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Presenting parabolic subgroups

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Consider a relatively hyperbolic group G. We prove that if G is finitely presented, so are its parabolic subgroups. Moreover, a presentation of the parabolic subgroups can be found algorithmically from a presentation of G, a solution of its word problem and generating sets of the parabolic subgroups. We also give an algorithm that finds parabolic subgroups in a given recursively enumerable class of groups.

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Consider a relatively hyperbolic group G with parabolic subgroups $H_1, \dots, H_n$. It is well known that if each H_i is finitely generated (or finitely presented), then so is G. Osin showed conversely that if G is finitely generated, then so are $H_1, \dots, H_n$ [9, Proposition 2.27]. Whether finite presentation of G implies finite presentation of $H_1, \dots, H_n$ is an important question raised by Osin in [9, Problem 5.1].

On the algorithmic side, given a finite presentation of a relatively hyperbolic group G and a generating set of the parabolic subgroups, can one find a presentation of the parabolic subgroups?

We give a positive answer to these two questions.

Theorem 1 *Let G be a finitely presented group. Assume that G is hyperbolic relative to $H_1, \dots, H_n$. Then each H_i is finitely presented.*

Theorem 2 *There exists an algorithm that takes as input a finite presentation of a group G, a solution to its word problem and a collection of finite subsets $S_1, \dots, S_n \subset G$, and that terminates if and only if G is hyperbolic relative to $\langle S_1 \rangle, \dots, \langle S_n \rangle$.*

In this case, the algorithm outputs a linear isoperimetry constant K for the corresponding relative presentation, a finite presentation for each of the parabolic subgroups $\langle S_i \rangle$, and says whether G is properly relative hyperbolic relative to $\langle S_1 \rangle, \dots, \langle S_n \rangle$ (ie $\langle S_i \rangle \subsetneq G$ for all i).

In this statement, the linear isoperimetry constant K is for the relative presentation X_∞ as defined in Section 1.2.

If one is not given generating sets of the parabolic subgroups, one can search for them, and require that they lie in some recursively enumerable class of groups.

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Theorem 3 *There exists an algorithm as follows. It takes as input a finite presentation of a group G, a solution for its word problem and a recursive class of finitely presented groups $\mathcal{C}$ (given by a Turing machine enumerating presentations of these groups).*

It terminates if and only if G is properly hyperbolic relative to subgroups that are in the class $\mathcal{C}$.

In this case, the algorithm outputs an isoperimetry constant K, a generating set and a finite presentation for each of the parabolic subgroups.

The Turing machine enumerating $\mathcal{C}$ is a machine that enumerates some finite presentations, each of which represents a group in $\mathcal{C}$, and such that every group in $\mathcal{C}$ has at least one presentation that is enumerated.

This paper can be seen as a continuation, extension and precision on the form and the substance of the first author's [3]. It is based on the analysis of some van Kampen diagrams in different *truncated* relative presentations. The main tool is Proposition 2.9, which says that if some relative presentation does not satisfy a linear isoperimetric inequality, then this shows up on some diagram of small area and small complexity.

Section 1 recalls definitions about isometric inequalities, introduces truncated relative presentations, and defines the complexity of a diagram. Section 2 contains the main technical results. Section 3 is devoted to corollaries. Theorems 1, 2 and 3 follow from Corollaries 3.3, 3.5 and 3.6.

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1 Context

1.1 Linear isoperimetric inequalities

A *presentation* is a pair $(S \mid \mathcal{R})$, where S is a set, and $\mathcal{R} \subset \mathbb{F}_S$ is a subset of the free group $\mathbb{F}_S$. The group defined by this presentation is $\langle S \mid \mathcal{R} \rangle = \mathbb{F}_S / \langle\langle \mathcal{R} \rangle\rangle$. If G is a given group and $\sigma\colon S \to G$ is a map from a set S to G, we say that $(S \mid \mathcal{R})$ is a presentation of G (with respect to the map σ) if $\sigma(S)$ generates G and $\mathcal{R}$ normally generates the kernel of the natural map $\mathbb{F}_S \to G$, ie if σ extends to an isomorphism $\langle S \mid \mathcal{R} \rangle \to G$. The elements of $\mathcal{R}$ are called *defining relations*, and we usually write $G = \langle S \mid \mathcal{R} \rangle$.

Note that we use distinct notations for the group and its presentation, we will usually denote by $X = (S \mid \mathcal{R})$ the presentation, and by $G = \langle S \mid \mathcal{R} \rangle$ the group defined by this presentation.

We denote by $\bar{s}$ the inverse of the basis element $s \in \mathbb{F}_S$ and we view an element of $\mathbb{F}_S$ as a reduced word over the alphabet $S \cup \overline{S}$. We say that a presentation is *triangular* if every defining relation has length at most 3 (as word over $S \cup \overline{S}$). If one allows to increase the generating set, it is not restrictive to consider triangular presentations: from an arbitrary finite presentation, one can effectively construct a triangular one.

If a word $w \in \mathbb{F}_S$ represents the trivial element of G (we write $w \overset{G}{=} 1$), the *area* of w for the presentation $X = (S \mid \mathcal{R})$, denoted by $\mathrm{Area}(w)$, is the minimal number n such that w is the product in $\mathbb{F}_S$ of n conjugates of elements of $\mathcal{R} \cup \mathcal{R}^{-1}$.

Given a word $w \in \mathbb{F}_S$ such that $w \overset{G}{=} 1$, a *van Kampen diagram* for w over the presentation $X = (S \mid \mathcal{R})$ is a simply connected planar 2–complex D such that oriented edges are labeled by elements of $S \cup \overline{S}$, such that reversing the orientation changes the label to its inverse, such that every 2–cell has its boundary labeled by a cyclically reduced word conjugate to an element of $\mathcal{R} \cup \mathcal{R}^{-1}$, and such that the topological boundary ∂D of D is labeled by w. Sometimes, we just say *cell* instead of 2–cell. It is well known that $\mathrm{Area}(w)$ is the minimal number of 2–cells of van Kampen diagrams for w. See Lyndon and Schupp [8, Section 5.1] for more details.

An *isoperimetric function* of a presentation $X = (S \mid \mathcal{R})$ is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $w \in \mathbb{F}_S$ representing the trivial element, $\mathrm{Area}(w) \le f(\mathrm{length}(w))$. Note that if S is infinite, there are infinitely many words of a given length, and it may happen that no such function (with finite values) exists.

Our approach is based on the fact that a group is relatively hyperbolic if and only if it has a presentation of a particular kind with a linear isoperimetric function [9]; see Theorem 1.2 below. Another important fact is that the failure of a specific linear isoperimetric inequality can be observed in a set of words of controlled area (Gromov [5], Bowditch [1] and Papasoglu [10]).

