Algebraic & Geometric Topology 14 (2014) 349–378

A spectral sequence for fusion systems

ANTONIO DÍAZ RAMOS

We build a spectral sequence converging to the cohomology of a fusion system with a strongly closed subgroup. This spectral sequence is related to the Lyndon–Hochschild–Serre spectral sequence and coincides with it for the case of an extension of groups. Nevertheless, the new spectral sequence applies to more general situations like finite simple groups with a strongly closed subgroup and exotic fusion systems with a strongly closed subgroup. We prove an analogue of a result of Stallings in the context of fusion preserving homomorphisms and deduce Tate's p–nilpotency criterion as a corollary.

55T10; 55R35, 20D20

1 Introduction

Let $K \trianglelefteq G$ be a normal subgroup of the finite group G and consider the extension

$$K \to G \to G/K.$$

The Lyndon–Hochschild–Serre spectral sequence of this short exact sequence is an important tool to analyze the cohomology of G with coefficients in the $\mathbb{Z}G$–module M. It has second page $E_2^{n,m} = H^n(G/K; H^m(K;M))$ with G/K acting on $H^m(K;M)$ and converges to $H^{n+m}(G;M)$.

Our aim in this work is to construct a related spectral sequence in the context of fusion systems. This concept was originally introduced by Puig and developed by Broto, Levi and Oliver in [3], to which we refer the reader for notation. It consists of a category $\mathcal{F}$ with objects the subgroups of a finite p–group S and morphisms bounded by axioms that mimic properties of conjugation morphisms.

In the setup of fusion systems the concept of a short exact sequence is an evasive one: Let $\mathcal{F}$ be a fusion system over the p–group S. For a strongly $\mathcal{F}$–closed subgroup T of S there is a quotient fusion system $\mathcal{F}/T$; Craven [7, 5.10]. Nevertheless, in general there is no normal fusion subsystem of $\mathcal{F}$ that would play the role of the kernel of the morphism of fusion systems $\mathcal{F} \to \mathcal{F}/T$; Aschbacher [1, 8.11 ff]. So the answer to Solomon and Stancu [16, Conjecture 11] is negative and one cannot expect to construct

Published: 9 January 2014 DOI: 10.2140/agt.2014.14.349

a Lyndon–Hochschild–Serre spectral sequence for fusion systems. Here we are able to construct a spectral sequence that converges to the cohomology of $\mathcal{F}$, $H^*(\mathcal{F};M)$, where M is a $\mathbb{Z}_{(p)}$–module with trivial action of S. Recall that $H^*(\mathcal{F};M)$ is defined [3, Section 5] as the following subring of $\mathcal{F}$–stable elements in $H^*(S;M)$:

$$H^*(S;M)^{\mathcal{F}} = \{z \in H^*(S;M) \mid \mathrm{res}(z) = \varphi^*(z) \text{ for each } \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)\},$$

where res: $H^*(S;M) \to H^*(P;M)$ is restriction in cohomology.

Theorem 1.1 *Let $\mathcal{F}$ be a fusion system over the p–group S, T a strongly $\mathcal{F}$–closed subgroup of S and M a $\mathbb{Z}_{(p)}$–module with trivial S–action. Then there is a first quadrant cohomological spectral sequence with second page*

$$E_2^{n,m} = H^n\big(S/T; H^m(T;M)\big)^{\mathcal{F}}$$

and converging to $H^{n+m}(\mathcal{F};M)$.

The notation $^{\mathcal{F}}$ for the second page will be fully described in Section 2, and must be thought as taking $\mathcal{F}$–stable elements in a similar way as explained for $H^*(\mathcal{F};M)$ above. Consider for each subgroup P of S the Lyndon–Hochschild–Serre spectral sequence of the extension

$$P \cap T \to P \to P/P\cap T \cong PT/T$$

converging to $H^*(P;M)$. A morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ induces a morphism φ^* between the spectral sequences corresponding to Q and P. Hence we have a contravariant functor from $\mathcal{F}$ to the category of spectral sequences. Recall that a morphism in this category from E' to E'' is a sequence of homomorphisms of differential bigraded $\mathbb{Z}_{(p)}$–modules, $f_k\colon E'_k \to E''_k$, $k \geq 0$, such that $H(f_k) \cong f_{k+1}$. The inverse limit spectral sequence or spectral sequence of $\mathcal{F}$–stable elements has $E_2^{n,m}$ entry equal to $H^n(S/T;H^m(T;M))^{\mathcal{F}}$, ie, the elements z from

$$H^n\big(S/T; H^m(T;M)\big)$$

such that $\varphi^*(z) = \mathrm{res}(z)$, where $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ and $\mathrm{res} = \iota^*$ is restriction in cohomology for the inclusion $P \overset{\iota}{\leq} S$. Hence $H^*(S/T;H^*(T;M))^{\mathcal{F}}$ is a differential graded subalgebra of the differential graded algebra $H^*(S/T;H^*(T;M))$ and its differential is just restriction of the differential of the latter. This should be useful in computations. The theorem states that the abutment of this spectral sequence is $H^*(\mathcal{F};M)$.

For the case of a normal subgroup $K \trianglelefteq G$ and $\mathcal{F} = \mathcal{F}_S(G)$ with $S \in \mathrm{Syl}_p(G)$ we have two spectral sequences converging to $H^*(G;M)$. Here, M is a $\mathbb{Z}_{(p)}$–module with trivial G–action (and hence trivial S–action). On the one hand, we have the Lyndon–Hochschild–Serre spectral sequence associated to $K \to G \to G/K$. On the other hand, we have the spectral sequence associated to $\mathcal{F}$ and the strongly $\mathcal{F}$–closed subgroup $T = K \cap S \in \mathrm{Syl}_p(K)$. In Section 5 we prove that the two spectral sequences are isomorphic. Note that, in particular, this shows that the Lyndon–Hochschild–Serre spectral sequence of the extension $K \to G \to G/K$ depends only on the intersection of K with a Sylow p–subgroup of G.

As an application of the spectral sequence in Theorem 1.1 we prove an analogue of a result of Stallings. Meanwhile the original theorem deals with a group homomorphism, here we replace that notion by that of a *fusion preserving* homomorphism. This is a group homomorphism $S_1 \to S_2$ between the Sylow p–subgroups of two fusion systems $\mathcal{F}_1$ and $\mathcal{F}_2$ such that morphisms of $\mathcal{F}_1$ are transformed into morphisms of $\mathcal{F}_2$ (see Section 6).

Theorem 1.2 (Stallings [17, page 170]) *Let $\mathcal{F}_i$ be a fusion system over the p–group S_i for $i = 1, 2$ and let $\phi\colon S_1 \to S_2$ be a fusion preserving homomorphism. If the induced map in cohomology $H^i(\mathcal{F}_2; \mathbb{F}_p) \to H^i(\mathcal{F}_1; \mathbb{F}_p)$ is an isomorphism for $i = 1$ and a monomorphism for $i = 2$ then $S_1/\mathcal{O}^p_{\mathcal{F}_1}(S_1) \cong S_2/\mathcal{O}^p_{\mathcal{F}_2}(S_2)$.*

The *hyperfocal subgroup* of $\mathcal{F}_i$, $O^p_{\mathcal{F}_i}(S_i)$, $(i = 1, 2)$ is defined as follows:

$$O^p_{\mathcal{F}_i}(S_i) = \langle [P, O^p(\mathrm{Aut}_{\mathcal{F}_i}(P))] \mid P \le S_i \rangle.$$

It is the smallest subgroup of S_i such that the quotient of $\mathcal{F}_i$ over that subgroup is a p–group; see Broto, Castellana, Grodal, Levi and Oliver [2]. Hence, the conclusion of the theorem is that the largest p–group quotients of $\mathcal{F}_1$ and $\mathcal{F}_2$ are isomorphic. For instance, when $\mathcal{F}_1$ and $\mathcal{F}_2$ are already p–groups, ie, $\mathcal{F}_i = \mathcal{F}_{S_i}(S_i)$, $i = 1, 2$, the conclusion is that S_1 and S_2 are isomorphic. This particular case is a variant of Stallings' result by Evens [10, 7.2.4]. We can also deduce fusion system versions of another result of Evens and Tate's p–nilpotency criterion:

Corollary 1.3 (Evens [10, 7.2.5]) *Let $\mathcal{F}$ be a fusion system over the p–group S. If the map $H^2(\mathcal{F}/E^p_{\mathcal{F}}(S); \mathbb{F}_p) \to H^2(\mathcal{F}; \mathbb{F}_p)$ is a monomorphism then $S/O^p_{\mathcal{F}}(S)$ is elementary abelian.*

Here, the *elementary focal subgroup* of $\mathcal{F}$ is defined as $E^p_{\mathcal{F}}(S) = \Phi(S) O^p_{\mathcal{F}}(S)$ (Díaz, Glesser, Park and Stancu [8]), where $\Phi(S)$ is the Frattini subgroup of S. The conclusion of this corollary is that the largest p–group quotient of $\mathcal{F}$ is elementary abelian.

Corollary 1.4 (Tate [18, Corollary on page 109]) *Let $\mathcal{F}$ be a fusion system over the p–group S. If the restriction map $H^1(\mathcal{F};\mathbb{F}_p)\to H^1(S;\mathbb{F}_p)$ is an isomorphism then $\mathcal{F}=\mathcal{F}_S(S)$.*

This last result was already proven in [8] using transfer for fusion systems and in Cantarero, Scherer and Viruel [5] by topological methods. Here the proof mimics Tate's original cohomological proof that relies on the five-term exact sequence associated to the Lyndon–Hochschild–Serre spectral sequence but uses instead the spectral sequence of Theorem 1.1.

There are situations where the Lyndon–Hochschild–Serre spectral sequence is not applicable while the spectral sequence from Theorem 1.1 can be used. For instance, a classical drawback of the Lyndon–Hochschild–Serre spectral sequence is that it cannot be applied to finite simple groups. Nevertheless there are finite simple groups that do have a strongly closed p–subgroup: Flores and Foote [11] classified all finite groups with a strongly closed p–subgroup, in particular such finite simple groups. Notice that even if $\mathcal{F}$ is induced from a nonsimple finite group $\mathcal{F}=\mathcal{F}_S(G)$ not every strongly closed $\mathcal{F}$–subgroup T of S is of the form $T=K\cap S$ for some normal subgroup $K\trianglelefteq G$ [1, Example 6.4]. This describes another circumstance where Lyndon–Hochschild–Serre does not apply but Theorem 1.1 does. As final example of this situation consider an exotic fusion system with a strongly closed p–subgroup. A family of such exotic fusion systems is described in Díaz, Ruiz and Viruel [9], where the authors classified all the fusion systems over p–groups of p–rank 2 (p odd).

This opens a new range of cohomology computations that can be carried out, some of which the author intends to perform in a subsequent paper. The main limitation here is that the spectral sequence from Theorem 1.1 requires knowledge of the Lyndon–Hochschild–Serre spectral sequence of the extension of p–groups $T\to S\to S/T$, and these computations do not abound.

Remark 1.5 Theorem 1.1 holds for the wider class of $\mathcal{F}$–stable $\mathbb{Z}_{(p)}S$–modules, ie, for $\mathbb{Z}_{(p)}S$–modules M such that for any morphism $\varphi\colon P\to S$ in $\mathcal{F}$ and any $p\in P$ we have $\varphi(p)\cdot m=p\cdot m$. Also, the Lyndon–Hochschild–Serre spectral sequence of $K\trianglelefteq G$ and the spectral sequence from Theorem 1.1 for $\mathcal{F}=\mathcal{F}_S(G)$ and $T=S\cap K$ coincide for G–stable $\mathbb{Z}_{(p)}G$–modules, ie, for $\mathbb{Z}_{(p)}G$–modules M such that $g^{-1}hg\cdot m=h\cdot m$ for any $h,g\in G$.

Organization of the paper

In Section 2, $\mathcal{F}$–stable elements and Mackey functors are defined and some related results introduced. In Section 3, we describe a particular cohomological Mackey functor that will play a central role in the construction of the spectral sequence. In Section 4, the spectral sequence is built and Theorem 1.1 is proven as Theorem 4.1. In Section 5 we compare the spectral sequence from Theorem 1.1 to the Lyndon–Hochschild–Serre spectral sequence and we give an example. In Section 6 we prove Stallings' result and some of its corollaries.

Acknowledgements I would like to thank A Viruel for several fruitful conversations when developing this paper. Also, I am grateful to P Symonds for showing me how to prove that the two spectral sequences coincide in the normal subgroup case (Theorem 5.1).

