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Growth of periodic quotients of hyperbolic groups

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Let G be a non-elementary torsion-free hyperbolic group. We prove that the exponential growth rate of the periodic quotient G/G^n tends to the one of G as n odd approaches infinity. Moreover, we provide an estimate for the rate at which the convergence is taking place.

20F65; 20F50, 20F67, 20F69

1 Introduction

A group G is said to have finite exponent n if for every $g \in G$, $g^n = 1$. At the beginning of the 20[th] century, W Burnside [4] posed the following problem (now known as the *Bounded Burnside Problem*). Is a finitely generated group with finite exponent necessarily finite? In order to study this question, the natural object to look at is the free Burnside group of rank k and exponent n denoted by $\mathbf{B}_k(n)$. It is the quotient of the free group of rank k, denoted by $\mathbf{F}_k$, by the (normal) subgroup $\mathbf{F}_k^n$ generated by the n^{th} power of all elements of $\mathbf{F}_k$. It is the largest group of rank k and exponent n.

For a long time it was only established that $\mathbf{B}_k(n)$ was finite for some small exponents ($n=2$ due to Burnside [4], $n=3$ due to Burnside [4] and Levi and van der Waerden [16], $n=4$ due to Sanov [22] and $n=6$ due to Hall [13]). The finiteness of $\mathbf{B}_2(5)$ is still open. In 1968, P S Novikov and S I Adian achieved a breakthrough. In a series of three articles [18], they provided the first examples of infinite free Burnside groups. More precisely, they proved the following result. If $k \geqslant 2$ and n is an odd integer larger than or equal to 4381, then $\mathbf{B}_k(n)$ is infinite. Their result has been improved in many directions. In particular, S V Ivanov [14] and I G Lysenok [17] solved the case of even exponents. Since free Burnside groups of sufficiently large exponents are infinite, a natural question is how "big" they are. This can be measured by the exponential growth rate.

Given a finitely generated group G endowed with the word metric with respect to some finite generating set of G, its *(exponential) growth rate* is defined to be

$$\lambda = \lim_{r\to\infty} \sqrt[r]{|B(r)|},$$

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where $|B(r)|$ denotes the cardinality of the ball of radius r of G. If $\lambda > 1$, one says that G has *exponential growth* (λ depends on the generating set, however having exponential growth is a property of the group G). Furthermore, if for every generating set, the corresponding growth rate is uniformly bounded away from 1, then G has *uniform exponential growth*.

In his book [1], S I Adian proved that free Burnside groups of sufficiently large odd exponents are not only infinite but also exponentially growing. Later D Osin showed that they are uniformly non-amenable, and therefore they have uniform exponential growth [21]. Another approach can be found in the paper by Atabekyan [3].

In 1991, using a diagrammatical description of graded small cancellation theory, A Y Ol'shanskiĭ proved an analogue of the Novikov–Adian Theorem [20] for hyperbolic groups.

Theorem 1.1 (Ol'shanskiĭ [20]) *Let G be a non-elementary torsion-free hyperbolic group. There exists a critical exponent n_0 such that for all odd integers $n \geqslant n_0$, the quotient G/G^n is infinite.*

Non-elementary hyperbolic groups are known to have uniform exponential growth (see Koubi [15]). On the other hand, hyperbolic groups are growth tight (see Arzhantseva and Lysenok [2]). This means that, given such a group G and a finite generating set A, for any infinite normal subgroup N of G, the exponential growth rate of G/N with respect to the natural image of A is strictly less than the one of G with respect to A. Therefore we were wondering what the growth rate of the periodic quotients G/G^n could be. In particular, is there a gap between the respective growth rates of G and G/G^n? The following theorem answers this question negatively: the growth rate of G/G^n converges to the one of G as n odd approaches infinity. Moreover, we provide an estimate for the rate at which this convergence is taking place.

Theorem 1.2 *Let G be a non-elementary torsion-free hyperbolic group and λ its exponential growth rate with respect to a finite generating set A of G. There exists a positive number κ such that for sufficiently large odd exponents n, the exponential growth rate of G/G^n with respect to the image of A is at least*

$$\lambda\left(1-\frac{\kappa}{n}\right).$$

In the case of free Burnside groups we even have a much more accurate estimate.

Theorem 1.3 *Let $k \geqslant 2$. Let A be a free generating set of $\mathbf{F}_k$ (ie with exactly k elements). There exists a positive number κ such that for sufficiently large odd* *exponents n, the exponential growth rate of $\mathbf{B}_k(n)$ with respect to the image of A is larger than*

$$(2k-1)\left(1-\frac{\kappa}{(2k-1)^{n/2}}\right).$$

Our proof extends the ideas of S I Adian. However, considering hyperbolic groups instead of free groups makes it much more complicated and requires new tools. Let us first recall the key argument of Adian's approach.

Main fact Let A be a free generating set of $\mathbf{F}_k$. Let w be a reduced word over the alphabet $A \cup A^{-1}$. We say that w *contains an m^{th} power* if there exists a non-trivial cyclically reduced word w_0 over $A \cup A^{-1}$ such that, as a concatenation of words, w can be written $w = w_- w_0^m w_+$. Let g be an element of $\mathbf{F}_k$. If the reduced word representing g does not contain a 16^{th} power, then g induces a non-trivial element of $\mathbf{B}_k(n)$ for every odd integer $n \geqslant 665$.

In particular, two distinct reduced words which do not contain an 8^{th} power induce different elements of $\mathbf{B}_k(n)$. Therefore, it is sufficient to estimate the growth rate of the set of reduced words without 8^{th} power. This is done by induction on the length of the words. The main steps of this proof are recalled in Section 4.1.

Consider now an arbitrary non-elementary torsion-free hyperbolic group G endowed with the word metric $|\cdot|$. Following A Y Ol'shanskiĭ, an (L,m)–power is an element of G that can be written $uv^m u'$ where u and u' have length at most L. An element g of G is (L,m)–aperiodic if it cannot be written $g = g_1 g_2 g_3$ where $|g| = |g_1| + |g_2| + |g_3|$ and g_2 is an (L,m)–power. The proof of Theorem 1.1 relies on the following fact. In [20], A Y Ol'shanskiĭ proved the existence of constants L, ϵ and n_0, which depend on G, with the following property. Let n be an odd integer larger than n_0. Then the set of $(L, \lfloor \epsilon n \rfloor)$–aperiodic elements embeds into G/G^n. The infiniteness of G/G^n follows from the one of the set of $(L, \lfloor \epsilon n \rfloor)$–aperiodic elements. Another approach based on techniques introduced by T Delzant and M Gromov [10] can be found in Coulon [8].

Hence one way to prove Theorem 1.2 is to compute the growth rate of the set of (L,m)–aperiodic elements. Instead of reasoning with words, we consider geodesic paths in the Cayley graph X of G. However, this definition of (L,m)–aperiodic elements does not behave well with the operations of extending geodesics or taking subgeodesics. While working with words, as in the free group, we can say the following. Let w be a reduced word and a a letter of $A \cup A^{-1}$. If wa does not contain an m^{th} power but wa does, then the word wa can be written $wa = w_- w_0^m$ (the m^{th} power occurs at the end of the word). A similar statement is not true for geodesics in X. Let g and h be two elements of G such that g lies on a geodesic σ between 1 and h. If g is (L,m)–aperiodic but h is not, then the (L,m)–power in h is not necessarily contained in the part of σ between g and h. Indeed, since X is not uniquely geodesic, the element g could contain a $(L+2\delta,m)$–power as illustrated in Figure 1.