Theorem 1.1 [10] *Let $X = (S \mid \mathcal{R})$ be an arbitrary (not necessarily finite) triangular presentation of an arbitrary group G and let $K \ge 1$.*

Assume that there is a word $w \in \mathbb{F}_S$ such that $w \overset{G}{=} 1$ and $\mathrm{Area}(w) > K\,\mathrm{length}(w)$. Then there exists a word $w' \in \mathbb{F}_S$ such that $w' \overset{G}{=} 1$ and such that:

- $\mathrm{Area}(w') \in \left[\frac{K}{2}, 240K\right]$
- $\mathrm{Area}(w') > \frac{1}{2\times 10^4}\,\mathrm{length}(w')^2$

As in [3], we use the fact that the statement and the proof of this theorem do not use that the presentation X is finite. Indeed, the argument considers a word of minimal area such that $\mathrm{Area}(w) > K\,\mathrm{length}(w)$, and extracts from a minimal diagram a subdiagram of controlled area with $\mathrm{Area}(D') > (1/2\times 10^4)\,\mathrm{length}(\partial D')^2$. Our constant K is K^2 in [10], where Papasoglu's K is assumed to be an integer in the statement, but only the inequality $K \geq 1$ is used in [10].

1.2 Truncated and exact relative presentations

Since finite generation of a relatively hyperbolic group implies finite generation of its maximal parabolic subgroups [9, Proposition 2.27], we always assume that relatively hyperbolic groups and their maximal parabolic subgroups are finitely generated.

Let us now define the *multiplication table* of a group H. This is the subset $\mathcal{T}(H)$ of the free group $\mathbb{F}_H$ consisting of all words on the alphabet $H \cup \overline{H}$ of length $1, 2$ or 3 that map to the trivial element in H under the morphism $\varphi\colon \mathbb{F}_H \to H$ induced by the identity map $H \to H$. Note that given $a \in H$ and its inverse $a^{-1} \in H$, the basis element a^{-1} of $\mathbb{F}_H$ is distinct from the inverse $\bar{a}$ in $\mathbb{F}_H$ of the basis element a. It is clear that the (usually infinite) presentation $(H \mid \mathcal{T}(H))$ is a presentation of H: the identity map $H \to H$ induces an isomorphism $\langle H \mid \mathcal{T}(H)\rangle \to H$.

Let G be a finitely presented group, and $H_1, \dots, H_n$ be finitely generated subgroups of G. For each i, let S_i be a finite symmetric generating set of H_i. Consider a finite triangular presentation $G = \langle S \mid \mathcal{R}\rangle$, where S is a finite symmetric generating set of G containing each S_i, and $\mathcal{R}$ is a finite set of triangular relations over S.

We are going to introduce a family of infinite presentations X_ρ, indexed by $\rho \in \mathbb{N} \cup \infty$. To make the definitions clearer, we first introduce the presentation X_∞. Let $\widetilde{H}_1, \dots, \widetilde{H}_n$ be some groups, isomorphic to H_i under an isomorphism $p_i\colon \widetilde{H}_i \to H_i$. We denote by $\widetilde{S}_i = p_i^{-1}(S_i)$ the corresponding generating set of $\widetilde{H}_i$. Consider the disjoint union

$$\widehat{S} = S \sqcup \widetilde{H}_1 \sqcup \cdots \sqcup \widetilde{H}_n,$$

and $\sigma\colon \widehat{S} \to G$ the map whose restriction to S is the inclusion and whose restriction to $\widetilde{H}_i$ is p_i. Since $S \subset \widehat{S}$, each relator in $\mathcal{R}$ can be viewed as an element of $\mathbb{F}_{\widehat{S}}$. To identify the generating set $\widetilde{S}_i$ of $\widetilde{H}_i$ with the corresponding subset of S, we consider for each $\tilde{s} \in \widetilde{S}_i$ the two-letter relator $\tilde{s}^{-1} p_i(\tilde{s}) \in \mathbb{F}_{\widehat{S}}$, where the first letter $\tilde{s}^{-1}$ lies in $\widetilde{H}_i$ and the second letter $p_i(\tilde{s})$ lies in S (because $S_i \subset S$). We define the finite subset $\mathcal{R}' \subset \mathbb{F}_{\widehat{S}}$ as the union of $\mathcal{R}$ with the set of all these two-letter relators. Finally, each element of the multiplication table $\mathcal{T}(\widetilde{H}_i)$ is naturally a word of length at most 3 in $\mathbb{F}_{\widehat{S}}$. Thus, one can define the *relative presentation* X_∞ as

$$X_\infty = \big(\widehat{S} \mid \mathcal{R}' \cup \mathcal{T}(\widetilde{H}_1) \cup \cdots \cup \mathcal{T}(\widetilde{H}_n)\big).$$

The triangular presentation X_∞ is a (usually infinite) presentation of G, with respect to σ.

Indeed, σ extends to a morphism φ from the group G' defined by X_∞ to G, and there is a morphism $\psi\colon G \to G'$ induced by the inclusion $S \subset \widehat{S}$. Since $\varphi \circ \psi = \mathrm{id}_G$, and since ψ is onto, φ and ψ are inverse of each other.

Theorem 1.2 [9, Theorem 1.7, Definition 2.29] *G is hyperbolic relative to the subgroups $H_1, \dots, H_n$ if and only if the relative presentation X_∞ satisfies a linear isoperimetric inequality.*

The subgroups $H_1, \dots, H_n$ of G are called the *maximal parabolic subgroups*. Since there is no risk of confusion, we will simply call them *parabolic subgroups*.

Remark 1.3 Osin includes all words of any length in the multiplication table. One easily checks that this does not change the result.

To introduce the *truncated* relative presentations X_ρ, we fix $\rho \in \mathbb{N} \cup \infty$. We are first going to define some auxiliary groups $\widetilde{H_i}^\rho$ with epimorphisms $p_i{}^\rho\colon \widetilde{H_i}^\rho \to H_i$. For each subgroup H_i, consider a copy $\widetilde{S_i}^\rho$ of S_i. Let $\mathcal{R}_\rho(S_i)$ be the set of all words in the alphabet $\widetilde{S_i}^\rho \cup \overline{\widetilde{S_i}^\rho}$, of length $\leq \rho$, whose image as words on the alphabet $S_i^{\pm 1}$ define trivial elements in H_i. Then we define $\widetilde{H_i}^\rho = \langle \widetilde{S_i}^\rho \mid \mathcal{R}_\rho(S_i) \rangle$, and denote by $p_i{}^\rho\colon \widetilde{H_i}^\rho \to H_i$ the obvious epimorphism.

Note that for $\rho = +\infty$, $p_i{}^\rho$ is an isomorphism, and $\widetilde{S_i}^\rho$ (resp. $\widetilde{H_i}^\rho$) is the set that we denoted $\widetilde{S}_i$ (resp. $\widetilde{H}_i$) above. For $\rho < \infty$, $\widetilde{H_i}^\rho$ is finitely presented.

The presentation X_ρ is analogous to X_∞, using $\widetilde{H}_i^\rho$ instead of $\widetilde{H}_i$. Let

$$\widehat{S}_\rho = S \sqcup \widetilde{H}_1^\rho \sqcup \cdots \sqcup \widetilde{H}_n^\rho,$$

and consider $\mathcal{R}'_\rho \subset \mathbb{F}_{\widehat{S}_\rho}$ consisting of $\mathcal{R}$ together with the set of two-letter words of the form $\widetilde{s}^{-1} p_i{}^\rho(\widetilde{s})$, where $s \in \widetilde{S_i}^\rho$. Then, we define the *truncated relative presentation* X_ρ as

$$X_\rho = \big(\widehat{S}_\rho \,\big|\, \mathcal{R}'_\rho \cup \mathcal{T}(\widetilde{H}_1^\rho) \cup \cdots \cup \mathcal{T}(\widetilde{H}_n^\rho)\big). \tag{1}$$

As above, this triangular presentation is still a presentation of G, with respect to the map $\sigma_\rho\colon \widehat{S}_\rho \to G$ that is the identity on S and restricts to $p_i{}^\rho$ on $\widetilde{H_i}^\rho$. Indeed σ_ρ extends to a morphism φ from the group G' defined by X_ρ to G and there is a morphism $\psi\colon G \to G'$ induced by the inclusion $S \subset \widehat{S}$. Since $\varphi \circ \psi = \mathrm{id}_G$, and since ψ is onto, φ and ψ are inverse to each other.