This work was supported by MCI grant RYC-2010-05663 and partially supported by FEDER-MCI grant MTM2010-18089 and Junta de Andalucía grant FQM-213

2 Cohomology and $\mathcal{F}$–stable elements

Throughout this section $\mathcal{F}$ denotes a fusion system over the p–group S. We start by introducing some notation: If $A\colon \mathcal{F} \to \mathcal{C}$ is a contravariant functor and $\mathcal{C}$ is any category then we denote the value $A(\varphi)$ by φ^*, where φ is a morphism in $\mathcal{F}$. For $\varphi = \iota_P^S$, the inclusion of P into S, we write $\operatorname{res} := \iota_P^{S*}$. If $\mathcal{C}$ is a complete category then we denote by $A^{\mathcal{F}}$ the inverse limit over $\mathcal{F}$ of the functor:

$$A^{\mathcal{F}} := \varprojlim_{\mathcal{F}} A.$$

For the complete category $\mathrm{CCh}(\mathbf{Ab})$ of (unbounded) cochain complexes we have the following favourable description of inverse limits:

Lemma 2.1 *Let $A\colon \mathcal{F} \to \mathrm{CCh}(\mathbf{Ab})$ be a contravariant functor. Then:*

$$A^{\mathcal{F}} = A(S)^{\mathcal{F}} := \{z \in A(S) \mid \operatorname{res}(z) = \varphi^*(z) \text{ for each } \varphi \in \operatorname{Hom}_{\mathcal{F}}(P,S)\} \subseteq A(S).$$

We call the elements in $A(S)^{\mathcal{F}}$ the *$\mathcal{F}$–stable elements* in $A(S)$. For such a functor we can consider the cohomology $H^*(A^{\mathcal{F}}) = H^*(A(S)^{\mathcal{F}})$ of $A(S)^{\mathcal{F}} \in \mathrm{CCh}(\mathbf{Ab})$. Notice that we also have functors $H^n(A)\colon \mathcal{F} \to \mathbf{Ab}$ obtained by taking cohomology in degree n. Hence we may also consider the inverse limits $H^*(A)^{\mathcal{F}} = H^*(A(S))^{\mathcal{F}}$. We are interested in functors A for which taking $\mathcal{F}$–stable elements and cohomology commute. We prove in this section (Proposition 2.8) that being a cohomological Mackey functor (Definition 2.2) with values in $\mathbb{Z}_{(p)}$–modules is sufficient for this.

Definition 2.2 Let $\mathcal{F}$ be a saturated fusion system over the p–group S and let $\mathcal{A}$ be an abelian category. A *cohomological Mackey functor* for $\mathcal{F}$ over $\mathcal{A}$ is a pair of functors $(A, B)\colon \mathcal{F} \to \mathcal{A}$ with $A\colon \mathcal{F} \to \mathcal{A}$ contravariant and $B\colon \mathcal{F} \to \mathcal{A}$ covariant such that:

(1) $A(P) = B(P)$ and $A(\varphi) = B(\varphi^{-1})$ for each $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, \varphi(P))$.

(2) (Identity) $A(c_p), B(c_p)\colon A(P) \to A(P)$ are the identity morphisms for every $p \in P \leq S$, where $c_p\colon P \to P$, $x \mapsto pxp^{-1}$ is conjugation by p.

(3) (Double coset formula)
$$A(\iota_Q^P) \circ B(\iota_R^P) = \sum_{x \in Q\backslash P/R} B(\iota_{Q\cap {}^xR}^{Q}) \circ A(\iota_{Q\cap {}^xR}^{{}^xR}) \circ A(c_{x^{-1}|{}^xR})$$
for $Q, R \leq P \leq S$, where $Q\backslash P/R$ are the double cosets.

(4) (Cohomological) $B(\iota_P^Q) \circ A(\iota_P^Q)\colon A(Q) \to A(Q)$ is multiplication by $|Q : P|$ for every $P \leq Q \leq S$.

See Webb [19] for the classical definition of Mackey functors and of cohomological Mackey functors for finite groups.

Remark 2.3 In Definition 2.2 we have omitted the familiar conditions:

- (Transitivity)
$$B(\iota_Q^R) \circ B(\iota_P^Q) = B(\iota_P^R) \quad \text{and} \quad A(\iota_P^Q) \circ A(\iota_Q^R) = A(\iota_P^R)$$
for $P \leq Q \leq R \leq S$.

- (Conjugation)
$$B(\iota_P^Q) \circ A(\varphi_{|P}) = A(\varphi) \circ B(\iota_{\varphi(P)}^{\varphi(Q)}),$$
$$B(\varphi_{|P}) \circ A(\iota_P^Q) = A(\iota_{\varphi(P)}^{\varphi(Q)}) \circ B(\varphi),$$
for $P \leq Q \leq S$, $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, \varphi(Q))$.

In fact, they are consequences of the functoriality of A and B and of Condition (1).

We will use several times in the paper that cohomology of finite groups is a cohomological Mackey functor. For a proof of this fact see, eg, Brown [4].

Remark 2.4 If the maps $B(\iota_P^Q)$ for the inclusions $\iota_P^Q\colon P \to Q$ with $P, Q \le S$ are given (these maps are called *transfer*) we can define B as follows: For any morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ define $B(\varphi)\colon A(P) \to A(Q)$ by $B(\varphi) = B(\iota_{\varphi(P)}^Q) \circ A(\widetilde{\varphi}^{-1})$ with $\widetilde{\varphi}\colon P \xrightarrow{\cong} \varphi(P)$. Such a B becomes functorial if for any $P \le Q \le R$ we have $B(\iota_Q^R) \circ B(\iota_P^Q) = B(\iota_P^R)$ and for any $P \le Q \xrightarrow{\varphi} \varphi(Q)$ we have

$$B(\iota_P^Q) \circ A(\varphi_{|P}) = A(\varphi) \circ B\big(\iota_{\varphi(P)}^{\varphi(Q)}\big).$$

Before proving the main result of this section we need to introduce (G, H)–bisets: Sets with commuting free right G–action and free left H–action. Every (G, H)–biset Ω can be decomposed into a disjoint union of transitive (G, H)–bisets of the form

$$H \times_\varphi G = H \times G/\sim,$$

with $K \le G$, $\varphi\colon K \to H$ a monomorphism and

$$(h, kg) \sim (h\varphi(k), g)$$

for $h \in H$, $g \in G$ and $k \in K$. A saturated fusion system gives rise to a special type of biset:

Proposition 2.5 [3, Proposition 5.5] *For any saturated fusion system $\mathcal{F}$ over a p–group S, there is an (S, S)–biset Ω with the following properties:*

(a) *Each transitive component of Ω is of the form $S \times_\varphi S$ for some $P \le S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$.*

(b) *For each $P \le S$ and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$, the (P, S)–biset Ω_P obtained by restricting the right action from S to P and the (P, S)–biset Ω_φ obtained by restricting the right action from S to P via φ are isomorphic as (P, S)–bisets.*

(c) $|\Omega|/|S| = 1 \bmod p$.

We call such an (S, S)–biset an *$\mathcal{F}$–stable (S, S)–biset*. Now let $(A, B)\colon \mathcal{F} \to \mathcal{A}$ be a cohomological Mackey functor for $\mathcal{F}$ over the abelian category $\mathcal{A}$. For each transitive (Q, R)–biset $R \times_\varphi Q$ with $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, R)$, $P \le Q \le S$, $R \le S$, we have the composition

$$A(R) \xrightarrow{A(\varphi)} A(P) = B(P) \xrightarrow{B(\iota_P^Q)} B(Q) = A(Q). \tag{2-1}$$

For each (Q, R)–biset Ω with

$$\Omega = \coprod R \times_\varphi Q$$

we can define a map $A(\Omega)\colon A(R) \to A(Q)$ by

$$A(\Omega) := \sum B(\iota) \circ A(\varphi). \tag{2-2}$$

Lemma 2.6 *Let $(A, B)\colon \mathcal{F} \to \mathcal{A}$ be a cohomological Mackey functor. Then:*

(1) *For each transitive (Q, R)–biset $R \times_\varphi Q$ the morphism (2-1) depends only on the isomorphism class of $R \times_\varphi Q$ as (Q, R)–biset.*

(2) *For any (Q, R)–biset Ω the morphism (2-2) depends only on the isomorphism class of Ω as (Q, R)–biset.*

(3) *For any (Q, R)–biset Ω and any monomorphism $\psi\colon P \to Q$ we have*

$$A(\psi) \circ A(\Omega) = A(\Omega_\psi),$$

where Ω_ψ is the (P, R)–biset obtained by restricting the right action of Ω from Q to P via ψ.

(4) *If $\mathcal{A} = \mathbb{Z}_{(p)}$–mod and Ω is an $\mathcal{F}$–stable (S, S)–biset then*

$$A(S)^{\mathcal{F}} = \operatorname{Im}\big(A(\Omega)\colon A(S) \to A(S)\big).$$

Proof of Lemma 2.6

Proof of (1) The transitive (Q, R)–bisets $R \times_{\varphi_1} Q$ and $R \times_{\varphi_2} Q$ with $\varphi_1\colon P_1 \to R$, $\varphi_2\colon P_2 \to R$, $P_1, P_2 \le Q$ are isomorphic as (Q, R)–bisets if and only if there exist elements $q \in Q$ and $r \in R$ such that the following diagram commutes:

$$\begin{tikzcd}
P_1 \arrow[r, "\varphi_1"] \arrow[d, "c_q"] & R \arrow[d, "c_r"] \\
P_2 \arrow[r, "\varphi_2"] & R
\end{tikzcd}$$

Hence both squares in the following diagram commute:

$$\begin{tikzcd}
A(R) \arrow[r, "A(\varphi_1)"] & A(P_1) \arrow[r, "B(\iota_{P_1}^Q)"] \arrow[d, shift left, "B(c_q)"] & A(Q) \arrow[d, "B(c_q)"] \\
A(R) \arrow[u, "A(c_r)"] \arrow[r, "A(\varphi_2)"] & A(P_2) \arrow[u, shift left, "A(c_q)"] \arrow[r, "B(\iota_{P_2}^Q)"] & A(Q)
\end{tikzcd}$$

Using Properties (1) and (2) from Definition 2.2 one finds that

$$B(\iota_{P_1}^Q) \circ A(\varphi_1) = B(\iota_{P_2}^Q) \circ A(\varphi_2).$$

Proof of (2) Any automorphism of Ω permutes its transitive components via isomorphisms. So we may apply Lemma 2.6(1) to each component.

Proof of (3) Write Ω as a disjoint union of transitive (Q,R)–bisets $\Omega = \coprod R \times_\varphi Q$. The transitive (Q,R) biset $R \times_\varphi Q$ with $\varphi\colon K \to R$, $K \le Q$ decomposes as a (P,R)–biset with P acting via ψ as follows:

$$R \times_\varphi Q = \bigcup_{\substack{q\in\\ \psi(P)\backslash Q/K}} R \times_{\varphi\circ c_{q^{-1}}\circ\psi} P,$$

with $P \ge P \cap \psi^{-1}({}^qK) \xrightarrow{\psi_|} \psi(P) \cap {}^qK \xrightarrow{c_{q^{-1}}} K \xrightarrow{\varphi} R$. Hence,

$$A(\Omega_\psi) = \sum_\varphi \sum_{\substack{q\in\\ \psi(P)\backslash Q/K}} B(\iota^P_{P\cap\psi^{-1}({}^qK)}) \circ A(\varphi\circ c_{q^{-1}}\circ\psi_|).$$

Using functoriality of A and B we get

$$A(\Omega_\psi) = A(\widetilde{\psi}) \circ \Bigg(\sum_\varphi \sum_{\substack{q\in\\ \psi(P)\backslash Q/K}} B\big(\iota^{\psi(P)}_{\psi(P)\cap{}^qK}\big) \circ A(c_{q^{-1}}) \circ A(\varphi) \Bigg),$$

with $\widetilde{\psi}\colon P \xrightarrow{\cong} \psi(P)$. Now the Mackey decomposition (3) from Definition 2.2 gives

$$A(\Omega_\psi) = \sum_\varphi A(\widetilde{\psi}) \circ A(\iota^Q_{\psi(P)}) \circ B(\iota^Q_K) \circ A(\varphi) = A(\psi) \circ A(\Omega).$$

Proof of (4) Let $z \in A(S)$. We want to see that $A(\Omega)(z) \in A(S)^{\mathcal{F}}$. So let ψ be a morphism in $\mathrm{Hom}_{\mathcal{F}}(P,S)$. Then

$$A(\psi)\big(A(\Omega)(z)\big) = \big(A(\psi)\circ A(\Omega)\big)(z) = A(\Omega_\psi)(z)$$

by Part (3). By Proposition 2.5(b), the (P,S)–bisets Ω_ψ and $\Omega_{\iota^S_P} = \Omega_P$ are isomorphic as (P,S)–bisets. Then by Part (2) we have $A(\Omega_\psi) = A(\Omega_{\iota^S_P})$. Hence,

$$A(\psi)\big(A(\Omega)(z)\big) = A(\Omega_\psi)(z) = A(\Omega_{\iota^S_P})(z) = A(\iota^S_P)\big(A(\Omega)(z)\big)$$

by Part (3). Thus $A(\Omega)(z) \in A(S)^{\mathcal{F}}$.

