Algebraic & Geometric Topology 14 (2014) 1395–1411

Heegaard splittings of distance exactly n

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In this paper, we show that, for any integers $n \geq 2$ and $g \geq 2$, there exist genus- g Heegaard splittings of compact 3–manifolds with distance exactly n.

57M27; 57M99

1 Introduction

For a closed orientable 3–manifold M, we say that $V_1 \cup_S V_2$ is a *Heegaard splitting* of M if V_1, V_2 are handlebodies such that $M = V_1 \cup V_2$ and $\partial V_1 = \partial V_2 = S$. Heegaard splittings of compact orientable 3–manifolds with nonempty boundaries can be defined similarly, using compression bodies for handlebodies (see Section 2). In [7], Hempel gave the definition of the *distance* of a Heegaard splitting by using the curve complex introduced by Harvey [6] and showed that, for any integer n, there exists some integer m such that the m^{th} power of a pseudo-Anosov map yields a Heegaard splitting of distance at least n by using a construction of Kobayashi [8]. Abrams and Schleimer [1] showed that the distance of the Heegaard splitting grows linearly with respect to m by using a result of Masur and Minsky [12]. Moreover, Evans [3] gave a combinatorial method to construct Heegaard splittings of high distance. The main purpose of this paper is to give an answer to the following question.

Question Given $n \geq 1$ and $g \geq 2$, does there exist a genus- g Heegaard splitting with distance exactly n?

For certain values, there are known examples that answer the above question affirmatively. For example, Berge and Scharlemann [2] showed that there exist 3–manifolds which admit genus-2 Heegaard splittings with distance exactly 3.

In this paper, for each integer $n \geq 2$, we first construct a Heegaard splitting of a compact orientable 3–manifold with nonempty boundary which has distance exactly n.

Published: 7 April 2014 DOI: 10.2140/agt.2014.14.1395

Theorem 1.1 *For any integers $n \geq 2$ and $g \geq 2$, there exists a genus-g Heegaard splitting $C_1 \cup_P C_2$ with distance exactly n, where C_1 and C_2 are compression bodies.*

To prove Theorem 1.1, we give a method of constructing a pair of curves on a closed surface with distance exactly n. In fact, Schleimer [14] gave a method of constructing a pair of curves with distance exactly four on the five-holed sphere by using *subsurface projection maps* defined by Masur and Minsky [13] (for the definition, see Section 2). In Section 4, we mimic the idea of Schleimer to construct a pair of curves with distance exactly n for any positive integer n. By using the pair of curves and the properties of a compression body obtained by adding a 1–handle to $S \times [0,1]$, where S is a closed surface (for details, see Section 3), we prove Theorem 1.1. As a consequence of Theorem 1.1, we have Corollary 1.2.

Corollary 1.2 *For any integers $n \geq 2$ and $g \geq 2$, there exists a genus-g Heegaard splitting of a closed 3–manifold with distance exactly n.*

Remark 1.3 In [4], Guo, Qiu and Zou prove a statement that includes Corollary 1.2. In fact, they show in [4, Theorem 1] that for each pair of integers $n \geq 1$ and $g \geq 2$ with $(n, g) \neq (1, 2)$, there is a genus-g Heegaard splitting of a closed 3–manifold with distance n. We note that the pair $(n, g) = (1, 2)$ is not realizable (see Thompson [15, Proposition 1]).

Acknowledgements After finishing the first version of this paper, Ruifeng Qiu informed us that there was a gap in the proof of the theorem which we used in the proof of Corollary 1.2 in the first version, and he sent us the paper by Ma, Qiu and Zou [11]. We tried to fill the gap, and could prove a partial result that suffices to give the corollary in our setting. Along the way, we learned from [11] that Li [10] gave a very sharp estimation of the radius of the image of the disk complex of a compact 3–manifold by subsurface projections, and this drastically improved the conclusion of Corollary 1.2 in the first version of this paper. When the new version was almost completed, we found that [4] was uploaded on the arXiv, and the main result of [4] completely covers Corollary 1.2 of this paper. Further, Yanqing Zou, who is one of the authors of [4], informed us that our arguments in the revised version mentioned above contained a gap and made some suggestions on how to fix it. Thanks to the suggestion, and in particular after consulting [4], we could give the formulation of Proposition 5.1 in this paper.

We thank Dr Michael Yoshizawa for many helpful discussions, particularly for teaching us the ideas of his dissertation [16] which include the existence of Heegaard splittings of distance $2n$ for each integer $n \geq 1$, and we also thank Professor Yo'av Rieck for giving us helpful information. We would like to especially thank Professor Ruifeng Qiu and Dr Yanqing Zou their help and for pointing us to the preprints [11; 4]. Finally, we would like to thank the referee for his/her careful reading of the manuscript and helpful suggestions.

2 Preliminaries

Let S be a compact connected orientable surface with genus g and p boundary components. A simple closed curve in S is *essential* if it does not bound a disk in S and is not parallel to a component of ∂S. An arc properly embedded in S is *essential* if it does not cobound a disk in S together with an arc on ∂S. We say that S is *sporadic* if $g = 0, p \leq 4$ or $g = 1, p \leq 1$. We say that S is *simple* if S contains no essential simple closed curves. We note that S is simple if and only if S is a 2–sphere with at most three boundary components. A subsurface X in S is *essential* if each component of ∂X is contained in ∂S or is essential in S.

Heegaard splittings A connected 3–manifold V is a *compression body* if there exists a closed (possibly empty) surface F and a 0–handle B such that V is obtained from $F \times [0,1] \cup B$ by adding 1–handles to $F \times \{1\} \cup \partial B$. The subsurface of ∂V corresponding to $F \times \{0\}$ is denoted by $\partial_- V$. Then $\partial_+ V$ denotes the subsurface $\partial V \setminus \partial_- V$ of ∂V. The genus of $\partial_+ V$ is the *genus* of the compression body V. A compression body V is called a *handlebody* if $\partial_- V = \varnothing$. A disk D properly embedded in V is called an *essential disk* if ∂D is an essential simple closed curve in $\partial_+ V$.

Let M be a compact orientable 3–manifold. We say that $C_1 \cup_P C_2$ is a *genus-g Heegaard splitting* of M if C_1 and C_2 are compression bodies of genus g in M such that $C_1 \cup C_2 = M$ and $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = P$. Alternatively, given a Heegaard splitting $C_1 \cup_P C_2$ of M, we may regard that there is a homeomorphism $f\colon \partial_+ C_1 \to \partial_+ C_2$ such that M is obtained from C_1 and C_2 by identifying $\partial_+ C_1$ and $\partial_+ C_2$ via f. When we take this viewpoint, we will denote the Heegaard splitting by the expression $C_1 \cup_f C_2$.