We say that the truncated presentation X_ρ is *exact* if for all i, ${p_i}^\rho\colon \widetilde{H}_i^{\,\rho} \to H_i$ is an isomorphism. By definition, X_∞ is always exact. If X_ρ is exact for some $\rho<\infty$, then all H_i are finitely presented. Conversely, if all H_i are finitely presented, then X_ρ is exact for ρ large enough (and X_ρ is exactly the same presentation as X_∞).

In Section 3, we are going to prove that if X_∞ satisfies a linear isoperimetric inequality, so does X_ρ for ρ large enough. This will easily imply that parabolic subgroups are finitely presented.

1.3 Complexities

Since X_ρ is an infinite presentation, it is convenient to have a measure of complexity for letters and words on $\widehat{S}$. Recall that $\widehat{S}_\rho = S \sqcup \widetilde{H}_1^{\,\rho} \sqcup \cdots \sqcup \widetilde{H}_n^\rho$. For $a \in \widetilde{H}_i^{\,\rho}$, we denote by $|\widetilde{a}|_{\widetilde{S}_i{}^\rho}$ the word length of a relative to the generating set $\widetilde{S}_i^{\,\rho}$. We define the *complexity* $\|a\|$ of $a\in\widehat{S}_\rho$ by $\|a\|=1$ if $a\in S$, and by $\|a\| = |a|_{\widetilde{S}_i{}^\rho}$ if $a\in\widetilde{H}_i^{\,\rho}$.

Given a word $w=a_1\cdots a_n$ over $\widehat{S}_\rho$, we define:

- $\operatorname{length}(w)=n$
- $\|w\|_1 = \sum_{i=1}^{n} \|a_i\|$
- $\|w\|_\infty = \max_{i=1,\dots,n} \|a_i\|$

Note that if w is a one-letter word, then $\|w\|_1=\|w\|_\infty=\|w\|$.

Similarly, if D is a diagram (or a path) whose edges are labeled by elements of $\widehat{S}_\rho$, we define $\|D\|_1$ and $\|D\|_\infty$ as the sum and the maximum of the complexities of the labels of its edges. For a labeled path p, $\operatorname{length}(p)$ denotes its number of edges, and $\operatorname{Area}(D)$ denotes the number of 2–cells of a diagram D.

2 Diagrams

The goal of this section is to prove that if X_ρ does not satisfy a linear isoperimetric inequality, this shows up on diagrams of small area and small complexity (Proposition 2.9).

2.1 Vocabulary

Thickness Let D be a van Kampen diagram over the presentation X_ρ (ρ being fixed in $\mathbb{N}\cup\{\infty\}$). We denote by $D_{\text{thick}}\subset D$ the union of all 2–cells, and of all vertices and edges that are contained in the boundary of a 2–cell. We say that D is *thick* if $D=D_{\text{thick}}$ ie if every edge lies in the boundary of a 2–cell.

Clusters We define cells of type $\mathcal{R}'$ (resp. of type $\widetilde{H}_i^{\rho}$) as those labeled by a word of $\mathcal{R}'$ (resp. by a word in $\mathcal{T}(\widetilde{H}_i^{\rho})$). Any cell having an edge labeled by an element of $\widetilde{H}_i^{\rho}$ is either a cell of type $\widetilde{H}_i^{\rho}$, in which case its other edges are also labeled by elements of $\widetilde{H}_i^{\rho}$, or is labeled by a two-letter word in $\mathcal{R}' \setminus \mathcal{R}$, and its unique other edge is labeled by an element of S. Note in particular that two cells of type $\widetilde{H}_i^{\rho}$ and $\widetilde{H}_j^{\rho}$ cannot share an edge if $i \neq j$.

Two cells of the same type $\widetilde{H}_i^{\rho}$ and sharing an edge are said to be *cluster-adjacent*. A *cluster* is an equivalence class for the transitive closure of this relation. All 2–cells of a cluster have the same type $\widetilde{H}_i^{\rho}$, which we define as the type of the cluster. We identify a cluster with the closure C of the 2–cells it is made of. Note that clusters are contained in D_{thick}.

If C is a cluster, we denote by ∂C its topological boundary, ie the union of closed edges of C that are in only one 2–cell of C.

Remark 2.1 Note that for any cluster C, any edge e in $\partial C \setminus \partial D$ has complexity $\|e\|_1 = \|e\|_\infty \leq 1$. Indeed, the 2–cell of $D \setminus C$ containing this edge is labeled by a relator $\widetilde{s}^{-1} p_i^{\rho}(\widetilde{s})$ for some $\widetilde{s} \in \widetilde{S}_i^{\rho}$.

2.2 Simply connected clusters, standard filling

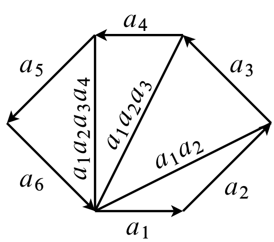

Figure 1: Standard filling

Note that a cluster C (as a subset of the plane) is simply connected if and only if C is a disk and ∂C is an embedded circle in the plane. We will mostly deal with diagrams whose clusters are simply connected.

Consider a simply connected cluster C, with ∂C labeled by the cyclic word $a_1, \dots, a_n$ (where each $a_j \in \widetilde{H}_i^{\rho} \cup \overline{\widetilde{H}_i^{\rho}}$). If $n \geq 3$, a *standard filling* of ∂C is a diagram with boundary ∂C, all whose vertices are in ∂C, and with $n-2$ triangles as in Figure 1. More precisely, if $v_1, \dots, v_n$ are the vertices of ∂C, and its edges are $e_1, \dots, e_n$ with e_j joining v_j to v_{j+1} (modulo n) and labeled $a_j \in \widetilde{H}_i^{\rho} \cup \overline{\widetilde{H}_i^{\rho}}$, then for all $j = 1, \dots, n-2$, there is a triangle joining v_1, v_{j+1}, v_{j+2}, where the edge joining v_1

to v_j is labeled by the image of the product $a_1 \cdots a_j$ in $\widetilde{H}_i^{\rho}$. If $n \leq 2$, the standard filling of ∂C is the diagram with boundary ∂C, and no other edge or vertex (it is a single cell that is a bigon or a monogon).