Now let $z \in \mathcal{A}(S)^{\mathcal{F}}$. Then

$$A(\Omega)(z) = \sum B(\iota)\big(A(\varphi)(z)\big) = \sum B(\iota)\big(A(\iota)(z)\big)$$

as z is $\mathcal{F}$–stable. Now by (4) of Definition 2.2 we get

$$A(\Omega)(z) = \Big(\sum |S:P|\Big)\cdot z$$

and by Proposition 2.5(c) the number $q = (\sum |S:P|) = |\Omega|/|S|$ is a p'–number. So $A(\Omega)(\frac{z}{q}) = z$ and hence $z \in \mathrm{Im}\, A(\Omega)$. □

For a fusion system $\mathcal{F}$ over the p–group S denote by $\mathrm{CohMack}_{\mathbb{Z}_{(p)}}(\mathcal{F})$ the abelian category with objects the cohomological Mackey functors with values in $\mathbb{Z}_{(p)}$–mod and morphisms the natural transformations commuting with both the contravariant and covariant parts. This means that if (A, B) and (A', B') are cohomological Mackey functors, a morphism η between them consists of a morphism of $\mathbb{Z}_{(p)}$–modules $\eta_P\colon A(P) \to A'(P)$ for each $P \leq S$ such that for $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ we have

$$A'(\varphi) \circ \eta_Q = \eta_P \circ A(\varphi) \quad \text{and} \quad \eta_Q \circ B(\varphi) = B'(\varphi) \circ \eta_P.$$

Lemma 2.7 *Let $\mathcal{F}$ be a fusion system over the p–group S. Then the functor*

$$\mathrm{CohMack}_{\mathbb{Z}_{(p)}}(\mathcal{F}) \xrightarrow{(-)^{\mathcal{F}}} \mathbb{Z}_{(p)}\text{–mod}$$

sending $(A, B) \mapsto A^{\mathcal{F}}$ is exact.

Proof Let

$$0 \Rightarrow (A_1, B_1) \Rightarrow (A_2, B_2) \overset{\eta}{\Rightarrow} (A_3, B_3) \Rightarrow 0$$

be an exact sequence in $\mathrm{CohMack}_{\mathbb{Z}_{(p)}}(\mathcal{F})$. We want to prove that

$$0 \to A_1^{\mathcal{F}} \longrightarrow A_2^{\mathcal{F}} \xrightarrow{\eta^{\mathcal{F}}} A_3^{\mathcal{F}} \longrightarrow 0$$

is exact in $\mathbb{Z}_{(p)}$–mod. The nontrivial assertion to prove is that the arrow $A_2^{\mathcal{F}} \to A_3^{\mathcal{F}}$ is an epimorphism. So let z be an $\mathcal{F}$–stable element in $A_3(S)$. Fix an (S, S)–biset Ω satisfying the properties of Proposition 2.5. By Lemma 2.6 (4) there exists an element $z' \in A_3(S)$ with $z = A_3(\Omega)(z')$. By hypothesis, the map

$$A_2(S) \xrightarrow{\eta_S} A_3(S)$$

is an epimorphism and hence there exists an element $y' \in A_2(S)$ with $\eta_S(y') = z'$. By Lemma 2.6 (4) again we have that

$$y \overset{\text{def}}{=} A_2(\Omega)(y')$$

belongs to $A_2^{\mathcal{F}}$. Because η commutes with the covariant and contravariant parts of (A_2, B_2) and (A_3, B_3), it is easy to see that

$$\eta^{\mathcal{F}}(y) = \eta^{\mathcal{F}}\big(A_2(\Omega)(y')\big) = A_3(\Omega)\big(\eta^{\mathcal{F}}(y')\big) = A_3(\Omega)(z') = z. \qquad \square$$

Proposition 2.8 *Let $\mathcal{F}$ be a fusion system over S and let $(A, B)\colon \mathcal{F} \to \mathrm{CCh}(\mathbb{Z}_{(p)})$ be a cohomological Mackey functor. Then*

$$H^*\big(A(S)^{\mathcal{F}}\big) \cong H^*\big(A(S)\big)^{\mathcal{F}}.$$

Proof This is a consequence of Lemma 2.7 and of the well-known fact that cohomology commutes with exact functors. □

Remark 2.9 Let $\mathcal{F}$ be a fusion system over the p–group S and let M be a trivial $\mathbb{Z}_{(p)}S$–module. By [3, Section 5] the cohomology of $\mathcal{F}$ is defined as

$$H^*(\mathcal{F}; M) = H^*(S; M)^{\mathcal{F}},$$

where $H^*(\,\cdot\,; M)\colon \mathcal{F} \to \mathbb{Z}_{(p)}$–modules is the cohomological Mackey functor with values $H^*(P; M)$. If one could choose cochains $C^*(\,\cdot\,; M)\colon \mathcal{F} \to \mathrm{CCh}(\mathbb{Z}_{(p)})$ such that $C^*(\,\cdot\,; M)$ was the contravariant part of a cohomological Mackey functor then Proposition 2.8 would give the computational-purposes formula

$$H^*(\mathcal{F}; M) = H^*\big(C^*(S; M)^{\mathcal{F}}\big).$$

In the next section some problems related to the functoriality of cochains will become apparent.

3 A Mackey functor

Let $\mathcal{F}$ be a fusion system over the p–group S, T a strongly $\mathcal{F}$–closed subgroup of S and M a $\mathbb{Z}_{(p)}$–module with trivial S–action. In this section we prove that for every $n, m \geq 0$ the functor $H^{n,m}\colon \mathcal{F} \to \mathbb{Z}_{(p)}$–mod sending the subgroup $P \leq S$ to $H^n(P/P \cap T; H^m(P \cap T; M))$ is the contravariant part of a cohomological Mackey functor $\mathcal{F} \to \mathrm{CCh}^2(\mathbb{Z}_{(p)})$ with values in double (cochain) complexes (Definition 2.2). Here, by double complexes we mean the abelian category with objects families of $\mathbb{Z}_{(p)}$–modules $\{A^{n,m}\}_{n,m\in\mathbb{Z}}$ together with maps d^h (horizontal differential) and d^v (vertical differential)

$$d^h\colon A^{n,m} \to A^{n+1,m} \quad \text{and} \quad d^v\colon A^{n,m} \to A^{n,m+1},$$

such that $d^h d^h = d^v d^v = d^h d^v + d^v d^h = 0$. A morphism from $\{A^{n,m}\}_{n,m\in\mathbb{Z}}$ to $\{A'^{n,m}\}_{n,m\in\mathbb{Z}}$ is a family of maps of $\mathbb{Z}_{(p)}$–modules $\{A^{n,m} \to A'^{n,m}\}_{n,m\in\mathbb{Z}}$ that commute with horizontal and vertical differentials.

For $P \leq S$ denote by $\overline{P}$ the group $P/P \cap T$. The bar resolutions $\mathcal{B}^*_P$ and $\mathcal{B}^*_{\overline{P}}$ for P and $\overline{P}$ respectively are projective resolutions of the trivial module $\mathbb{Z}_{(p)}$ over $\mathbb{Z}_{(p)}P$ and $\mathbb{Z}_{(p)}\overline{P}$ respectively. Recall that the bar resolution is functorial (covariant) over finite groups and homomorphisms. Define $A^{*,*}(P)$ as the double complex associated to the short exact sequence

$$0 \to P \cap T \to P \to \overline{P} \cong PT/T \to 0.$$

More precisely, for $n \geq 0$ and $m \geq 0$, we define

$$A^{n,m}(P) = \mathrm{Hom}_P\big(\mathcal{B}^n_{\overline{P}} \otimes \mathcal{B}^m_P, M\big),$$

where P acts on $\mathcal{B}^n_{\overline{P}} \otimes \mathcal{B}^m_P$ by $p(y \otimes x) = \bar{p}y \otimes px$ for $y \in \mathcal{B}^n_{\overline{P}}$ and $x \in \mathcal{B}^m_P$.

As the action of P on M is trivial the cochains in $A^{n,m}(P)$ are the homomorphisms $f \in \mathrm{Hom}(\mathcal{B}^n_{\overline{P}} \otimes \mathcal{B}^m_P, M)$ such that

$$f(\bar{p}y \otimes px) = f(y \otimes x)$$

for all $y \in \mathcal{B}^n_{\overline{P}}$, $x \in \mathcal{B}^m_P$ and $p \in P$.

To obtain a double complex we consider the following horizontal and vertical differentials for $f \in A^{n,m}(P)$:

$$d^h(f)(y \otimes x) = (-1)^{n+m+1} f(d(y) \otimes x), \quad y \in \mathcal{B}^{n+1}_{\overline{P}},\ x \in \mathcal{B}^m_P,$$

$$d^v(f)(y \otimes x) = (-1)^{m+1} f\big(y \otimes d(x)\big), \qquad y \in \mathcal{B}^n_{\overline{P}},\ x \in \mathcal{B}^{m+1}_P,$$

where we are using the differential d of the complexes $\mathcal{B}^*_{\overline{P}}$ and $\mathcal{B}^*_P$. We choose the signs as given by Mac Lane [13, XI.10.1] to ensure that $d^h d^v + d^v d^h = 0$. We will obtain the functor $H^{n,m}$ by taking vertical cohomology followed by horizontal cohomology in $A^{n,m}$.

To define A on morphisms notice that any morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ takes $P \cap T$ to $Q \cap T$ as T is strongly $\mathcal{F}$–closed. Hence it induces a homomorphism

$$\overline{\varphi}\colon \overline{P} \to \overline{Q}.$$

Thus for any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ we may define

$$A^{n,m}(Q) \xrightarrow{A^{n,m}(\varphi)} A^{n,m}(P)$$

mapping the cochain $f \in A^{n,m}(Q)$ to the cochain in $A^{n,m}(P)$ that takes $y \in \mathcal{B}^n_{\overline{P}}$ and $x \in \mathcal{B}^m_P$ to

$$f\big(\mathcal{B}^n(\overline{\varphi})(y) \otimes \mathcal{B}^m(\varphi)(x)\big),$$

where $\mathcal{B}^n(\overline{\varphi})$ and $\mathcal{B}^m(\varphi)$ are the usual morphisms between bar resolutions. They commute with differentials and satisfy

$$\mathcal{B}^n(\overline{\varphi})(\bar{p} \cdot y) = \overline{\varphi}(\bar{p}) \cdot \mathcal{B}^n(\overline{\varphi})(y)$$

for every $y \in \mathcal{B}^n_{\overline{P}}$ and every $\bar{p} \in \overline{P}$ and

$$\mathcal{B}^m(\varphi)(p \cdot x) = \varphi(p) \cdot \mathcal{B}^m(\varphi)(x)$$

for every $x \in \mathcal{B}^m_P$ and $p \in P$. It is straightforward that $A^{n,m}(\varphi)(f) \in A^{n,m}(P)$ and that the family of morphisms $\{A^{n,m}(\varphi)\}_{n,m \geq 0}$ commutes with the horizontal and vertical differentials of the double complexes $A^{*,*}(Q)$ and $A^{*,*}(P)$.

Remark 3.1 By definition the fusion system $\mathcal{F}/T$ is defined over the p–group S/T. For $T \leq P, Q \leq S$ the morphisms in $\mathrm{Hom}_{\mathcal{F}/T}(P/T, Q/T)$ are those homomorphisms $\overline{\psi}\colon P/T \to Q/T$ induced on the quotient from $\psi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$.