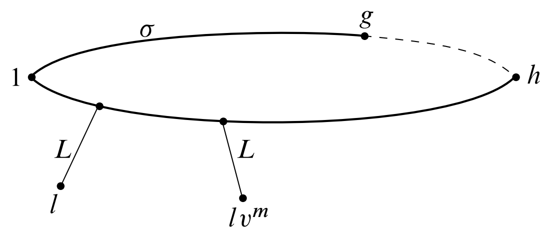

Figure 1: Extending aperiodic elements

To avoid this difficulty, we focus on a particular set of geodesics. We fix a spanning tree in X such that for every $h \in G$, the path σ_h joining 1 to h in this tree is geodesic. We call such a path a *selected geodesic*. In particular, if $g \in G$ lies on σ_h, then σ_g is the subpath of σ_h between 1 and g. Then we adopt the following definition. An element $g \in G$ contains an (L,m)–power if there are $l \in G$ and a non-trivial cyclically reduced element $v \in G$ such that both l and lv^m belong to the L–neighborhood of σ_g.

This adaptation leads to another difficulty. Given an element $g \in G$, we need to be sure that σ_g has sufficiently many selected extensions. However this could be impossible (see Figure 2). This question is handled in Section 3. For every r, we construct a subset K of G which, among others, satisfies the following property. For all $g \in K$, the number of elements $h \in K$ such that σ_h extends σ_g by a length r is at least $\kappa_1\lambda^r$, where κ_1 is some positive constant which depends on G and A and λ is the exponential growth rate of G. Our proof uses as a tool the Cannon cone types [5].

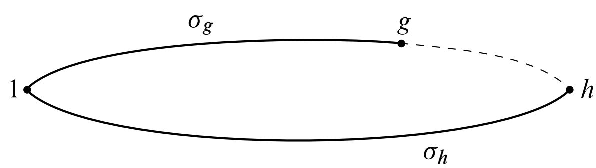

Figure 2: Extending selected geodesics

Finally, we prove that the set of (L,m)–aperiodic elements of K grows exponentially with a rate at least $\lambda(1-a/m)$ (see Section 4). Our theorem follows then from Ol'shanskiĭ's work (see Section 5).

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2 Hyperbolic geometry

In this section we fix notations and review some of the standard facts on hyperbolic spaces and hyperbolic groups (in the sense of Gromov). For more details we refer the reader to the original paper of M Gromov [12] or to Coornaert, Delzant and Papadopoulos [7] or Ghys and de la Harpe [11].

Let G be a group generated by a finite set A. We denote by X the Cayley graph of G with respect to A. The vertices of X are the elements of G. For every $g \in G$ and $a \in A \cup A^{-1}$, g is joined to ga by an edge labeled by a. The group G acts on the left by isometries on X.

Given two points x and x' in X, $|x-x'|$ stands for the distance between them. The Gromov product of three points $x, y, z \in X$ is defined by

$$\langle x, y \rangle_z = \frac{1}{2}\left\{ |x-z| + |y-z| - |x-y| \right\}.$$

The space X is said to be δ–hyperbolic if for all $x, y, z, t \in X$,

$$\langle x, y \rangle_t \geqslant \min\left\{ \langle x, z \rangle_t , \langle z, y \rangle_t \right\} - \delta.$$

Remark The constant δ depends on A. Nevertheless for a group, being hyperbolic (for some δ) does not depend on the generating set. In this article we fix once for all the generating set A. Therefore the hyperbolicity constant δ of X is fixed as well. Beside the hyperbolicity constant, the proof of the main theorem involves many parameters. To help the reader, greek letters will represent constants which only depend on G and A and not on the other objects that will be introduced later (eg in Sections 3.2 and 4.2).

In the rest of the article we assume that G is torsion-free and non-elementary, ie non virtually cyclic.

As a consequence of hyperbolicity, the geodesic triangles of X are 4δ–thin, ie for every $x, y, z \in X$, for every p (respectively q) lying on a geodesic between x and y (respectively between x and z), if $|x-p| = |x-q| \leqslant \langle y, z \rangle_x$ then $|p-q| \leqslant 4\delta$.

Let $g \in G$. For simplicity of notation, $|g|$ stands for the distance $|g-1|$. This is exactly the word length of g with respect to A. To measure the action of g on X, we define two quantities: the *translation length* $[g]$ and the *stable translation length* $[g]^\infty$.

$$[g] = \inf_{x \in X} |gx - x| \,; \quad [g]^\infty = \lim_{n \to \infty} \frac{1}{n} \left| g^n x - x \right|.$$

They are related as explained in the following proposition.

Proposition 2.1 (Coornaert, Delzant and Papadopoulos [7, Chapitre 10, Proposition 6.4]) *For every element $g \in G$, we have*

$$[g]^\infty \leqslant [g] \leqslant [g]^\infty + 50\delta.$$

A *cyclically reduced element* is an element $g \in G$ such that $[g] = |g|$. Every conjugacy class of G contains such an element. The set of all non-trivial cyclically reduced elements is denoted by C.

If $[g]^\infty$ is positive then g is called *hyperbolic*. It is known that every element of G either is hyperbolic or has finite order (in the latter case it is said to be *elliptic*). For our purpose we assumed that G was torsion-free. Therefore every non-trivial element is hyperbolic.

In a hyperbolic group, the range of stable translation lengths is discrete:

Theorem 2.2 (Delzant [9, Proposition 3.1]) *There exists a constant $\tau \in \mathbb{Q}_+^*$ such that for all $g \in G$, $[g]^\infty \in \tau\mathbb{N}$. In particular, for every hyperbolic element $g \in G$, $[g]^\infty \geqslant \tau$.*

Given $r \in \mathbb{R}$, we write $B(r)$ for the *closed ball* of G of center 1 and radius r, ie the set of elements $g \in G$ such that $|g| \leqslant r$. If r is an integer, the *sphere of radius* r, denoted by $S(r)$, is the set of elements $g \in G$ such that $|g| = r$.

If P is a finite subset of G, $|P|$ stands for its cardinality. In order to estimate the size of an infinite subset of G, we use the exponential growth rate.

Definition 2.3 Let P be a subset of G. The *(exponential) growth rate* of P is the quantity

$$\limsup_{r \to +\infty} \sqrt[r]{|P \cap B(r)|}.$$

We denote by λ the growth rate of G. Since the map $r \to |B(r)|$ is submultiplicative, λ satisfies in fact

$$\lambda = \lim_{r\to+\infty} \sqrt[r]{|B(r)|} = \inf_{r\in\mathbb{N}^*} \sqrt[r]{|B(r)|}.$$

In particular, for all $r \in \mathbb{N}$, $|B(r)| \geqslant \lambda^r$. The next proposition gives an upper bound for $|B(r)|$.

Proposition 2.4 (Coornaert [6]) *There exists $\alpha \geqslant 1$ such that for all $r \in \mathbb{R}_+$, $|B(r)| \leqslant \alpha\lambda^r$.*

3 Growth of cone types

3.1 Essential cone types

Definition 3.1 Let $g \in G$. The *cone type* of g is the set of elements $u \in G$ such that there exists a geodesic of X between 1 and gu that passes through g. We denote it by T_g.

We write $\mathcal{T}$ for the set of all cone types. One important feature of hyperbolic groups is given by the following proposition.

Proposition 3.2 (Coornaert, Delzant and Papadopoulos [7, Chapitre 12, Théorème 3.2]) *If G is a hyperbolic group, then the set $\mathcal{T}$ of all cone types is finite.*

Definition 3.3 We say that a cone type $T \in \mathcal{T}$ is *essential* if

$$\exists c > 0,\ \forall r \in \mathbb{N},\ \exists s \in \mathbb{N} \cap [r, +\infty), \quad |T \cap B(s)| \geqslant c\lambda^s,$$

where λ is the growth rate of G.

Notation An element $g \in G$ is *essential* if its cone type T_g is essential. The set of all essential elements is denoted by E. We write $\mathcal{T}_E$ for the set of all essential cone types.

Remark It follows easily from the definition that the growth rate of an essential cone type is exactly λ. Roughly speaking, the essential elements are the ones who are responsible for the growth of G.