Curve complexes Except in sporadic cases, the *curve complex* $\mathcal{C}(S)$ is defined as follows: each vertex of $\mathcal{C}(S)$ is the isotopy class of an essential simple closed curve on S, and a collection of $k+1$ vertices forms a k–simplex of $\mathcal{C}(S)$ if they can be realized by mutually disjoint curves in S. In sporadic cases, we need to modify the definition of the curve complex slightly, as follows. Note that the surface S is simple unless S is a torus, a torus with one boundary component, or a sphere with 4 boundary components. When S is a torus or a torus with one boundary component (resp. a sphere with 4 boundary components), a collection of $k+1$ vertices forms a

k–simplex of $\mathcal{C}(S)$ if they can be realized by curves in S which mutually intersect exactly once (resp. twice). The *arc-and-curve complex* $\mathcal{AC}(S)$ is defined similarly, as follows: each vertex of $\mathcal{AC}(S)$ is the isotopy class of an essential properly embedded arc or an essential simple closed curve on S, and a collection of $k+1$ vertices forms a k–simplex of $\mathcal{AC}(S)$ if they can be realized by mutually disjoint arcs or simple closed curves in S. Throughout this paper, for a vertex $x \in \mathcal{C}(S)$ we often abuse notation and use x to represent (the isotopy class of) a geometric representative of x. The symbol $\mathcal{C}^0(S)$ (resp. $\mathcal{AC}^0(S)$) denotes the 0–skeleton of $\mathcal{C}(S)$ (resp. $\mathcal{AC}(S)$).

For two vertices a, b of $\mathcal{C}(S)$, we define the *distance* $d_{\mathcal{C}(S)}(a,b)$ between a and b, which will be denoted by $d_S(a,b)$ in brief, as the minimal number of 1–simplexes of a simplicial path in $\mathcal{C}(S)$ joining a and b. For two subsets A, B of $\mathcal{C}^0(S)$, we define $\mathrm{diam}_S(A,B) :=$ the diameter of $A \cup B$. Similarly, we can define $d_{\mathcal{AC}(S)}(a,b)$ for $a,b \in \mathcal{AC}^0(S)$ and $\mathrm{diam}_{\mathcal{AC}(S)}(A,B)$ for $A, B \subset \mathcal{AC}^0(S)$.

For a sequence $a_0, a_1, \ldots, a_n$ of vertices in $\mathcal{C}(S)$ with $a_i \cap a_{i+1} = \varnothing$, $i = 0, 1, \ldots, n-1$, we denote by $[a_0, a_1, \ldots, a_n]$ the path in $\mathcal{C}(S)$ with vertices $a_0, a_1, \ldots, a_n$ in this order. We say that a path $[a_0, a_1, \ldots, a_n]$ is a *geodesic* if $n = d_S(a_0, a_n)$.

Let V be a compression body. Then the *disk complex* $\mathcal{D}(V)$ is the subset of $\mathcal{C}^0(\partial_+ V)$ consisting of the vertices with representatives bounding essential disks of V. For a genus-$g(\geq 2)$ Heegaard splitting $C_1 \cup_P C_2$, the (Hempel) *distance* of $C_1 \cup_P C_2$ is $d_P(\mathcal{D}(C_1), \mathcal{D}(C_2)) = \min\{d_P(x,y) \mid x \in \mathcal{D}(C_1), y \in \mathcal{D}(C_2)\}$.

Subsurface projection maps Let $\mathcal{P}(Y)$ denote the power set of a set Y. Suppose that X is an essential subsurface of S, where X is not a simple surface or a torus. We call the composition $\pi_0 \circ \pi_A$ of maps $\pi_A \colon \mathcal{C}^0(S) \to \mathcal{P}(\mathcal{AC}^0(X))$ and $\pi_0 \colon \mathcal{P}(\mathcal{AC}^0(X)) \to \mathcal{P}(\mathcal{C}^0(X))$ a *subsurface projection* if they satisfy the following: for a vertex α, take a representative of α so that $|\alpha \cap X|$ is minimal, where $|\cdot|$ is the number of connected components. Then

- $\pi_A(\alpha)$ is the set of all isotopy classes of the components of $\alpha \cap X$,
- $\pi_0(\{\alpha_1, \ldots, \alpha_n\})$ is the union, for all $i = 1, \ldots, n$, of the set of all isotopy classes of the components of $\partial N(\alpha_i \cup \partial X)$ which are essential in X, where $N(\alpha_i \cup \partial X)$ is a regular neighborhood of $\alpha_i \cup \partial X$ in X.

We denote the subsurface projection $\pi_0 \circ \pi_A$ by π_X. We say that α *misses* X (resp. α *cuts* X) if $\alpha \cap X = \varnothing$ (resp. $\alpha \cap X \neq \varnothing$).

Lemma 2.1 *Let X be as above. Let $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ be a path in $\mathcal{C}(S)$ such that every α_i cuts X. Then* $\mathrm{diam}_X(\pi_X(\alpha_0), \pi_X(\alpha_n)) \leq 2n$.

Proof Since $d_S(\alpha_i, \alpha_{i+1}) = 1$ and every α_i cuts X, we have

$$\mathrm{diam}_{\mathcal{AC}(X)}(\pi_A(\alpha_i), \pi_A(\alpha_{i+1})) \leq 1$$

for every $i = 0, 1, \ldots, n-1$. This together with [13, Lemma 2.2] implies

$$\mathrm{diam}_X(\pi_0(\pi_A(\alpha_i)), \pi_0(\pi_A(\alpha_{i+1})))(= \mathrm{diam}_X(\pi_X(\alpha_i), \pi_X(\alpha_{i+1}))) \leq 2.$$

Since $\mathrm{diam}_X(\pi_X(\alpha_0), \pi_X(\alpha_n)) \leq \sum_{i=0}^{n-1} \mathrm{diam}_X(\pi_X(\alpha_i), \pi_X(\alpha_{i+1}))$, this implies

$$\mathrm{diam}_X(\pi_X(\alpha_0), \pi_X(\alpha_n)) \leq 2n. \qquad \square$$

Remark 2.2 Let X be an essential subsurface of S. Suppose that X is disconnected, and at least two components of X are nonsimple. Then for any pair of curves α, α' on S we have $\mathrm{diam}_X(\pi_X(\alpha), \pi_X(\alpha')) \leq 2$. To be precise, let X_1 be one of the nonsimple components of X, and X_2 the union of the others. Let a and a' be elements of $\pi_X(\alpha)$ and $\pi_X(\alpha')$, respectively. If both a and a' are contained in X_i for some $i = 1, 2$, say X_1, then we can find a curve on X_2 that is disjoint from $a \cup a'$, which implies $d_X(a, a') \leq 2$. If $a \subset X_1$ and $a' \subset X_2$ (or $a \subset X_2$ and $a' \subset X_1$), we have $d_X(a, a') \leq 1$. Thus $d_X(a, a') \leq 2$ for any pair of elements $a \in \pi_X(\alpha)$ and $a' \in \pi_X(\alpha')$, and hence we have $\mathrm{diam}_X(\pi_X(\alpha), \pi_X(\alpha')) \leq 2$.