For $P, Q \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ we have a morphism $\overline{\varphi}\colon \overline{P} \to \overline{Q}$. Then we have a commutative diagram

$$\begin{tikzcd} \overline{P} \arrow[r, "\overline{\varphi}"] \arrow[d, "\cong"'] & \overline{Q} \arrow[d, "\cong"] \\ PT/T \arrow[r, "\overline{\varphi}"] & QT/T, \end{tikzcd}$$

where the $\overline{\varphi}$ are induced by φ and where the vertical arrows are the natural isomorphisms. According to [7, 5.10] bottom morphism $\overline{\varphi}$ belongs to $\mathcal{F}/T$, ie, there exists $\psi \in \mathrm{Hom}_{\mathcal{F}}(PT, QT)$ such that the induced map $\overline{\psi}\colon PT/T \to QT/T$ coincides with the given one.

Remark 3.2 The construction of $A^{n,m}$ is clearly functorial and hence so far we have a contravariant functor $A^{*,*}\colon \mathcal{F} \to \mathrm{CCh}^2(\mathbb{Z}_{(p)})$ with values in double complexes.

Now we define $B^{n,m}(P) = A^{n,m}(P)$ for every $P \leq S$ and $n, m \geq 0$. For each morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ we will define a morphism of double complexes

$$B^{n,m}(\varphi)\colon A^{n,m}(P) \to A^{n,m}(Q).$$

This will not make B into a covariant functor $\mathcal{F} \to \mathrm{CCh}^2(\mathbb{Z}_{(p)})$ as the definition depends on a choice of representatives. Nevertheless, B will become functorial once we pass to cohomology.

To define $B^{*,*}(\psi)$ on $\psi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$, write $\psi = \iota \circ \widetilde{\psi}$, where $\widetilde{\psi}\colon P \to \psi(P)$ is an isomorphism and ι is the inclusion $\psi(P) \leq Q$, and set

$$B^{*,*}(\psi) = B^{*,*}(\iota) \circ A^{*,*}(\widetilde{\psi}^{-1}). \tag{3-1}$$

So we just need to define B on inclusions.

So let ι be the inclusion between subgroups $P \leq Q$ of S. There are maps of $\mathbb{Z}_{(p)}P$–chain complexes and of $\mathbb{Z}_{(p)}\overline{P}$–chain complexes respectively

$$\tau_*^{Q,P}\colon \mathcal{B}_Q^* \to \mathcal{B}_P^*, \quad \overline{\tau}_*^{Q,P}\colon \mathcal{B}_{\overline{Q}}^* \to \mathcal{B}_{\overline{P}}^*,$$

built as in [4, (D), page 82]. More precisely, the map $\tau_*^{Q,P}$ is induced by a map of left P–sets $Q \xrightarrow{\rho} P$ defined as follows: fix a set of representatives for the right cosets $P\backslash Q$, then $\rho(q) = q\overline{q}^{-1}$, where $\overline{q}$ is the representative with $Pq = P\overline{q}$. The map

$\overline{\tau}_*^{Q,P}$ is defined analogously choosing representatives for the right cosets $\overline{P}\backslash\overline{Q}$. These choices of representatives prevent $B^{p,q}$ from being functorial.

We define the map

(3-2)
$$B^{n,m}(\iota)\colon \operatorname{Hom}_P(\mathcal{B}^n_{\overline{P}}\otimes\mathcal{B}^m_P, M)\to \operatorname{Hom}_Q(\mathcal{B}^n_{\overline{Q}}\otimes\mathcal{B}^m_Q, M),$$
$$B^{n,m}(\iota)(f)(y\otimes x)=\sum_{w\in Q/P} f\big(\overline{\tau}_n^{Q,P}(\overline{w}^{-1}y)\otimes\tau_m^{Q,P}(w^{-1}x)\big),$$

where w runs over a set of representatives of the left cosets Q/P. This formula can be thought as a relative transfer formula for twisted coefficients. Clearly its definition does not depend on the representatives w chosen and $B^{n,m}(\iota)(f)\in A^{n,m}(Q)$. Moreover, $B^{n,m}(\iota)$ commutes with both the horizontal and vertical differentials as τ_* and $\overline{\tau}_*$ do and so it is a map of double complexes.

Remark 3.3 By Park [14] there are finite groups G and $\overline{G}$ such that S is a p–subgroup of G (not necessarily a Sylow p–subgroup), $\overline{S}=S/T$ is a p–subgroup of $\overline{G}$ (not necessarily a Sylow p–subgroup) and with $\mathcal{F}=\mathcal{F}_S(G)$ and $\mathcal{F}/T=\mathcal{F}_{\overline{S}}(\overline{G})$. Let B^*_G and $B^*_{\overline{G}}$ be the bar resolutions of G and $\overline{G}$ respectively. Then we could have defined for $P\leq S$

$$A^{n,m}(P)=\operatorname{Hom}_P\big(\mathcal{B}^n_{\overline{G}}\otimes\mathcal{B}^m_G, M\big),$$

where P acts on $\mathcal{B}^n_{\overline{G}}\otimes\mathcal{B}^m_G$ by restricting the actions of G on $\mathcal{B}^*_G$ and of $\overline{G}$ on $\mathcal{B}^*_{\overline{G}}$. This means that $p(y\otimes x)=\overline{p}y\otimes px$ for $p\in P$. In this setup clearly one can define a functorial $B^{n,m}$ on inclusions. On the other hand, to realize a morphism $\varphi\colon P\to Q$ we need to choose $g\in N_G(P,Q)$ with $\varphi=c_g$ and $\overline{g}\in N_{\overline{G}}(\overline{P},\overline{Q})$ with $\overline{\varphi}=c_{\overline{g}}$ and then define

$$A^{n,m}(\varphi)(y\otimes x)=(\overline{g}y\otimes gx).$$

It is clear that in general $A^{n,m}$ defined this way will not be functorial on morphisms. If one could choose $A^{n,m}$ and $B^{n,m}$ such that $(A^{n,m},B^{n,m})\colon\mathcal{F}\to\mathrm{CCh}^2(\mathbb{Z}_{(p)})$ was a Mackey functor then the proof of Theorem 4.1 would be simpler.

On each double complex $A^{*,*}(P)$ with $P\leq S$ we may take vertical cohomology followed by horizontal cohomology to obtain $H^*(\overline{P};H^*(P\cap T;M))$ [13, Equation (10.2), page 352]. For any homomorphism $\varphi\in\operatorname{Hom}_{\mathcal{F}}(P,Q)$ the maps $A^{*,*}(\varphi)$ and $B^{*,*}(\varphi)$ are maps of double complexes and hence they induce maps

$$H^{n,m}(A)(\varphi)\colon H^n\big(\overline{Q};H^m(Q\cap T;M)\big)\to H^n\big(\overline{P};H^m(P\cap T;M)\big),$$
$$H^{n,m}(B)(\varphi)\colon H^n\big(\overline{P};H^m(P\cap T;M)\big)\to H^n\big(\overline{Q};H^m(Q\cap T;M)\big).$$

Lemma 3.4 *For $\varphi\colon P \to Q$ the map $H^{n,m}(A)(\varphi)$ factors as*

$$H^n\big(\overline{Q}; H^m(Q\cap T; M)\big) \xrightarrow{H^n(\overline{\varphi})} H^n\big(\overline{P}; H^m(Q\cap T; M)\big) \xrightarrow{H^m(\varphi)} H^n\big(\overline{P}; H^m(P\cap T; M)\big),$$

where

- $H^n(\overline{\varphi})$ *is the map induced by* $\overline{\varphi}$ *in cohomology with* $H^m(Q\cap T; M)$*–coefficients,*
- $H^m(\varphi)$ *is the map induced by the change of coefficients*

$$H^m(\varphi)\colon H^m(Q\cap T; M) \to H^m(P\cap T; M).$$

This map is a map of $\mathbb{Z}_{(p)}\overline{P}$*–modules where* $\overline{P}$ *acts on* $H^m(Q\cap T; M)$ *via* $\overline{P} \xrightarrow{\overline{\varphi}} \overline{\varphi(P)} \le \overline{Q}$.

Proof By construction. □

Lemma 3.5 *If $P \le Q$ and ι denotes the inclusion then the map $H^{n,m}(B)(\iota)$ factors as*

$$H^n\big(\overline{P}; H^m(P\cap T; M)\big) \xrightarrow{H^m(\mathrm{tr}')} H^n\big(\overline{P}; H^m(Q\cap T; M)\big) \xrightarrow{H^n(\mathrm{tr})} H^n\big(\overline{Q}; H^m(Q\cap T; M)\big),$$

where

- $H^n(\mathrm{tr})$ *is the transfer map in cohomology with* $H^m(Q\cap T; M)$*–coefficients,*
- $H^m(\mathrm{tr}')$ *is the map induced by the change of coefficients given by the transfer map in cohomology:*

$$H^m(\mathrm{tr}')\colon H^m(P\cap T; M) \to H^m(Q\cap T; M)$$

This map is a map of $\mathbb{Z}_{(p)}\overline{P}$*–modules where* $\overline{P}$ *acts on* $H^m(Q\cap T; M)$ *via* $\overline{P} \le \overline{Q}$.

Proof Choose representatives $z_i \in \overline{Q}$ of the left cosets $\overline{Q}/\overline{P}$ and representatives $t_j \in Q\cap T$ of the left cosets $(Q\cap T)/(P\cap T)$. Choose also representatives $q_k \in Q$ of the left cosets $\overline{Q} = Q/(Q\cap T)$. Then each $z_i \in \overline{Q}$ is represented as $z_i = \overline{q_{k_i}}$ for a unique k_i. It is an exercise to prove that the set of elements of $Q\, q_{k_i} t_j$ for all i and j is a set of representatives of Q/P. Then we can rewrite Equation (3-2) as

$$\sum_{z_i \in \overline{Q}/\overline{P}} \ \sum_{t_j \in (Q\cap T)/(P\cap T)} f\Big(\overline{\tau}_n^{Q,P}(\overline{q_{k_i} t_j}^{-1} y) \otimes \tau_m^{Q,P}\big((q_{k_i} t_j)^{-1} x\big)\Big),$$

Because $t_j \in Q \cap T$ then $\overline{q_{k_i} t_j} = \overline{q_{k_i}}$ and the formula simplifies to

$$\sum_{z_i \in \overline{Q}/\overline{P}} \sum_{t_j \in (Q\cap T)/(P\cap T)} f\Big(\overline{\tau}_n^{Q,P}(\overline{q_{k_i}}^{-1} y) \otimes \tau_m^{Q,P}(t_j^{-1} q_{k_i}{}^{-1} x)\Big).$$

This coincides with the composition in the statement of the lemma. □

Lemma 3.5 proves in particular that the definition of $H^{n,m}(B)(\iota)$ does not depend on the representatives chosen to construct the maps $\tau_*^{Q,P}$ and $\overline{\tau}_*^{Q,P}$. (Although $B^{n,m}(\iota)$ do depends on them.)

Corollary 3.6 *For $n, m \geq 0$ the assignment*

$$H^{n,m}(B)\colon \mathcal{F} \to \mathbb{Z}_{(p)}\text{–mod}$$

taking P to $H^n(\overline{P}; H^m(P\cap T; M))$ and taking $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ to $H^{n,m}(B)(\varphi)$ is a functor.