Proposition 3.4 *There exists $\beta > 0$ such that for all $T \in \mathcal{T}_E$, for all $r \in \mathbb{N}$,*

$$|T \cap B(r)| \geqslant \beta\lambda^r.$$

Proof Note that $\mathcal{T}_E$ is finite. Hence it is sufficient to prove the following statement:

$$\forall T \in \mathcal{T}_E,\ \exists \beta > 0,\ \forall r \in \mathbb{N}, \quad |T \cap B(r)| \geqslant \beta\lambda^r.$$

Let T be an essential cone type. By definition there exists $c > 0$ such that for all $r \in \mathbb{N}$ there is an integer $s \geqslant r$ satisfying $|T \cap B(s)| \geqslant c\lambda^s$. Let $r \in \mathbb{N}$. We denote by s an integer greater than or equal to r such that $|T \cap B(s)| \geqslant c\lambda^s$. Every element of $T \cap B(s)$ can be written uv where $u \in T \cap B(r)$ and $v \in B(s-r)$. Consequently, $|T \cap B(s)| \leqslant |T \cap B(r)| \cdot |B(s-r)|$. Using Proposition 2.4 we obtain

$$c\lambda^s \leqslant |T \cap B(s)| \leqslant \alpha\lambda^{s-r}\, |T \cap B(r)|.$$

Thus for all $r \in \mathbb{N}$, $|T \cap B(r)| \geqslant \alpha^{-1}c\lambda^r$. □

Lemma 3.5 *Let $g \in G$. Let $u \in T_g$. If gu is essential then so is g.*

Proof By definition of the cone type, uT_{gu} is a subset of T_g. Hence for all integers $r \geqslant |u|$, $\left|T_g \cap B(r)\right| \geqslant \left|T_{gu} \cap B(r-|u|)\right|$. However, gu is essential. It follows from Proposition 3.4 that for all integers $r \geqslant |u|$,

$$\left|T_g \cap B(r)\right| \geqslant \left|T_{gu} \cap B(r-|u|)\right| \geqslant \beta\lambda^{-|u|}\lambda^r.$$

Thus g is essential. □

Let $g \in G$ and $u \in T_g$. According to the previous lemma, if gu is essential, so is g. The converse statement is not necessarily true. Nevertheless, the next proposition gives a lower bound for $\left|g(T_g \cap S(r)) \cap E\right|$ which is the number of elements $u \in T_g \cap S(r)$ such that gu is essential. (Recall that $S(r)$ is the sphere of radius r defined in Section 2.)

Proposition 3.6 *There exists $\gamma \in (0,1)$ such that for all $g \in E$ and for all $r \in \mathbb{N}$,*

$$\left|g(T_g \cap S(r)) \cap E\right| \geqslant \gamma\lambda^r.$$

Proof Let $\gamma \in (0,1)$. Suppose the proposition were false. There exist an essential element $g \in E$ and $r \in \mathbb{N}$ such that $\left|g(T_g \cap S(r)) \cap E\right| < \gamma\lambda^r$. Negating the definition of essential cone types, we have

$$\forall T \in \mathcal{T} \setminus \mathcal{T}_E,\ \exists s \in \mathbb{N},\ \forall t \in \mathbb{N} \cap [s, +\infty), \quad |T \cap B(t)| < \gamma\lambda^t.$$

Recall that the set of cone types $\mathcal{T}$ is finite. Thus we have in fact

(1) $$\exists s \in \mathbb{N},\ \forall T \in \mathcal{T} \setminus \mathcal{T}_E,\ \forall t \in \mathbb{N} \cap [s, +\infty), \quad |T \cap B(t)| < \gamma\lambda^t.$$

Let $t \geqslant s$ be an integer. It follows from the definition of cone types that

$$T_g \cap B(r+t) \subset T_g \cap B(r-1) \cup \Bigg(\bigcup_{h \in g(T_g \cap S(r))} g^{-1}h(T_h \cap B(t)) \Bigg).$$

Since g is essential, Proposition 3.4 yields

$$\beta\lambda^{r+t} \leqslant \left|T_g \cap B(r+t)\right| \leqslant \left|T_g \cap B(r-1)\right| + \sum_{h \in g(T_g \cap S(r))} |T_h \cap B(t)|\,.$$

Let $h \in g(T_g \cap S(r))$. If h is not essential, applying (1), we get $|T_h \cap B(t)| < \gamma\lambda^t$. On the other hand, if h is essential, then Proposition 2.4 leads to $|T_h \cap B(t)| \leqslant |B(t)| \leqslant \alpha\lambda^t$. It follows that

$$\left|T_g \cap B(r+t)\right| \leqslant \left|T_g \cap B(r-1)\right| + \gamma\lambda^t \left|g(T_g \cap S(r)) \setminus E\right| + \alpha\lambda^t \left|g(T_g \cap S(r)) \cap E\right|.$$

However, by assumption, $\left|g(T_g \cap S(r)) \cap E\right| < \gamma\lambda^r$. Moreover, by Proposition 2.4 we have $\left|T_g \cap B(r-1)\right| \leqslant \alpha\lambda^{r-1}$ and $\left|g(T_g \cap S(r)) \setminus E\right| \leqslant \alpha\lambda^r$. Thus for all integers $t \geqslant s$, we get

$$\beta\lambda^{r+t} \leqslant \alpha\lambda^{r-1} + 2\alpha\gamma\lambda^{r+t}.$$

Therefore $0 < \beta \leqslant 2\alpha\gamma$. This inequality holds for all sufficiently small $\gamma > 0$, which is impossible. □

3.2 Selected cone types

From now on we fix a spanning tree in the Cayley graph X of G such that for every $h \in G$, the path σ_h joining 1 to h in this tree is geodesic in X. The path σ_h is called a *selected geodesic*. Note that if $g \in G$ lies on σ_h, then σ_g is exactly the subpath of σ_h between 1 and g.

Remark Such a tree can be obtained in the following way. Let us fix an arbitrary order on $A \cup A^{-1}$. Let $g, h \in G$. Using the labeling of the edges of X, any geodesic joining g to h can be identified with a word over the alphabet $A \cup A^{-1}$ representing $g^{-1}h$. Thus the set of geodesics inherits the lexicographic order (we read the words from the left to the right). For all $h \in G$, σ_h is the geodesic joining 1 to h which is the smallest for the lexicographic order.

Definition 3.7 Let $g \in G$. The *selected cone type* of g is the set of elements $u \in G$ such that σ_{gu} passes through g. We denote it by L_g.

Remark It follows from the definition that L_g is a subset of T_g. Contrary to $\mathcal{T}$, it is not clear whether or not the set of all selected cone types is finite.

Our goal is to construct a subset K of G such that its elements satisfy analogues for the selected cone types of Lemma 3.5 and Proposition 3.6. To that end, we need the following lemma.

Lemma 3.8 (Arzhantseva and Lysenok [2, Lemma 5]) *There exists a constant $\rho \in (0,1)$ satisfying the following. For every finite subset P of G, there is a subset P' of P such that $|P'| \geq \rho |P|$ and for all distinct g, g' in P', $|g - g'| > 20\delta$.*

Recall that γ is the constant given by Proposition 3.6. Let us put $\kappa_1 = \rho\gamma\, |B(4\delta)|^{-1}$. Note that κ_1 belongs to $(0,1)$. Let r be an integer larger than 10δ. The set K that we are going to build will depend on the parameter r, which represents a distance. However, for simplicity, we do not mention the dependence on r in the notation. First, we construct by induction a sequence (H_i) of subsets of G.

- Put $H_0 = G$.
- Let $i \in \mathbb{N}$. Assume that H_i is already defined. The set H_{i+1} is

$$H_{i+1} = \left\{ g \in G \;\middle|\; \left| g(L_g \cap S(r)) \cap H_i \right| \geq \kappa_1 \lambda^r \right\}.$$

The set H is defined to be the intersection of all H_i's.