3 Disk complexes

Let $\mathcal{D}(V)$ $(\subset \mathcal{C}^0(\partial_+ V))$ be the disk complex of a compression body V. We have a decomposition $\mathcal{D}(V) = \mathcal{D}_{\mathrm{nonsep}}(V) \sqcup \mathcal{D}_{\mathrm{sep}}(V)$, where $\mathcal{D}_{\mathrm{nonsep}}(V)$ (resp. $\mathcal{D}_{\mathrm{sep}}(V)$) denotes the subset of $\mathcal{D}(V)$ consisting of the vertices with representatives bounding nonseparating (resp. separating) essential disks of V. In this section, we prove the following proposition.

Proposition 3.1 *Let V be a compression body obtained by adding a 1–handle to $F \times [0,1]$, where F is a genus-$(g-1)$ closed orientable surface ($g \geq 2$). Then we have the following:*

(1) *$\mathcal{D}_{\mathrm{nonsep}}(V)$ consists of a single vertex, say c_0.*

(2) *For each element c_α of $\mathcal{D}_{\mathrm{sep}}(V)$, there is a 1–simplex in $\mathcal{C}(\partial_+ V)$ joining c_0 and c_α.*

Remark 3.2 In fact, we can see that $\mathcal{D}_{\mathrm{sep}}(V)$ is a countable, infinite set and that there is no 1–simplex between c_α and $c_{\alpha'}$ for each pair $c_\alpha, c_{\alpha'} \in \mathcal{D}_{\mathrm{sep}}(V)$.

In the remainder of this section, V denotes a compression body obtained by adding a 1–handle to $F\times[0,1]$, where F is a genus-$(g-1)$ closed orientable surface ($g\geq 2$). Then D denotes the essential disk of V corresponding to the cocore of the 1–handle. Proposition 3.1 follows from Lemmas 3.3 and 3.4 below.

Lemma 3.3 *Any nonseparating disk properly embedded in V is ambient isotopic to D.*

Proof Let D' be a nonseparating disk in V. Assume that D and D' intersect transversely, and $|D\cap D'|$ is minimized up to ambient isotopy class of D'.

Suppose $|D\cap D'|=0$, ie, $D\cap D'=\varnothing$. Then D' is properly embedded disk in the manifold obtained from V by cutting along D, that is, $F\times[0,1]$. Since any disk properly embedded in $F\times[0,1]$ is boundary parallel and D' is nonseparating in V, we see that $D\cup D'$ bounds a product region, and hence D' is ambient isotopic to D.

Suppose $|D\cap D'|>0$. By standard innermost disk arguments, we can see that $D\cap D'$ has no loop components. Note that there are at least two components of $D'\setminus D$ which are outermost in D'. Take a pair of such outermost components, say Δ_1 and Δ_2, which are next to each other, ie, there is a subarc $\beta\subset\partial D'$ such that $\beta\cap\Delta_1$ is an endpoint of β and $\beta\cap\Delta_2$ is the other endpoint of β, and β does not intersect any other outermost disk of $D'\setminus D$. Note that we can retrieve $F\times[0,1]$ by cutting V along D. Let D^+, D^- be the copies of D in $F\times\{1\}$, and let $\overline{\Delta}_1$ (resp. $\overline{\Delta}_2$) be the closure of the image of Δ_1 (resp. Δ_2) in $F\times[0,1]$. Note that $\overline{\Delta}_1$ and $\overline{\Delta}_2$ are disks properly embedded in $F\times[0,1]$, and $\overline{\Delta}_i\cap(D^+\cup D^-)$ consists of an arc properly embedded in $D^+\cup D^-$. Let Γ_i $(i=1,2)$ be the disk in $F\times\{1\}$ such that $\partial\Gamma_i=\partial\overline{\Delta}_i$. Without loss of generality, we may suppose $\overline{\Delta}_1\cap(D^+\cup D^-)=\overline{\Delta}_1\cap D^+$. Note that if D^- is not contained in Γ_1, we can isotope D' in V via the product region between $\overline{\Delta}_1$ and Γ_1 to reduce $|D\cap D'|$, a contradiction. Hence, D^- is contained in Γ_1. Let β be the arc in $\partial D'$ as above. Then $\beta\cap D$ consists of finite number of points, say $p_0,p_1,\dots,p_n$, where $\partial\beta=\{p_0,p_n\}$, $p_0\in\partial\overline{\Delta}_1$, $p_n\in\partial\overline{\Delta}_2$, and $p_0,p_1,\dots,p_n$ are arrayed on β in this order. Then a small neighborhood of p_0 in β is contained in a small neighborhood of D^- in $F\times[0,1]$. If the other endpoint of the subarc $\overline{p_0p_1}$ of β is contained in ∂D^-, then we see that the subarc $\overline{p_0p_1}$ is an inessential arc in $\mathrm{Cl}(F\times\{1\}\setminus(D^+\cup D^-))$. This shows that we can reduce $|D\cap D'|$ by an isotopy on D', a contradiction. By applying the same argument successively, we see that each subarc $\overline{p_{i-1}p_i}$ $(i=1,2,\dots,n)$ joins D^+ and D^-, and particularly, a small neighborhood of p_n in β is contained in a small neighborhood of D^+. This shows that $\overline{\Delta}_2\cap(D^+\cup D^-)=\overline{\Delta}_2\cap D^-$. Then we see that D^+ is not contained in Γ_2, hence we have a contradiction by using the argument as above. □

Let D' be a separating essential disk properly embedded in V. By an argument similar to that in the proof of Lemma 3.3, we can see that D' is ambient isotopic to a disk disjoint from D. Hence, we have the following lemma.

Lemma 3.4 *Any separating essential disk properly embedded in V can be isotoped to be disjoint from the nonseparating disk D.*

4 A pair of curves with distance exactly *n*

In this section, for each integer $n \geq 3$, we construct pairs of curves with distance exactly n. Let S be a closed connected orientable surface with genus greater than or equal to 2. We first prove Propositions 4.1 and 4.4. Then we describe the constructions of paths in $\mathcal{C}(S)$ of length n and show that they are geodesics in $\mathcal{C}(S)$.

Proposition 4.1 *For an integer $n(\geq 4)$, let $[\alpha_0, \alpha_1, \dots, \alpha_n]$ be a path in $\mathcal{C}(S)$ satisfying the following:*

(H1) $[\alpha_0, \dots, \alpha_{n-2}]$ *and* $[\alpha_{n-2}, \alpha_{n-1}, \alpha_n]$ *are geodesics in* $\mathcal{C}(S)$.

(H2) $\mathrm{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_{n-4}), \pi_{X_{n-2}}(\alpha_n)) \geq 4n$, *where* $X_{n-2} = \mathrm{Cl}(S \setminus N(\alpha_{n-2}))$.

Then $[\alpha_0, \alpha_1, \dots, \alpha_n]$ *is a geodesic in* $\mathcal{C}(S)$.

Remark 4.2 In Proposition 4.1, we note that X_{n-2} is connected, ie, α_{n-2} is nonseparating in S. This can be shown by using Remark 2.2 together with the condition (H2).

