High distance Heegaard splittings via Dehn twists

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In 2001, J Hempel proved the existence of Heegaard splittings of arbitrarily high distance by using a high power of a pseudo-Anosov map as the gluing map between two handlebodies. We show that lower bounds on distance can also be obtained when using a high power of a suitably chosen Dehn twist. In certain cases, we can then determine the exact distance of the resulting splitting. These results can be seen as a natural extension of work by A Casson and C Gordon in 1987 regarding strongly irreducible Heegaard splittings.

57M50; 57M25

1 Introduction

Since Hempel [6] introduced the notion of the distance of a Heegaard splitting in 2001, a number of results have linked distance with the topology of the ambient 3–manifold. For example, work by Thompson [20] and Hempel [6] showed that if a manifold admits a Heegaard splitting of distance greater than or equal to 3, then the manifold is atoroidal and not Seifert fibered, hence hyperbolic by the geometrization conjecture. Hartshorn [4] found that if a manifold admits a distance d Heegaard splitting, then the genus g of an orientable incompressible surface must satisfy $2g \geq d$. Scharlemann and Tomova [16] generalized this result to show that a Heegaard splitting of genus g and distance d such that $2g < d$ is the unique splitting of minimal genus for that manifold.

Concurrently, there have been many efforts to construct examples of Heegaard splittings of high distance. Many approaches make use of the fact that a Heegaard splitting can be described by a homeomorphism between two handlebodies of equal genus. One of the first major achievements was made by Hempel [6] who, adapting an argument of Kobayashi [9], proved the existence of Heegaard splittings of arbitrarily high distance via the use of a high power of a pseudo-Anosov map as the gluing map between two handlebodies. However, the use of pseudo-Anosov maps meant no concrete examples were provided.

This can be rectified by considering Heegaard splittings determined by a Dehn twist map, rather than one that is pseudo-Anosov. Casson and Gordon [1] introduced what

is now referred to as the Casson–Gordon rectangle condition (published by Kobayashi in [8]) that would ensure a Heegaard splitting has distance greater than or equal to 2. Moreover, they provided methods of generating Heegaard splittings using Dehn twists that satisfied this condition. Two of their results can be stated as the following. Note that in the following statements, $D(H)$ denotes the disk set of a handlebody H and distance refers to the distance in the curve complex of ∂H (these definitions and more details appear in Section 4).

Theorem 1.1 (Casson and Gordon [1]) *Suppose H is a genus g handlebody. Let γ be a simple closed curve that is distance greater than or equal to 2 from $D(H)$. Then gluing H to a copy of itself via greater than or equal to 2 Dehn twists about γ determines a genus g Heegaard splitting of distance greater than or equal to 2.*

Theorem 1.2 (Casson and Gordon [1]) *Suppose H_1 and H_2 are genus g handlebodies with $\partial H_1 = \partial H_2$ and the distance between $D(H_1)$ and $D(H_2)$ is at most 1. Let γ be a simple closed curve that is distance greater than or equal to 2 from both $D(H_1)$ and $D(H_2)$. Then gluing H_1 to H_2 via greater than or equal to 6 Dehn twists about γ determines a genus g Heegaard splitting of distance greater than or equal to 2.*

More recently, Berge developed a modified rectangle condition that would guarantee a genus 2 splitting has distance greater than or equal to 3 (this criterion is described by Scharlemann in [15]) and constructs examples of genus 2 splittings satisfying this condition. Hempel [6] used Dehn twists and the notion of stacks to construct Heegaard splittings of distance greater than or equal to 3 and Evans [2] extended this result to Heegaard splittings of distance greater than or equal to d for any $d \geq 2$. Lustig and Moriah [12] used Dehn twists and derived train tracks to produce another class of examples of splittings with distance greater than or equal to d. In [10], Tao Li utilized the techniques of Lustig and Moriah to construct the first examples of high distance Heegaard splittings that are known to be non-Haken. However, all of these results only provided a lower bound on the distance of the constructed splittings; except for low distances, the exact distances of the constructed Heegaard splittings were typically unknown.

In fact, until recently there was no proof that there existed Heegaard splittings with distance exactly d for every $d \in \mathbb{N}$. This question has been settled by Ido, Jang and Kobayashi [7] and Qiu, Zou and Guo [14]. Both (over a similar time frame) developed examples of high distance Heegaard splittings with distance exactly d.

We provide concrete examples of Heegaard splittings with an exact known distance using a different approach that relies exclusively on Dehn twist maps. Our results, shown

below, can be seen as an extension of the work by Casson and Gordon (Theorems 1.1 and 1.2).

Theorem 1.3 *Suppose H is a genus g handlebody. Let γ be a simple closed curve that is distance d from $D(H)$ for some $d \geq 2$. Then gluing H to a copy of itself via greater than or equal to $2d - 2$ Dehn twists about γ determines a genus g Heegaard splitting of distance exactly $2d - 2$.*

Theorem 1.4 *Suppose H_1 and H_2 are genus g handlebodies with $\partial H_1 = \partial H_2$ and $n = \max\{1, d(D(H_1), D(H_2))\}$. Let γ be a simple closed curve that is distance d_1 from $D(H_1)$ and d_2 from $D(H_2)$ where $d_1 \geq 2$, $d_2 \geq 2$ and $d_1 + d_2 - 2 > n$. Then gluing H_1 to H_2 via greater than or equal to $n + d_1 + d_2$ Dehn twists about γ determines a genus g Heegaard splitting whose distance is at least $d_1 + d_2 - 2$ and at most $d_1 + d_2$.*

In Section 2 we define terms, in Section 3 we review Dehn twists and define twisting number, in Section 4 we review diskbusting curves and results of Starr [18], and in Section 5 we prove the main theorems.

2 Standard cut systems and pants decompositions

Throughout this paper, Σ will denote a closed orientable surface with genus g such that $g \geq 2$.

Definition 2.1 A *standard cut system of* Σ is a collection of g essential simple closed curves X in Σ such that $\Sigma - X$ is a $2g$–punctured sphere.

Definition 2.2 Let H be a genus g handlebody. A *standard set of compressing disks of H* is a collection of disjoint compressing disks $\mathcal{D} \subset H$ so that $\partial \mathcal{D}$ is a standard cut system in ∂H. A *standard set of meridians of H* is any collection of curves in ∂H that bounds a standard set of compressing disks of H. (Thus a standard set of meridians in ∂H is a standard cut system, but not vice versa.)

Definition 2.3 A *pants decomposition* $\mathcal{P}$ of Σ is a collection of $3g - 3$ essential simple closed curves in Σ such that $\Sigma - \mathcal{P}$ is a collection of pairs of pants (ie three punctured spheres).

Definition 2.4 A *complete collection of compressing disks* for a handlebody H is a collection of disjoint compressing disks whose boundary is a pants decomposition of ∂H. The boundary of such a complete collection of disks will be called a pants decomposition of H (ie it is a pants decomposition of ∂H and each curve bounds a disk in H).

Definition 2.5 Let $\mathcal{P}$ be a pants decomposition of Σ. If P is the closure of a pair of pants component of $\Sigma - \mathcal{P}$, then

- a *seam* of P is an essential properly embedded arc in P that has endpoints on distinct components of ∂P,

- a *wave* of P is an essential properly embedded arc in P that has endpoints on the same component of ∂P.
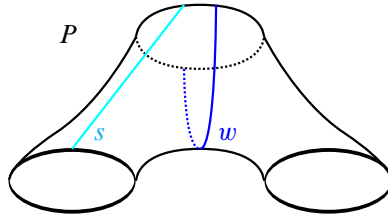

Figure 1: In the pair of pants component P, w is a wave and s is a seam.

Definition 2.6 Let a and b be simple closed curves or arcs with fixed endpoints in Σ. Then a and b admit a *bigon* if there is an embedded disk D in Σ whose boundary is the endpoint union of a subarc of a and a subarc of b.

Definition 2.7 Two essential simple closed curves in Σ (or two properly embedded arcs with fixed endpoints in an essential subsurface of Σ) *intersect efficiently* if they do not admit a bigon.

The condition that two curves a and b intersect efficiently is equivalent to the condition that the number of intersection points between a and b is minimal, up to (proper) isotopy (for a proof see Farb and Margalit [3]).

Definition 2.8 Let $\mathcal{P}$ be a pants decomposition of Σ. An embedded curve γ is *k–seamed with respect to* $\mathcal{P}$ if it intersects $\mathcal{P}$ efficiently and for each pair of pants component P of $\Sigma - \mathcal{P}$, there are at least k arcs of $\gamma \cap P$ representing each of the three distinct isotopy classes of seams of P.

The notion of a 1–seamed curve was introduced by Kobayashi [9], who denoted such curves to be of *full type*. Note that if a curve γ is 1–seamed with respect to a pants decomposition $\mathcal{P}$ of Σ then neither γ, nor any simple closed curve disjoint from γ, can contain a wave in any pants component of $\Sigma - \mathcal{P}$.