Lemma 3.9 *The sequence (H_i) is non-increasing for the inclusion.*

Proof The proof is by induction on i. First note that H_1 is contained in $H_0 = G$. Assume now that H_i is contained in H_{i-1}. Given $g \in H_{i+1}$ we have

$$\left| g(L_g \cap S(r)) \cap H_{i-1} \right| \geq \left| g(L_g \cap S(r)) \cap H_i \right| \geq \kappa_1 \lambda^r.$$

Thus g belongs to H_i. Consequently, H_{i+1} is a subset of H_i. □

Corollary 3.10 *For all $g \in H$, $\left| g(L_g \cap S(r)) \cap H \right| \geq \kappa_1 \lambda^r$.*

Proposition 3.11 *For all $g \in E$, there exists g' in $gB(4\delta) \cap H$ such that $|g'| = |g|$.*

Proof According to the definition of H, it is sufficient to prove the following statement. For all $i \in \mathbb{N}$, for all $g \in E$, there exists g' in $gB(4\delta) \cap H_i$ such that $|g'| = |g|$. To that end, we proceed by induction on i. If $i = 0$, the statement follows from the fact that $H_0 = G$. Assume now that the statement holds for $i \in \mathbb{N}$. Let $g \in E$. According to the choice of γ, $\left| g(T_g \cap S(r)) \cap E \right| \geq \gamma \lambda^r$ (see Proposition 3.6). By the choice of ρ, there exists a subset P of $g(T_g \cap S(r)) \cap E$ such that

(i) $|P| \geq \rho \left| g(T_g \cap S(r)) \cap E \right| \geq \rho\gamma\lambda^r$,

(ii) for all distinct h_1, h_2 in P, $|h_1 - h_2| > 20\delta$.

(See Lemma 3.8.) All elements of P are essential and have length $|g| + r$. According to the induction assumption, for every $h \in P$, the ball $hB(4\delta)$ contains an element h' of H_i of length $|g| + r$. We denote by P' the set of all elements h' obtained in this way (see Figure 3). Recall that two elements of P are a distance at least 20δ apart, therefore P' contains at least $|P|$ elements.

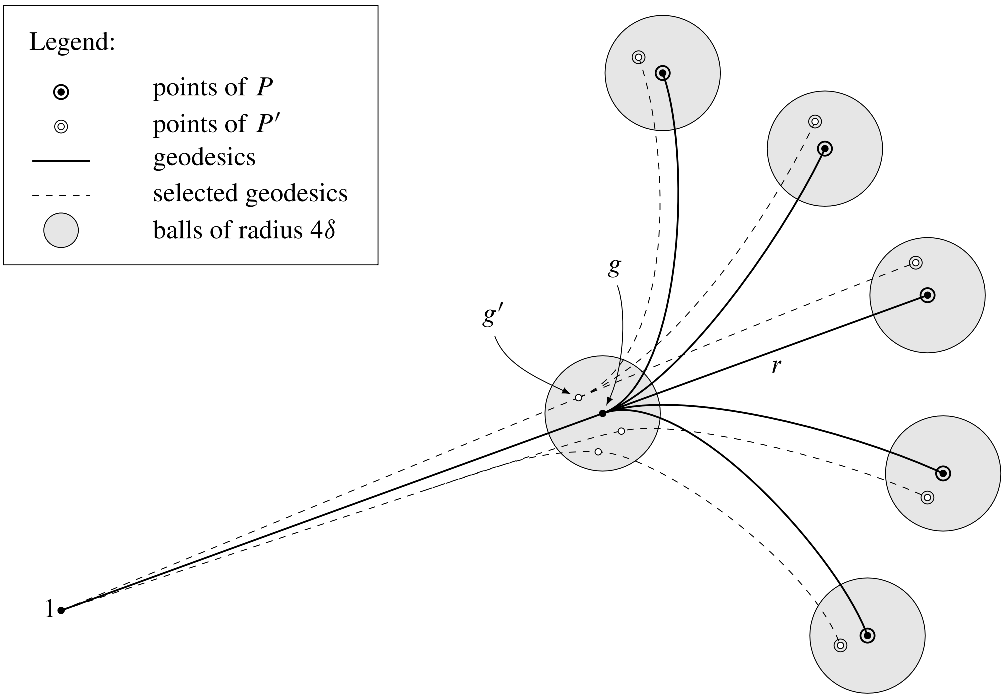

Figure 3: Elements of P and P'

Let $h \in P$ and h' be an element of P' contained in the ball $hB(4\delta)$. We denote by g' the point of $\sigma_{h'}$ such that $|g'| = |g|$. In this way h' belongs to $g'(L_{g'} \cap S(r))$. By construction g (respectively g') lies on a geodesic between 1 and h (respectively h'). Moreover, $|h - h'| \leqslant 4\delta$, thus

$$\langle h, h' \rangle_1 \geqslant \tfrac{1}{2}\big(|h| + |h'|\big) - 2\delta = |g| + r - 2\delta \geqslant |g| = |g'|.$$

The triangle with vertices 1, h and h' being 4δ–thin, $|g - g'| \leqslant 4\delta$. Consequently, P' is a subset of H_i contained in the union of all $g'(L_{g'} \cap S(r))$ where $g' \in gB(4\delta) \cap S(|g|)$. In particular, there is $g' \in gB(4\delta) \cap S(|g|)$ such that

$$\left|g'\left(L_{g'} \cap S(r)\right) \cap H_i\right| \geqslant |B(4\delta)|^{-1} \left|P'\right| \geqslant |B(4\delta)|^{-1} |P| \geqslant \rho\gamma \, |B(4\delta)|^{-1} \lambda^r = \kappa_1 \lambda^r.$$

By definition of H_{i+1}, g' belongs to H_{i+1}. Thus the statement holds for $i+1$. □

Corollary 3.12 *The element* 1 *belongs to* H.

Proof Since $T_1 = G$, 1 is essential. According to Proposition 3.11 there exists an element of length 0 in H, ie $1 \in H$. □

Corollary 3.10 is an analogue for the selected cone types of Proposition 3.6. However, given $g \in G$ and $u \in L_g$, if gu lies in H, there is no reason that g should also belong to H. That is why we have to consider a subset K of H which will in addition satisfy an analogue of Lemma 3.5 (see Proposition 3.13 (iii)). To that end we proceed by induction.

- Put $K_0 = \{1\}$.
- Let $i \in \mathbb{N}$ such that K_i is already defined. The set K_{i+1} is given by

$$K_{i+1} = \bigcup_{g \in K_i} g\big(L_g \cap S(r)\big) \cap H.$$

Finally, the set K is the union of all the K_i's. Note that K is a subset of H. Moreover, for all $g \in K$, $g(L_g \cap S(r)) \cap H$ lies inside K. Therefore by Corollary 3.10 we have

$$\forall g \in K, \quad \left|g\big(L_g \cap S(r)\big) \cap K\right| \geqslant \kappa_1 \lambda^r.$$

On the other hand, a proof by induction shows the following fact. Given $h \in K$, there exist an integer i and elements $1 = g_0, g_1, \dots, g_i = h$ ordered in this way on σ_h such that for every $j \in \{0, \dots, i\}$, g_j belongs to K_j and the distance between two consecutive g_j is exactly r. Finally, we have proved the following result.

Proposition 3.13 *There is* $\kappa_1 \in (0,1)$ *such that for all* $r > 10\delta$, *there exists a subset* K *of* G *satisfying the following properties:*

(i) 1 *belongs to* K,

(ii) *for all* $g \in K$, $\left|g\big(L_g \cap S(r)\big) \cap K\right| \geqslant \kappa_1 \lambda^r$,

(iii) *for all* $h \in K$, *for all* $x \in \sigma_h$, *there exists* $g \in K$ *which lies on* σ_h *between* 1 *and* x *such that* $|x - g| \leqslant r$.

4 Avoiding large powers

The goal here is to estimate the growth rate of a subset of G "without large power". This section entails many parameters. As a warmup we start with the case of free groups (Section 4.1). We present briefly the ideas used by S I Adian in [1]. This particular case only involves counting arguments and does not require Section 3. The estimation that we obtain will also be useful in Section 5.

4.1 The case of free groups

We assume here that A is a free generating set of $\mathbf{F}_k$, ie it contains exactly k elements. Consequently, the exponential growth rate of $\mathbf{F}_k$ with respect to A is $\lambda = 2k-1$. Let $m \in \mathbb{N}$. An element of $\mathbf{F}_k$ is said to be *m–aperiodic* if the reduced word over the alphabet $A \cup A^{-1}$ representing it does not contain an m^{th} power (see Main fact in the introduction). We denote by K_m the set of m–aperiodic elements of $\mathbf{F}_k$.