Lemma 2.2 *If C is an arbitrary simply connected cluster, then*

$$\|\partial C\|_1 \leq 3\,\mathrm{Area}(D) + \|\partial D\|_1.$$

If C is standardly filled, then

$$\mathrm{Area}(C) = \max\{1, \mathrm{length}(\partial C) - 2\} \quad \textit{and} \quad \|C\|_\infty \leq \|\partial C\|_1.$$

Proof Each edge of $\partial C \setminus \partial D$ has complexity at most 1 by Remark 2.1, and there are at most $\mathrm{length}(\partial C) \leq 3\,\mathrm{Area}(D)$ such edges. The sum of complexities of the edges in ∂D is bounded by $\|\partial D\|_1$. This proves the first assertion. The second assertion is clear from the definition of a standard filling. □

Remark 2.3 If C is any cluster, then $\mathrm{Area}(C) \geq \mathrm{length}(\partial C) - 2$. Indeed, denote by F the number of 2–cells of C, and by E_{ext}, E_{int}, the number of edges in ∂C and in $C \setminus \partial C$ respectively. Then, by connectedness of the dual graph, $F - 1 \leq E_{\mathrm{int}}$. Since cells of C have at most 3 sides, $2E_{\mathrm{int}} + E_{\mathrm{ext}} \leq 3F$. It follows that $E_{\mathrm{ext}} \leq F + 2$ as required.

The following lemma shows that in many situations, clusters are simply connected.

Lemma 2.4 *Let w be a word over $\widehat{S}_\rho$ defining the trivial element in G. Let D be a minimal van Kampen diagram for w over the presentation X_ρ. Assume that $\rho \geq 3\,\mathrm{Area}(D)$.*

If D is chosen among diagrams for w over X_ρ to minimize successively the area, and the number of 2–cells of type $\mathcal{R}'$, then every cluster of D is simply connected.

Assume either that D is as above and that all its clusters are standardly filled, or that D minimizes successively the area, the number of 2–cells of type $\mathcal{R}'$ and $\|D\|_\infty$. Then

$$\|D\|_\infty \leq 3\,\mathrm{Area}(D) + \|w\|_1.$$

Proof Assume for contradiction that there exists a cluster C of type $\widetilde{H}_i^{\rho}$ that is not simply connected. Then there is a simply connected subdiagram $D' \subset D$ such that edges of $\partial D'$ are all in $\partial C \setminus \partial D$. Since edges of $\partial D'$ lie in a 2–cell, $\mathrm{length}(\partial D') \leq 3\,\mathrm{Area}(D)$. Moreover $\|\partial D'\|_\infty = 1$, since by Remark 2.1, every edge in $\partial C \setminus \partial D$ has complexity 1. Thus, $\|\partial D'\|_1 \leq 3\,\mathrm{Area}(D)$. Since $\rho \geq 3\,\mathrm{Area}(D)$, the definition of X_ρ says that the word labeled by $\partial D'$ is trivial in $\widetilde{H}_i$. One can then replace the subdiagram bounded by $\partial D'$ by a standardly filled diagram (with cells of type $\widetilde{H}_i{}^{\rho}$), that has smaller or equal area. This contradicts the minimality of D for the number of 2–cells of type $\mathcal{R}'$. It follows that all clusters of D are simply connected.

Assume now that all clusters are standardly filled. By Lemma 2.2, for each cluster C, $\|C\|_\infty \leq \|\partial C\|_1 \leq 3\,\mathrm{Area}(D) + \|w\|_1$. Since each edge of D_{thick} of complexity at least 2 is contained in a cluster, this implies that $\|D_{\mathrm{thick}}\|_\infty \leq 3\,\mathrm{Area}(D) + \|w\|_1$.

Finally, assume that D minimizes successively the area, the number of 2–cells of type $\mathcal{R}'$ and $\|D\|_\infty$. Since clusters of D are simply connected, we can modify D to a diagram D' whose clusters are standardly filled, and having the same area and the same number 2–cells of type $\mathcal{R}'$ as D. In particular, $\|D\|_\infty \leq \|D'\|_\infty$. By the argument above, $\|D'\|_\infty \leq 3\,\mathrm{Area}(D) + \|w\|_1$ which concludes the proof. □

2.3 Complicated clusters

A cluster C is said to be *complicated* if $\partial C \cap \partial D$ contains at least two edges.

Lemma 2.5 *Assume that D is a van Kampen diagram, and $C \subset D$ is a simply connected cluster.*

If C is not complicated, then $\|\partial C\|_\infty \leq \mathrm{length}(\partial C)$, $\|\partial C\|_1 \leq 2\,\mathrm{length}(\partial C)$.

Proof Denote by $\widetilde{H}_i{}^{\rho}$ the type of the cluster C, so that edges of C are labeled by elements of $\widetilde{H}_i{}^{\rho}$. If C is not complicated, all edges of ∂C but one have complexity 1. The cluster being simply connected, the label of the remaining edge has the same image in $\widetilde{H}_i{}^{\rho}$ as a product of $\mathrm{length}(\partial C) - 1$ elements of $(\widetilde{S}_i{}^{\rho})^{\pm 1}$. Therefore, this edge has complexity at most $\mathrm{length}(\partial C) - 1$. It follows that $\|\partial C\|_\infty \leq \mathrm{length}(\partial C)$, and $\|\partial C\|_1 \leq (\mathrm{length}(\partial C) - 1) + \sum_{e \in \partial C} 1$. This proves the lemma. □

Lemma 2.6 (See also [9, Lemma 2.27]) *Let D be a van Kampen diagram whose clusters are simply connected, noncomplicated and standardly filled.*

Then $\|D_{\mathrm{thick}}\|_\infty \leq 6\,\mathrm{Area}(D)$.

Proof Any edge of D_{thick} is either contained in a cell of type $\mathcal{R}'$ (it has complexity 1) or in a cluster C. Since the number of edges of D that lie in the boundary of a 2–cell is bounded by $3 \times \mathrm{Area}(D)$, we have $\mathrm{length}(\partial C) \leq 3 \times \mathrm{Area}(D)$. Since C is not complicated, $\|C\|_\infty \leq 6 \times \mathrm{Area}(D)$ by Lemma 2.5. The lemma follows. □

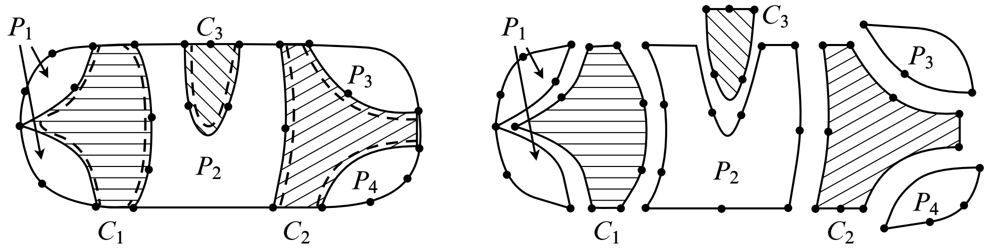

Figure 2: 3 complicated clusters, 4 regular pieces and 6 cluster arcs

2.4 Cluster arcs and pieces

In this section, we explain how to cut D along the boundary components of the complicated clusters (we do not touch the noncomplicated clusters).

Consider a diagram D whose clusters are simply connected. A *cluster arc* is a maximal subpath $c \subset \partial C$ for some complicated cluster C that does not contain any edge of ∂D (see Figure 2). Since ∂C is an embedded circle, each cluster arc c is an embedded arc with endpoints in ∂D, and $c \cap \partial D$ contains no edge, but it may contain vertices distinct from its endpoints.