Definition 2.9 Let $\mathcal{P}$ be a pants decomposition of Σ. A collection of essential simple closed curves $Y = \{y_1, y_2, \ldots, y_t\}$ is *collectively k–seamed with respect to* $\mathcal{P}$ if Y intersects $\mathcal{P}$ efficiently and for each pair of pants component P of $\Sigma - \mathcal{P}$, there are at least k arcs of $\bigcup_{1 \le i \le t} y_i \cap P$ representing each of the three distinct isotopy classes of seams of P.

Lemma 2.10 *Let $\mathcal{D}$ be a complete collection of compressing disks for H and $\mathcal{P} = \partial\mathcal{D} \subset \partial H$. If γ is an essential simple closed curve in ∂H that bounds a compressing disk in H and intersects $\mathcal{P}$ efficiently, then γ is either isotopic to a curve in $\mathcal{P}$ or γ contains at least two waves of components of $\partial H - \mathcal{P}$.*

Proof Let $D \subset H$ be the disk that γ bounds. Isotope the interior of D to intersect $\mathcal{D}$ minimally. A standard innermost circle argument, exploiting the irreducibility of H, guarantees that each component of $D \cap \mathcal{D}$ is an arc. If $D \cap \mathcal{D} = \varnothing$ then γ is isotopic to a curve in $\mathcal{P}$. So suppose $D \cap \mathcal{D} \ne \varnothing$.

Let α be an outermost arc of $D \cap \mathcal{D}$ in D. Then α and an arc $\beta \subset \gamma$ together bound a disk in D that is disjoint from $\mathcal{D}$ in its interior. As γ intersects $\mathcal{P}$ efficiently, β is an essential arc in a component of $\Sigma - \mathcal{P}$ and is therefore a wave. Since $D - \mathcal{D}$ contains at least two distinct disks cut off by outermost arcs, γ contains at least two distinct waves (which may be parallel). $\square$

Note that Lemma 2.10 implies that any meridian of H intersects a k–seamed curve in at least $2k$ points.

3 Dehn twists and twisting number

We first reproduce a definition of a standard Dehn twist as described in [3].

Definition 3.1 Let A be an annulus $S^1 \times [0, 1]$ embedded in the (θ, r)–plane by the map $(\theta, r) \to (\theta, r+1)$ and let the standard orientation of the plane induce an orientation on A. Let $T: A \to A$ be the *left twist map* of A given by $T(\theta, r) = (\theta + 2\pi r, r)$. Similarly, let the *right twist map* be given by $T^{-1}(\theta, r) = (\theta - 2\pi r, r)$.

Definition 3.2 Let y be a simple closed curve in Σ and N an annular neighborhood of y. Define ϕ to be an orientation-preserving homeomorphism from A to N. Then the *Dehn twist operator along* y is the homeomorphism $\tau_y: \Sigma \to \Sigma$ given by

$$\tau_y = \begin{cases} \phi \circ T \circ \phi^{-1}(p) & \text{if } p \in N, \\ p & \text{if } p \in \Sigma - N. \end{cases}$$

The homeomorphism τ_y is well defined up to isotopy.

Remark 3.3 In this paper, we have picked the convention that all positive powers of the Dehn twist operator will apply a left twist.

Definition 3.4 Suppose $Y = \{y_1, y_2, \ldots, y_t\}$ is a collection of pairwise disjoint simple closed curves in Σ. Let $\tau_Y: \Sigma \to \Sigma$ denote the composition of Dehn twists $\tau_{y_1} \circ \tau_{y_2} \circ \cdots \circ \tau_{y_t}$. Since the y_i are pairwise disjoint, it is easy to see that τ_Y is independent of the ordering of $\{y_i\}$.

Definition 3.5 Suppose $N \subset \Sigma$ is an annulus with a specified I–fibration in an oriented surface Σ, and c is a properly embedded arc in N with fixed endpoints that intersects each I–fiber of N efficiently. Let p be a point of intersection between c and ∂N. Then denote v_c to be the inwards tangent vector of c based at p. Similarly, let v_I be the inwards tangent vector based at p of the I–fiber of N that has p as an endpoint. Then c *turns left in N at p* (resp. *turns right in N at p*) if the orientation determined by the pair $\langle v_I, v_c \rangle$ (resp. the pair $\langle v_c, v_I \rangle$) matches the corresponding orientation of the tangent space of Σ at p.
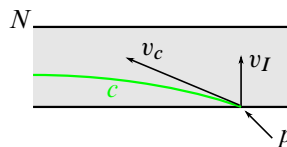

Figure 2: The arc c turns left in N.

Definition 3.6 Let N be an annulus and let $X = \{x_1, x_2, \ldots, x_s\}$ be a collection of pairwise disjoint simple closed curves in Σ. Then N is *fibered with respect to X* if N has an I–fibration such that $N \cap X$ consists of I–fibers of N.

One way to arrange for N to be fibered with respect to X is to isotope N so that the core curve of N is transverse to X and then make the fibers of N sufficiently short.

The definition of the twisting number between two simple closed curves was first introduced by Lustig and Moriah in [12]. To handle some additional subtleties required for our arguments, we provide a modified definition. We first give the definition of the twisting number for a properly embedded essential arc with fixed endpoints in an annulus and later generalize to the twisting number between two simple closed curves.

Definition 3.7 Let $X = \{x_1, x_2, \ldots, x_s\}$ be a collection of pairwise disjoint curves in Σ. Suppose N is an annulus in Σ that is fibered with respect to X. Let c be an essential properly embedded arc in N with fixed endpoints disjoint from $X \cap N$ that intersects each I–fiber of N efficiently. Set $m = \frac{|c \cap X|}{|N \cap X|} \in \mathbb{Q}$. Then c has *twisting number m* (resp. *twisting number $(-m)$*) in N with respect to X if it turns left (resp. turns right) in N at its endpoints.

In the following lemmas, it is helpful to work in the universal cover of the annulus. Recall that A is an annulus $S^1 \times [0, 1]$ embedded in the (θ, r)–plane by the map $(\theta, t) \to (\theta, t + 1)$. Then $B = [0, 1] \times \mathbb{R} \subset \mathbb{R}^2$ is the universal cover of A with covering map $\psi \colon B \to A$ given by $\psi(x, y) = (2\pi y, x + 1)$. Note that we have flipped the factors between A and B as this allows for a natural relation between an increase in (left) twisting in A and an increase of slope in B. The left twist map $T \colon A \to A$ induces a homeomorphism $\tilde{T} \colon B \to B$ given by $\tilde{T}(x, y) = (x, y + x)$ (see Figure 3) so that the following diagram commutes:

$$
\begin{array}{ccc}
B & \xrightarrow{\ \tilde{T}\ } & B \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\
A & \xrightarrow{\ T\ } & A
\end{array}
$$

If N is an embedded annulus in Σ, we can compose ψ with an orientation-preserving homeomorphism $\phi \colon A \to N$ to consider B as the universal cover of N. Choose ϕ so that the I–fibers of N lift to horizontal arcs in B. A properly embedded arc in N that intersects the I–fibers of N efficiently can be isotoped relative to its endpoints so that it lifts to a collection of straight line segments in B, each with the same slope, that differ by a vertical translation of an integral distance. Applying the induced left twist map $\tilde{T}$ (resp. right twist map) to a straight line segment in B then increases (resp. decreases) the slope by 1.

Now let $X = \{x_1, x_2, \ldots, x_s\}$ be a collection of pairwise disjoint simple closed curves on Σ that intersect N in I–fibers. Suppose $n = |N \cap X|$ and then, after possibly
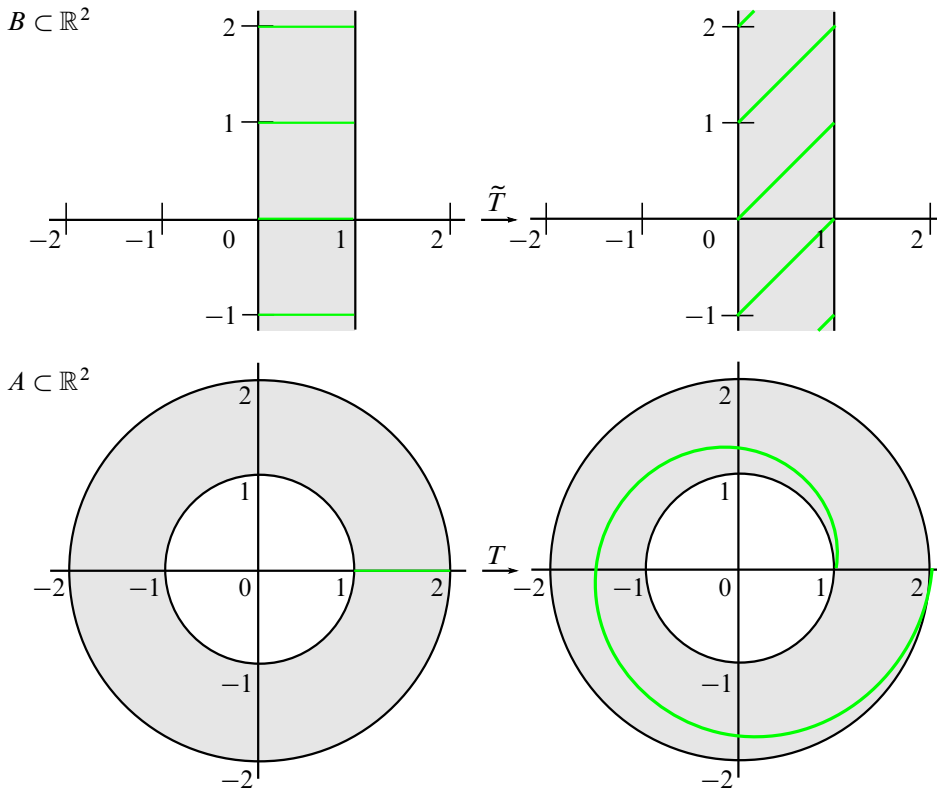
Figure 3: The left twist map T on the annulus A and the induced map $\tilde{T}$ on $B = [0, 1] \times \mathbb{R}$