Proof of Proposition 4.1 Let $[\beta_0, \beta_1, \dots, \beta_m]$ be a geodesic in $\mathcal{C}(S)$ such that $\beta_0 = \alpha_0$, $\beta_m = \alpha_n$. Then note that $m \leq n$.

Claim 4.3 $\beta_j = \alpha_{n-2}$ *for some* $j \in \{0, 1, \dots, m\}$.

Proof Assume on the contrary that $\beta_j \neq \alpha_{n-2}$ for any j. Then every β_j cuts X_{n-2} (ie, $\beta_j \cap X_{n-2} \neq \varnothing$) since $X_{n-2} = \mathrm{Cl}(S \setminus N(\alpha_{n-2}))$. By Lemma 2.1, we have $\mathrm{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\beta_0), \pi_{X_{n-2}}(\beta_m)) \leq 2m$. On the other hand, since $[\alpha_0, \alpha_1, \dots, \alpha_{n-2}]$ is a geodesic, no α_i $(0 \leq i \leq n-3)$ is isotopic to α_{n-2}. Hence each α_i $(0 \leq i \leq n-3)$ cuts X_{n-2}. By Lemma 2.1, $\mathrm{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_0), \pi_{X_{n-2}}(\alpha_{n-4})) \leq 2(n-4) < 2n$. These imply

$$\begin{aligned}\mathrm{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_{n-4}), \pi_{X_{n-2}}(\alpha_n)) &\leq \mathrm{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_{n-4}), \pi_{X_{n-2}}(\alpha_0)) \\ &\qquad + \mathrm{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_0), \pi_{X_{n-2}}(\alpha_n)) \\ &< 2n + \mathrm{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\beta_0), \pi_{X_{n-2}}(\beta_m)) \\ &\leq 2n + 2m \leq 4n.\end{aligned}$$

This contradicts the hypothesis (H2). □

By Claim 4.3 and the hypothesis (H1), we have the equalities

$$j = d_S(\beta_0, \beta_j) = d_S(\alpha_0, \alpha_{n-2}) = n-2,$$
$$m-j = d_S(\beta_j, \beta_m) = d_S(\alpha_{n-2}, \alpha_n) = 2.$$

By combining the above equalities, we have $m = n$. Recall that $[\beta_0, \beta_1, \dots, \beta_m]$ is a geodesic in $\mathcal{C}(S)$ with $\beta_0 = \alpha_0$ and $\beta_m = \alpha_n$. Hence, $[\alpha_0, \alpha_1, \dots, \alpha_n]$ is a geodesic in $\mathcal{C}(S)$. □

Proposition 4.4 *For an integer $n (\geq 3)$, let $[\alpha_0, \alpha_1, \dots, \alpha_n]$ be a path in $\mathcal{C}(S)$ satisfying the following:*

(H1′) $[\alpha_0, \dots, \alpha_{n-1}]$ *and* $[\alpha_{n-2}, \alpha_{n-1}, \alpha_n]$ *are geodesics in* $\mathcal{C}(S)$.

(H2′) $\alpha_{n-2} \cup \alpha_{n-1}$ *is nonseparating in* S, *and* $\mathrm{diam}_{S'}(\pi_{S'}(\alpha_0), \pi_{S'}(\alpha_n)) > 2n$, *where* $S' = \mathrm{Cl}(S \setminus N(\alpha_{n-2} \cup \alpha_{n-1}))$.

Then $[\alpha_0, \alpha_1, \dots, \alpha_n]$ *is a geodesic in* $\mathcal{C}(S)$.

Proof Let $[\beta_0, \beta_1, \dots, \beta_m]$ be a geodesic in $\mathcal{C}(S)$ such that $\beta_0 = \alpha_0$, $\beta_m = \alpha_n$. Then note that $m \leq n$.

Claim 4.5 *There exists $j \in \{0, 1, \dots, m\}$ such that $\beta_j = \alpha_{n-2}$ or $\beta_j = \alpha_{n-1}$.*

Proof Suppose that $\beta_j \neq \alpha_{n-2}$ and $\beta_j \neq \alpha_{n-1}$ for any j. Since $\alpha_{n-2} \cup \alpha_{n-1}$ is nonseparating in S, each β_j cuts S'. Hence, by Lemma 2.1, we have

$$\mathrm{diam}_{S'}(\pi_{S'}(\beta_0), \pi_{S'}(\beta_m)) \leq 2m \leq 2n.$$

On the other hand, by (H2′), $\mathrm{diam}_{S'}(\pi_{S'}(\beta_0), \pi_{S'}(\beta_m)) > 2n$, a contradiction. □

Suppose $\beta_j = \alpha_{n-2}$. Then we have the equalities

$$j = d_S(\beta_0, \beta_j) = d_S(\alpha_0, \alpha_{n-2}) = n-2,$$
$$m-j = d_S(\beta_j, \beta_m) = d_S(\alpha_{n-2}, \alpha_n) = 2.$$

By combining the above equalities, we have $n = m$. Hence, $[\alpha_0, \alpha_1, \dots, \alpha_n]$ is a geodesic in $\mathcal{C}(S)$. We can use a similar argument for the case when $\beta_j = \alpha_{n-1}$. This completes the proof of Proposition 4.4. □

Construction 4.6 (The case when n is even) We first assume that n is an even integer with $n \geq 4$. Let α_0, α_2 be essential nonseparating simple closed curves on S which intersect transversely in one point, and let α_1 be an essential simple closed curve on S which is disjoint from $\alpha_0 \cup \alpha_2$. Let $X_2 = \mathrm{Cl}(S \setminus N(\alpha_2))$. Note that $[\alpha_0, \alpha_1, \alpha_2]$ is a geodesic of length two in $\mathcal{C}(S)$. Choose a homeomorphism $\varphi_2\colon S \to S$ such that $\varphi_2(N(\alpha_2)) = N(\alpha_2)$ and that $\mathrm{diam}_{X_2}(\pi_{X_2}(\alpha_0), \pi_{X_2}(\varphi_2(\alpha_0))) \geq 4n$. This is possible by [12, Proposition 4.6]. Let $\alpha_3 = \varphi_2(\alpha_1)$ and $\alpha_4 = \varphi_2(\alpha_0)$. Note that $[\alpha_2, \alpha_3, \alpha_4]$ is a geodesic of length two in $\mathcal{C}(S)$, and $|\alpha_2 \cap \alpha_4| = 1$.