We define *regular pieces* of D as the connected components of $D \setminus \mathring{\mathcal{C}}$, where $\mathring{\mathcal{C}}$ denotes the interior in D of the union of all complicated clusters in D (edges in $\partial D \cap \partial C$ for some complicated cluster are in $\mathring{\mathcal{C}}$); see Figure 2. Regular pieces and complicated clusters are called *pieces*.

Here is an alternative description. For each complicated cluster C, consider properly embedded arcs with endpoints in ∂D, that are very close and parallel to each cluster arc, obtained by pushing inside C the cluster arcs. Let $\mathcal{A}$ be the union of such embedded arcs when C ranges over all complicated clusters. Then connected components of $D \setminus \mathcal{A}$ are in one-to-one correspondence with pieces. On Figure 2, $\mathcal{A}$ is represented by dotted lines.

Although we won't need it, we note that this also makes sense if D is not thick: edges of $D \setminus D_{\text{thick}}$ are contained in regular pieces.

Clearly, the set of pieces induces a partition of the set of 2–cells of D. There is a natural *incidence graph* $\mathcal{G}$ for this partition, whose vertices are the pieces, whose edges are the cluster arcs, the two endpoints of an edge being the cluster and the regular piece on both sides of the corresponding cluster arc.

Lemma 2.7 *Let D be a van Kampen diagram and assume that any cluster of D is simply connected. The incidence graph $\mathcal{G}$ is a bipartite tree and the degree of a vertex*

v associated to a complicated cluster C is at most the number of edges in $\partial D \cap \partial C$, with strict inequality when v is a leaf of the tree $\mathcal{G}$.

Proof The graph is bipartite by definition. It is connected because D is connected. Since every cluster arc separates D, every edge of the incidence graph disconnects it. This proves that $\mathcal{G}$ is a tree.

Consider a vertex v associated to a complicated cluster C. The degree of v is, by definition, the number of cluster arcs on ∂C. Since C is simply connected, ∂C is an embedded circle, and since C is complicated, ∂C contains an edge of ∂D. By maximality in the definition of cluster arcs, each such arc is followed in ∂C (with a chosen fixed orientation) by an edge of $\partial C \cap \partial D$. This association, which is clearly one-to-one, ensures the bound on the degree.

Finally, if v is a leaf of $\mathcal{G}$, its degree is 1 and $\partial D \cap \partial C$ contains at least 2 edges because C is complicated. □

The following result of [3] was, to some extent, left to the reader. We include a proof.

Lemma 2.8 *Let D be a van Kampen diagram. If every cluster is simply connected, then the number of pieces, and the number of cluster arcs, are both bounded by* $\operatorname{length}(\partial D)$.

Proof The number N of pieces is the number of vertices of the incidence graph $\mathcal{G}$. Since $\mathcal{G}$ is a tree, $N = E + 1$, where E is the number of edges of $\mathcal{G}$, ie the number of cluster arcs. Denote by v_C the vertex corresponding to a cluster C, by $d(v_C)$ its degree, and by V_{cl} the set of all vertices of $\mathcal{G}$ corresponding to clusters. Since $\mathcal{G}$ is bipartite, $E = \sum_{v_C \in V_{cl}} d(v_C)$. By Lemma 2.7, $d(v_C)$ is bounded by the number $e(C)$ of edges of $\partial C \cap \partial D$. Therefore $E \le \sum_{v_C \in V_{cl}} e(C) \le \operatorname{length}(\partial D)$.

Finally, if some v_C is a leaf of $\mathcal{G}$, this last inequality is a strict inequality, which yields $N = E + 1 \le \operatorname{length}(\partial D)$. There remains the case where some leaf of $\mathcal{G}$ is a regular piece B. This means that $\partial B = \alpha \cup \beta$, where α is a cluster arc, and β is a path in ∂D. Since clusters are simply connected, the endpoints of α are distinct, so β contains at least an edge. This implies that $\sum_{v_C \in V_{cl}} e(C) < \operatorname{length}(\partial D)$, and concludes the proof. □

2.5 Reduction to diagrams of small complexity

We are now ready to state and prove the main statement of this section. It claims that if X_ρ does not satisfy a linear isoperimetric inequality, this shows up on diagrams of small area (this is Papasoglu's Theorem) and small complexity.

Proposition 2.9 [3, Proposition 1.5] *Let $K \geq 10^6$ and $\rho \in \mathbb{N} \cup \{\infty\}$, $\rho \geq 3 \times 240K$. Assume that X_ρ fails to satisfy a linear isoperimetric inequality of constant K (that is, there exists a word w over the alphabet $\widehat{S}_\rho$ such that $\mathrm{Area}(w) > K\,\mathrm{length}(w)$).*

Then, there exists a word w'' over the alphabet $\widehat{S}_\rho$, and a minimal van Kampen diagram D'' (over X_ρ) for w'', such that:

(1) $\mathrm{Area}(D'') \leq 240K$
(2) $\|D''\|_\infty \leq 2.10^6 K^2$
(3) $\mathrm{Area}(D'') > \frac{\sqrt{K}}{600}\,\mathrm{length}(\partial D'')$

Proof The first step is to apply Papasoglu's Theorem 1.1 to the presentation X_ρ to obtain a word w' over $\widehat{S}_\rho$ for which

$$K/2 \leq \mathrm{Area}(w') \leq 240K \quad \text{and} \quad \mathrm{Area}(w') > \frac{1}{2 \times 10^4}\,\mathrm{length}(w')^2.$$

Using $\sqrt{\mathrm{Area}(w')} > \mathrm{length}(w')/\sqrt{2 \times 10^4}$ and $\mathrm{Area}(w') \geq K/2$, we get

$$\mathrm{Area}(w') > \sqrt{\frac{\mathrm{Area}(w')}{2 \times 10^4}} \times \mathrm{length}(w') \geq \frac{\sqrt{K}}{200} \times \mathrm{length}(w').$$

Choose a diagram D' among minimal area diagrams over X_ρ for w' so that the number of 2–cells of type $\mathcal{R}'$ is minimal. We claim that up to changing w', we can assume that D' is thick, ie all edges lie in the boundary of a 2–cell. Indeed, if all connected components $A'_1, \dots, A'_l$ of D'_{thick} satisfy $\mathrm{Area}(A'_i) \leq (\sqrt{K}/200) \times \mathrm{length}(\partial A'_i)$, then

$$\mathrm{Area}(D') = \sum_i \mathrm{Area}(A'_i) \leq \frac{\sqrt{K}}{200} \sum_i \mathrm{length}(\partial A'_i) \leq \frac{\sqrt{K}}{200} \times \mathrm{length}(w'),$$

which is a contradiction. It follows that some component A'_i satisfies $\mathrm{Area}(A'_i) > (\sqrt{K}/200) \times \mathrm{length}(\partial A'_i)$. Obviously, $\mathrm{Area}(A'_i) \leq \mathrm{Area}(D') \leq 240K$, and A'_i is a diagram for $\partial A'_i$ that minimizes the area and the number of cells of type $\mathcal{R}'$ (if not, substituting a diagram of smaller area for $\partial A'_i$ in D' contradicts minimality of D'). This proves that we can assume that D' is thick.