Proposition 4.1 *For all integers $m \geqslant 2$, for all $s \in \mathbb{N}$,*

$$|K_m \cap B(s+1)| \geqslant \lambda\, |K_m \cap B(s)| - \frac{2k}{2k-1} \sum_{j \geqslant 1} \lambda^j \, |K_m \cap B(s+1-mj)| .$$

The proof is based on this observation. An m–aperiodic word of length $s+1$ can always be written as an m–aperiodic word of length s followed by one letter. However, all such words are not necessarily m–aperiodic. Indeed, a power could occur at the end of the word. We need to exclude them when counting the number of m–aperiodic elements of length $s+1$. More precisely, the proof works as follows.

Proof Let $s \in \mathbb{N}$. An m–aperiodic word w of length $s+1$ can be written $w = w'a$ where w' is an m–aperiodic word of length s and a an element of $A \cup A^{-1}$. Since w is reduced, the number of possible choices for a is $2k-1$.

Consider now a reduced word of the form $w'a$ where $w' \in K_m \cap B(s)$ and $a \in A \cup A^{-1}$. If such a word is not m–aperiodic, then there exist $j \in \mathbb{N}^*$, $w_0 \in \mathbf{F}_k$ with $|w_0| = j$ and $w_- \in K_m \cap B(s+1-mj)$ such that $w'a = w_- w_0^m$. The number of words of this last form is bounded above by

$$|K_m \cap B(s+1-mj)| \cdot |S(j)| \leqslant 2k(2k-1)^{j-1}\, |K_m \cap B(s+1-mj)| .$$

Therefore the number of reduced words of the form $w'a$ ($w' \in K_m \cap B(s)$ and $a \in A \cup A^{-1}$) which are m–aperiodic is bounded below by

$$(2k-1)\, |K_m \cap B(s)| - 2k \sum_{j \geqslant 1} (2k-1)^{j-1}\, |K_m \cap B(s+1-mj)| ,$$

which gives the desired conclusion. □

Proposition 4.2 *Let $k \geqslant 2$. For every $a > 2k$, there exists a number m_0 such that for every integer $m \geqslant m_0$, the exponential growth rate of K_m is at least $\lambda\,(1 - a\lambda^{-m})$.*

Proof We consider the function $f_m\colon (\sqrt[m]{\lambda}, \lambda) \to \mathbb{R}$ defined by

$$f_m(\mu) = \lambda - \frac{2k\mu}{2k-1}\sum_{j\geqslant 1}\left(\frac{\lambda}{\mu^m}\right)^j = \lambda - \frac{2k\mu}{\mu^m - \lambda}.$$

Let $a > 2k$. We put $\mu_m = \lambda\,(1 - a\lambda^{-m})$. The sequence (μ_m) tends to $\lambda > 1$ as m approaches infinity. Therefore

$$f_m(\mu_m) = \lambda\left(1 - \frac{2k}{\lambda^m} + \underset{m\to+\infty}{\mathrm{o}}\left(\frac{1}{\lambda^m}\right)\right).$$

Since $a > 2k$, there exists a number m_0 such that for every integer $m \geqslant m_0$, $f_m(\mu_m) \geqslant \mu_m$. Fix $m \geqslant m_0$. For simplicity of notation, we write μ for μ_m. We now prove by induction that for every $s \in \mathbb{N}$, $|K_m \cap B(s)| \geqslant \mu\,|K_m \cap B(s-1)|$. The statement is true for $s = 0$. Assume that it holds for all integers less or equal to s. In particular, for every $j \geqslant 1$, $|K_m \cap B(s+1-mj)| \leqslant \mu^{1-mj}\,|K_m \cap B(s)|$. It follows then from Proposition 4.1 that

$$|K_m \cap B(s+1)| \geqslant \left[\lambda - \frac{2k\mu}{(2k-1)}\sum_{j\geqslant 1}\left(\frac{\lambda}{\mu^m}\right)^j\right]|K_m \cap B(s)|.$$

The expression between the brackets is exactly $f_m(\mu)$. Therefore the assertion holds for $s+1$. A second induction proves that for every $s \in \mathbb{N}$, $|K_m \cap B(s)| \geqslant \mu^s$, which leads to the result. □

4.2 The general case

We now deal with the case of hyperbolic groups. Recall that a cyclically reduced element is an element $g \in G$ such that $[g] = |g|$. The set of non-trivial cyclically reduced elements is denoted by C (see Section 2). Let $L > 0$ and $m \in \mathbb{N}^*$. Given $g \in G$, we say that g contains an (L,m)*–power* if there exists $(l, v) \in G \times C$ such that both l and lv^m belong to the L–neighborhood of σ_g. If g does not contain any (L,m)–power, it is called (L,m)*–aperiodic*.

Remark Our definition of (L,m)–aperiodic elements is a slightly weaker form of the one of A Y Ol'shanskiĭ [20]. However, it is sufficient to apply Ol'shanskiĭ's results (see the remark following Theorem 5.4).

Let $h \in G$ and g be an element of G which lies on σ_h. If h is (L,m)–aperiodic, so is g.

Given a subset K of G, we denote by $K_{L,m}$ the set of elements $g \in K$ which are (L,m)–aperiodic. Our aim is to give a lower bound for the growth rate of $G_{L,m}$. More precisely, we prove the following result.

Proposition 4.3 *Let $L > 0$. There exist $a > 0$ and $m_0 \in \mathbb{N}$ satisfying the following property. For all integers $m \geqslant m_0$, the exponential growth rate of $G_{L,m}$ is larger than or equal to $\lambda\,(1 - a/m)$.*

The rest of this section is dedicated to the proof of Proposition 4.3. We will not compute directly the growth of $G_{L,m}$ but provide an estimate for the growth of $K_{L,m}$ for a well chosen subset K of G. To that end, we first need to fix some parameters. The constants τ and κ_1 are the ones respectively given by Theorem 2.2 and Proposition 3.13. Let $L > 0$. Let r be an integer larger than 10δ. According to Proposition 3.13, there exists a subset K of G containing 1, such that

(i) for all $g \in K$, $\left|g\left(L_g \cap S(r)\right) \cap K\right| \geqslant \kappa_1 \lambda^r$,

(ii) for all $h \in K$, for all $x \in \sigma_h$, there exists $g \in K$ which lies on σ_h between 1 and x such that $|x - g| \leqslant r$.

Let m be an integer such that $m\tau \geqslant 2L + r$. We now define auxiliary subsets of G.

- $Z = \left\{h \in g\left(L_g \cap S(r)\right) \cap K \;\middle|\; g \in K_{L,m}\right\}$.
- For all $v \in G \setminus \{1\}$, Z_v is the set of elements $h \in G$ that can be written $h = guv^m u'$ where
  (i) g belongs to $K_{L,m}$,
  (ii) $|h| \geqslant |g| + |v^m| - 2L$,
  (iii) $|u|, |u'| \leqslant L + r$.

Roughly speaking, Z denotes the set of elements of K which are a selected extension of an (L,m)–aperiodic element of K by a length r. On the other hand, Z_v contains the elements which extend an (L,m)–aperiodic element by an almost m^{th} power of v. The idea of the next proposition is the following. An (L,m)–aperiodic element of length $s + r$ can be obtained by extending a (L,m)–aperiodic element of length s. However, this extension should not involve an m^{th} power.

Proposition 4.4 *The set $Z \setminus \bigcup_{v \in C} Z_v$ is contained in $K_{L,m}$.*

Proof Equivalently, we prove that $Z \setminus K_{L,m} \subset \bigcup_{v \in C} Z_v$. Let h be an element of $Z \setminus K_{L,m}$. In particular it belongs to $K \setminus K_{L,m}$. Therefore there exist $l \in G$ and $v \in C$

such that l and lv^m belong to the L–neighborhood of σ_h. We are going to prove that h belongs to Z_v.