Proof By Remarks 2.4 and 3.2 and Equation (3-1) it is enough to prove that for any $P \leq Q \leq R$ we have

$$H^{n,m}(B)\big(\iota_Q^R\big) \circ H^{n,m}(B)\big(\iota_P^Q\big) = H^{n,m}(B)\big(\iota_P^R\big)$$

and for any $P \leq Q \xrightarrow{\varphi} \varphi(Q)$ we have

$$H^{n,m}(B)\big(\iota_P^Q\big) \circ H^{n,m}(A)(\varphi_{|P}) = H^{n,m}(A)(\varphi) \circ H^{n,m}(B)\big(\iota_{\varphi(P)}^{\varphi(Q)}\big).$$

We can check both conditions at the level of cochains: For the first condition, the definitions (3-2) of $B^{n,m}(\iota_P^Q)$, $B^{n,m}(\iota_Q^R)$ and $B^{n,m}(\iota_P^R)$ depend upon choices of representatives for the right cosets

$$P\backslash Q \text{ and } \overline{P}\backslash\overline{Q}, \quad Q\backslash R \text{ and } \overline{Q}\backslash\overline{R}, \quad P\backslash R \text{ and } \overline{P}\backslash\overline{R}$$

respectively. Fix choices of representatives for the first four right cosets. Then the bijections $P\backslash Q \times Q\backslash R \to P\backslash R$ and $\overline{P}\backslash\overline{Q} \times \overline{Q}\backslash\overline{R} \to \overline{P}\backslash\overline{R}$ provide choices for the last two right cosets. With these choices we have

$$B^{n,m}\big(\iota_Q^R\big) \circ B^{n,m}\big(\iota_P^Q\big) = B^{n,m}\big(\iota_P^R\big).$$

For the second condition, the maps $B^{n,m}(\iota_P^Q)$ and $B^{n,m}(\iota_{\varphi(P)}^{\varphi(Q)})$ depend on choices of representatives for the right cosets

$$P\backslash Q \text{ and } \overline{P}\backslash\overline{Q}, \quad \varphi(P)\backslash\varphi(Q) \text{ and } \overline{\varphi(P)}\backslash\overline{\varphi(Q)}$$

respectively. Fix representatives in $P\backslash Q$ and $\overline{P}\backslash\overline{Q}$ and force the other choices via the bijections

$$P\backslash Q \xrightarrow{\varphi} \varphi(P)\backslash\varphi(Q) \quad \text{and} \quad \overline{P}\backslash\overline{Q} \xrightarrow{\overline{\varphi}} \overline{\varphi(P)}\backslash\overline{\varphi(Q)}.$$

Then we have

$$B^{n,m}\big(\iota_P^Q\big) \circ A^{n,m}(\varphi_{|P}) = A^{n,m}(A)(\varphi) \circ B^{n,m}\big(\iota_{\varphi(P)}^{\varphi(Q)}\big). \qquad \square$$

Proposition 3.7 *For each $p, q \geq 0$ the functor $\mathcal{F} \to \mathbb{Z}_{(p)}$–mod with values*

$$H^p(\overline{P}; H^q(P \cap T; M))$$

and taking $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ to $H^{p,q}(A)(\varphi)$ is a cohomological Mackey functor with covariant part taking $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ to $H^{p,q}(B)(\varphi)$.

Proof Property (1) from Definition 2.2 holds by Equation (3-1). Property (2) follows from Property (1), the well known fact that conjugation induces the identity on cohomology, from Lemma 3.4 and from $\overline{c_p} = c_{\overline{p}}$ for $p \in P \leq S$. Now we check Property (3), also known as the Mackey condition or double coset formula. So let $Q, R \leq P \leq S$. We will prove this condition at the level of cochains, ie

$$A^{n,m}(\iota_Q^P) \circ B^{n,m}(\iota_R^P) = \sum_{x \in Q\backslash P/R} B^{n,m}(\iota_{Q\cap {}^xR}^Q) \circ A^{n,m}(\iota_{Q\cap {}^xR}^{{}^xR}) \circ A^{n,m}(c_{x^{-1}|{}^xR}).$$

So let $f \in A^{n,m}(R) = \mathrm{Hom}_R(\mathcal{B}_{\overline{R}}^n \otimes \mathcal{B}_R^m, M)$, $y \in \mathcal{B}_{\overline{Q}}^n$ and $x \in \mathcal{B}_Q^m$. Then

$$\begin{aligned} A^{n,m}\big(\iota_Q^P\big)\big(B^{n,m}\big(\iota_R^P\big)(f)\big)(y \otimes x) &= B^{n,m}\big(\iota_R^P\big)(f)\big(\overline{\iota_Q^P}(y) \otimes \iota_Q^P(x)\big) \\ &= B^{n,m}\big(\iota_R^P\big)(f)(y \otimes x). \end{aligned}$$

This equals

$$\sum_{w \in P/R} f\big(\overline{\tau}_n^{P,R}(\overline{w}^{-1}y) \otimes \tau_m^{P,R}(w^{-1}x)\big),$$

where w runs over a set of representatives of the left cosets P/R, $\overline{\tau}_n^{P,R}\colon \mathcal{B}_{\overline{P}}^n \to \mathcal{B}_{\overline{R}}^n$ and $\tau_m^{P,R}\colon \mathcal{B}_P^m \to \mathcal{B}_R^m$. Now we let Q acts on the left on P/R and we group together the terms corresponding to a given Q–orbit in P/R:

$$\sum_{p \in Q\backslash P/R} \ \sum_{q \in Q/Q\cap {}^pR} f\big(\overline{\tau}_n^{P,R}(\overline{qp}^{-1}y) \otimes \tau_m^{P,R}\big((qp)^{-1}x\big)\big),$$

where now p runs over a set of representatives for the double cosets $Q\backslash P/R$ and q runs over a set of representatives of the left cosets $Q/Q \cap {}^nR$. This equals

$$\sum_{p \in Q\backslash P/R} \ \sum_{q \in Q/Q\cap {}^pR} f\big(\overline{\tau}_n^{P,R}(\overline{p}^{-1}\overline{q}^{-1}y) \otimes \tau_m^{P,R}(p^{-1}q^{-1}x)\big).$$

The right-hand side of the Mackey formula is

$$\sum_{p\in Q\backslash P/R}\ \sum_{q\in Q/Q\cap {}^pR} f\big(\bar{p}^{-1}\bar{\tau}_n^{Q,Q\cap {}^pR}(\bar{q}^{-1}y)\bar{p}\otimes p^{-1}\tau_q^{Q,Q\cap {}^pR}(q^{-1}x)p\big)$$

with

$$\bar{\tau}_n^{Q,Q\cap {}^pR}\colon \mathcal{B}^n_{\overline{Q}}\to\mathcal{B}^n_{\overline{Q\cap {}^pR}},\quad \tau_m^{Q,Q\cap {}^pR}\colon \mathcal{B}^m_{Q}\to\mathcal{B}^m_{Q\cap {}^pR}$$

and where we have realized $c_{p^{-1}|{}^pR}$ at the level of cochains as

$$A^{n,m}(c_{p^{-1}|{}^pR})(y\otimes x)=(\bar{p}^{-1}y\bar{p}\otimes p^{-1}xp).$$

The map $\tau_m^{P,R}$ depends on a choice of representatives for the right cosets $R\backslash P$. Similarly, for any representative $p\in Q\backslash P/R$, the map $\tau_m^{Q,Q\cap {}^pR}$ is built out of a set of representatives of $Q\cap {}^pR\backslash Q$. We want to choose representatives of $R\backslash P$ and of $Q\cap {}^pR\backslash Q$ for each double coset QpR such that

$$\begin{tikzcd} Q \arrow[r, "\rho_p"] \arrow[d, "q\mapsto p^{-1}q"'] & Q\cap {}^pR \arrow[d, "q\mapsto p^{-1}qp"] \\ P \arrow[r, "\rho"] & R \end{tikzcd}$$

commutes for each double coset QpR. For this is enough to choose arbitrary representatives q of $Q\cap {}^pR\backslash Q$ for each double coset QpR and build the representatives in $R\backslash P$ via the bijection

$$\coprod_{p\in Q\backslash P/R} Q\cap {}^pR\backslash Q\to R\backslash P$$

that takes $(Q\cap {}^pR)q$ to $Rp^{-1}q$. The same argument for $\bar{\tau}_n^{P,R}$ and the maps $\bar{\tau}_n^{Q,Q\cap {}^pR}$ finishes the proof of Property (3).

To prove Property (4) we go back to the level of cohomology. Let $P\le Q\le S$. By Lemmas 3.4 and 3.5 the composition $H^{n,m}(B)(\iota_P^Q)\circ H^{n,m}(A)(\iota_P^Q)$ is equal to

$$H^n(\mathrm{tr})\circ H^m(\mathrm{tr}')\circ H^m(\iota_P^Q)\circ H^n\big(\iota_P^{\overline{Q}}\big).$$

Because cohomology over finite groups is a cohomological Mackey functor we know that $H^m(\mathrm{tr}')\circ H^m(\iota_P^Q)$ is multiplication by $|Q\cap T|/|P\cap T|$. Moving out this factor we are left with

$$H^n(\mathrm{tr})\circ H^n\big(\iota_P^{\overline{Q}}\big).$$

As $\iota_P^{\overline{Q}}=\iota_{\overline{P}}^{\overline{Q}}$ we obtain again by properties of cohomology for finite groups that this composition is multiplication by $|\overline{Q}|/|\overline{P}|$. So finally we obtain that

$$H^{n,m}(B)\big(\iota_P^Q\big)\circ H^{p,q}(A)\big(\iota_P^Q\big)$$

is multiplication by
$$\frac{|Q\cap T|}{|P\cap T|}\frac{|\overline{Q}|}{|\overline{P}|}=\frac{|Q|}{|P|}.$$
□

4 Construction of the spectral sequence

In this section we prove the main theorem of this paper:

Theorem 4.1 *Let $\mathcal{F}$ be a fusion system over the p–group S, T a strongly $\mathcal{F}$–closed subgroup of S and M a $\mathbb{Z}_{(p)}$–module with trivial S–action. Then there is a first quadrant cohomological spectral sequence with second page*
$$E_2^{n,m}=H^n\big(S/T;H^m(T;M)\big)^{\mathcal{F}}$$
and converging to $H^{n+m}(\mathcal{F};M)$.

Proof For each subgroup $P\leq S$ we have the short exact sequence
$$P\cap T\to P\to \overline{P}=P/P\cap T.$$
The construction of the Lyndon–Hochschild–Serre spectral sequence in [13, XI.10.1] associates to this short exact sequence a double complex naturally isomorphic to the double complex
$$A^{n,m}(P)=\mathrm{Hom}_P(\mathcal{B}^n_{\overline{P}}\otimes\mathcal{B}^m_P,M)$$
defined in Section 3. This double complex we can filter either by columns or rows. If we filter by columns we obtain a spectral sequence $\{{}^cE_k^{*,*}(P),d_k\}_{0\leq k\leq\infty}$ whose second page is ${}^cE_2^{n,m}(P)=H^n(\overline{P};H^m(P\cap T;M))$. If we filter by rows we obtain a spectral sequence $\{{}^rE_k^{*,*}(P),d_k\}_{0\leq k\leq\infty}$ whose second page collapses as ${}^rE_2^{n,m}(P)=H^m(P;M)$ for $n=0$ and ${}^rE_2^{n,m}(P)=0$ for $n>0$.

For each morphism $\varphi\in\mathrm{Hom}_{\mathcal{F}}(P,Q)$ we have morphisms of double complexes
$$A^{n,m}(\varphi)\colon A^{n,m}(Q)\to A^{n,m}(P)\quad\text{and}\quad B^{n,m}(\varphi)\colon A^{n,m}(P)\to A^{n,m}(Q)$$
defined in Section 3. These morphisms of double complexes induce morphisms of spectral sequences consisting of a sequence of morphisms of differential bigraded $\mathbb{Z}_{(p)}$–modules
$$\begin{aligned}
{}^cE_k^{*,*}(A)(\varphi)&\colon {}^cE_k^{*,*}(Q)\to {}^cE_k^{*,*}(P),\\
{}^cE_k^{*,*}(B)(\varphi)&\colon {}^cE_k^{*,*}(P)\to {}^cE_k^{*,*}(Q),\\
{}^rE_k^{*,*}(A)(\varphi)&\colon {}^rE_k^{*,*}(Q)\to {}^rE_k^{*,*}(P),\\
{}^rE_k^{*,*}(B)(\varphi)&\colon {}^rE_k^{*,*}(P)\to {}^rE_k^{*,*}(Q),
\end{aligned}$$

for $0 \leq k \leq \infty$. We deal now with the filtration by columns spectral sequences. The second page ${}^cE_2^{*,*}$ is obtained by computing vertical cohomology followed by horizontal cohomology in the double complex $A^{*,*}$. Hence we have

$$\begin{aligned} {}^cE_2^{n,m}(P) &= H^n\big(\overline{P}; H^m(P \cap T; M)\big), \\ {}^cE_2^{n,m}(A)(\varphi) &= H^{n,m}(A)(\varphi), \\ {}^cE_2^{n,m}(B)(\varphi) &= H^{n,m}(B)(\varphi), \end{aligned}$$

for $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, where $H^{n,m}(A)$ and $H^{n,m}(B)$ are *functors* $\mathcal{F} \to \mathbb{Z}_{(p)}$–mod by Remark 3.2 and Corollary 3.6 respectively. Hence, for each $2 \leq k \leq \infty$, we have a contravariant functor

$${}^cE_k^{*,*}(A)\colon \mathcal{F} \to \text{ differential bigraded } \mathbb{Z}_{(p)}\text{–modules}$$

and a covariant functor

$${}^cE_k^{*,*}(B)\colon \mathcal{F} \to \text{ differential bigraded } \mathbb{Z}_{(p)}\text{–modules.}$$