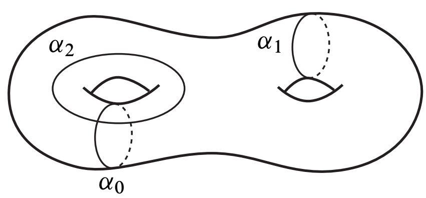

Figure 1

We repeat this process to construct a path $[\alpha_0, \alpha_1, \dots, \alpha_n]$ inductively as follows. Suppose we have constructed a path $[\alpha_0, \alpha_1, \dots, \alpha_i]$ with $|\alpha_{i-2} \cap \alpha_i| = 1$ for an even integer $i\,(< n)$. Let $X_i = \mathrm{Cl}(S \setminus N(\alpha_i))$. Choose a homeomorphism $\varphi_i\colon S \to S$ such that $\varphi_i(N(\alpha_i)) = N(\alpha_i)$ and that

(4-1) $$\mathrm{diam}_{X_i}(\pi_{X_i}(\alpha_{i-2}), \pi_{X_i}(\varphi_i(\alpha_{i-2}))) \geq 4n.$$

Then we let $\alpha_{i+1} = \varphi_i(\alpha_{i-1})$ and $\alpha_{i+2} = \varphi_i(\alpha_{i-2})$. Note that $[\alpha_i, \alpha_{i+1}, \alpha_{i+2}]$ is a geodesic of length two in $\mathcal{C}(S)$, and we have obtained the path $[\alpha_0, \alpha_1, \dots, \alpha_i, \alpha_{i+1}, \alpha_{i+2}]$ with $|\alpha_i \cap \alpha_{i+2}| = 1$.

Assertion 4.7 *For each $k \in \{2, 4, \dots, n\}$, the path $[\alpha_0, \alpha_1, \dots, \alpha_k]$ in $\mathcal{C}(S)$ is a geodesic.*

Proof We prove the proposition by mathematical induction on k. It is clear that $[\alpha_0, \alpha_1, \alpha_2]$ is a geodesic in $\mathcal{C}(S)$. Hence, Assertion 4.7 holds for $k = 2$. Assume that $[\alpha_0, \alpha_1, \dots, \alpha_k]$ is a geodesic in $\mathcal{C}(S)$ for some $k \in \{2, 4, \dots, n-2\}$. We note that $[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]$ is a geodesic in $\mathcal{C}(S)$. Furthermore, by the inequality (4-1), we have $\mathrm{diam}_{X_k}(\pi_{X_k}(\alpha_{k-2}), \pi_{X_k}(\alpha_{k+2})) \geq 4n \geq 4(k+2)$. Hence, by Proposition 4.1, the path $[\alpha_0, \alpha_1, \dots, \alpha_{k+2}]$ is a geodesic in $\mathcal{C}(S)$, which shows that Assertion 4.7 holds for $k+2$. This completes the proof of Assertion 4.7. □

Construction 4.8 (The case when n is odd) Suppose that n is an odd integer with $n \geq 3$. Let $[\alpha_0, \alpha_1, \dots, \alpha_{n-1}]$ be a geodesic in $\mathcal{C}(S)$ as in the previous subsection. Here, in addition, we assume that each α_i is a nonseparating curve. (It is easy to see that this holds if we take a nonseparating curve in S for α_1 at the beginning of the construction of the geodesic.) Note that α_{n-3} intersects α_{n-1} transversely in one point and is disjoint from α_{n-2}. It is easy to see that these imply that $\alpha_{n-1}\cup\alpha_{n-2}$ is nonseparating. Choose a nonseparating essential simple closed curve γ on S such that $\gamma\cap\alpha_{n-1}=\varnothing$ and γ intersects α_{n-2} transversely in one point. Let $S'=\mathrm{Cl}(S\setminus N(\alpha_{n-2}\cup\alpha_{n-1}))$. By [12, Theorem 1.1], the diameter of $\mathcal{C}(S')$ is infinite. This shows that there exists $\gamma'\in\mathcal{C}^0(S')$ such that $d_{S'}(\gamma',\pi_{S'}(\alpha_0))>2n+2$. If $g>2$, it is easy to find $\gamma''\in\mathcal{C}^0(S')$ such that $d_{S'}(\gamma'',\gamma')\le 2$ and that γ'' cuts off a pair of pants P with $\partial N(\alpha_{n-2})\subset\partial P$. If $g=2$, then γ' separates S' into two pairs of pants P_1 and P_2. If $\partial N(\alpha_{n-2})\subset\partial P_i$ ($i=1$ or 2), then set $\gamma''=\gamma'$, otherwise, take any essential simple closed curve γ'' in S' such that $\gamma''\cap\gamma'$ consists of two points (ie, $d_{S'}(\gamma'',\gamma')=1$) and that γ'' cuts off a pair of pants P with $\partial N(\alpha_{n-2})\subset\partial P$. Since γ'' cuts off a pair of pants P with $\partial N(\alpha_{n-2})\subset\partial P$ in either case, there is a simple closed curve $\alpha_n(\subset S)$ intersecting α_{n-2} in one point such that $\alpha_n\cap\alpha_{n-1}=\varnothing$ and that $\pi_{S'}(\alpha_n)=\gamma''$. Then we have

$$\begin{aligned}\mathrm{diam}_{S'}(\pi_{S'}(\alpha_0),\pi_{S'}(\alpha_n)) &\ge \mathrm{diam}_{S'}(\pi_{S'}(\alpha_0),\gamma')-d_{S'}(\gamma'',\gamma')\\ &> (2n+2)-2=2n.\end{aligned}$$

On the other hand, since $\alpha_n\cap\alpha_{n-1}=\varnothing$ and α_n intersects α_{n-2} transversely in one point, $[\alpha_{n-2},\alpha_{n-1},\alpha_n]$ is a geodesic in $\mathcal{C}(S)$. Further, $[\alpha_0,\dots,\alpha_{n-1}]$ is also a geodesic in $\mathcal{C}(S)$. Hence, by Proposition 4.4 together with the above inequality $\mathrm{diam}_{S'}(\pi_{S'}(\alpha_0),\pi_{S'}(\alpha_n))>2n$, we see that the path $[\alpha_0,\alpha_1,\dots,\alpha_n]$ is a geodesic in $\mathcal{C}(S)$.

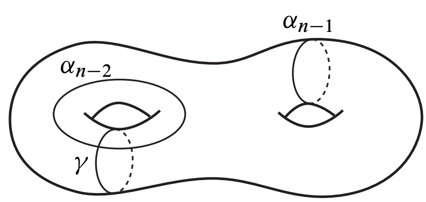

Figure 2

5 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1 Let C_1 and C_2 be copies of the compression body obtained by adding a 1–handle to $F\times[0,1]$, where F is a genus-$(g-1)$ closed orientable surface ($g\ge 2$). Let α_0 be the boundary of the nonseparating essential disk D_1

properly embedded in C_1 and α_2 a simple closed curve on ∂_+C_1 which intersects α_0 transversely in one point. Then we construct a geodesic $[\alpha_0, \alpha_1, \dots, \alpha_{n+2}]$ on ∂_+C_1 as in Section 4. Note that α_{n+2} intersects α_n transversely in one point by the construction. Take any homeomorphism $f\colon \partial_+C_1 \to \partial_+C_2$ such that $f(\alpha_{n+2}) = \partial D_2$, where D_2 is the nonseparating essential disk properly embedded in C_2. We identify the boundary components ∂_+C_1 and ∂_+C_2 by f, and let $P = \partial_+C_1 = f^{-1}(\partial_+C_2)$. Then $C_1 \cup_P C_2$ is a genus-g Heegaard splitting of a compact orientable 3–manifold.