We denote by p and q respective projections of l and lv^m on σ_h. By replacing if necessary v by v^{-1}, one can assume that 1, p, q and h are ordered in this way on σ_h. We claim that $|h-q| \leqslant r$. Assume on the contrary that this assertion is false. Since h belongs to Z, there is $g_1 \in K_{L,m} \cap \sigma_h$ such that $|h-g_1| = r < |h-q|$. It follows that p and q both belong to σ_{g_1}. In particular, l and lv^m lie in the L–neighborhood of σ_{g_1}. This contradicts the fact that g_1 is (L,m)–aperiodic.

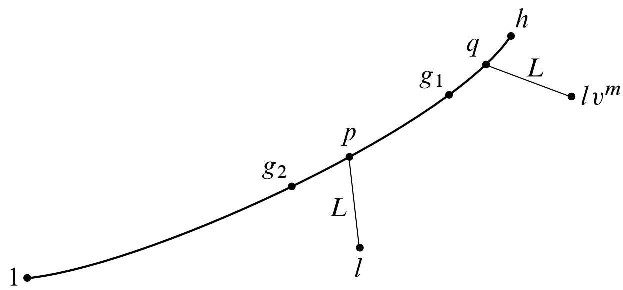

Figure 4: Positions of the points on σ_h

Recall that h belongs to K. The point p is on σ_h. Therefore there is a point $g_2 \in K$ which lies on σ_h between 1 and p such that $|g_2 - p| \leqslant r$ (see Figure 4). We put $u = g_2^{-1} l$ and $u' = v^{-m} l^{-1} h$. Hence $h = g_2 u v^m u'$. The triangle inequality combined with our previous claim gives $|u|, |u'| \leqslant L + r$ and $|h| \geqslant |g_2| + |v^m| - 2L$. It only remains to prove that g_2 is (L,m)–aperiodic. We assumed that $m\tau \geqslant 2L + r$. The previous inequality becomes

$$|h| \geqslant |g_2| + |v^m| - 2L \geqslant |g_2| + m\tau - 2L \geqslant |g_2| + r.$$

Hence g_2 lies on σ_g between 1 and g_1. Since g_1 is (L,m)–aperiodic, so is g_2. □

Corollary 4.5 *For all $s \in \mathbb{N}$,*

$$\left|K_{L,m} \cap B(s)\right| \geqslant |Z \cap B(s)| - \sum_{v \in C} |Z_v \cap B(s)|.$$

Lemma 4.6 *For all $s \in \mathbb{N}$, $|Z \cap B(s+r)| \geqslant \kappa_1 \lambda^r \left|K_{L,m} \cap B(s)\right|$.*

Remark Recall that κ_1 is given by Proposition 3.13, whereas r is the radius that we fixed at the beginning of this section.

Proof By definition of Z,

$$Z \cap B(s+r) = \bigcup_{g \in K_{L,m} \cap B(s)} g\big(L_g \cap S(r)\big) \cap K.$$

We claim that this union is in fact a disjoint union. Assume on the contrary that this assertion is false. There are two distinct elements $g, g' \in K_{L,m} \cap B(s)$ and $u, u' \in S(r)$ such that g and g' lie on the selected geodesic from 1 to $gu = g'u'$. Since u and u' have the same length, $|g - g'| = ||g| - |g'|| = 0$, thus $g = g'$. Contradiction. Therefore we have

$$|Z \cap B(s+r)| = \sum_{g \in K_{L,m} \cap B(s)} \left|g\big(L_g \cap S(r)\big) \cap K\right|.$$

It follows from Proposition 3.13, that for all $g \in K_{L,m} \cap B(s)$, $\left|g\big(L_g \cap S(r)\big) \cap K\right| \geqslant \kappa_1 \lambda^r$. Consequently,

$$|Z \cap B(s+r)| \geqslant \kappa_1 \lambda^r \left|K_{L,m} \cap B(s)\right|. \qquad \square$$

Lemma 4.7 *Let $v \in G \setminus \{1\}$. For all $s \geqslant 0$,*

$$|Z_v \cap B(s+r)| \leqslant \alpha^2 \lambda^{2(L+r)} \left|K_{L,m} \cap B(s+r+2L-m\,[v]^\infty)\right|.$$

Proof Let h be an element of $Z_v \cap B(s+r)$. By definition there are $g \in K_{L,m}$ and $u, u' \in B(L+r)$ such that $h = guv^m u'$. Moreover,

$$|h| \geqslant |g| + |v^m| - 2L \geqslant |g| + m\,[v]^\infty - 2L.$$

Consequently, g belongs to the ball of center 1 and radius $s+r+2L-m\,[v]^\infty$. Hence $Z_v \cap B(s+r)$ is a subset of

$$\big[K_{L,m} \cap B(s+r+2L-m\,[v]^\infty)\big] B(L+r) v^m B(L+r).$$

The conclusion follows from Proposition 2.4. $\square$

Let us summarize. Let $s \in \mathbb{N}$. By Corollary 4.5 and Lemma 4.6,

$$\left|K_{L,m} \cap B(s+r)\right| \geqslant |Z \cap B(s+r)| - \sum_{v \in C} |Z_v \cap B(s+r)| \tag{2}$$

$$\geqslant \kappa_1 \lambda^r \left|K_{L,m} \cap B(s)\right| - \sum_{v \in C} |Z_v \cap B(s+r)|. \tag{3}$$

Let $j \in \mathbb{N}^*$. We consider $v \in C$ such that $[v]^\infty = j\tau$ (see Theorem 2.2). By Lemma 4.7,

$$|Z_v \cap B(s+r)| \leqslant \alpha^2 \lambda^{2(L+r)} \left|K_{L,m} \cap B(s+r+2L-mj\tau)\right|.$$

Since v is cyclically reduced, it satisfies $|v| = [v] \leqslant [v]^\infty + 50\delta$. Hence such an element belongs to $B(j\tau + 50\delta)$. The number of elements $v \in C$ such that $[v]^\infty = j\tau$ is therefore bounded above by $|B(j\tau + 50\delta)| \leqslant \alpha\lambda^{j\tau+50\delta}$. Consequently,

$$\sum_{v\in C} |Z_v \cap B(s+r)| \leqslant \alpha^3 \lambda^{2(L+r+25\delta)} \sum_{j\geqslant 1} \lambda^{j\tau} \left|K_{L,m} \cap B(s+r+2L-mj\tau)\right|.$$

Note that in the sum on the right-hand side all but finitely many terms vanish. Combining this last inequality with (3), we get

$$\begin{aligned}&\left|K_{L,m} \cap B(s+r)\right| \\ &\quad \geqslant \kappa_1 \lambda^r \left|K_{L,m} \cap B(s)\right| - \alpha^3 \lambda^{2(L+r+25\delta)} \sum_{j\geqslant 1} \lambda^{j\tau} \left|K_{L,m} \cap B(s+r+2L-mj\tau)\right|.\end{aligned}$$

Finally we have proved the following proposition.