rechoosing ϕ, the collection of all lifts of the arcs of $N \cap X$ are exactly the horizontal arcs in B of the form $[0, 1] \times \frac{z}{n}$ for all $z \in \mathbb{Z}$. Then a properly embedded straight line segment α in B with endpoints disjoint from lifts of $N \cap X$ and with slope m projects to a properly embedded arc in N with twisting number $\frac{a}{n}$ with respect to X, where $a \in \mathbb{Z}$ is chosen so that $|a|$ is the total number of intersections between α and lifts of $N \cap X$, and a and m share the same sign (see Figure 4).

With this perspective we can make the following observations.

Lemma 3.8 Let $X = \{x_1, x_2, \ldots, x_s\}$ be a collection of pairwise disjoint curves in Σ. Suppose N is an annulus in Σ that is fibered with respect to X. Let c_1 and c_2 be disjoint properly embedded arcs in N with twisting numbers m_1 and m_2 in N respectively with respect to X. Then $|m_1 - m_2| \leq 1$.

Proof Let $n = |N \cap X|$ and let $B = [0, 1] \times \mathbb{R}$ denote the universal cover of N with covering map as defined previously so that the lifts of the arcs $N \cap X$ is the
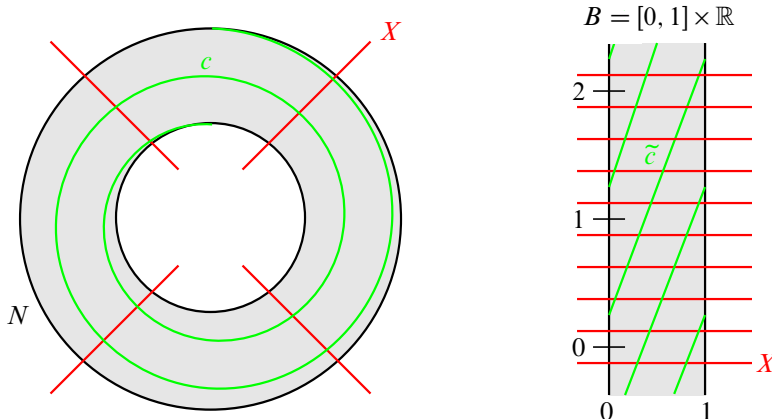
Figure 4: An arc c with twisting number 2 with respect to X lifts to arcs with slope 2 in $B = [0, 1] \times \mathbb{R}$.

set $\{[0, 1] \times \frac{z}{n} | z \in \mathbb{Z}\}$. Then c_1 and c_2 can be isotoped so that the collection of all of their lifts, $\tilde{c_1}$ and $\tilde{c_2}$ respectively, are a collection of linear arcs in B with slopes equal to their twisting numbers, denoted as m_1 and m_2. Since c_2 is disjoint from c_1, any lift of c_2 must lie in a parallelogram-shaped region R of $B - \tilde{c_1}$. The vertical distance between consecutive lifts of c_1 in B is equal to 1, so the slope m_2 of c_2 must satisfy $m_1 - 1 \leq m_2 \leq m_1 + 1$. $\square$

Lemma 3.9 *Let $X = \{x_1, x_2, \ldots, x_s\}$ be a collection of pairwise disjoint curves in Σ. Suppose y is a simple closed curve in Σ and N is an annular neighborhood of y that is fibered with respect to X. Let c be a properly embedded arc in N with twisting number m with respect to X. Then $\tau_y^k(c)$ has twisting number $m + k$ in N with respect to X.*

Proof Again let $n = |N \cap X|$ and let $B = [0, 1] \times \mathbb{R}$ be the universal cover of N such that the set of all lifts of $N \cap X$ is equal to $\{[0, 1] \times \frac{z}{n} \mid z \in \mathbb{Z}\}$. Then c can be isotoped so that the collection of all of its lifts in B, denoted as $\tilde{c}$, is a collection of linear arcs in B with some slope m. Applying $\tilde{T}^k$ to B sends each linear arc of $\tilde{c}$ to a linear arc with slope $m + k$. Projecting a component of $\tilde{T}^k(\tilde{c})$ back down to N yields a properly embedded arc in N that agrees with $\tau_y^k(c)$ and has twisting number $m + k$ with respect to X. $\square$

The next goal is to define the twisting number for a simple closed curve γ about a simple closed curve y that γ intersects efficiently. This is more complicated than defining the twisting number for an arc in an annulus; given an annular neighborhood N of y, ambient isotopies of γ can modify the twisting number of any component of $\gamma \cap N$.

Let $X = \{x_1, x_2, \ldots, x_s\}$ and $Y = \{y_1, y_2, \ldots, y_t\}$ be collections of pairwise disjoint curves such that the components of X and Y intersect efficiently, and suppose that $N = \{N_1, N_2, \ldots, N_t\}$ is a collection of pairwise disjoint annuli such that each N_i is an annular neighborhood of y_i and N_i is fibered with respect to X.

Definition 3.10 If γ is a simple closed curve that intersects X, Y, ∂N and each I–fiber of N efficiently, then we will say γ is in *efficient position* with respect to (X, Y, N).

We will also require the use of the following helpful fact (see Hass and Scott [5] for a proof).

Lemma 3.11 *Let $Y_1, \ldots, Y_{n-1}$ each be a collection of disjoint essential simple closed curves in Σ so that each pair Y_i, Y_j intersects efficiently. If Y_n is another collection of disjoint essential simple closed curves, then Y_n can be isotoped to intersect each of the other sets efficiently without disturbing the efficient intersection of the others. Moreover, as long as no component of Y_n is parallel to a component of any Y_i, any two embeddings of Y_n that intersect $Y_1, \ldots, Y_{n-1}$ efficiently are isotopic through an isotopy which keeps all intersections efficient (though during the isotopy Y_n may pass over intersection points of Y_i with Y_j).*

For our purposes, Lemma 3.11 implies that as long as γ is not parallel to a component of X or Y, then between any two embeddings of γ that are in efficient position with respect to (X, Y, N), there exists an isotopy f_t (with $t \in [0, 1]$) such that $f_t(\gamma)$ intersects X, Y and ∂N efficiently for all t. Moreover, after perturbing f_t in N, we can assume $f_t(\gamma)$ is in efficient position with respect to (X, Y, N) for all t. Then f_t induces an isotopy f_t^N of Σ that is transverse to $\partial N \cup Y$ and that can restrict to an isotopy on N. Let $c_1, c_2, \ldots, c_r$ denote the components of $\gamma \cap N$ with twisting numbers $m_1, m_2, \ldots, m_r$ respectively in N. We can associate $c_1, c_2, \ldots, c_r$ with their images under f_t^N and then track the corresponding changes in their twisting numbers. Using such isotopies, we want to isotope γ to maximize $\sum_j |m_j|$.

Definition 3.12 Let γ be a simple closed curve that is in efficient position with respect to (X, Y, N). Suppose there exists a triangle E in Σ with a side s_X in X, a side s_N in ∂N, and a side s_γ in γ, such that $\overset{\circ}{E}$ is disjoint from N. Then the triangle E will be called an *outer triangle of N* (see Figure 5).

Given an outer triangle E of N, we can perform an isotopy of γ supported in an open neighborhood of E that pushes s_γ and all the other arcs of $\overset{\circ}{E} \cap \gamma$ into N (see Figure 6)
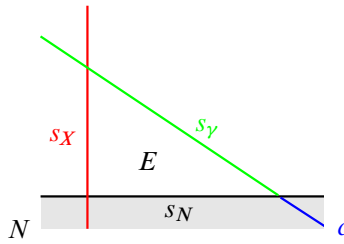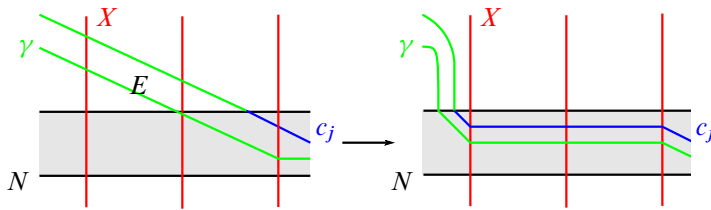
Figure 5: E is an outer triangle of N.