Let D_1' be a separating essential disk in C_1 disjoint from α_2 obtained as follows. Let D_1^+ and D_1^- be the components of $\mathrm{Cl}(\partial N(D_1) \setminus \partial_+C_1)$, where $N(D_1)$ is a regular neighborhood of D_1 in C_1. Take the subarc of α_2 lying outside of the product region $N(D_1)$ between D_1^+ and D_1^-. Then D_1' is obtained from $D_1^+ \cup D_1^-$ by adding a band along the subarc of α_2. Similarly, we can obtain a separating essential disk D_2' in C_2 disjoint from α_n, by using D_2 and α_n. On the other hand, we have $d_P(\alpha_2, \alpha_n) = n-2$ since $[\alpha_0, \alpha_1, \dots, \alpha_{n+2}]$ is a geodesic in $\mathcal{C}(P)$. Hence

$$\begin{aligned} d_P(\partial D_1', \partial D_2') &\le d_P(\partial D_1', \alpha_2) + d_P(\alpha_2, \alpha_n) + d_P(\alpha_n, \partial D_2') \\ &= 1 + (n-2) + 1 = n. \end{aligned}$$

Let $D_1'' \subset C_1$ and $D_2'' \subset C_2$ be any essential disks. By Proposition 3.1, we have $d_P(\partial D_i'', \partial D_i) \le 1$ for $i = 1, 2$. This implies

$$\begin{aligned} d_P(\partial D_1, \partial D_2) &\le d_P(\partial D_1, \partial D_1'') + d_P(\partial D_1'', \partial D_2'') + d_P(\partial D_2'', \partial D_2) \\ &\le 1 + d_P(\partial D_1'', \partial D_2'') + 1, \end{aligned}$$

and hence

$$\begin{aligned} d_P(\partial D_1'', \partial D_2'') &\ge d_P(\partial D_1, \partial D_2) - 2 \\ &= d_P(\alpha_0, \alpha_{n+2}) - 2 = (n+2) - 2 = n. \end{aligned}$$

Therefore, $d_P(\partial D_1'', \partial D_2'') \ge n$ for any pair of essential disks $D_1'' \subset C_1$ and $D_2'' \subset C_2$, which implies $d_P(\mathcal{D}(C_1), \mathcal{D}(C_2)) \ge n$. Since $d_P(\partial D_1', \partial D_2') \le n$, we have that $d_P(\mathcal{D}(C_1), \mathcal{D}(C_2)) = n$. □

In the remainder of the paper, we prove Corollary 1.2 by using the following proposition. (Throughout this paper, given an embedding $\varphi\colon X \to Y$ between compact surfaces X and Y, we abuse notation and use φ to denote the map $\mathcal{C}^0(X) \to \mathcal{C}^0(Y)$ or $\mathcal{P}(\mathcal{C}^0(X)) \to \mathcal{P}(\mathcal{C}^0(Y))$ induced by $\varphi\colon X \to Y$.)

Proposition 5.1 *Let $V_1 \cup_f V_2$ be a genus-g (≥ 2) Heegaard splitting, where V_1 and V_2 are handlebodies. Let D_0 be a separating essential disk in V_1, and let $\mathcal{D}_2$ be either $\mathcal{D}(V_2)$ or a finite subset of $\mathcal{D}(V_2)$. Assume that $d_{\partial V_2}(f(\partial D_0), \mathcal{D}_2) = n \ge 3$. Then there exists a homeomorphism $g\colon \partial V_1 \to \partial V_1$ such that $d_{\partial V_2}(fg(\mathcal{D}(V_1)), \mathcal{D}_2) = n$.*

Proof Let V_1^1 and V_1^2 be the closures of the two components of $V_1 \setminus D_0$. For $i = 1, 2$, let F_i be the subsurface $\partial V_1^i \cap \partial V_1$ of ∂V_1, and let $\pi_{F_i} = \pi_0 \circ \pi_A^i \colon \mathcal{C}^0(\partial V_1) \to \mathcal{P}(\mathcal{AC}^0(F_i)) \to \mathcal{P}(\mathcal{C}^0(F_i))$ be the subsurface projection introduced in Section 2. Let $P_i \colon F_i \to F_i \cup D_0$ be the inclusion map. Since D_0 is separating, the image of any essential simple closed curve in F_i by P_i is essential in $F_i \cup D_0$. This immediately implies the following.

Claim 5.2 *For any nonempty subset E of $\mathcal{C}^0(F_i)$, we have that*

- *$P_i(E)$ is nonempty,*
- $\mathrm{diam}_{F_i \cup D_0}(P_i(E)) \leq \mathrm{diam}_{F_i}(E)$.

We note that there exists a constant N such that

$$\mathrm{diam}_{F_i}(\pi_{F_i} f^{-1}(\mathcal{D}_2)) \leq N \quad (i = 1, 2). \tag{5-1}$$

In fact, if $\mathcal{D}_2$ is a finite subset of $\mathcal{D}(V_2)$, this is clear. Thus assume $\mathcal{D}_2 = \mathcal{D}(V_2)$. We claim that the condition $d_{\partial V_2}(f(\partial D_0), \mathcal{D}_2) \geq 3$ implies the pair $(V_2, f(F_1) \cup f(F_2))$ is not homeomorphic to a ([0, 1]–bundle, the associated ∂[0, 1]–bundle). In fact, otherwise, we have a contradiction as follows. Take an essential arc α and an essential simple closed curve l in $f(F_1)$. If $(V_2, f(F_1) \cup f(F_2))$ is not homeomorphic to a [0, 1]–bundle, then $f(\partial D_0)$, l and $\alpha \times [0,1]$ give a path of length 2. Further, $\alpha \times [0,1]$ is an essential disk in V_2, a contradiction to $d_{\partial V_2}(f(\partial D_0), \mathcal{D}_2) \geq 3$. Hence, by [10, Theorem 1] together with the assumption $d_{\partial V_2}(f(\partial D_0), \mathcal{D}_2) \geq 3$, we see that $\mathrm{diam}_{f(F_i)}(\pi_{f(F_i)}(\mathcal{D}_2)) \leq 12$, which means $\mathrm{diam}_{F_i}(\pi_{F_i} f^{-1}(\mathcal{D}_2)) \leq 12$. By Claim 5.2, the inequality (5-1) implies

$$\mathrm{diam}_{F_i \cup D_0}(P_i \pi_{F_i} f^{-1}(\mathcal{D}_2)) \leq N \quad (i = 1, 2). \tag{5-2}$$