Proposition 4.8 *There exist positive constants τ, κ_1 and κ_2 with $\kappa_1 < 1$ satisfying the following property. Let $L > 0$. Let r be an integer larger than 10δ. There exists a subset K of G containing 1 such that for all integers m satisfying $m\tau \geqslant 2L + r$, for all $s \in \mathbb{N}$,*

$$\begin{aligned}&\left|K_{L,m} \cap B(s+r)\right| \\ &\qquad \geqslant \kappa_1 \lambda^r \left|K_{L,m} \cap B(s)\right| - \kappa_2 \lambda^{2(L+r)} \sum_{j\geqslant 1} \lambda^{j\tau} \left|K_{L,m} \cap B(s+r+2L-mj\tau)\right|.\end{aligned}$$

Before proving Proposition 4.3, we introduce a family of auxiliary maps. For all $L > 0$, for all $m \in \mathbb{N}^*$, the function $f_{L,m}\colon (\sqrt[m]{\lambda}, \lambda) \times \mathbb{R}_+ \to \mathbb{R}$ is given by

$$\begin{aligned} f_{L,m}(\mu, r) &= \kappa_1 \lambda^r - \kappa_2 \lambda^{4(L+r)} \sum_{j\geqslant 1} \left(\frac{\lambda}{\mu^m}\right)^{j\tau} \\ &= \kappa_1 \lambda^r - \kappa_2 \lambda^{4(L+r)} \frac{\lambda^\tau}{\mu^{m\tau} - \lambda^\tau}. \end{aligned}$$

Proposition 4.9 *Let $L > 0$. Let $b \in (0, \tau/3)$ and $a > -\ln(\kappa_1)/b$. There exists a number m_0 such that for all integers $m \geqslant m_0$,*

$$f_{L,m}\left(\lambda\left(1 - \frac{a}{m}\right), mb\right) \geqslant \left[\lambda\left(1 - \frac{a}{m}\right)\right]^{mb}.$$

Remark Recall that $\kappa_1 < 1$, thus $a > 0$.

Proof For every $m \in \mathbb{N}^*$, we put $\mu_m = \lambda\,(1-a/m)$. Note that if m is sufficiently large, $\mu_m \in (\sqrt[m]{\lambda}, \lambda)$. For every $d > 0$, we have the following asymptotic behavior

$$(\mu_m)^{md} = \lambda^{md}\Big[e^{-ad} + \underset{m\to+\infty}{\mathrm{o}}(1)\Big].$$

In particular,

$$(\mu_m)^{mb} = \lambda^{mb}\Big[e^{-ab} + \underset{m\to+\infty}{\mathrm{o}}(1)\Big], \tag{4}$$

and

$$\kappa_2\lambda^{4(L+mb)}\frac{\lambda^\tau}{(\mu_m)^{m\tau}-\lambda^\tau} = \lambda^{m(4b-\tau)}\Big[\kappa_2 e^{a\tau}\lambda^{4L+\tau} + \underset{m\to+\infty}{\mathrm{o}}(1)\Big].$$

Since $b < \tau/3$, the previous quantity is asymptotically dominated by λ^{mb}. Consequently,

$$f_{L,m}(\mu_m, mb) = \kappa_1\lambda^{mb} - \kappa_2\lambda^{4(L+mb)}\frac{\lambda^\tau}{(\mu_m)^{m\tau}-\lambda^\tau} = \lambda^{mb}\Big[\kappa_1 + \underset{m\to+\infty}{\mathrm{o}}(1)\Big].$$

By construction, $e^{-ab} < \kappa_1$. By (4), there exists m_0 such that for every $m \geqslant m_0$,

$$f_{L,m}\,(\mu_m, mb) \geqslant (\mu_m)^{mb}. \qquad \square$$

Proof of Proposition 4.3 Let $L > 0$. According to Proposition 4.9, there are positive numbers a, b and $m_0 \in \mathbb{N}$ such that for all $m \geqslant m_0$,

(i) $mb > 10\delta$,

(ii) $m\tau \geqslant 2L + mb$,

(iii) $m\ln\left(\lambda\left(1-\frac{a}{m}\right)\right) > \ln\lambda$,

(iv) $f_{L,m}\left(\lambda\left(1-\frac{a}{m}\right), mb\right) \geqslant \left[\lambda\left(1-\frac{a}{m}\right)\right]^{mb}$.

Let $m \geqslant m_0$. For simplicity of notation, we write $\mu = \lambda\,(1-a/m)$ and $r = mb$. Hence, the previous inequalities can be written $r > 10\delta$, $m\tau \geqslant 2L + r$, $\lambda\mu^{-m} < 1$ and $f_{L,m}(\mu, r) \geqslant \mu^r$. By Proposition 4.8, there exists a subset K of G containing 1 such that for all $s \geqslant 0$,

$$\begin{aligned}&\left|K_{L,m}\cap B(s+r)\right| \\ &\qquad \geqslant \kappa_1\lambda^r\left|K_{L,m}\cap B(s)\right| - \kappa_2\lambda^{2(L+r)}\sum_{j\geqslant 1}\lambda^{j\tau}\left|K_{L,m}\cap B(s+r+2L-mj\tau)\right|.\end{aligned}$$

We now prove by induction that for all $i \in \mathbb{N}$,

$$\left|K_{L,m}\cap B(ir)\right| \geqslant \mu^r\left|K_{L,m}\cap B((i-1)r)\right|. \tag{$\mathcal{H}_i$}$$

$(\mathcal{H}_0)$ is obviously true. Assume the induction hypothesis holds for every integer smaller than or equal to i. In particular, for all $t \geqslant 0$, we have

$$\left|K_{L,m} \cap B(ir - t)\right| \leqslant \mu^{-\lfloor \frac{t}{r} \rfloor r} \left|K_{L,m} \cap B(ir)\right| \leqslant \mu^{r-t} \left|K_{L,m} \cap B(ir)\right|.$$

By construction of m_0, for all $j \geqslant 1$, we have $mj\tau - 2L - r \geqslant 0$, thus

$$\begin{aligned} \lambda^{j\tau} \left|K_{L,m} \cap B(ir + r + 2L - mj\tau)\right| &\leqslant \left(\frac{\lambda}{\mu^m}\right)^{j\tau} \mu^{2(r+L)} \left|K_{L,m} \cap B(ir)\right| \\ &\leqslant \left(\frac{\lambda}{\mu^m}\right)^{j\tau} \lambda^{2(r+L)} \left|K_{L,m} \cap B(ir)\right|. \end{aligned}$$

Note that m_0 has been chosen in such a way that $\lambda\mu^{-m} < 1$. Hence by summing these inequalities, we obtain

$$\begin{aligned} \left|K_{L,m} \cap B((i+1)r)\right| &\geqslant \left[\kappa_1 \lambda^r - \kappa_2 \lambda^{4(L+r)} \sum_{j \geqslant 1} \left(\frac{\lambda}{\mu^m}\right)^{j\tau}\right] \left|K_{L,m} \cap B(ir)\right| \\ &\geqslant f_{L,m}(\mu, r) \left|K_{L,m} \cap B(ir)\right|. \end{aligned}$$

However, by construction, $f_{L,m}(\mu, r) \geqslant \mu^r$. Consequently, $(\mathcal{H}_{i+1})$ holds. A second induction shows that for all $i \in \mathbb{N}$, $\left|K_{L,m} \cap B(ir)\right| \geqslant \mu^{ir}$. Therefore the growth rate of $K_{L,m}$, and thus the one of $G_{L,m}$, is at least $\mu = \lambda\,(1 - a/m)$. □

5 Growth of periodic quotients

In the introduction we pointed out the main fact used by S I Adian to prove that free Burnside groups of large odd exponents have exponential growth: if two distinct elements of $\mathbf{F}_k$ can be represented by reduced words that do not contain an 8^{th} power, then they induce distinct elements in the Burnside group. Therefore one can estimate the growth rate of the free Burnside group by computing the one of the set of 8–aperiodic words. Actually, a stronger statement is true. It will allow us to estimate more accurately the growth rate of $\mathbf{B}_k(n)$.

Theorem 5.1 (Coulon [8, Theorem 4.5]) *Let $k \geqslant 2$. Let A be a free generating set of $\mathbf{F}_k$. There exist numbers n_0 and ξ such that for every odd exponent $n \geqslant n_0$, the following holds. Let w_1 and w_2 be reduced words over $A \cup A^{-1}$. If they do not contain an $\lfloor n/2 - \xi \rfloor$–power, then they represent distinct elements of $\mathbf{B}_k(n)$.*

Remark A similar statement can be found in Adian [1, Chapter IV, Proposition 2.16] or Ol'shanskiĭ [19, Lemma 5.5] respectively for $n/150$– and $n/3$–aperiodic elements. Their method could also be adapted to get the previous theorem.