Figure 6: An isotopy of γ across an outer triangle E of N that increases $|m_j|$

such that γ will be in efficient position with respect to (X, Y, N) throughout the isotopy. Note that this isotopy increases $\sum_j |m_j|$.

As $|\gamma \cap X|$ is finite, a finite number of isotopies of γ across outer triangles of N will yield an embedding of γ in efficient position with respect to (X, Y, N) that admits no such outer triangles. We claim that for such embeddings, $\sum_j |m_j|$ is maximal over all representatives of the isotopy class of γ that are in efficient position with respect to (X, Y, N).

Proposition 3.13 *Suppose γ is a simple closed curve in efficient position with respect to (X, Y, N) and γ is not parallel to a component of Y. Let $c_1, c_2, \ldots, c_r$ denote the components of $\gamma \cap N$ and let m_j be the twisting number of c_j. Then $\sum_j |m_j|$ is maximal over all embeddings of γ that are in efficient position with respect to (X, Y, N) if and only if γ admits no outer triangles of N.*

Proof Note that in the case that γ is parallel to a component of X, then γ admits no outer triangles (since it is disjoint from X) and $\sum_j |m_j| = 0$ is maximal.

If γ is an embedding in efficient position with respect to (X, Y, N) such that $\sum_j |m_j|$ is maximal, then γ admits no outer triangles, as otherwise an isotopy across such a triangle would increase $\sum_j |m_j|$.

Conversely, suppose that γ admits no outer triangles of N, but $\sum_j |m_j|$ is not maximal. Then there exists an isotopy f_t of γ that increases $\sum_j |m_j|$. Moreover, by Lemma 3.11 we can assume $f_t(\gamma)$ intersects X, Y and ∂N efficiently for all $t \in [0, 1]$. This implies f_t induces isotopies f_t^X and f_t^N which are transverse to X and $Y \cup \partial N$ respectively such that $f_t^X(\gamma) = f_t^N(\gamma) = f_t(\gamma)$ for all t.

Since f_t increases $\sum_j |m_j|$, there exists a point $p \in \gamma \cap X$ such that p lies off of N, but $f_1^X(p)$ lies in N. Let t_* denote the smallest value such that $f_{t_*}^X(p) \in \partial N$. As $f_{t_*}^X(p)$ lies in ∂N, $f_{t_*}^X(p) = f_{t_*}^N(q)$ for some point $q \in \gamma \cap \partial N$.

Since we chose t_* to be the smallest value such that $f_{t_*}^X(p) \in \partial N$, there exists a subarc s of γ connecting p to q such that $\mathring{s}$ is disjoint from N. Then as f_t^X pushes p along X across ∂N and f_t^N pushes q along ∂N across X, s must form the side of an outer triangle E of N, where the other two sides of E consist of a subarc of X connecting p to $f_{t_*}^X(p)$ and a subarc of ∂N connecting q to $f_{t_*}^N(q)$. This contradicts our assumption that γ admitted no such outer triangles of N, and hence $\sum_j |m_j|$ is maximal.   □

We can now provide the definition of twisting number about a simple closed curve.

Definition 3.14 Let y be a simple closed curve that intersects X efficiently and N be an annular neighborhood of y that is fibered with respect to X. Suppose γ is an essential simple closed curve that intersects y nontrivially. Then define the *twisting number of γ in N with respect to X*, denoted as $tw(\gamma, N, X)$, to be the maximum twisting number of a component of $\gamma \cap N$ with respect to X over all representatives of the isotopy class of γ that are in efficient position with respect to (X, y, N) and admit no outer triangles of N.

We require γ to intersect X efficiently as otherwise there would be no maximum to the twisting number of components of $\gamma \cap N$. We include the assumption that γ does not admit outer triangles of N as otherwise we could push any "negative twisting" of γ about y to lie outside of N and therefore $tw(\gamma, N, X)$ would always be nonnegative.

Suppose that N' is another choice of annular neighborhood of y that is fibered with respect to X. Then there exists an isotopy g_t that sends N to N' such that I–fibers of N are sent to I–fibers of N' and X is fixed. In particular, this means for any component c of $\gamma \cap N$, $g_1(c)$ is a component of $\gamma \cap N'$. Moreover, if γ is in efficient position with respect to (X, y, N) and does not admit any outer triangles of N, then $g_1(\gamma)$ is in efficient position with respect to (X, y, N') and does not admit any outer triangles of N'. Hence we can conclude that $tw(\gamma, N, X) = tw(\gamma, N', X)$ and twisting

number is independent of the choice of annular neighborhood of y. We will thus denote the twisting number as $tw(\gamma, y, X)$.

Given our definition of twisting number and Lemma 3.9, one would expect there to be a natural relationship between $tw(\gamma, y, X)$ and $tw(\tau_y(\gamma), y, X)$. If an embedding of γ admits no outer triangles of N, by the following lemma we can then obtain an embedding of $\tau_y^k(\gamma)$ via arc replacements of the components of $\gamma \cap N$ that, for k sufficiently large, continues to intersect X efficiently. This result and its corollary are integral to proving Lemma 3.18, which provides the foundation for our later results.

Lemma 3.15 *We continue to define X, Y and N as before (Y is allowed to have multiple components). Suppose γ is a simple closed curve in efficient position with respect to (X, Y, N) and admits no outer triangles of N. Denote the components of $\gamma \cap N$ as $c_1, c_2, \ldots, c_r$ and let m_j be the twisting number of c_j in the component of N that contains c_j. Let $k \in \mathbb{Z}$ such that either for all j, $k + m_j \geq 0$ or for all j, $k + m_j \leq 0$. Then there exists an embedding of $\tau_Y^k(\gamma)$ in efficient position with respect to (X, Y, N) that admits no outer triangles of N and the components of $\tau_Y^k(\gamma) \cap N$ consist of arcs $\{c_1', c_2', \ldots, c_r'\}$, where c_j' has twisting number $m_j + k$ with respect to X for each j.*

Proof Let $\gamma' = \tau_Y^k(\gamma)$. By Lemma 3.9, we can obtain an embedding of γ' by fixing γ off of N and replacing each component c_j of $\gamma \cap N$ with a properly embedded arc c_j' in N that has twisting number $m_j + k$. We replace these arcs carefully so that this embedding of γ' intersects Y and each I–fiber of N efficiently. As γ intersects ∂N efficiently, this embedding of γ' will as well. So it remains to show that this embedding of γ' intersects X efficiently.

Suppose otherwise. Then there exists a component b' of $\gamma' - X$ that cobounds a bigon D with a subarc of some $x_\ell \in X$. As γ intersects X efficiently, b' cannot be contained in the complement of N since it would then coincide with γ. Moreover, as the components of $\gamma' \cap N$ intersects each I–fiber of N efficiently, b' cannot be contained in N. So b' and ∂N must intersect at least once and we can consider the two distinct components β_1 and β_2 of $b' - \partial N$ which connect x_ℓ with ∂N and have interiors disjoint from ∂N.

Suppose the interior of β_1 lies off of N. Then β_1 forms a side of an outer triangle of N. However, as this embedding of γ' agrees with γ off of N, γ' also does not admit any outer triangles of N. Hence we have a contradiction and β_1 must instead be contained in N. Repeating the argument with β_2 implies β_2 must also lie in N.

So both β_1 and β_2 are subarcs of components of $\gamma' \cap N$. By hypothesis, k was chosen so every component of $\gamma' \cap N$ has twisting number with the same sign. On the other

hand, the arcs of $\partial N \cap D$ must have one endpoint on x_ℓ and one endpoint on b', as otherwise ∂N would not intersect x_ℓ or b' efficiently. This means that β_1 and β_2 have twisting numbers of opposite sign in N (see Figure 7), a contradiction.

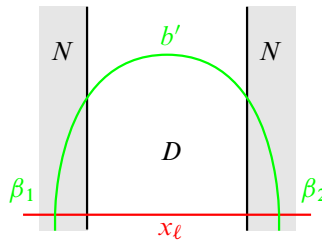Hence γ' must have efficient intersection with X. □

Figure 7: In this case, β_1 has positive twisting number in N and β_2 has negative twisting number in N.

In particular, Lemma 3.15 provides a lower bound for the twisting number of $\tau_y^k(\gamma)$ about y with respect to X for appropriately chosen k.

Corollary 3.16 Let X, Y, N and γ be as in Lemma 3.15. Let $m_1, m_2, \ldots, m_r$ be the twisting numbers of the components of $\gamma \cap N$ and choose $k \in \mathbb{Z}$ so that for all j, $k + m_j \geq 0$. If a component of $\gamma \cap N_\ell$ has twisting number equal to m, for some $N_\ell \in N$, then $tw(\tau_Y^k(\gamma), y_\ell, X) \geq m + k$.

Proof Suppose c is the component of $\gamma \cap N_\ell$ with twisting number equal to m in N_ℓ. Then by Lemma 3.15, we obtain an embedding of $\tau_Y^k(\gamma)$ that is in efficient position with respect to (X, Y, N) by replacing c with an arc c' in N_ℓ that has twisting number equal to $m + k$. Hence $m + k$ is a lower bound for $tw(\tau_Y^k(\gamma), y_\ell, X)$. □

We now replace the arbitrary collection of pairwise disjoint curves X in Σ with a pants decomposition $\mathcal{P}$ of Σ. The twisting number of γ around a 1–seamed curve y can then provide a lower bound on the number of seams γ contains by the following natural relationship.