Let $\mathcal{D}'(V_1^i)$ be the subset of $\mathcal{C}^0(F_i)$ consisting of simple closed curves that bound disks in V_1^i $(i = 1, 2)$. By the inequality (5-2) and [7] (see also [1]), we see that there exists a homeomorphism $g \colon \partial V_1 \to \partial V_1$ such that $g(\partial D_0) = \partial D_0$ and

$$d_{F_i \cup D_0}(P_i(\mathcal{D}'(V_1^i)), \widehat{g}_i^{-1}(P_i \pi_{F_i} f^{-1}(\mathcal{D}_2))) \geq 2n \tag{5-3}$$

for each $i = 1, 2$, where $\widehat{g}_i \colon F_i \cup D_0 \to F_i \cup D_0$ is a homeomorphism obtained by extending $g|_{F_i} \colon F_i \to F_i$. (We note that $g|_{F_i} \colon F_i \to F_i$ extends to a homeomorphism $\widehat{g}_i \colon F_i \cup D_0 \to F_i \cup D_0$ in a unique way up to isotopy in D_0 by Alexander's trick.) Since $g(\partial D_0) = \partial D_0$, it is easy to see that $\widehat{g}_i^{-1}(P_i \pi_{F_i} f^{-1}(\mathcal{D}_2)) = P_i (g|_{F_i})^{-1} \pi_{F_i} f^{-1}(\mathcal{D}_2) = P_i \pi_{F_i} g^{-1} f^{-1}(\mathcal{D}_2) = P_i \pi_{F_i} (fg)^{-1}(\mathcal{D}_2)$. We denote the map $P_i \pi_{F_i}(fg)^{-1}(=\hat{g}_i^{-1} P_i \pi_{F_i} f^{-1})\colon \mathcal{C}^0(\partial V_2) \to \mathcal{P}(\mathcal{C}^0(F_i \cup D_0))$ by Φ_i. Then, by the inequality (5-3), we have

$$d_{F_i \cup D_0}(P_i(\mathcal{D}'(V_1^i)), \Phi_i(\mathcal{D}_2)) \geq 2n \quad (i = 1, 2). \tag{5-4}$$

Note that $d_{\partial V_2}(fg(\mathcal{D}(V_1)), \mathcal{D}_2) \leq n$ since $f(\partial D_0) = fg(\partial D_0) \in fg(\mathcal{D}(V_1))$ and $d_{\partial V_2}(f(\partial D_0), \mathcal{D}_2) = n$ by the assumption. To prove $d_{\partial V_2}(fg(\mathcal{D}(V_1)), \mathcal{D}_2) = n$, assume on the contrary that $d_{\partial V_2}(fg(\mathcal{D}(V_1)), \mathcal{D}_2) < n$, or equivalently, $d_{\partial V_1}(\mathcal{D}(V_1), (fg)^{-1}(\mathcal{D}_2)) < n$. Then there exist $a \in \mathcal{D}(V_1)$ and $b \in \mathcal{D}_2$ such that

$$d_{\partial V_1}(a, (fg)^{-1}(b)) = m < n. \tag{5-5}$$

Let $[\gamma_0, \gamma_1, \dots, \gamma_m]$ be a geodesic in $\mathcal{C}(\partial V_1)$ from a to $(fg)^{-1}(b)$.

Claim 5.3 *Every γ_j $(j = 1, 2, \dots, m)$ cuts both F_1 and F_2.*

Proof Assume that γ_j does not cut F_i for some $j \in \{1, 2, \dots, m\}$ and some $i \in \{1, 2\}$. Then γ_j is disjoint from $\partial D_0 (= \partial F_1 = \partial F_2)$, and hence we have

$$\begin{aligned} n &= d_{\partial V_1}(\partial D_0, (fg)^{-1}(\mathcal{D}_2)) \\ &\leq d_{\partial V_1}(\partial D_0, \gamma_j) + d_{\partial V_1}(\gamma_j, (fg)^{-1}(\mathcal{D}_2)) \\ &\leq d_{\partial V_1}(\partial D_0, \gamma_j) + d_{\partial V_1}(\gamma_j, \gamma_m) \leq 1 + (m - j) < 1 + n - j, \end{aligned}$$

a contradiction. □

Let D_a be a disk in V_1 bounded by a. We may assume that $|D_a \cap D_0|$ is minimal. By using innermost disk arguments, we see that $D_a \cap D_0$ has no loop components.

Case 1: $|D_a \cap D_0| \neq 0$ Let Δ be a component of $D_a \setminus D_0$ that is outermost in D_a. Then $\Delta \subset V_1^i$ for some $i = 1, 2$. Without loss of generality, we may assume that $\Delta \subset V_1^1$, which implies $a(=\gamma_0)$ cuts F_1. This, together with Claim 5.3, shows that every γ_j $(j = 0, 1, \dots, m)$ in the geodesic $[\gamma_0, \gamma_1, \dots, \gamma_m]$ from a to $(fg)^{-1}(b)$ cuts F_1. Hence, by Lemma 2.1, we have

$$\operatorname{diam}_{F_1}(\pi_{F_1}(a), \pi_{F_1}(fg)^{-1}(b)) \leq 2m < 2n, \tag{5-6}$$

which implies, by Claim 5.2,

$$\operatorname{diam}_{F_1 \cup D_0}(P_1 \pi_{F_1}(a), \Phi_1(b)) < 2n. \tag{5-7}$$

Note that $\partial\Delta \cap F_1$ is an element of the image of a by $\pi_A^1\colon \mathcal{C}^0(\partial V_1) \to \mathcal{P}(\mathcal{AC}^0(F_1))$. Further, by the minimality of $|D_a \cap D_0|$, the disk Δ is essential in V_1^1. Let D_0^1 and D_0^2 be the two components of $D_0 \setminus \Delta$, and let Δ' be one of the disks properly embedded in V_1^1 which is parallel to $D_0^1 \cup \Delta$ or $D_0^2 \cup \Delta$. Then we have $\partial\Delta' \in P_1(\mathcal{D}'(V_1^1))$, and also $\partial\Delta' \in P_1\pi_0(\partial\Delta \cap F_1) \subset P_1\pi_0\pi_A^1(a) = P_1\pi_{F_1}(a)$. These, together with the inequality (5-7), imply

$$\begin{aligned} d_{F_1\cup D_0}(P_1(\mathcal{D}'(V_1^1)), \Phi_1(\mathcal{D}_2)) &\le d_{F_1\cup D_0}(\partial\Delta', \Phi_1(b)) \\ &\le \operatorname{diam}_{F_1\cup D_0}(P_1\pi_{F_1}(a), \Phi_1(b)) < 2n, \end{aligned}$$

which contradicts the inequality (5-4).

Case 2: $|D_a \cap D_0| = 0$ In this case, the arguments in Case 1 work with regarding $D_a = \Delta'$ to have a contradiction.

The above contradictions give $d_{\partial V_2}(fg(\mathcal{D}(V_1)), \mathcal{D}_2) = n$. □

Remark 5.4 If we pose the assumption that the distance $d(V_1 \cup_f V_2)$ of the genus-g Heegaard splitting $V_1 \cup_f V_2$ is greater than or equal to 2 in Proposition 5.1, then the statement of the proposition can be strengthened as in the following form.