On the one hand, we can take invariants for each $2 \leq k \leq \infty$ to obtain a differential bigraded $\mathbb{Z}_{(p)}$–module

$${}^cE_k^{*,*\mathcal{F}} = \{z \in {}^cE_k^{*,*}(S) \mid {}^cE_k^{*,*}(A)(\varphi)(z) = {}^cE_k^{*,*}(A)(\iota_P^S)(z) \text{ for } P \xrightarrow{\varphi} S\}.$$

On the other hand, for $k=2$, we have by Proposition 3.7 that $({}^cE_2^{*,*}(A), {}^cE_2^{*,*}(B))$ is a cohomological Mackey functor. Because ${}^cE_{k+1}^{*,*} = H^*({}^cE_k^{*,*}, d_k)$ and because passing to cohomology preserves cohomological Mackey functors we deduce that ${}^cE_k^{*,*}(A)$ is a cohomological Mackey functor with covariant part ${}^cE_k^{*,*}(B)$ for $2 \leq k < \infty$. By Proposition 2.8 we obtain then that

$$ {}^cE_{k+1}^{*,*\ \mathcal{F}} = H^*\big({}^cE_k^{*,*}, d_k\big)^{\mathcal{F}} = H^*\big({}^cE_k^{*,*\mathcal{F}}, d_k\big) \tag{4-1}$$

for $2 \leq k < \infty$. Fix now $n \geq 0$ and $m \geq 0$. For each subgroup $P \leq S$ we have

$${}^cE_k^{n,m}(P) = {}^cE_{k+1}^{n,m}(P) = \cdots = {}^cE_\infty^{n,m}(P)$$

for k big enough. Because there are a finite number of subgroups of S we deduce that

$${}^cE_k^{n,m\mathcal{F}} = {}^cE_{k+1}^{n,m\ \mathcal{F}} = \cdots = {}^c\,E_\infty^{n,m\mathcal{F}}$$

for k big enough. Hence Equation (4-1) also holds for $k=\infty$ and we have obtained a spectral sequence

$$\{{}^cE_k^{*,*\mathcal{F}}, d_k\}_{2 \leq k \leq \infty}.$$

To study whether this spectral sequence converges recall that for $P \leq S$ the spectral sequence $\{{}^cE_k^{*,*}(P), d_k\}_{0\leq k\leq\infty}$ converges to $H^*(P;M)$. Hence we have short exact sequences

$$0 \longrightarrow F^n H^{n+m}(P;M) \longrightarrow F^{n+1} H^{n+m}(P;M) \longrightarrow {}^cE_\infty^{n,m}(P) \longrightarrow 0,$$

where

$$0 \subseteq \cdots \subseteq F^n H^{n+m}(P;M) \subseteq F^{n+1} H^{n+m}(P;M) \subseteq \cdots \subseteq H^{n+m}(P;M)$$

is the filtration induced on $H^*(P;M)$ by the filtration by columns on the double complex $A^{*,*}(P)$. This short exact sequence is natural with respect to morphisms of double complexes. Hence for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ we have morphisms of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & F^n H^{n+m}(Q;M) \arrow[r] \arrow[d, "F^n H^{n+m}(A)(\varphi)"] & F^{n+1} H^{n+m}(Q;M) \arrow[r] \arrow[d, "F^{n+1} H^{n+m}(A)(\varphi)"] & {}^cE_\infty^{n,m}(Q) \arrow[r] \arrow[d, "{}^cE_\infty^{n,m}(A)(\varphi)"] & 0 \\
0 \arrow[r] & F^n H^{n+m}(P;M) \arrow[r] & F^{n+1} H^{n+m}(P;M) \arrow[r] & {}^cE_\infty^{n,m}(P) \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & F^n H^{n+m}(P;M) \arrow[r] \arrow[d, "F^n H^{n+m}(B)(\varphi)"] & F^{n+1} H^{n+m}(P;M) \arrow[r] \arrow[d, "F^{n+1} H^{n+m}(B)(\varphi)"] & {}^cE_\infty^{n,m}(P) \arrow[r] \arrow[d, "{}^cE_\infty^{n,m}(B)(\varphi)"] & 0 \\
0 \arrow[r] & F^n H^{n+m}(Q;M) \arrow[r] & F^{n+1} H^{n+m}(Q;M) \arrow[r] & {}^cE_\infty^{n,m}(Q) \arrow[r] & 0.
\end{tikzcd}$$

We want to show that the morphism $H^n(A)(\varphi)\colon H^n(Q;M) \to H^n(P;M)$ and the morphism $H^n(B)(\varphi)\colon H^n(P;M) \to H^n(Q;M)$ induced by A and B on the targets of the spectral sequences are the usual maps in cohomology of groups. We consider the total complex of the double complex $A^{*,*}(P)$ defined as usual by

$$\mathrm{Tot}^s(A) = \bigoplus_{n+m=s} A^{n,m}(P)$$

and with total differential $d^h + d^v$. There is a chain map given by

$$\zeta\colon \mathrm{Hom}_{\boldsymbol{P}}(\mathcal{B}_{\boldsymbol{P}}^*, M) \to \mathrm{Tot}^*(A)$$

sending $f \in \mathrm{Hom}_{\boldsymbol{P}}(\mathcal{B}_{\boldsymbol{P}}^m, M)$ to $\zeta(f) \in A^{0,m}$ defined by

$$\zeta(f)(\overline{p} \otimes x) = f(x), \quad \overline{p} \in \overline{P},\ x \in \mathcal{B}_{\boldsymbol{P}}^m.$$

The map ζ induces an isomorphism between the cohomology of the total complex and $H^*(P;M)$; cf [13, page 352]. Now, from the definitions of the maps

$$A^{0,m}(\varphi)\colon A^{0,m}(Q)\to A^{0,m}(P) \quad \text{and} \quad B^{0,m}(\iota_P^Q)\colon A^{0,m}(P)\to A^{0,m}(Q)$$

it is easy to check that $H^n(A)(\varphi)$ and $H^n(B)(\iota_P^Q)$ are the usual maps in cohomology of groups; see [4, (D), page 82].

By properties of cohomology for finite groups $(H^n(A),H^n(B))\colon \mathcal{F}\to\mathbb{Z}_{(p)}$–mod is a cohomological Mackey functor for each $n\geq 0$. Hence so are the functors $(F^nH^{n+m}(A),F^nH^{n+m}(B))\colon \mathcal{F}\to\mathbb{Z}_{(p)}$–mod induced in the filtration for $n,m\geq 0$. By the arguments above also the pair $({}^cE_\infty^{n,m}(A),{}^cE_\infty^{n,m}(B))\colon \mathcal{F}\to\mathbb{Z}_{(p)}$–mod is a cohomological Mackey functor for $n,m\geq 0$. Then by Lemma 2.7 we have a short exact sequence of $\mathbb{Z}_{(p)}$–modules

$$0\longrightarrow (F^nH^{n+m})^{\mathcal{F}}\longrightarrow (F^{n+1}H^{n+m})^{\mathcal{F}}\longrightarrow {}^cE_\infty^{n,m\,\mathcal{F}}\longrightarrow 0.$$

It is immediate that taking invariants and filtering commute and hence we have

$$0\longrightarrow F^n(H^{n+m\,\mathcal{F}})\longrightarrow F^{n+1}(H^{n+m\,\mathcal{F}})\longrightarrow {}^cE_\infty^{n,m\,\mathcal{F}}\longrightarrow 0$$

for the filtration of $H^{n+m\,\mathcal{F}}=H^{n+m}(S)^{\mathcal{F}}$ given by

$$F^n(H^{n+m\,\mathcal{F}})=F^n(H^{n+m}(S))\cap H^{n+m}(S)^{\mathcal{F}}.$$

This finishes the proof. □

Remark 4.2 We have seen in the proof that for each $2\leq k\leq\infty$ the pair

$$\big({}^cE_k^{*,*}(A),{}^cE_k^{*,*}(B)\big)\colon \mathcal{F}\to \text{ differential bigraded } \mathbb{Z}_{(p)}\text{–modules}$$

is a cohomological Mackey functor. Moreover, $\{({}^cE_k^{*,*}(A),{}^cE_k^{*,*}(B))\}_{2\leq k\leq\infty}$ is a spectral sequence of Mackey functors that converges as a Mackey functor to the usual cohomology of finite groups Mackey functor $(H^*(A),H^*(B))\colon \mathcal{F}\to\mathbb{Z}_{(p)}$–mod.

5 Comparison

In this section we compare our spectral sequence and Lyndon–Hochschild–Serre spectral sequence. Let G be a finite group, $K\trianglelefteq G$ and $S\in \mathrm{Syl}_p(G)$. Then $T=K\cap S$ is a Sylow p–subgroup of K. Moreover, T is strongly $\mathcal{F}_S(G)$–closed. Fix a $\mathbb{Z}_{(p)}$–module M with trivial G–action. The Lyndon–Hochschild–Serre spectral sequence $E_{*,G}$ of the extension $K\to G\to G/K$ is

$$H^n\big(G/K;H^m(K;M)\big)\Rightarrow H^{n+m}(G;M)$$

meanwhile the spectral sequence E_* from Theorem 1.1 associated to T is

$$H^n\big(S/T; H^m(T;M)\big)^{\mathcal{F}_S(G)} \Rightarrow H^{n+m}(\mathcal{F};M).$$

Note that by the classical stable elements theorem, attributed to Tate by Cartan and Eilenberg [6, XII.10.1], $H^*(G;M) = H^*(\mathcal{F};M)$ and both spectral sequences converge to the same target. Recall that, by construction, E_* is a subspectral sequence of the Lyndon–Hochschild–Serre spectral sequence $E_{*,S}$ of $T \to S \to S/T$.

Theorem 5.1 *The spectral sequences $E_{*,G}$ and E_* are isomorphic.*

Proof Consider the category $\mathcal{F}_G(G)$ with objects the subgroups of G and morphisms given by $\mathrm{Mor}_{\mathcal{F}_G(G)}(H,I) = \mathrm{Hom}_G(H,I)$. Clearly $\mathcal{F}_S(G)$ is a full subcategory of $\mathcal{F}_G(G)$. For each subgroup $H \leq G$ we have a short exact sequence

$$H \cap K \to H \to \overline{H} = H/H \cap K.$$

If $\varphi = c_g\colon H \to I$ is a morphism in $\mathcal{F}_G(G)$ then, as K is normal in G, conjugation by $g \in G$ takes $H \cap K \to H \to \overline{H}$ to $I \cap K \to I \to \overline{I}$. Exactly the same construction of Section 3 gives a cohomological Mackey functor $(A,B)\colon \mathcal{F}_G(G) \to \mathrm{CCh}^2(\mathbb{Z}_{(p)})$ with values $H \mapsto A^{n,m}(H) = \mathrm{Hom}_H(\mathcal{B}^n_{\overline{H}} \otimes \mathcal{B}^m_H, M)$, where $\mathcal{B}^*_H$ and $\mathcal{B}^*_{\overline{H}}$ are the bar resolutions for H and $\overline{H}$ respectively. Moreover, for $H \leq S$, as $T = K \cap S$, we have $H \cap K = H \cap T$ and this functor over $\mathcal{F}_G(G)$ extends the one built in Section 3 over $\mathcal{F}_S(G)$.

