge S^n) \wedge S^n$; we omit the details.

Since the multiplication $\mu\colon R \otimes_{\mathbb{Z}} R \to R$ is an isomorphism, R is a *solid ring* (defined and studied by Bousfield and Kan in [1]). Applying the Künneth theorem to compute A_1 (which we know to be zero), we find that $\mathrm{Tor}_{\mathbb{Z}}(R, R) = 0$.

Lemma 10 *If R is a solid ring with $\mathrm{Tor}_{\mathbb{Z}}(R, R) = 0$, then R is isomorphic to a subring of $\mathbb{Q}$.*

Proof Let $T \subseteq R$ be the torsion subgroup. We cannot have $T = R$, for then $R \cong \mathbb{Z}/d$ by [1, Lemma 3.6], and this forces $\mathrm{Tor}_{\mathbb{Z}}(R, R) \neq 0$. Thus we apply [1, Lemma 3.10] to the short exact sequence

$$0 \to T \longrightarrow R \longrightarrow R/T \to 0$$

to discover that R/T is isomorphic to a subring $S \subseteq \mathbb{Q}$ (and is therefore a flat $\mathbb{Z}$–module). But also T must be a sum of cyclic groups, which implies that if $T \neq 0$, then $\mathrm{Tor}_{\mathbb{Z}}(T, T) \neq 0$. Since $R \cong S \oplus T$ as abelian groups, we may compute

$$\begin{aligned} \mathrm{Tor}_{\mathbb{Z}}(R, R) &\cong \mathrm{Tor}_{\mathbb{Z}}(S, S) \oplus \mathrm{Tor}_{\mathbb{Z}}(S, T) \oplus \mathrm{Tor}_{\mathbb{Z}}(T, S) \oplus \mathrm{Tor}_{\mathbb{Z}}(T, T) \\ &\cong \mathrm{Tor}_{\mathbb{Z}}(T, T), \end{aligned}$$

showing that if $\mathrm{Tor}_{\mathbb{Z}}(R, R) = 0$, then $T = 0$ and $R \cong S \subseteq \mathbb{Q}$. □

Lemma 10 implies that $L_f(S^n) \sim M(R, n)$ where R is a unital subring of $\mathbb{Q}$. As we observed in the beginning of this section, the unit $\mathbb{Z} \to R$ of this subring is induced by $\iota_{S^n}\colon S^n \to L_f(S^n)$. It follows that ι_{S^n} is R–localization for $n \geq N$.

2.4 Finishing the proof of Theorem 1

Write $q = \Sigma^{N-1} q_R$. To finish the proof of Theorem 1, we show that $L_f(q)$ is a weak equivalence, so that there is a comparison transformation $\xi\colon L_q \to L_f$. Our work so far implies ξ_{S^n} is a weak equivalence for $n \geq N$ and since both L_q and L_f commute with cofiber sequences of finite $(N-1)$–connected complexes, an easy induction shows that ξ_K is a weak equivalence for every finite $(N-1)$–connected complex K.

Let p be a prime that is invertible in R, and consider $L_f(p_n)$ with $n \geq N$. This map is understood using the diagram

$$\begin{tikzcd} S^n \arrow[r, "p_n"] \arrow[d, "\iota"'] & S^n \arrow[d, "\iota"] & \\ M(R,n) \arrow[r, "L_f(p_n)"] & M(R,n) \arrow[r, "\mathrm{id}/p"] & M(R,n). \end{tikzcd}$$

Since everything in this diagram is in the stable range, we have

$$(\mathrm{id}/p)\circ\iota\circ p_n \simeq (p\cdot\iota)/p \simeq \iota.$$

Now Lemma 3(b) implies that $(\mathrm{id}/p)\circ L_f(p_n)\simeq \mathrm{id}_{M(R,n)}$, so that $L_f(p_n)$ is a weak equivalence provided p is invertible in R.