Let D_0 be a separating essential disk in V_1, and let $\mathcal{D}_2$ be any subset of $\mathcal{D}(V_2)$. If $d_{\partial V_2}(f(\partial D_0), \mathcal{D}_2) = n$, then there exists a homeomorphism $g\colon \partial V_1 \to \partial V_1$ such that $d_{\partial V_2}(fg(\mathcal{D}(V_1)), \mathcal{D}_2) = n$.

In fact, the statement can be proved basically by using the arguments of the proof of Proposition 5.1. The difference is the proof of inequality (5-1). We should replace the argument with the following.

Note that $f(\partial D_0)(= f(\partial F_1) = f(\partial F_2))$ intersects with every essential loop in $\mathcal{D}(V_2)$, since $d_{\partial V_2}(f(\partial D_0), \mathcal{D}(V_2)) \ge d(V_1 \cup_f V_2) \ge 2$. By [10, Theorem 1], either

$$\operatorname{diam}_{F_i}(\pi_{F_i} f^{-1}(\mathcal{D}_2)) \le \operatorname{diam}_{F_i}(\pi_{F_i} f^{-1}(\mathcal{D}(V_2))) \le 12 \tag{5-8}$$

or V_2 is a $[0,1]$–bundle over $f(F_1)$. In the latter case, it is easy to see that g must be even and that the union of V_2 and $N(D_0)$ is homeomorphic to a $[0,1]$–bundle over a closed surface, say S, of genus $g/2$. Note that the exterior of the union of V_2 and $N(D_0)$ is $\mathrm{Cl}(V_1 \setminus N(D_0))$ and consists of two handlebodies of genus $g/2$. Thus, S is a Heegaard surface of genus $g/2$, and $\partial V_2(= f(\partial V_1))$ is a stabilization of S. This implies $d(V_1 \cup_f V_2) = 0$, a contradiction. Hence, we have the inequality (5-8).

Proof of Corollary 1.2 We first note that the proof of the corollary for the case when $n = 2$ is exceptional, and we give it in the Appendix of this paper, and in this proof we show the corollary for the case $n \ge 3$. Let $C_1 \cup_P C_2 = C_1 \cup_f C_2$ be a genus-g Heegaard splitting with distance $n(\ge 3)$ obtained in Theorem 1.1. By the proof of Theorem 1.1, there are separating essential disks D_1 and D_2 in C_1 and C_2, respectively, such that $d_{\partial_+ C_2}(f(\partial D_1), \partial D_2) = n$. Let H_i $(i = 1, 2)$ be a handlebody of genus $(g-1)$. Take and fix any homeomorphism $h_i\colon \partial H_i \to \partial_- C_i$, and put $V_i := C_i \cup_{h_i} H_i$ (hence, V_i is a handlebody of genus g). Then $V_1 \cup_f V_2$ is a genus-g Heegaard splitting.

By Proposition 5.1, there exists a homeomorphism $g_1\colon \partial V_1 \to \partial V_1$ such that

$$d_{\partial V_2}(fg_1(\mathcal{D}(V_1)), \partial D_2) = n.$$

By applying Proposition 5.1 again to $V_2 \cup_{(fg_1)^{-1}} V_1$, we see that there exists a homeomorphism $g_2\colon \partial V_2 \to \partial V_2$ such that

$$d_{\partial V_1}((fg_1)^{-1} g_2(\mathcal{D}(V_2)), \mathcal{D}(V_1)) = n.$$

That is, the distance of the Heegaard splitting $V_1 \cup_{g_2^{-1} fg_1} V_2$ is exactly n. □

Appendix: A construction of distance 2 examples

In this Appendix, we show for each $g \geq 2$, there is a genus-g Heegaard splitting of a closed 3–manifold with distance 2. The examples are given by using the construction of strongly irreducible Heegaard splittings by Kobayashi and Rieck in [9]. For the description of the construction we will use the notation $(H, A_1 \cup A_2)$, N, R etc from [9, Section 2.1].

For the case when $g = 2$, let F be an annulus, and let $R = F \times [0, 1]$. For the case when $g \geq 3$, let F be a genus-$(g-2)$ nonorientable surface (connected sum of $g-2$ copies of projective planes) with two holes, and let R be the orientable twisted $[0, 1]$–bundle over F. Note that F is homotopy equivalent to a bouquet of $g-1$ circles, hence R is homeomorphic to the genus-$(g-1)$ handlebody. Let R' be a copy of R. Then let $\mathcal{A}_1 \cup \mathcal{A}_2$ (resp. $\mathcal{A}_1' \cup \mathcal{A}_2'$) be the union of annuli in ∂R (resp. $\partial R'$) corresponding to the $[0, 1]$–bundle over ∂F. Then let N be the manifold obtained from R and R' by identifying the subsurfaces of the boundaries corresponding to the associated $\partial[0, 1]$–bundle. It is easy to see that the manifolds N, R and R' satisfy [9, page 639, Conditions (1)–(3)].

Recall from [9] that H is a genus-2 handlebody, and $\{A_1, A_2\}$ is a pair of primitive annuli in ∂H. Let $(H', A_1' \cup A_2')$ be a copy of $(H, A_1 \cup A_2)$. Then it is observed in [9] that for any 2–bridge link L in S^3 there is a homeomorphism $h\colon \mathrm{Cl}(\partial H \setminus (A_1 \cup A_2)) \to \mathrm{Cl}(\partial H' \setminus (A_1' \cup A_2'))$ such that the manifold obtained from H and H' by identifying $\mathrm{Cl}(\partial H \setminus (A_1 \cup A_2))$ and $\mathrm{Cl}(\partial H' \setminus (A_1' \cup A_2'))$ by h is homeomorphic to the exterior $E(L)$ of L. Then let M be the 3–manifold obtained from $E(L)$ and N by identifying their boundaries by an orientation-reversing homeomorphism such that $\mathcal{A}_i$ (resp. $\mathcal{A}_i'$) is identified with A_i (resp. A_i'). Then it is shown in [9, Section 2.1] that $H \cup R$ and $H' \cup R'$ are genus-g handlebodies, and these handlebodies give a Heegaard splitting of M.

Then we have the following.

Assertion *Suppose that the 2–bridge link L is not a trivial link or a Hopf link, then the distance of the Heegaard splitting $(H \cup R) \cup (H' \cup R')$ is exactly 2.*

Proof Since L is not a trivial link or a Hopf link, we see, by [9, Proposition 2.1], that $(H \cup R) \cup (H' \cup R')$ is strongly irreducible, ie the distance of the Heegaard splitting is greater than or equal to 2. On the other hand, since $\partial E(L)$ $(= \partial N) \subset M$ is an essential torus, we see, by Hartshorn [5], that the distance of any Heegaard splitting of M is at most 2, and this together with the above shows that the distance of the Heegaard splitting $(H \cup R) \cup (H' \cup R')$ is exactly 2. □

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Received: 19 August 2013 Revised: 6 October 2013