Remark 3.17 Let $\mathcal{P}$ be a pants decomposition of Σ and y a simple closed curve that is 1–seamed with respect to $\mathcal{P}$. If γ is a simple closed curve such that $|tw(\gamma, y, \mathcal{P})| > k$, then γ is k–seamed with respect to $\mathcal{P}$.

When applied in this setting, Corollary 3.16 gives the following result.

Lemma 3.18 *Let H be a handlebody with $\partial H = \Sigma$ and $\mathcal{P}$ a pants decomposition for H. Let $Y = \{y_1, y_2, \ldots, y_t\}$ be a collection of pairwise disjoint curves in Σ such that $\mathcal{P}$ and Y intersect efficiently and y_i is 1–seamed with respect to $\mathcal{P}$ for each i. Suppose γ is a simple closed curve that intersects $\mathcal{P}$ and Y efficiently. If γ bounds a disk in H, then for every $y_i \in Y$ and $k \geq 1$,*

$$tw(\tau_Y^k(\gamma), y_i, \mathcal{P}) \geq k.$$

Proof Let $N = \{N_1, N_2, \ldots, N_t\}$ be a collection of pairwise disjoint annuli such that N_i contains y_i and N_i is fibered with respect to $\mathcal{P}$. Isotope γ to be in efficient position with respect to $(\mathcal{P}, Y, N)$ and so it admits no outer triangles of N. Let $c_1, c_2, \ldots, c_r$ be the components of $\gamma \cap N$ with twisting numbers $m_1, m_2, \ldots, m_r$ respectively.

By Lemma 2.10, γ is either parallel to a component of $\mathcal{P}$ or contains a wave of a component of $\Sigma - \mathcal{P}$. If γ is parallel to a component of $\mathcal{P}$, then for every $y_i \in Y$, y_i intersects γ and $\gamma \cap N_i$ consists of arcs with twisting number 0. Therefore, for $k \geq 1$, Corollary 3.16 implies $tw(\tau_Y^k(\gamma), y_i, \mathcal{P}) \geq k$.

So suppose γ instead contains a wave w of a component of $\Sigma - \mathcal{P}$. Then since every 1–seamed curve must intersect w, each y_i intersects w and moreover $w \cap N_i$ consists of properly embedded arcs in N_i with twisting number 0. This implies $m_1, m_2, \ldots, m_r$ are all greater than or equal to -1 by Lemma 3.8. So, for $k \geq 1$, Corollary 3.16 implies $tw(\tau_Y^k(\gamma), y_i, \mathcal{P}) \geq k$. □

4 Heegaard splittings and Hempel distance

Definition 4.1 A collection of pairwise disjoint essential simple closed curves Z in Σ is a *full set of curves* if $\Sigma - Z$ is a collection of punctured spheres.

In particular, both a standard cut system and a pants decomposition of Σ are full.

Suppose Z is a full set of curves in Σ. Consider the 3–manifold obtained from $\Sigma \times I$ by attaching 2–handles to $\Sigma \times \{1\}$, one along each curve in Z. Since $\Sigma - Z$ consists of a collection of punctured spheres, the boundary of the 3–manifold consists of $\Sigma \times \{0\}$ together with a collection of spheres. Attaching a 3–ball to each of the spheres creates a handlebody whose boundary is Σ. Denote this handlebody as V_Z, with $\partial V_Z = \Sigma$.

Definition 4.2 A genus g *Heegaard splitting* of a closed 3–manifold M is a decomposition of M into the union of two genus g handlebodies, identified along their boundaries.

Remark 4.3 Let Z and Z' be two full sets of curves in Σ. Then V_Z and $V_{Z'}$ are handlebodies and $M = V_Z \cup_\Sigma V_{Z'}$ is a Heegaard splitting of the closed orientable manifold M.

Given two full sets of curves Z and Z' in Σ such that $\Sigma - Z$ and $\Sigma - Z'$ are homeomorphic (for example, if both Z and Z' are standard cut systems), there exists a surface automorphism of Σ that sends Z to Z' (for more details see [3]). This is the map that identifies the boundary of V_Z to $V_{Z'}$. Conversely, if we start with just Z and h is a surface automorphism of Σ, then we can obtain a 3–manifold with Heegaard splitting $V_Z \cup_\Sigma V_{h(Z)}$. In this way, every surface automorphism of Σ induces a Heegaard splitting of some closed 3–manifold.

Definition 4.4 The *curve complex* for Σ, denoted as $C(\Sigma)$, is the complex whose vertices are isotopy classes of essential simple closed curves in Σ and a set of distinct vertices $\{v_0, v_1, \ldots, v_k\}$ determines a k–simplex if they are pairwise disjoint.

Note that in this paper we only need to consider the 1–skeleton of the curve complex.

Definition 4.5 Given two collections of simple closed curves $\mathcal{Z}$ and $\mathcal{Z}'$ in Σ, the *distance between $\mathcal{Z}$ and $\mathcal{Z}'$*, denoted $d(\mathcal{Z}, \mathcal{Z}')$, is the minimal number of edges in a path in $C(\Sigma)$ between a vertex in $\mathcal{Z}$ and a vertex in $\mathcal{Z}'$.

Definition 4.6 Let H be a handlebody with $\partial H = \Sigma$. Then the *disk complex of H*, denoted as $D(H)$, is the subcomplex of $C(\Sigma)$ determined by all essential simple closed curves that bound embedded disks in H. If Z is a full set of curves in Σ, we will let K_Z denote $D(V_Z)$.

The following definition was first introduced by Hempel [6].

Definition 4.7 Let Z and Z' be full sets of curves in Σ. Then the *Hempel distance*, or *distance*, of the Heegaard splitting $V_Z \cup V_{Z'}$ is equal to $d(K_Z, K_{Z'})$.

We now introduce the useful term "diskbusting."

Definition 4.8 Let H be a handlebody with $\partial H = \Sigma$. A collection of pairwise disjoint essential curves $Y = \{y_1, y_2, \ldots, y_t\}$ is *diskbusting* on H if every meridian of H intersects some $y_i \in Y$.

Observe that a simple closed curve y is diskbusting on H if and only if $d(D(H), y) \geq 2$. On the other hand, there exist collections of curves Y such that Y is diskbusting on H and $d(D(H), y_i) = 1$ for all $y_i \in Y$. For an example, see the collection Y_2 in Figure 8.

Definition 4.9 Let X be a standard cut system and $Y = \{y_1, y_2, \ldots, y_t\}$ a collection of essential curves in Σ that intersect X efficiently. Then $\Sigma - X$ is a $2g$–punctured sphere and Y is a collection of properly embedded essential arcs in $\Sigma - X$. Obtain a planar graph $\Gamma_X(Y)$ by having each puncture of $\Sigma - X$ correspond to a vertex and each arc of Y in $\Sigma - X$ represent an edge.

The graph $\Gamma_X(Y)$ is known as the *Whitehead graph* (for more details, see Stallings [17]).

Definition 4.10 A graph Γ is 2–connected if it is connected and Γ does not contain a vertex whose removal would disconnect the graph (ie Γ does not contain a cut vertex).

The following theorems characterize diskbusting sets of curves on the surface of a handlebody and were first proven by Starr [18]. Alternate proofs have been given by Wu [21, Theorem 1.2], Strong [19, Theorem 3] and Luo [11, Theorem 3.1].

Theorem 4.11 (Starr [18, Theorem 1]) *Let H be a handlebody with outer boundary Σ. Suppose $Y = \{y_1, y_2, \ldots, y_t\}$ is a collection of essential simple closed curves in Σ. Then Y is diskbusting on H if and only if there exists a standard set of meridians X of H such that $\Gamma_X(Y)$ is 2–connected.*

Theorem 4.12 (Starr [18, Theorem 2]) *Let H be a handlebody with outer boundary Σ. Suppose $Y = \{y_1, y_2, \ldots, y_t\}$ is a collection of essential simple closed curves in Σ. Then Y is diskbusting on H if and only if there exists a pants decomposition $\mathcal{P}$ of H such that Y is collectively 1–seamed with respect to $\mathcal{P}$.*

Note that in the proof of Theorem 4.12, Starr proved that if Y is a collection of essential simple closed curves such that $\Gamma_X(Y)$ is 2–connected (and therefore Y is diskbusting), then there exists a pants decomposition $\mathcal{P}$ of H such that $X \subset \mathcal{P}$ and Y is collectively 1–seamed with respect to $\mathcal{P}$.

The following definition is a generalization of the notion of diskbusting to distance 3. The term "nearly fills" was chosen since "almost fills" is already used by Hempel in [6] for an alternate purpose.