Write $q=\Sigma^{N-1}q_R=\bigvee_{p\in\mathcal{P}(R)}p_N$. Since $L_f(p_N)$ is a weak equivalence for every $p\in\mathcal{P}(R)$, we see that every f–local space is also q–local, and so there is a comparison transformation $\xi\colon L_q\to L_f$. Applied to S^n with $n\geq N$ we have

$$\begin{tikzcd} & S^n \arrow[dl] \arrow[dr] & \\ L_q(S^n) \arrow[rr, "\xi_{S^n}"] & & L_f(S^n) \end{tikzcd}$$

in which both diagonal maps are R–localizations, forcing ξ_{S^n} to be a weak equivalence. Finally, suppose ξ_K is a weak equivalence for all CW complexes with at most k cells. If L has at most $k+1$ cells, then there is a cofiber sequence $S^n\to K\to L\to S^{n+1}\to\Sigma K$ in which K and ΣK have at most k cells. Applying ξ we obtain the diagram

$$\begin{tikzcd} L_q(S^n) \arrow[r] \arrow[d, "\xi_{S^n}"] & L_q(K) \arrow[r] \arrow[d, "\xi_K"] & L_q(L) \arrow[r] \arrow[d, "\xi_L"] & L_q(S^{n+1}) \arrow[r] \arrow[d, "\xi_{S^{n+1}}"] & L_q(\Sigma K) \arrow[d, "\xi_{\Sigma K}"] \\ L_f(S^n) \arrow[r] & L_f(K) \arrow[r] & L_f(L) \arrow[r] & L_f(S^{n+1}) \arrow[r] & L_f(\Sigma L). \end{tikzcd}$$

The inductive hypothesis shows that the maps ξ_{S^n}, ξ_K, $\xi_{S^{n+1}}$ and $\xi_{\Sigma K}$ are weak equivalences. Since all the spaces are simply connected and the rows are cofiber sequences, ξ_L is also a weak equivalence, completing the proof of Theorem 1. $\square$

3 Proof of Theorem 2

First of all, it is well known that the three conditions of Theorem 2 hold for the rationalization of simply connected spaces.

To prove the converse, suppose L_f satisfies those conditions. Then Theorem 1 gives, for some $R\subseteq\mathbb{Q}$, a comparison transformation $\xi\colon L_{\Sigma q_R}\to L_f$ which evaluates to weak equivalences on simply connected finite complexes. But R must be $\mathbb{Q}$, for if the prime p is not invertible in R, then the Moore space $M(\mathbb{Z}/p,n)$ is R–local and cannot split as copies of $L_f(S^{n_\alpha})=M(R,n_\alpha)$, even after repeated suspension.

Let $\mathcal{L}$ denote the collection of f–local spaces. We know that $S^n_{\mathbb{Q}} \in \mathcal{L}$ for $n \geq 2$, and in particular $K(\mathbb{Q}, 2n+1) \sim S^{2n+1}_{\mathbb{Q}} \in \mathcal{L}$ for all $n \geq 1$; then we obtain that $K(\mathbb{Q}, 2n) \sim \Omega K(\mathbb{Q}, 2n+1) \in \mathcal{L}$ by [3, 1.A.8(e.1)]. Therefore

$$\prod_{\mathcal{I}} K(\mathbb{Q}, n) \sim K\left(\textstyle\prod_{\mathcal{I}} \mathbb{Q}, n\right) \in \mathcal{L}$$

for all index sets $\mathcal{I}$. Since every rational vector space V is a retract of a vector space of this form, $K(V, n) \in \mathcal{L}$ for any rational vector space V. We deduce that every simply connected rational Postnikov piece is in $\mathcal{L}$, and since $\mathcal{L}$ is closed under homotopy limits [3, 1.A.8(e.3)], every simply connected rational space is f–local.

Lemma 7 implies that if X is simply connected, then $X \to L_{\Sigma q_{\mathbb{Q}}}(X)$ is rationalization, and in particular that $L_{\Sigma q_{\mathbb{Q}}}(X)$ is a simply connected rational space, and hence is f–local. Therefore Proposition 6 implies that $\xi_X \colon L_{\Sigma q_{\mathbb{Q}}}(X) \to L_f(X)$ is a weak equivalence for simply connected X, which completes the proof of Theorem 2. □

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Received: 21 October 2012 Revised: 15 February 2013