We do not have any control on the complexity of a diagram filling w' yet. Since $\rho \geq 3 \times 240K$, Lemma 2.4 shows that the clusters of D' are simply connected. We can modify D' and assume that all clusters are standardly filled. By Remark 2.3, D' still minimizes area and the number of cells of type $\mathcal{R}'$. By Lemma 2.8, the number of pieces in the decomposition into complicated clusters and regular pieces is at most $\mathrm{length}(\partial D')$.

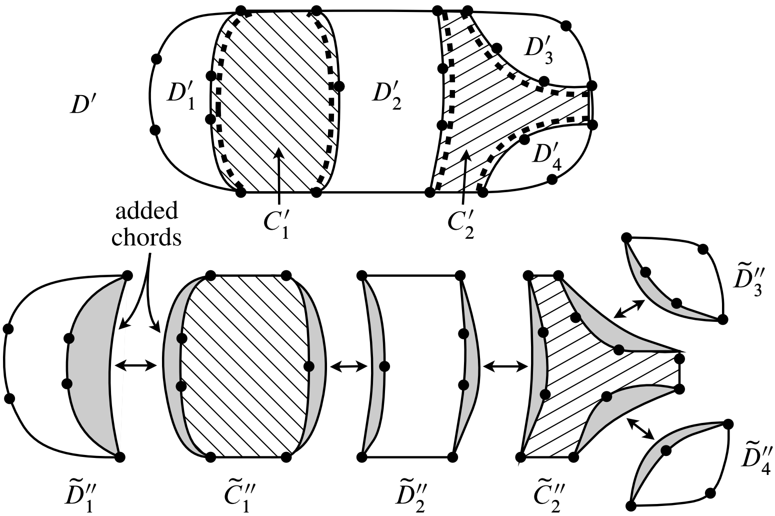

Figure 3: Adding chords to the pieces of D', and regluing them together

Let $C'_1, \dots, C'_s$ be the complicated clusters of D', and $D'_1, \dots, D'_r$, be the regular pieces. We construct new diagrams C''_i, D''_j, and $\widetilde{C}''_i$, $\widetilde{D}''_j$ from C'_i, D'_j by first *adding chords*, then by changing the triangulation as follows (see Figure 3).

Fix a complicated cluster C'_k of D', and denote by $\widetilde{H_i}^\rho$ its type. Its boundary $\partial C'_k$ is a union of pairwise disjoint cluster arcs, together with arcs in $\partial D'$. Consider a cluster arc $c \subset \partial C'_k$ whose edges are labeled by elements $a_1, \dots, a_n$ of $\widetilde{H_i}^\rho$, and let $a_c = a_1 \cdots a_n \in \widetilde{H_i}^\rho$ be their product. We glue along c a standardly filled disk with boundary labeled by $a_1, \dots, a_n, a_c^{-1}$. We name the new edge labeled by a_c^{-1} a *chord*. Performing this operation for each cluster arc, we get a disk C''_k made of cells of type $\widetilde{H_i}^\rho$. Finally, we change the triangulation of this disk to a standard filling, and we call $\widetilde{C}''_k$ the obtained diagram. Note that $\mathrm{Area}(\widetilde{C}''_k) \le \mathrm{length}(\partial \widetilde{C}''_k) - 2$.

Now, we perform a similar operation for each regular piece D'_j. For each cluster arc $c \subset \partial D'_j$ labeled by $a_1, \dots, a_n \in \widetilde{H_i}^\rho$ (now, the type $\widetilde{H_i}^\rho$ may depend on c), we define $a_c = a_1 \dots a_n \in \widetilde{H_i}^\rho$, and glue to C'_k along c a new cluster of type $\widetilde{H_i}^\rho$, standardly filled, whose boundary is labeled by $a_1, \dots, a_n, a_c^{-1}$. Since the filling is standard, the area of the added cluster is $(n+1) - 2 = \mathrm{length}(c) - 1$. Performing this operation for each cluster arc, we get the new diagram D''_j. Finally, we take for $\widetilde{D}''_j$ a diagram with boundary $\partial D''_j$, and minimizing successively the area and the number of 2–cells of type $\mathcal{R}'$.

We are going to bound $\|\widetilde{D}''_j\|_\infty$ by first bounding $\|D''_j\|_\infty$. Since all complicated clusters of D' are $C'_1, \dots, C'_s$, D''_j has no complicated cluster coming from D'. The newly created clusters in D''_j have just one edge in $\partial D''_j$, so are not complicated. Therefore, clusters of D''_j are not complicated, simply connected and standardly filled. Since D' is thick, so is D''_j. Applying Lemma 2.6 to D''_j, we get

$$\|D''_j\|_\infty \le 6 \times \mathrm{Area}(D''_j) \le 6 \times 240K.$$

In particular, $\|\partial \widetilde{D}''_j\|_\infty = \|\partial D''_j\|_\infty \le 6 \times 240K$, and since D''_j is thick, $\|\partial D''_j\|_1 \le 3\,\mathrm{Area}(D''_j)\|\partial D''_j\|_\infty \le 18 \times (240K)^2$. Applying Lemma 2.4 to $\widetilde{D}''_j$, we get

$$\|\widetilde{D}''_j\|_\infty \le 3\,\mathrm{Area}(D''_j) + \|\partial D''_j\|_1 \le 3 \times 240K + 18 \times (240K)^2 \le 2.10^6 K^2.$$

This proves that for all $j \in \{1, \dots, r\}$, $\widetilde{D}''_j$ satisfies assertions (1) and (2) of the proposition.

We now prove that one of the diagrams $\widetilde{D}''_j$, $j = 1, \dots, r$ must satisfy (3). Assume by contradiction that for all $j \in \{1, \dots, r\}$, $\mathrm{Area}(\widetilde{D}''_j) \le (\sqrt{K}/600)\,\mathrm{length}(\partial \widetilde{D}''_j)$. Note that $\widetilde{C}''_k$ satisfies this inequality as well. Indeed, $\mathrm{Area}(\widetilde{C}''_k) \le \mathrm{length}(\partial \widetilde{C}''_k)$, and by assumption, $K \ge 10^6$ so $(\sqrt{K}/600) \ge 1$.

Gluing together the diagrams $\widetilde{D}''_1, \dots, \widetilde{D}''_r$ and $\widetilde{C}''_1, \dots, \widetilde{C}''_s$ pairwise along the two chords corresponding to a given cluster arc as shown on Figure 3, we get another (not necessarily minimal) van Kampen diagram $\widetilde{D}'$ for w'.