Definition 4.13 Suppose Y is a finite set of disjoint essential simple closed curves in the boundary Σ of a handlebody H. Then Y *nearly fills* H if for every meridian d of H, the union of d and Y fills Σ. That is, $\Sigma - (Y \cup d)$ is a collection of 2–cells.
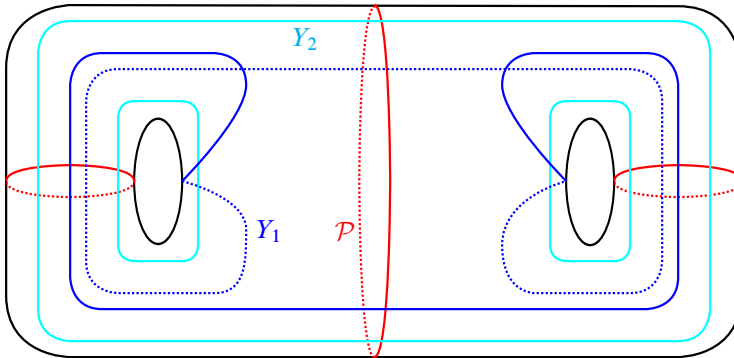
Figure 8: If $\mathcal{P}$ is a pants decomposition of H, then Y_1 (a single curve) and Y_2 (a collection of 3 curves) are both diskbusting on H.

Proposition 4.14 *Let H be a handlebody with $\partial H = \Sigma$. Then*

(1) *a finite collection Y of pairwise disjoint essential curves in Σ nearly fills H if and only if for every essential curve γ that is disjoint from Y, $d(D(H), \gamma) \geq 2$;*

(2) *a single simple closed curve y in Σ nearly fills H if and only if $d(D(H), y) \geq 3$.*

Proof We begin with the proof of (1). Suppose Y nearly fills H. Let β be an essential curve disjoint from Y and d an arbitrary meridian of H. If β is disjoint from d, then $d \cup Y$ does not fill Σ, which contradicts our assumption. Hence β must intersect every meridian of H and therefore $d(D(H), \gamma) \geq 2$.

Conversely, suppose Y is a collection of pairwise disjoint essential simple closed curves with the property that any essential curve γ disjoint from Y satisfies $d(D(H), \gamma) \geq 2$. Let d be an arbitrary meridian of H. If $d \cup Y$ does not fill Σ, then there exists some essential curve β in Σ that lies disjoint from d and Y. Since $d \in D(H)$, $d(D(H), \beta) < 2$ and we have a contradiction.

For the proof of (2), suppose y is a single simple closed curve that nearly fills H. By part (1), any curve γ disjoint from y satisfies $d(D(H), \gamma) \geq 2$. Hence the length of any path between y to $D(H)$ must be at least 3. □

5 Main theorems

Suppose H is a handlebody with $\partial H = \Sigma$ and consider the Heegaard splitting induced by the identity map on Σ. The resulting 3–manifold is homeomorphic to $\#_g(S^2 \times S^1)$ and the Heegaard splitting has distance 0, as the disk sets of the two handlebodies are identical. Hempel [6] replaced the identity map with a high power of a pseudo-Anosov

map to prove the existence of Heegaard splittings with arbitrarily high distance. We give a similar result by providing a lower bound on the distance of the resulting Heegaard splitting when the identity map is replaced with a high power of a Dehn twist map. As shown later in Lemma 5.4, Dehn twisting about a curve γ that is distance d from $D(H)$ determines a Heegaard splitting of distance at most $2d - 2$. We show that for a sufficiently high power of Dehn twists about γ the resulting Heegaard splitting is exactly distance $2d - 2$.

We first require the following utility lemma, which is a generalization of Lemma 3.8.

Lemma 5.1 *Let $X = \{x_1, x_2, \ldots, x_s\}$ be a collection of pairwise disjoint curves in Σ. Suppose y is a simple closed curve in Σ with annular neighborhood N that is fibered with respect to X. Let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be a collection of essential simple closed curves such that each intersects y, each is in efficient position with respect to (X, y, N), and α_j and α_{j+1} are disjoint for $0 \le j \le n - 1$. Assume a component of $\alpha_n \cap N$ has twisting number equal to m_n with respect to X. If m_i is the twisting number of a component of $\alpha_i \cap N$ for some $0 \le i \le n - 1$, then $|m_n - m_i| \le |n - i|$.*

Proof Let b_n be the component of $\alpha_n \cap N$ that is assumed to have twisting number m_n. For a fixed i such that $0 \le i \le n - 1$, let m_i be the twisting number of a component of $\alpha_i \cap N$. Then for $i < j < n$, there exists some component b_j of $\alpha_j \cap N$ that is a properly embedded arc in N since each α_j nontrivially intersects y. Let m_j denote the twisting number of b_j in N with respect to X. Then by Lemma 3.8, as b_{j+1} and b_j are disjoint we have that $|m_{j+1} - m_j| \le 1$ for each j. So by the triangle inequality, $|m_n - m_i| \le |n - i|$. $\square$

We now prove an extension of a result of Casson and Gordon [1], whose result is the following.

Theorem 5.2 (Casson and Gordon [1]) *Let H be a handlebody with $\partial H = \Sigma$ and γ a simple closed curve such that $d(D(H), \gamma) \ge 2$. Let $\mathcal{P}$ be a pants decomposition of H (so $K_\mathcal{P} = D(H)$) such that γ is 1–seamed with respect to $\mathcal{P}$. Then for $k \ge 2$, $d(K_\mathcal{P}, K_{\tau_\gamma^k(\mathcal{P})}) \ge 2$.*

The above result of Casson and Gordon can be proved by showing that Dehn twisting at least twice about a 1–seamed curve will yield a Heegaard diagram that satisfies the Casson–Gordon rectangle condition (see [8, Example 3] for details).

Note that if $d(D(H), \gamma) \le 1$, then there exists some meridian α in H that is disjoint from γ, and therefore fixed by τ_γ. Consequently, the Heegaard splitting induced by τ_γ^k

will always have distance 0. So if we are interested in creating Heegaard splittings of nonzero distance, we need to assume that $d(D(H), \gamma) \geq 2$. Then by Theorem 4.12, there is a pants decomposition $\mathcal{P}$ of H such that γ is 1–seamed with respect to $\mathcal{P}$.

Lemma 5.3 *Let H be a handlebody with $\partial H = \Sigma$ and γ a simple closed curve such that $d(D(H), \gamma) \geq d$ for $d \geq 2$. Let $\mathcal{P}$ be a pants decomposition of H (so $K_{\mathcal{P}} = D(H)$) such that γ is 1–seamed with respect to $\mathcal{P}$. Then*

$$d(K_{\mathcal{P}}, K_{\tau_\gamma^k(\mathcal{P})}) \geq \begin{cases} k & \text{if } 2 \leq k \leq 2d - 2, \\ 2d - 2 & \text{if } k \geq 2d - 2. \end{cases}$$

Proof   To simplify notation, let $\mathcal{P}' = \tau_\gamma^k(\mathcal{P})$.

Suppose, for the sake of a contradiction, that $d(K_{\mathcal{P}}, K_{\mathcal{P}'}) = \ell$ for some $\ell < k$ and $\ell < 2d - 2$. Then there exists a sequence of simple closed curves $\alpha_0, \alpha_1, \ldots, \alpha_\ell$ such that consecutive curves are disjoint, $\alpha_0 \in K_{\mathcal{P}}$ and $\alpha_\ell \in K_{\mathcal{P}'}$.

Note that $d(K_{\mathcal{P}}, \gamma) = d(K_{\mathcal{P}'}, \gamma)$ since any path in $C(\Sigma)$ between $K_{\mathcal{P}}$ and γ can be sent by τ_γ^k to a path between $K_{\mathcal{P}'}$ and γ (since τ_γ^k fixes γ) and conversely, any path between $K_{\mathcal{P}'}$ and γ can be sent to a path between $K_{\mathcal{P}}$ and γ via τ_γ^{-k}. Therefore $d(K_{\mathcal{P}}, \gamma) \geq d$ implies that $d(K_{\mathcal{P}'}, \gamma) \geq d$.

Since $\ell < 2d - 2$, $j \leq d - 2$ or $\ell - j \leq d - 2$ for any j such that $0 \leq j \leq \ell$. Consequently, α_j is a distance at most $d - 2$ from at least one of α_0 or α_ℓ in $C(\Sigma)$. As $\alpha_0 \in K_{\mathcal{P}}$ and $\alpha_\ell \in K_{\mathcal{P}'}$, we have that $d(K_{\mathcal{P}}, \alpha_j) \leq d - 2$ or $d(K_{\mathcal{P}'}, \alpha_j) \leq d - 2$. Then γ being at least distance d from both $K_{\mathcal{P}}$ and $K_{\mathcal{P}'}$ implies that α_j must be at least distance 2 from γ. Hence, α_j intersects γ for each $0 \leq j \leq \ell$.