The inclusion of the short exact sequence $T \to S \to S/T$ into $K \to G \to G/K$ gives a morphism $\{\mathrm{res}_r\}_{r\geq 2}$ of spectral sequences from $E_{*,G}$ into $E_{*,S}$. The morphism of differential graded algebras $\mathrm{res}_2\colon E_{2,G} \to E_{2,S}$ coincides with the morphism induced in cohomology by the functor A applied to the inclusion morphism $S \leq G$ of $\mathcal{F}_G(G)$, $H^{*,*}(A)(\iota_S^G)$. Applying the functor B to the same inclusion $S \leq G$ we get another morphism going in the opposite direction (transfer)

$$\begin{tikzcd} H^n\big(G/K; H^m(K;M)\big) \arrow[rr, bend left, "H^{n,m}(A)(\iota_S^G)"] && H^n\big(S/T; H^m(T;M)\big). \arrow[ll, bend left, "H^{n,m}(B)(\iota_S^G)"] \end{tikzcd}$$

Recall that $E_2 \leq E_{2,S}$ are exactly the $\mathcal{F}$–stable elements $H^p(S/T; H^q(T;M))^{\mathcal{F}_S(G)}$. Because conjugation by $g \in G$ induces the identity on $H^p(G/K; H^q(K;M))$ it is straightforward that $\mathrm{res}_2(E_{2,G}) \leq E_2$. Hence $\{\mathrm{res}_r\}_{r\geq 2}$ is a morphism of spectral sequences $E_{*,G} \to E_*$. If we prove that $\mathrm{res}_2(E_{2,G}) = E_2$ then res_2 is an isomorphism and hence res_r is an isomorphism for each $r \geq 2$ and we are done. To see that $\mathrm{res}_2(E_{2,G}) = E_2$ we proceed as usual when there is a Mackey functor available; cf [4, Theorem III.10.3]. Let $z \in H^n(S/T; H^m(T; M))^{\mathcal{F}_S(G)}$ and consider $w = H^{n,m}(B)(\iota_S^G)(z) \in H^n(G/K; H^m(K; M))$. By the double coset formula 2.2(3) and the cohomological condition 2.2(4) and because z is $\mathcal{F}_S(G)$–stable we obtain

$$\begin{aligned} H^{n,m}(A)\big(\iota_S^G\big)(w) &= \sum_{x \in S\backslash G/S} B\big(\iota_{S\cap {}^xS}^S\big)A\big(\iota_{S\cap {}^xS}^{{}^xS}\big)c_{x^{-1}}^*(z) \\ &= \sum_{x \in S\backslash G/S} B\big(\iota_{S\cap {}^xS}^S\big)A\big(\iota_{S\cap {}^xS}^S\big)(z) \\ &= \sum_{x \in S\backslash G/S} |S : S \cap {}^xS| z = |G : S| z. \end{aligned}$$

As $q = |G : S|$ is a p'–number it follows that $z = \mathrm{res}(\frac{w}{q})$. □

Example 5.2 Consider the symmetric group on 6 letters S_6. It has Sylow 2–subgroup $S = C_2 \times D_8$, where D_8 is the dihedral group of order 8. Because $A_6 \trianglelefteq S_6$, the subgroup $T = S \cap A_6 = D_8$ is strongly closed in $\mathcal{F} = \mathcal{F}_S(S_6)$. In this example we describe the Lyndon–Hochshild–Serre spectral sequence of $A_6 \to S_6 \to C_2$ interpreted as the spectral sequence $E_*^{*,*}$ of Theorem 1.1 applied to $\mathcal{F}$ and T. This demonstrates how the new spectral sequence works.

In the fusion system $\mathcal{F}$ there are three $\mathcal{F}$–centric an $\mathcal{F}$–radical subgroups, namely, S, $P = C_2^3$ and $Q = C_2^3$. The intersections $P \cap T$ and $Q \cap T$ are the two Klein subgroups of $T = D_8$. The automorphisms are $\mathrm{Aut}_{\mathcal{F}}(S) = 1$ and $\mathrm{Aut}_{\mathcal{F}}(P) \cong \mathrm{Aut}_{\mathcal{F}}(Q) \cong S_3$, the symmetric group on 3 letters.

Denote by $E_{*,S}^{*,*}$, $E_{*,P}^{*,*}$ and $E_{*,Q}^{*,*}$ the Lyndon–Hochschild–Serre spectral sequences of the extensions $T \to S \to C_2$, $P \cap T \to P \to C_2$ and $Q \cap T \to Q \to C_2$ respectively. All three extensions are direct products and hence all differentials are 0 and the three spectral sequences collapse at the second page. In particular, the ring $H^*(S; \mathbb{F}_2)$ is isomorphic as a ring to $E_{2,S}^{*,*}$ and hence $H^*(S_6; \mathbb{F}_2)$ is isomorphic as a ring to $E_2^{*,*}$. Moreover, for the invariants we have

$$E_2^{*,*} = {E_{2,S}^{*,*}}^{\mathcal{F}} = E_{2,S}^{*,*} \cap \big(\mathrm{res}_P^S\big)^{-1}\big({E_{2,P}^{*,*}}^{S_3}\big) \cap \big(\mathrm{res}_Q^S\big)^{-1}\big({E_{2,Q}^{*,*}}^{S_3}\big),$$

because it is enough to consider invariants with respect to $\mathcal{F}$–centric and $\mathcal{F}$–radical subgroups by Alperin's fusion theorem. Here,

$$\mathrm{res}_P^S \colon E_{2,S}^{*,*} \to E_{2,P}^{*,*} \quad \text{and} \quad \mathrm{res}_Q^S \colon E_{2,S}^{*,*} \to E_{2,Q}^{*,*}$$

are the restriction maps. Denoting by subscripts the degrees we have the following:

$$E_{2,S}^{*,*} = H^*(D_8;\mathbb{F}_2) \otimes H^*(C_2;\mathbb{F}_2) = \mathbb{F}_2[x_1, y_1, w_2]/(xy) \otimes \mathbb{F}_2[u],$$
$$E_{2,P}^{*,*} = H^*(C_2^2;\mathbb{F}_2) \otimes H^*(C_2;\mathbb{F}_2) = \mathbb{F}_2[x_1, x_1'] \otimes \mathbb{F}_2[u],$$
$$E_{2,Q}^{*,*} = H^*(C_2^2;\mathbb{F}_2) \otimes H^*(C_2;\mathbb{F}_2) = \mathbb{F}_2[y_1, y_1'] \otimes \mathbb{F}_2[u].$$

Restrictions are given by

$$\mathrm{res}_P^S(x) = x, \quad \mathrm{res}_P^S(y) = 0, \quad \mathrm{res}_P^S(w) = xx' + x'^2, \quad \mathrm{res}_P^S(u) = u,$$
$$\mathrm{res}_Q^S(x) = 0, \quad \mathrm{res}_Q^S(y) = y, \quad \mathrm{res}_Q^S(w) = yy' + y'^2, \quad \mathrm{res}_Q^S(u) = u.$$

Now $S_3 = \mathrm{Aut}_{S_6}(P)$ acts on $P \cap T = C_2^2$ and on the quotient $C_2 = P/P \cap T$. The induced action on $H^*(C_2^2;\mathbb{F}_2)$ is the natural one and on $H^*(C_2;\mathbb{F}_2)$ the only possibility is the trivial action. Hence, the invariants are given by

$$E_{2,P}^{*,*\,S_3} = \mathbb{F}_2[x_1, x_1']^{S_3} \otimes \mathbb{F}_2[u] = \mathbb{F}_2[d_2, d_3] \otimes \mathbb{F}_2[u],$$

where $d_2 = x^2 + x'^2 + xx'$ and $d_3 = (x + x')xx'$ are Dickson's invariants. Analogously, we have that

$$E_{2,Q}^{*,*\,S_3} = \mathbb{F}_2[e_2, e_3] \otimes \mathbb{F}_2[u]$$

with $e_2 = y^2 + y'^2 + yy'$ and $e_3 = (y + y')yy'$. It is straightforward that

$$d_2 = \mathrm{res}_P^S(x^2 + w), \quad d_3 = \mathrm{res}_P^S(xw), \quad e_2 = \mathrm{res}_Q^S(y^2 + w) \quad \text{and} \quad e_3 = \mathrm{res}_Q^S(yw).$$

From this, it is immediate that $\mathbb{F}_2[x^2 + y^2 + w, xw, yw] \otimes \mathbb{F}_2[u] \subseteq E_2^{*,*}$.

To check the reversed inclusion we first consider stable elements in the polynomial algebras $\mathbb{F}_2[x, w]$ and $\mathbb{F}_2[y, w]$. As for Long [12, Lemma 1.4.6], the restrictions $\mathrm{res}_P^S|_{\mathbb{F}_2[x,w]}$ and $\mathrm{res}_Q^S|_{\mathbb{F}_2[y,w]}$ are injective, therefore

$$\mathbb{F}_2[x, w] \cap (\mathrm{res}_P^S)^{-1}(E_{2,P}^{*,*\,S_3}) = \mathbb{F}_2[x^2 + w, xw],$$
$$\mathbb{F}_2[y, w] \cap (\mathrm{res}_Q^S)^{-1}(E_{2,Q}^{*,*\,S_3}) = \mathbb{F}_2[y^2 + w, yw].$$

A class v of $H^n(D_8;\mathbb{F}_2)$ can be written as follows, where we set $k = [\frac{n}{2}]$:

$$v = \sum_{i=0}^{k} \alpha_i w^i x^{n-2i} + \beta_i w^i y^{n-2i}.$$

From the discussion above we have that if v is $\mathcal{F}$–invariant then

$$v = \epsilon w^k + \sum_{2i+3j=n} \gamma_i (x^2 + w)^i (xw)^j + \delta_i (y^2 + w)^i (yw)^j,$$

where $\epsilon = 0$ for n odd and $\epsilon = \alpha_k + \beta_k = \gamma_k = \delta_k$ if n is even. If n is odd, then the equalities $(x^2+y^2+w)(xw) = (x^2+w)(xw)$ and $(x^2+y^2+w)(yw) = (y^2+w)(yw)$ give that $v \in \mathbb{F}_2[x^2+y^2+w, xw, yw]$. If n is even, then the only term left to consider is

$$\gamma_k(x^2+w)^k + \delta_k(y^2+w)^k + \epsilon w^k = \gamma_k\big((x^2+w)^k + (y^2+w)^k + w^k\big)$$

and an easy induction shows that $(x^2+w)^k + (y^2+w)^k + w^k = (x^2+y^2+w)^k$. So $E_2^{*,*} = \mathbb{F}_2[x^2+y^2+w, xw, yw] \otimes \mathbb{F}_2[u]$. The corner of $E_2^{*,*}$ is

$$\begin{array}{c|ccc}
xw, yw & xwu, ywu & xwu^2, ywu^2 \\
x^2+y^2+w & (x^2+y^2+w)u & (x^2+y^2+w)u^2 \\
0 & 0 & 0 \\
\hline
1 & u & u^2
\end{array}$$

and we deduce that $H^*(S_6; \mathbb{F}_2) = \mathbb{F}_2[u_1, a_2, b_3, c_3]/(bc)$ with generators

$$a = x^2+y^2+w, \quad b = xw \quad \text{and} \quad c = yw.$$

6 Stallings' Theorem

Associated to every first quadrant spectral sequence there is a five-term exact sequence. In the case of the Lyndon–Hochschild–Serre spectral sequence for $K \trianglelefteq G$ and the G–module M we obtain the inflation-restriction exact sequence:

$$\text{(6-1)} \quad 0 \to H^1(G/K; M^K) \to H^1(G; M) \\ \to H^1(K, M)^{G/K} \to H^2(G/K; M^K) \to H^2(G; M),$$

where the second arrow from the right is the transgression. Before introducing the five-term exact sequence for the spectral sequence of Theorem 1.1 we introduce some notation. So let $\mathcal{F}$ be a fusion system over the p–group S with a strongly closed $\mathcal{F}$–subgroup T. Set $[T, \mathcal{F}] = \langle [t, \varphi] \mid t \in T, \varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle u \rangle, T) \rangle \le T$, where $[t, \varphi] = t\varphi(t^{-1})$, $T^p = \langle t^p, t \in T \rangle$, which is characteristic in T, and the commutator subgroup $[T, S] = \langle t^{-1}s^{-1}ts \mid t \in T \text{ and } s \in S \rangle \trianglelefteq T$. Because the element-wise product $T^p[T, S]$ is a normal subgroup of T, the element-wise product $T^p[S, T]R$ is a subgroup of T for any $R \le T$. For instance, $T^p[T, S][T, \mathcal{F}] = T^p[T, \mathcal{F}] \le T$.

The five-term exact sequence for the spectral sequence of Theorem 1.1 for $\mathcal{F}$, T and the $\mathbb{Z}_{(p)}$–module M with trivial S–action is the following:

$$\text{(6-2)} \quad 0 \to H^1(S/T; M)^{\mathcal{F}} \to H^1(\mathcal{F}; M) \\ \to H^1(T; M)^{\mathcal{F}} \to H^2(S/T; M)^{\mathcal{F}} \to H^2(\mathcal{F}; M),$$

where the arrow $H^1(T;M)^{\mathcal{F}} \to H^2(S/T;M)^{\mathcal{F}}$ is the transgression.

For coefficients $M = \mathbb{F}_p$ we have

$$H^1(\mathcal{F};\mathbb{F}_p) = H^1(S;\mathbb{F}_p)^{\mathcal{F}} = \mathrm{Hom}(S/S^p[S,\mathcal{F}],\mathbb{F}_p), \tag{6-3}$$

$$H^1(T;\mathbb{F}_p)^{\mathcal{F}} = \mathrm{Hom}(T/T^p[T,\mathcal{F}],\mathbb{F}_p). \tag{6-4}$$

We also have

$$\begin{aligned} H^1(S/T;M)^{\mathcal{F}} &= H^1(S/T;M)^{\mathcal{F}/T}, \\ H^2(S/T;M)^{\mathcal{F}} &= H^2(S/T;M)^{\mathcal{F}/T}, \end{aligned} \tag{6-5}$$

by Remark 3.1.