We have

$$\begin{aligned}\mathrm{Area}(D') \le \mathrm{Area}(\widetilde{D}') &= \sum_j \mathrm{Area}(\widetilde{D}''_j) + \sum_k \mathrm{Area}(\widetilde{C}''_k) \\ &\le \frac{\sqrt{K}}{600}\Big(\sum_j \mathrm{length}(\partial \widetilde{D}''_j) + \sum_k \mathrm{length}(\partial \widetilde{C}''_k)\Big) \\ &\le \frac{\sqrt{K}}{600}\big(\mathrm{length}(\partial D') + 2n_a\big),\end{aligned}$$

where n_a is the number of cluster arcs in D'. By Lemma 2.8, $n_a \le \mathrm{length}(\partial D')$, so $\mathrm{Area}(D') \le (\sqrt{K}/200) \times \mathrm{length}(\partial D')$, thus contradicting the property of D' established at the beginning of the proof. □

3 Consequences

Corollary 3.1 *Assume that X_∞ satisfies a linear isoperimetric inequality of factor $K \ge 10^6$. Let $K' = (600K)^2$ and $\rho_0(K) = 10^{26} K^5$ Then for all $\rho \ge \rho_0(K)$, X_ρ satisfies a linear isoperimetric inequality of factor K'.*

Before proving the corollary, we need to relate more explicitly the presentations X_ρ and X_∞. Recall that $\widehat{S}_\rho = S \sqcup \widetilde{H}_1{}^\rho \sqcup \cdots \sqcup \widetilde{H}_n^\rho$ and $\widehat{S}_\infty = S \sqcup \widetilde{H}_1 \sqcup \cdots \sqcup \widetilde{H}_n$ the corresponding generating sets, and that we have morphisms $p_i{}^\rho \colon \widetilde{H}_i{}^\rho \to H_i$ and isomorphisms $p_i \colon \widetilde{H}_i \to H_i$. The morphisms

$$q_i = p_i^{-1} \circ p_i{}^\rho \colon \widetilde{H}_i{}^\rho \to \widetilde{H}_i$$

induce an obvious map $p\colon \widehat{S}_\rho \to \widehat{S}_\infty$ that is the identity on S and maps $\widetilde{H}_i{}^\rho$ to $\widetilde{H}_i$ through $p_i^{-1} \circ p_i{}^\rho$. If $w = a_1 \cdots a_n$ is a word over $\widehat{S}_\rho$, we denote by $p(w) = p(a_1)\cdots p(a_n)$ the corresponding word over $\widehat{S}_\infty$. Clearly, if w is any relator of the presentation X_ρ, then $p(w)$ is a relator of X_∞. It follows that given any diagram D over X_ρ for a word w, one gets a new diagram $p_*(D)$ for $p(w)$ over X_∞ by applying the map p to all the labels of all edges of D.

On the other hand, q_i induces a bijection between the balls of radius $\rho/2$ of $\widetilde{H}_i{}^\rho$ and $\widetilde{H}_i$, whose inverse we denote by q_i^{-1}. Similarly, we denote by p^{-1} the inverse of the restriction of $p\colon \widehat{S}_\rho \to \widehat{S}_\infty$ to the set of elements of complexity at most $\rho/2$. Now, if $a, b, c \in H_i$ are in the ball of radius $\rho/3$ of H_i and satisfy $abc = 1$ in $\widetilde{H}_i$, then $q_i^{-1}(a) q_i^{-1}(b) q_i^{-1}(c) = 1$ in $\widetilde{H}_i{}^\rho$. This means that if some diagram D over X_∞ for w satisfies $\|D\|_\infty \le \rho/3$, then the diagram $p_*^{-1}(D)$ (with obvious notations) is a diagram over X_ρ for $p^{-1}(w)$.

Proof of Corollary 3.1 Assume that X_ρ fails to satisfy the predicted isoperimetric inequality (of factor K'), and argue towards a contradiction. By Proposition 2.9, there is a word w'' representing the trivial element, with a diagram D'', minimal over the presentation X_ρ, of area at most $240K'$, and complexity $\|D''\|_\infty \le 2.10^6 K'^2$ and such that $\mathrm{Area}(D'') > (\sqrt{K'}/600) \times \mathrm{length}(w'') = K \times \mathrm{length}(w'')$.

Consider the map $p\colon \widehat{S}_\rho \to \widehat{S}_\infty$ described above. Choose D_0'' among diagrams for $p(w'')$ in the presentation X_∞, in order to minimize successively the area, the number of 2–cells of type $\mathcal{R}'$, and the complexity $\|D_0''\|_\infty$. Since X_∞ satisfies a linear isoperimetric inequality of factor K, $\mathrm{Area}(D_0'') \le K \times \mathrm{length}(w'')$. It follows from the estimate of the previous paragraph, that $\mathrm{Area}(D_0'') < \mathrm{Area}(D'')$ which is itself $\le 240K'$.

By Lemma 2.4, $\|D_0''\|_\infty \le 720K' + \|p(w'')\|_1$. Let us estimate the terms:

$$\begin{aligned} \|p(w'')\|_1 \le \|w''\|_1 \le \mathrm{length}(w'')\|D''\|_\infty &\le \mathrm{length}(w'') \times 2.10^6 K'^2 \\ &\le \frac{1}{K}\,\mathrm{Area}(D'') \times 2.10^6 K'^2 \\ &\le \frac{1}{K} \times 240K' \times 2.10^6 K'^2 \le 3.10^{25} K^5. \end{aligned}$$

Also, since $K \geq 10^6$, one has $720K' \leq 10^9 K^2 \leq K^5$. By hypothesis on ρ, we see that $\|D_0''\|_\infty \leq \rho/3$. It follows that $p_*^{-1}(D_0'')$ is a diagram over X_ρ for w''. We already noticed that it has area $<$ Area(D''), a contradiction to the minimality of D'' over the presentation X_ρ. □

Lemma 3.2 *Assume that X_ρ satisfies a linear isoperimetric inequality of factor K' with $\rho \geq \max(3K', 2)$.*

Then $p_i{}^\rho\colon \widetilde{H}_i{}^\rho \to H_i$ is an isomorphism. In particular, H_i is finitely presented, with a presentation whose defining relations are of length $\leq \rho$.

Proof Assume for contradiction that $p_i\colon \widetilde{H}_i{}^\rho \to H_i$ is not injective, and consider $a \in \ker p_i \setminus \{1\}$. Then a is a generator of the presentation X_ρ that represents the trivial element of G. Note that since $\rho > 1$, $a \notin \widetilde{S}_i{}^\rho$. Therefore, there exists a van Kampen diagram D over X_ρ whose boundary consists of a single edge e labeled a, and whose area is at most K'. We choose a diagram for a over X_ρ in order to minimize successively the area, the number of 2–cells of type $\mathcal{R}'$ and $\|D\|_\infty$. Since $\rho \geq 3K'$, Lemma 2.4 implies that clusters of D are simply connected. Since $a \notin \widetilde{S}_i{}^\rho$, e lies in a cluster C of type $\widetilde{H}_i{}^\rho$. But since C is simply connected, and since a cluster of type $\widetilde{H}_i^\rho$ involves only relations of $\widetilde{H}_i{}^\rho$, we get that a is trivial in $\widetilde{H}_i{}^\rho$, a contradiction. □

Corollary 3.3 *Assume that X_∞ satisfies a linear isoperimetric inequality of factor K. Let ρ_0 be the function defined in Corollary 3.1.*

Then the subgroups H_i are finitely presented, with a presentation whose defining relations are of length $\leq \rho_0(\max(K, 10^6))$.

Proof Without loss of generality, we can assume $K \geq 10^6$. By Corollary 3.1, $X_{\rho_0(K)}$ satisfies a linear isoperimetric inequality of factor $K' = (600K)^2$. Lemma 3.2 concludes. □

Lemma 3.4 (See also [9, Lemma 5.4]) *Assume that X_∞ satisfies a linear isoperimetric inequality of factor K.*

If $s \in S$ represents an element a of $\widetilde{H}_i$, then $\|a\| \leq 12K$.