As α_ℓ is a component of $K_{\mathcal{P}'}$, $\alpha_\ell = \tau_\gamma^k(\beta)$ for some $\beta \in K_{\mathcal{P}}$. Lemma 3.18 then implies $tw(\alpha_\ell, \gamma, \mathcal{P}) \geq k$. Let N be an annular neighborhood of γ that is fibered with respect to $\mathcal{P}$ and assume $\alpha_0, \alpha_1, \ldots, \alpha_\ell$ have all been isotoped so that they are each in efficient position with respect to $(\mathcal{P}, \gamma, N)$, intersect each other efficiently, and a component of $\alpha_\ell \cap N$ has twisting number greater than or equal to k in N with respect to $\mathcal{P}$. Then by Lemma 5.1, each component of $\alpha_1 \cap N$ must have a twisting number of at least $k - (\ell - 1) \geq 2$ in N with respect to $\mathcal{P}$. So α_1 is 1–seamed with respect to $\mathcal{P}$ (since γ is 1–seamed) and therefore $d(K_{\mathcal{P}}, \alpha_1) \geq 2$ by Theorem 4.12. But this is a contradiction as $d(K_{\mathcal{P}}, \alpha_1) = 1$. Hence, $d(K_{\mathcal{P}}, K_{\mathcal{P}'}) \geq \min\{k, 2d - 2\}$.   □

One of the key arguments needed in the proof of Lemma 5.3 is confirming that, for any sequence $\alpha_0, \alpha_1, \ldots, \alpha_\ell$ of simple closed curves that form a path in $C(\Sigma)$ between $K_{\mathcal{P}}$ and $K_{\tau_\gamma^k(\mathcal{P})}$ of length less than $2d - 2$, each α_j must intersect γ. This allows us to use Lemma 5.1 to show that the twisting numbers of each α_j about γ are dependent

on k. In particular, their twisting numbers will increase as k increases. Therefore, we would expect that for sufficiently large values of k, the shortest path between $K_{\mathcal{P}}$ and $K_{\tau_\gamma^k(\mathcal{P})}$ must include a curve that is disjoint from γ. The following lemma shows it is not hard to construct a path of length $2d-2$ between $K_{\mathcal{P}}$ and $K_{\tau_\gamma^k(\mathcal{P})}$ that indeed includes such a disjoint curve.

Lemma 5.4 *Let $\mathcal{P}$ be a pants decomposition of Σ. Suppose γ is a simple closed curve such that $d(K_{\mathcal{P}}, \gamma) = d$ for some $d \geq 1$. Then for any $k \in \mathbb{Z}$, $d(K_{\mathcal{P}}, K_{\tau_\gamma^k(\mathcal{P})}) \leq 2d-2$.*

Proof For the duration of this proof, let $\mathcal{P}' = \tau_\gamma^k(\mathcal{P})$.

If $d(K_{\mathcal{P}}, \gamma) = 1$, there exists some $\alpha \in K_{\mathcal{P}}$ that is disjoint from γ. Then τ_γ fixes α and therefore $\alpha \in K_{\mathcal{P}'}$ and $d(K_{\mathcal{P}}, K_{\mathcal{P}'}) = 0$.

So suppose $d > 1$. By definition, $d(K_{\mathcal{P}}, \gamma) = d$ implies there exists a sequence of simple closed curves $\alpha_0, \alpha_1, \ldots, \alpha_d$ such that $\alpha_j \cap \alpha_{j+1} = \varnothing$ for $0 \leq j < d$, $\alpha_0 \in K_{\mathcal{P}}$ and $\alpha_d = \gamma$.

As α_{d-1} is disjoint from both γ and α_{d-2}, α_{d-1} is also disjoint from $\tau_\gamma^k(\alpha_{d-2})$. We then have the following path in $C(\Sigma)$ of $2d-2$ curves between $K_{\mathcal{P}}$ and $K_{\mathcal{P}'}$ (see Figure 9):

$$\alpha_0, \ \alpha_1, \ \ldots, \ \alpha_{d-2}, \ \alpha_{d-1}, \ \tau_\gamma^k(\alpha_{d-2}), \ \ldots, \ \tau_\gamma^k(\alpha_1), \ \tau_\gamma^k(\alpha_0).$$

Hence $d(K_{\mathcal{P}}, K_{\mathcal{P}'}) \leq 2d-2$. $\qquad\square$
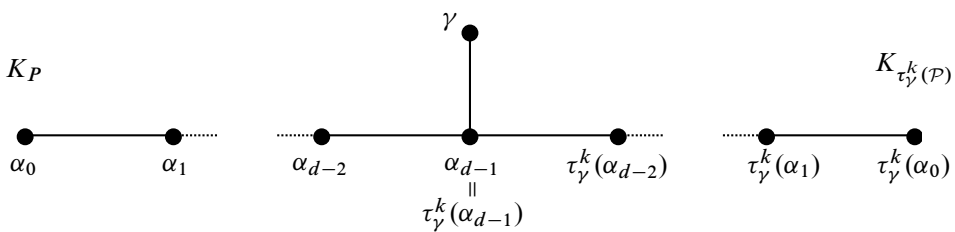

Figure 9: Path in $C(\Sigma)$ between $K_{\mathcal{P}}$ and $K_{\mathcal{P}'}$

Combining the above two lemmas then gives the following result.

Main Theorem 1 *Let H be a handlebody with $\partial H = \Sigma$ and γ a simple closed curve such that $d(D(H), \gamma) = d$ for $d \geq 2$. Let $\mathcal{P}$ be a pants decomposition of H (so $K_{\mathcal{P}} = D(H)$) such that γ is 1–seamed with respect to $\mathcal{P}$. Then for $k \geq 2d-2$, $d(K_{\mathcal{P}}, K_{\tau_\gamma^k(\mathcal{P})}) = 2d-2$.*

Proof By Lemma 5.3, for $k \geq 2d - 2$, $d(K_{\mathcal{P}}, K_{\tau_\gamma^k(\mathcal{P})}) \geq 2d - 2$. On the other hand, Lemma 5.4 shows that Dehn twisting about a curve γ such that $d(K_{\mathcal{P}}, \gamma) = d$ implies that $d(K_{\mathcal{P}}, K_{\tau_\gamma^k(\mathcal{P})}) \leq 2d - 2$. Hence we achieve the desired equality.   □

If we started with a simple closed curve γ such that $d(D(H), \gamma) = 3$, the above theorem shows that for $k \geq 4$, $V_{\mathcal{P}} \cup_\Sigma V_{\tau_\gamma^k(\mathcal{P})}$ is a 3–manifold that admits a Heegaard splitting of distance exactly 4. While we lack the nice characterization of distance 3 curves that the work of Starr [18] (Theorems 4.11 and 4.12) and others provide for diskbusting curves, some distance greater than or equal to 3 Heegaard splitting criteria like those provided in [15; 6] can be adapted to help identify distance 3 curves. So suppose we have identified such a curve γ and that $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are simple closed curves that form a minimum length path in $C(\Sigma)$ such that $\alpha_0 \in D(H)$ and $\alpha_3 = \gamma$. Then Main Theorem 1 shows that Dehn twisting about γ a total of 4 times will produce a distance 4 Heegaard splitting. However, the arguments of Lemmas 5.3 and 5.4 also show that $d(D(H), \tau_\gamma^k(\alpha_0)) = 4$ for $k \geq 4$. So if we let $\beta = \tau_\gamma^k(\alpha_0)$ and consider Dehn twisting about β, we can produce a distance 6 Heegaard splitting and also $d(D(H), \tau_\beta^{k'}(\alpha_0)) = 6$ for $k' \geq 6$. Repeating this process allows us to produce Heegaard splittings of arbitrarily high distance that have a known exact distance.

Even if one was to obtain a curve with distance exactly d for any d, Main Theorem 1 will only produce Heegaard splittings of even distance. See [7; 14] for constructions of Heegaard splittings with exactly odd distance.

As the examples from Main Theorem 1 are obtained by Dehn twisting about a single curve, the following result of Evans implies these resulting 3–manifolds are all Haken.

Theorem 5.5 (Evans [2, Corollary 3.5])   *Let H be a handlebody with $\partial H = \Sigma$ and Z a collection of pairwise disjoint simple closed curves in Σ that bound disks in H. Suppose $\mathcal{P}$ is a pants decomposition of H (so $H = V_{\mathcal{P}}$). If Z is not a full set of curves, then for any nonzero k, the closed 3–manifold $V_{\mathcal{P}} \cup_\Sigma V_{\tau_Z^k(\mathcal{P})}$ is Haken.*

The manifolds generated by Main Theorem 1 can be considered the result of $(\frac{1}{k})$-Dehn surgery along a curve γ in the double of a handlebody. Casson and Gordon [1] generalized their Theorem 5.2 to the case where Dehn surgery is performed in a manifold admitting a weakly reducible (distance less than or equal to 1) Heegaard splitting. Their result can be restated as the following (see the appendix of Moriah and Schultens [13] for a proof).

Theorem 5.6 (Casson and Gordon [1]) *Let H_1 and H_2 be handlebodies such that $\partial H_1 = \partial H_2 = \Sigma$ and $d(D(H_1, D(H_2)) \leq 1$. Suppose γ is a simple closed curve such that $d(D(H_1, \gamma)) \geq 2$ and $d(D(H_2, \gamma)) \geq 2$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be pants decompositions of H_1 and H_2 respectively (so $K_{\mathcal{P}_1} = D(H_1)$ and $K_{\mathcal{P}_2} = D(H_2)$) such that γ is 1–seamed with respect to both $\mathcal{P}_1$ and $\mathcal{P}_2$. Then for $k \geq 6$, $d(K_{\mathcal{P}_1}, K_{\tau_\gamma^k(\mathcal{P}_2)}) \geq 2$.*

We can extend this result of Casson and Gordon with the following lemma.