If $\mathcal{F}_i$ is a fusion system over the p–group S_i for $i = 1,2$, a homomorphism of groups $\phi\colon S_1 \to S_2$ is fusion preserving if for each $\varphi \in \mathrm{Hom}_{\mathcal{F}_1}(P,S_1)$ there exists $\widehat{\varphi} \in \mathrm{Hom}_{\mathcal{F}_2}(\phi(P),S_2)$ such that $\phi\circ\varphi = \widehat{\varphi}\circ\phi$. It is easy to see that such a homomorphism induces a map in cohomology $H^*(\mathcal{F}_2;\mathbb{F}_p) \to H^*(\mathcal{F}_1;\mathbb{F}_p)$. In fact, by the work of Ragnarsson [15], it induces a map even at the level of stable classifying spaces. Assume, in addition, that ϕ induces a map of short exact sequences

$$\begin{tikzcd} T_1 \arrow[r] \arrow[d] & S_1 \arrow[r] \arrow[d, "\phi"] & S_1/T_1 \arrow[d] \\ T_2 \arrow[r] & S_2 \arrow[r] & S_2/T_2, \end{tikzcd}$$

where T_i is strongly closed in S_i with respect to $\mathcal{F}_i$ for $i = 1,2$. This is equivalent to assume that $\phi(T_1) \le T_2$. Denote by E_i the spectral sequence from Theorem 1.1 applied to the strongly closed subgroup T_i for $i = 1,2$. Then ϕ induces a morphism of spectral sequences $E_2 \to E_1$ and, in particular, a map of five terms exact sequences.

Theorem 6.1 (Stallings [17]) *Let $\mathcal{F}_i$ be a fusion system over the p–group S_i for $i = 1,2$ and let $\phi\colon S_1 \to S_2$ be a fusion preserving homomorphism. Define*

$$S_{i,0} = S_i \quad \textit{and} \quad S_{i,n+1} = S_{i,n}^p[S_{i,n},\mathcal{F}_i] \quad \textit{for } i = 1,2 \textit{ and } n \ge 0.$$

If the induced map in cohomology $H^i(\mathcal{F}_2;\mathbb{F}_p) \to H^i(\mathcal{F}_1;\mathbb{F}_p)$ is isomorphism for $i = 1$ and monomorphism for $i = 2$ then $S_1/S_{1,n} \cong S_2/S_{2,n}$ for each $n \ge 1$. In particular, for n big enough we obtain that $S_1/\mathcal{O}^p_{\mathcal{F}_1}(S_1) \cong S_2/\mathcal{O}^p_{\mathcal{F}_2}(S_2)$.

Proof We will prove by induction that $S_1/S_{1,n} \cong S_2/S_{2,n}$ and that $S_{i,n}$ is strongly $\mathcal{F}_i$–closed and contains $O^p_{\mathcal{F}_i}(S_i)$ for $i = 1,2$. For the base case $n = 1$, we have that $S_{i,1}$ contains $O^p_{\mathcal{F}_i}(S_i)$ and is strongly $\mathcal{F}_i$–closed by [8, Corollary A.6] $(i = 1,2)$.

Moreover, by hypothesis, $H^1(\mathcal{F}_2;\mathbb{F}_p)\cong H^1(\mathcal{F}_1;\mathbb{F}_p)$ and then by Equation (6-3) we get $S_1/S_{1,1}\cong S_2/S_{2,1}$.

Now let $n\geq 1$. As Φ is fusion preserving it is easy to see that $\phi(S_{1,n})\leq S_{2,n}$. Then we have the following map of short exact sequences:

$$\begin{tikzcd}
S_{1,n} \arrow[r] \arrow[d] & S_1 \arrow[r] \arrow[d, "\phi"] & S_1/S_{1,n} \arrow[d] \\
S_{2,n} \arrow[r] & S_2 \arrow[r] & S_2/S_{2,n}
\end{tikzcd}$$

By the induction hypothesis, $S_{1,n}$ and $S_{2,n}$ are strongly closed in $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. Then by the discussion before the theorem we have a map of five-term short exact sequences in cohomology with trivial $\mathbb{F}_p$–coefficients:

$$\begin{tikzcd}
0 \arrow[r] & H^1(S_1/S_{1,n})^{\mathcal{F}_1} \arrow[r] \arrow[d, "f_1"] & H^1(\mathcal{F}_1) \arrow[r] \arrow[d, "g_1"] & H^1(S_{1,n})^{\mathcal{F}_1} \arrow[r] \arrow[d, "h_1"] & H^2(S_1/S_{1,n})^{\mathcal{F}_1} \arrow[r] \arrow[d, "f_2"] & H^2(\mathcal{F}_1) \arrow[d, "g_2"] \\
0 \arrow[r] & H^1(S_2/S_{2,n})^{\mathcal{F}_2} \arrow[r] & H^1(\mathcal{F}_2) \arrow[r] & H^1(S_{2,n})^{\mathcal{F}_2} \arrow[r] & H^2(S_2/S_{2,n})^{\mathcal{F}_2} \arrow[r] & H^2(\mathcal{F}_2)
\end{tikzcd}$$

Because $O^p_{\mathcal{F}_i}(S_i)$ is contained in $S_{i,n}$ the quotient $\mathcal{F}/S_{i,n}$ is a p–group, ie,

$$\mathcal{F}/S_{i,n}=\mathcal{F}_{S/S_{i,n}}(S/S_{i,n})\quad \text{for } i=1,2.$$

Then, by Equation (6-5), the maps f_1 and f_2 are isomorphisms as $S_1/S_{1,n}\cong S_2/S_{2,n}$. Now, by hypothesis, g_1 is an isomorphism and g_2 is a monomorphism. Hence by the five lemma h_1 is an isomorphism. Then by Equation (6-4) we obtain that $S_{1,n}/S_{1,n+1}\cong S_{2,n}/S_{2,n+1}$ and hence $S_1/S_{1,n+1}\cong S_2/S_{2,n+1}$.

To finish the induction step, denote by $\mathcal{F}_{i,n}$ the unique p–power index fusion subsystem of $\mathcal{F}_i$ on $S_{i,n}$ [2, Theorem 4.3]. Then using [8, Lemma A.5] we obtain that $S_{i,n+1}=S^p_{i,n}[S_{i,n},S_i]O^p_{\mathcal{F}_{i,n}}(S_{i,n})$ and hence, by [8, Corollary A.14],

$$S_{i,n+1}=S^p_{i,n}[S_{i,n},S_i]O^p_{\mathcal{F}_i}(S_i).$$

Then $S_{i,n+1}$ contains $O^p_{\mathcal{F}_i}(S_i)$ and by [8, Proposition A.7(1)] $S_{i,n+1}$ is strongly $\mathcal{F}_i$–closed for $i=1,2$.

For the second part of the statement recall that for any finite p–group R the series $R_0=R$, $R_n=R^p_{n-1}[R_{n-1},R]$ $(n\geq 1)$ becomes trivial for n big enough. Then, considering the image of $S_{i,n}$ in $S_i/O^p_{\mathcal{F}}(S_i)$, it is easy to see that $S_{i,n}=O^p_{\mathcal{F}}(S_i)$ for n big enough and $i=1,2$. □

Corollary 6.2 (Evens [10, 7.2.5]) *Let $\mathcal{F}$ be a fusion system over the p–group S. If the map $H^2(\mathcal{F}/E^p_{\mathcal{F}}(S);\mathbb{F}_p) \to H^2(\mathcal{F};\mathbb{F}_p)$ is a monomorphism then $S/O^p_{\mathcal{F}}(S)$ is elementary abelian.*

Proof Set $\mathcal{F}_1 = \mathcal{F}$ and $\mathcal{F}_2 = \mathcal{F}/E^p_{\mathcal{F}}(S)$ and consider the fusion preserving quotient map $\mathcal{F}_1 \to \mathcal{F}_2$. By Equation (6-3) and because $E^p_{\mathcal{F}}(S) = \Phi(S)O^p_{\mathcal{F}}(S) = S^p[S,\mathcal{F}]$, the quotient map induces an isomorphism in degree-1 cohomology. Then Theorem 6.1 gives that $O^p_{\mathcal{F}}(S) = E^p_{\mathcal{F}}(S)$ and we are done. □

Corollary 6.3 (Tate [18]) *Let $\mathcal{F}$ be a fusion system over the p–group S. If the restriction map $H^1(\mathcal{F};\mathbb{F}_p) \to H^1(S;\mathbb{F}_p)$ is an isomorphism then $\mathcal{F} = \mathcal{F}_S(S)$.*

Proof Consider $\mathcal{F}_1 = \mathcal{F}_S(S)$, $\mathcal{F}_2 = \mathcal{F}$ and the fusion preserving morphism given by inclusion $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Then $H^1(\mathcal{F};\mathbb{F}_p) \to H^1(S;\mathbb{F}_p)$ is isomorphism by hypothesis and $H^2(\mathcal{F};\mathbb{F}_p) \to H^2(S;\mathbb{F}_p)$ is monomorphism by definition. Then from Theorem 6.1 we obtain $O^p_{\mathcal{F}}(S) = 1$. Thus there are no p'–automorphisms in $\mathcal{F}$ and $\mathcal{F} = \mathcal{F}_S(S)$. □

References

[1] **M Aschbacher**, *Normal subsystems of fusion systems*, Proc. Lond. Math. Soc. 97 (2008) 239–271 MR2434097

[2] **C Broto**, **N Castellana**, **J Grodal**, **R Levi**, **B Oliver**, *Extensions of p–local finite groups*, Trans. Amer. Math. Soc. 359 (2007) 3791–3858 MR2302515

[3] **C Broto**, **R Levi**, **B Oliver**, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. 16 (2003) 779–856 MR1992826

[4] **K S Brown**, *Cohomology of groups*, Grad. Texts in Math. 87, Springer, New York (1982) MR672956

[5] **J Cantarero**, **J Scherer**, **A Viruel**, *Nilpotent p–local finite groups* (2011) `arXiv:1107.5158v1`

[6] **H Cartan**, **S Eilenberg**, *Homological algebra*, Princeton University Press (1956) MR0077480

[7] **D A Craven**, *Control of fusion and solubility in fusion systems*, J. Algebra 323 (2010) 2429–2448 MR2602388

[8] **A Díaz**, **A Glesser**, **S Park**, **R Stancu**, *Tate's and Yoshida's theorems on control of transfer for fusion systems*, J. Lond. Math. Soc. 84 (2011) 475–494 MR2835340

[9] **A Díaz**, **A Ruiz**, **A Viruel**, *All p–local finite groups of rank two for odd prime p*, Trans. Amer. Math. Soc. 359 (2007) 1725–1764 MR2272147

[10] **L Evens**, *The cohomology of groups*, Clarendon Press, Oxford (1991) MR1144017

[11] **R J Flores**, **R M Foote**, *Strongly closed subgroups of finite groups*, Adv. Math. 222 (2009) 453–484 MR2538017

[12] **J H Long**, *The cohomology rings of the special affine group of* $\mathbb{F}_p^2$ *and of* $\mathrm{PSL}(3, p)$, PhD thesis, University of Maryland (2008) MR2712167 Available at http://search.proquest.com//docview/304560975

[13] **S Mac Lane**, *Homology*, Springer, Berlin (1995) MR1344215

[14] **S Park**, *Realizing a fusion system by a single finite group*, Arch. Math. (Basel) 94 (2010) 405–410 MR2643975

[15] **K Ragnarsson**, *Classifying spectra of saturated fusion systems*, Algebr. Geom. Topol. 6 (2006) 195–252 MR2199459

[16] **R Solomon**, **R Stancu**, *Conjectures on finite and p–local groups*, L'Enseignement Mathématique 54 (2008) 171–176

[17] **J Stallings**, *Homology and central series of groups*, J. Algebra 2 (1965) 170–181 MR0175956

[18] **J Tate**, *Nilpotent quotient groups*, Topology 3 (1964) 109–111 MR0160822

[19] **P Webb**, *A guide to Mackey functors*, from: "Handbook of algebra", (M Hazewinkel, editor), North-Holland, Amersterdam (2000) 805–836 MR1759612

Departamento de Álgebra, Geometría y Topología
Campus de Teatinos, Universidad de Málaga, 29071, Málaga, Spain

adiaz@agt.cie.uma.es

http://agt.cie.uma.es/~adiaz

Received: 13 December 2012 Revised: 22 May 2013