Proof The word $w = s^{-1}a$ is a word of length 2 over X_∞. If it represents the trivial element in G, then there is a van Kampen diagram D over X_∞ whose boundary is a path of length 2 labeled $s^{-1}a$, and whose area is at most $2K$. We choose D among minimal area diagrams over X_∞ for w so that the number of 2–cells of type $\mathcal{R}'$ is minimal. Lemma 2.4 (applied with $\rho = \infty$) implies that clusters of D are simply connected, and we can assume that they are standardly filled.

Note that there is no complicated cluster as only the edge labeled a of ∂D can be in a cluster. By Lemma 2.6, this implies that $\|D_{\text{thick}}\|_\infty \le 12K$, so $\|a\| \le 12K$. □

We obtain the following improvement of [3]:

Corollary 3.5 *There exists an algorithm that takes as input a finite presentation of a group G, a solution of its word problem and a collection of finite subsets $S_1, \dots, S_n \subset G$, and that terminates if and only if G is hyperbolic relative to $\langle S_1\rangle, \dots, \langle S_n\rangle$.*

In this case, produces an isoperimetry constant K for the presentation X_∞, a finite presentation for each of the parabolic subgroups, and says whether G is parabolic (ie $G = \langle S_i\rangle$ for some i).

Proof For a fixed $K \ge 10^6$, we consider all diagrams D over X_∞ such that $\|D\|_\infty \le B = 2.10^6 K^2$ and $\text{Area}(D) \le 240K$. There are only finitely many. The word problem in G allows to list all relators of $\langle S_i\rangle$ of length at most $3B$, to compute the ball of radius $3B$ in the Cayley graph of H_i with respect to the generating set S_i and hence to list all these diagrams. Out of this list, we make the list $\mathcal{W}(K)$ of words labeling the boundaries of these diagrams.

We claim that given $w \in \mathcal{W}(K)$, we can compute $\text{Area}(w)$. Indeed, let D' be a diagram for w chosen to minimize area, the number of cells of type $\mathcal{R}'$, and $\|D'\|_\infty$. By Lemma 2.4, $\|D'\|_\infty \le 3\,\text{Area}(D') + \|w\|_1 \le 720K + \|w\|_1$. We can compute the upper bound $M = 720K + \|w\|_1$ for $\|D'\|_\infty$, and we can list all diagrams D' with $\text{Area}(D') \le 240K$ and $\|D'\|_\infty \le M$ whose boundary is w. We can then compute $\text{Area}(w)$ as the minimal area of these diagrams.

Now we can check whether $\text{Area}(w) \le (\sqrt{K}/600)\,\text{length}(w)$ for all $w \in \mathcal{W}(K)$. If this is not the case, the algorithm increments K and starts over.

If this is the case, then by Proposition 2.9, X_∞ satisfies isoperimetric inequality of factor K, and the algorithm stops. It outputs K, and gives as set of relators for $\langle S_i\rangle$, the set of all words of length $\le \rho_0(K)$ that are trivial in G; this can be done using the word problem in G, and this is indeed a presentation of $\langle S_i\rangle$ by Corollary 3.3. To check whether $G = \langle S_i\rangle$, one needs to check whether each $s \in S$ represents an element $a \in \langle S_i\rangle$. Lemma 3.4 bounds the complexity of a, and we can try all possibilities for a using the word problem.

If X_∞ does satisfy a linear isoperimetric inequality of factor K_0, then the process will obviously stop when K will reach a value greater than $(600K_0)^2$. □

Corollary 3.6 *There exists an algorithm as follows. It takes as input a finite presentation of a group G, a solution for its word problem and a recursive class of finitely presented groups $\mathcal{C}$ (given by a Turing machine enumerating them). It terminates if and only if G is properly hyperbolic relative to subgroups that are in the class $\mathcal{C}$.*

In this case, the algorithm produces an isoperimetry constant K, a generating set and a finite presentation for each of the parabolic subgroups.

Proof First, enumerate all possible presentations of groups in $\mathcal{C}$ using the Turing machine given as input, and Tietze transformations. Denote by L the set of currently proposed presentations. In a second parallel process, list all possible families of finite subsets $\mathcal{S} = (S_1, \dots, S_n)$ of G. For each of them, run in parallel the algorithm of Corollary 3.5 that stops if G is hyperbolic relative to $\langle S_1\rangle, \dots, \langle S_n\rangle$ and outputs a presentation of $\langle S_i\rangle$ in this case, and says whether G is parabolic. For each $\mathcal{S}$ such that G is hyperbolic relative to $\langle S_1\rangle, \dots, \langle S_n\rangle$ and is not parabolic, record the obtained tuple of presentations $P_{\mathcal{S}} = (\mathcal{P}_1, \dots, \mathcal{P}_n)$.

We thus have two enumerations: one of the presentations of the groups in $\mathcal{C}$ (the list L), and one of the proper relatively hyperbolic structures of G, with presentations of the parabolic subgroups. At each step compare the set of recorded tuples $P_{\mathcal{S}}$ with the presentations in the list L. If for some $\mathcal{S}$ all the presentations of the tuple $P_{\mathcal{S}}$ are listed in L, then stop. □

4 A geometric proof of the finite presentation

After listening to a talk by V Gerasimov, we realized that Theorem 1 about the finite presentation of parabolic subgroups can be proved using the following geometric argument which is developed further in Gerasimov and Potyagailo's [4].

We recall a construction of a proper hyperbolic space X for G (see Bowditch [2], Groves and Manning [6]; see also Hruska [7, Theorem 4.4]). Let G be relatively hyperbolic relative to finitely generated subgroups $H_1, \dots, H_n$. We take a finite generating set S_i for each H_i and we take Y a Cayley 2–complex of G for a generating set S containing each S_i. We denote by C_i the Cayley graph of H_i with respect to S_i, which we view as a subgraph of Y. For each i, consider a combinatorial horoball based on C_i: this is a graph with vertices $V(C_i) \times \mathbb{N}$, where there is an edge between (x, i) and $(x, i+1)$, and an edge between (x, i) and (y, i) if $d_{C_i}(x, y) \le 2^i$. We view C_i as the subset $C_i \times \{0\}$ of B_i. Then we define X by gluing on Y a copy of B_i on each gC_i for each $g \in G/H_i$. If G is hyperbolic relative to $H_1, \dots, H_n$, then X is a Gromov-hyperbolic space [6].

Let $p\colon X \to B_i$ be the closest point projection for the metric in X. Clearly, $p(Y) \subset C_i$. Quasiconvexity of B_i shows that p is coarsely Lipshitz. The projection of each 2–cell of Y is a uniformly bounded subset of C_i (for the metric on X, hence for its intrinsic metric). Let K be a bound on the diameter in C_i of the projection of a 2–cell of Y, and let C_i' be the corresponding Rips complex of C_i, where one adds a simplex on a set of vertices $S \subset C_i$ whenever $\mathrm{diam}(S) \leq K$.

Let c be any cycle in C_i. Thus c can be viewed as a path in Y. Since G is finitely presented, c bounds a disk D in Y. Then the projection of each 2–cell of Y can be filled by a 2–cell in C_i', so c is nullhomotopic in C_i'. Since this argument is independent of the choice of the path c, this proves that C_i' is simply connected. Since C_i'/H_i is compact, it follows that H_i is finitely presented.

This argument can be refined to show that if G has a finite classifying space (resp. is of type FP_n), then so are its parabolic subgroups.

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