Lemma 5.7 *Let H_1 and H_2 be handlebodies satisfying $\partial H_1 = \partial H_2 = \Sigma$ and let $n = \max\{1, d(D(H_1), D(H_2))\}$. Suppose d_1, d_2 are integers such that $d_1, d_2 \geq 2$ and $d_1 + d_2 - 2 > n$ and let γ be a simple closed curve such that $d(D(H_1), \gamma) \geq d_1$ and $d(D(H_2), \gamma) \geq d_2$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be pants decompositions of H_1 and H_2 respectively (so $K_{\mathcal{P}_1} = D(H_1)$ and $K_{\mathcal{P}_2} = D(H_2)$) such that γ is 1–seamed with respect to both $\mathcal{P}_1$ and $\mathcal{P}_2$. Then*

$$d(K_{\mathcal{P}_1}, K_{\tau_\gamma^{k+n+2}(\mathcal{P}_2)}) \geq \begin{cases} k & \text{if } 2 \leq k \leq d_1 + d_2 - 2, \\ d_1 + d_2 - 2 & \text{if } k \geq d_1 + d_2 - 2. \end{cases}$$

Proof Assume that $\mathcal{P}_1$, $\mathcal{P}_2$ and γ intersect efficiently. Let $\bar{n} = d(D(H_1), D(H_2))$. Then there exists a sequence of simple closed curves $\alpha_0, \ldots, \alpha_{\bar{n}}$ such that consecutive curves are disjoint, $\alpha_0 \in D(H_1)$ and $\alpha_{\bar{n}} \in D(H_2)$. Suppose N is an annular neighborhood of γ that is fibered with respect to $\mathcal{P}_1$ and $\mathcal{P}_2$ and $\alpha_0, \ldots, \alpha_{\bar{n}}$ are all in efficient position with respect to $(\mathcal{P}_1, \gamma, N)$ and $(\mathcal{P}_2, \gamma, N)$ and also intersect each other efficiently. Let $m_1, m_2, \ldots, m_r$ denote the twisting numbers of the components of $\alpha_{\bar{n}} \cap N$ with respect to $\mathcal{P}_1$.

Claim 1 We have $\max\{|m_1|, \ldots, |m_r|\} \leq n$.

Proof of Claim 1 As $\bar{n} < d_1 + d_2 - 2$, either $j \leq d_1 - 2$ or $\bar{n} - j \leq d_2 - 2$ for all j such that $0 \leq j \leq \bar{n}$. If $j \leq d_1 - 2$, then $d(\alpha_j, D(H_1)) \leq d_1 - 2$. On the other hand, $d(\gamma, D(H_1)) = d_1$, so it follows that $d(\alpha_j, \gamma) \geq 2$. Similarly, the second case would imply that $d(\alpha_j, D(H_2)) \leq d_2 - 2$ and therefore $d(\alpha_j, \gamma) \geq 2$. In either case, α_j intersects γ for each $0 \leq j \leq \bar{n}$.

Suppose, for the sake of a contradiction, that some component of $\alpha_{\bar{n}} \cap \gamma$ has twisting number m^* with respect to $\mathcal{P}_1$ such that $|m^*| > n$. For $\bar{n} \geq 1$, Lemma 5.1 implies the twisting numbers of the components of $\alpha_1 \cap N$ will have absolute value strictly greater than 1. So α_1 would be diskbusting on H_1, which implies that $d(\alpha_1, D(H_1)) \geq 2$, a contradiction. If $\bar{n} = 0$, then $n = 1$ and $|m^*| > 1$ would imply that α_0 is diskbusting, also a contradiction. Hence $\max\{|m_1|, \ldots, |m_r|\} \leq n$. This concludes the proof of Claim 1. □

Let β be an arbitrary meridian of H_2. Isotope β to be in efficient position with respect to both $(\mathcal{P}_1, \gamma, N)$ and $(\mathcal{P}_2, \gamma, N)$, intersects $\alpha_{\bar{n}}$ efficiently, and let $b_1, b_2, \ldots, b_s$ denote the twisting numbers of the components of $\beta \cap N$ with respect to $\mathcal{P}_1$.

Claim 2   We have $\max\{|b_1|, \ldots, |b_s|\} \leq n + 2$.

Proof of Claim 2   Lemma 2.10 implies that any curve that bounds a disk in H_2 is either parallel to a component of $\mathcal{P}_2$ or contains a wave with respect to $\mathcal{P}_2$. It follows that any meridian of H_2 that is in efficient position with respect to $(\mathcal{P}_2, \gamma, N)$ contains a component in N that has twisting number 0 with respect to $\mathcal{P}_2$. Therefore, as $\alpha_{\bar{n}}$ and β are both meridians of H_2, there exists a component of $\alpha_{\bar{n}} \cap N$ and a component of $\beta \cap N$ that are disjoint in N. So by Lemma 3.8, their twisting numbers must differ by at most 1. Moreover, as the components of $\beta \cap N$ are all disjoint, $b_1, b_2, \ldots, b_s$ must all be within 1 of each other. It then follows from Claim 1 that $\max\{|b_1|, \ldots, |b_s|\} \leq n + 2$. This concludes the proof of Claim 2.   □

Claim 2 implies that the lower bound on the twisting number of any component of $\beta \cap N$ is $-(n+2)$. So by Corollary 3.16, for $k \geq 2$, $tw(\tau_\gamma^{k+n+2}(\beta), \gamma, \mathcal{P}_1) \geq k$. As β was an arbitrary meridian of H_2, this implies that every element $\beta' \in K_{\tau_\gamma^{k+n+2}(\mathcal{P}_2)}$ satisfies $tw(\beta', \gamma, \mathcal{P}_1) \geq k$.

Suppose, for the sake of contradiction, that $\zeta_0, \zeta_1, \ldots, \zeta_t$ is a sequence of simple closed curves such that consecutive curves are disjoint, $\zeta_0 \in K_{\mathcal{P}_1}$ and $\zeta_t \in K_{\tau_\gamma^{k+n+2}(\mathcal{P}_2)}$, such that $t < d_1 + d_2 - 2$ and $t < k$. Then $t < d_1 + d_2 - 2$, combined with our assumption that $d(D(H_1), \gamma) \geq d_1$ and $d(D(H_2), \gamma) \geq d_2$, implies that ζ_ℓ intersects γ for each $0 \leq \ell \leq t$. As ζ_t is an element of $K_{\tau_\gamma^{k+n+2}(\mathcal{P}_2)}$, we have $tw(\zeta_t, \gamma, \mathcal{P}_1) \geq k$. So Lemma 5.1 and $t < k$ imply $tw(\zeta_1, \gamma, \mathcal{P}_1) \geq 2$ and therefore ζ_1 is diskbusting on H_1, which is a contradiction. Hence, $d(K_{\mathcal{P}_1}, K_{\tau_\gamma^{k+n+2}(\mathcal{P}_2)}) \geq \min\{k, d_1 + d_2 - 2\}$.   □

We can then combine the above lemma with an upper bound on the distance of the resulting splitting to show that Dehn surgery on a sufficiently complicated curve in a closed 3–manifold can produce a Heegaard splitting with lower and upper bounds on its distance that differ by 2.

Main Theorem 2   *Let H_1 and H_2 be handlebodies with $\partial H_1 = \partial H_2 = \Sigma$ and let $n = \max\{1, d(D(H_1), D(H_2))\}$. Suppose d_1, d_2 are integers such that $d_1, d_2 \geq 2$ and $d_1 + d_2 - 2 > n$ and let γ be a simple closed curve such that $d(D(H_1), \gamma)) = d_1$ and $d(D(H_2), \gamma)) = d_2$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be pants decompositions of H_1 and H_2 respectively (so $K_{\mathcal{P}_1} = D(H_1)$ and $K_{\mathcal{P}_2} = D(H_2)$) such that γ is 1–seamed with respect to both $\mathcal{P}_1$ and $\mathcal{P}_2$. Then for $k \geq d_1 + d_2 - 2$,*

$$d_1 + d_2 - 2 \leq d(K_{\mathcal{P}_1}, K_{\tau_\gamma^{k+n+2}(\mathcal{P}_2)}) \leq d_1 + d_2.$$

Proof The lower bound of $d_1 + d_2 - 2$ is provided by Lemma 5.7. As $d(\gamma, K_{\mathcal{P}_2}) = d_2$ implies that $d(\gamma, K_{\tau_\gamma^{k+n+2}(\mathcal{P}_2)}) = d_2$, we have

$$d(K_{\mathcal{P}_1}, K_{\tau_\gamma^{k+n+2}(\mathcal{P}_2)}) \leq d(K_{\mathcal{P}_1}, \gamma) + d(\gamma, K_{\tau_\gamma^{k+n+2}(\mathcal{P}_2)}) \leq d_1 + d_2$$

by the triangle inequality. □

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