Theorem 5.2 *Let $k \geqslant 2$. Let A be a free generating set of $\mathbf{F}_k$. There exists a positive number κ such that for sufficiently large odd exponents n, the exponential growth rate of $\mathbf{B}_k(n)$ with respect to the image of A is at least*

$$(2k-1)\left(1-\frac{\kappa}{(2k-1)^{n/2}}\right).$$

Proof The constants n_0 and ξ are the ones given by Theorem 5.1. We fix $a > 2k$. According to Proposition 4.2, there exists an integer m_0 such that for every $m \geqslant m_0$, the exponential growth rate of the set of m–aperiodic words is at least

$$(2k-1)\left(1-\frac{a}{(2k-1)^{m}}\right).$$

Let $n \geqslant \max\{n_0, 2m_0+2\xi+2\}$ be an odd integer. By Theorem 5.1, the natural map $\mathbf{F}_k \to \mathbf{B}_k(n)$ restricted to the set of $\lfloor n/2-\xi \rfloor$–aperiodic elements is one-to-one. Therefore the growth rate of $\mathbf{B}_k(n)$ is larger than or equal to the one of this set. In particular, it is at least

$$(2k-1)\left(1-\frac{a(2k-1)^{\xi+1}}{(2k-1)^{n/2}}\right). \qquad \square$$

In [20], A Y Ol'shanskiĭ solved the Burnside problem for hyperbolic groups.

Theorem 5.3 (Ol'shanskiĭ [20]) *Let G be a non-elementary torsion-free hyperbolic group. There exists an integer n_0 such that for all odd exponents $n \geqslant n_0$, the quotient G/G^n is infinite.*

The proof relies on the following fact. If n is large enough, then the restriction of the canonical projection $G \twoheadrightarrow G/G^n$ to a set of sufficiently aperiodic elements (which is infinite) is injective. More precisely, he showed the following statement.

Theorem 5.4 (Ol'shanskiĭ [20, Proof of the main theorem, page 540]) *Let G be a non-elementary torsion-free hyperbolic group. There exist constants L, ϵ and n_0 with the following property. Let $n \geqslant n_0$ be an odd integer. Then the restriction of $G \twoheadrightarrow G/G^n$ to the set of $(L, \lfloor \epsilon n \rfloor)$–aperiodic elements is one-to-one.*

Remarks The definition of aperiodic elements used by A Y Ol'shanskiĭ is slightly different from ours (see Section 4). He says that an element $g \in G$ contains an (L,m)–power if there is $(l,v) \in G \times C$ such that both l and lv^n belong to the L–neighborhood of some geodesic between 1 and g (not necessarily σ_g). However, in a hyperbolic space, two geodesics joining the same extremities are 2δ–close one from the other.

Hence an (L, m)–aperiodic element in the sense of Ol'shanskiĭ is (L, m)–aperiodic in our sense. Conversely, an $(L + 2\delta, m)$–aperiodic element in our sense is (L, m)–aperiodic in the sense of Ol'shanskiĭ. Therefore the statement of Theorem 5.4 with one or the other definition are equivalent. Another approach based on the work of T Delzant and M Gromov [10] can be found in Coulon [8].

Theorem 5.5 *Let G be a non-elementary torsion-free hyperbolic group and λ its exponential growth rate with respect to a finite generating set A. There exists a positive number κ such that for sufficiently large odd exponents n, the exponential growth rate of G/G^n with respect to the image of A is at least $\lambda\,(1-\kappa/n)$.*

Proof Let the parameters L, ϵ and n_0 be given by Theorem 5.4. The constants a and m_0 are then provided by Proposition 4.3. Let $n \geqslant \max\left\{\epsilon^{-1}(m_0+1), n_0, 2\epsilon^{-1}\right\}$ be an odd integer. We put $m = \lfloor \epsilon n \rfloor$. According to Proposition 4.3, the exponential growth rate of $G_{L,m}$ is at least $\lambda\,(1-a/m) \geqslant \lambda\,(1-2a/\epsilon n)$. By Theorem 5.4, the restriction of $G \twoheadrightarrow G/G^n$ to $G_{L,m}$ is one-to-one. On the other hand, for every $g \in G$, the length of g with respect to A is not smaller than the length of its image in G/G^n with respect to the image of A. Therefore for all $r \geqslant 0$, the ball of radius r in G/G^n contains at least $\left|G_{L,m} \cap B(r)\right|$ elements. Thus the exponential growth rate of G/G^n is not less than the one of $G_{L,m}$. In particular, it is at least $\lambda\,(1-2a/\epsilon n)$. □

References

[1] **S I Adian**, *The Burnside problem and identities in groups*, Ergeb. Math. Grenzgeb. 95, Springer, Berlin (1979) MR537580

[2] **G N Arzhantseva**, **I G Lysenok**, *Growth tightness for word hyperbolic groups*, Math. Z. 241 (2002) 597–611 MR1938706

[3] **V S Atabekyan**, *Uniform nonamenability of subgroups of free Burnside groups of odd period*, Mat. Zametki 85 (2009) 516–523 MR2549414

[4] **W Burnside**, *On an unsettled question in the theory of discontinuous groups*, Quart. J. Pure Appl. Math. 33 (1902) 230–238

[5] **J W Cannon**, *The combinatorial structure of cocompact discrete hyperbolic groups*, Geom. Dedicata 16 (1984) 123–148 MR758901

[6] **M Coornaert**, *Mesures de Patterson–Sullivan sur le bord d'un espace hyperbolique au sens de Gromov*, Pacific J. Math. 159 (1993) 241–270 MR1214072

[7] **M Coornaert**, **T Delzant**, **A Papadopoulos**, *Géométrie et théorie des groupes*, Lecture Notes in Mathematics 1441, Springer, Berlin (1990) MR1075994

[8] **R Coulon**, *Detecting trivial elements of Burnside groups* `arXiv:1211.4267`

[9] **T Delzant**, *Sous-groupes distingués et quotients des groupes hyperboliques*, Duke Math. J. 83 (1996) 661–682 MR1390660

[10] **T Delzant**, **M Gromov**, *Courbure mésoscopique et théorie de la toute petite simplification*, J. Topol. 1 (2008) 804–836 MR2461856

[11] **E Ghys**, **P de la Harpe**, **Editors**, *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Mathematics 83, Birkhäuser, Boston, MA (1990) MR1086648

[12] **M Gromov**, *Hyperbolic groups*, from: "Essays in group theory", Math. Sci. Res. Inst. Publ. 8, Springer, New York (1987) 75–263 MR919829

[13] **M Hall, Jr**, *Solution of the Burnside problem of exponent 6*, Proc. Nat. Acad. Sci. U.S.A. 43 (1957) 751–753 MR0089847

[14] **S V Ivanov**, *The free Burnside groups of sufficiently large exponents*, Internat. J. Algebra Comput. 4 (1994) ii+308 MR1283947

[15] **M Koubi**, *Croissance uniforme dans les groupes hyperboliques*, Ann. Inst. Fourier (Grenoble) 48 (1998) 1441–1453 MR1662255

[16] **F Levi**, **B L van der Waerden**, *Über eine besondere Klasse von Gruppen*, Abh. Math. Sem. Univ. Hamburg 9 (1933) 154–158 MR3069591

[17] **I G Lysënok**, *Infinite Burnside groups of even period*, Izv. Ross. Akad. Nauk Ser. Mat. 60 (1996) 3–224 MR1405529

[18] **S P Novikov**, *Adams operators and fixed points*, Izv. Akad. Nauk SSSR Ser. Mat. 32 (1968) 1245–1263 MR0256390

[19] **A Y Ol'shanskiĭ**, *The Novikov–Adyan theorem*, Mat. Sb. 118(160) (1982) 203–235, 287 MR658789

[20] **A Y Ol'shanskiĭ**, *Periodic quotient groups of hyperbolic groups*, Mat. Sb. 182 (1991) 543–567 MR1119008

[21] **D V Osin**, *Uniform non-amenability of free Burnside groups*, Arch. Math. (Basel) 88 (2007) 403–412 MR2316885

[22] **I N Sanov**, *Solution of Burnside's problem for exponent 4*, Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. 10 (1940) 166–170 MR0003397

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