Heegaard splittings of distance exactly *n*

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In this paper, we show that, for any integers $n \geq 2$ and $g \geq 2$, there exist genus-g Heegaard splittings of compact 3–manifolds with distance exactly n.

1 Introduction

For a closed orientable 3–manifold M, we say that $V_1 \cup_S V_2$ is a *Heegaard splitting* of M if V_1, V_2 are handlebodies such that $M = V_1 \cup V_2$ and $\partial V_1 = \partial V_2 = S$. Heegaard splittings of compact orientable 3–manifolds with nonempty boundaries can be defined similarly, using compression bodies for handlebodies (see Section 2). In [7], Hempel gave the definition of the *distance* of a Heegaard splitting by using the curve complex introduced by Harvey [6] and showed that, for any integer n, there exists some integer m such that the mth power of a pseudo-Anosov map yields a Heegaard splitting of distance at least n by using a construction of Kobayashi [8]. Abrams and Schleimer [1] showed that the distance of the Heegaard splitting grows linearly with respect to m by using a result of Masur and Minsky [12]. Moreover, Evans [3] gave a combinatorial method to construct Heegaard splittings of high distance. The main purpose of this paper is to give an answer to the following question.

Question Given $n \geq 1$ and $g \geq 2$, does there exist a genus-g Heegaard splitting with distance exactly n?

For certain values, there are known examples that answer the above question affirmatively. For example, Berge and Scharlemann [2] showed that there exist 3–manifolds which admit genus-2 Heegaard splittings with distance exactly 3.

In this paper, for each integer $n \geq 2$, we first construct a Heegaard splitting of a compact orientable 3–manifold with nonempty boundary which has distance exactly n.

Theorem 1.1 *For any integers $n \geq 2$ and $g \geq 2$, there exists a genus-g Heegaard splitting $C_1 \cup_P C_2$ with distance exactly n, where C_1 and C_2 are compression bodies.*

To prove Theorem 1.1, we give a method of constructing a pair of curves on a closed surface with distance exactly n. In fact, Schleimer [14] gave a method of constructing a pair of curves with distance exactly four on the five-holed sphere by using *subsurface projection maps* defined by Masur and Minsky [13] (for the definition, see Section 2). In Section 4, we mimic the idea of Schleimer to construct a pair of curves with distance exactly n for any positive integer n. By using the pair of curves and the properties of a compression body obtained by adding a 1–handle to $S \times [0, 1]$, where S is a closed surface (for details, see Section 3), we prove Theorem 1.1. As a consequence of Theorem 1.1, we have Corollary 1.2.

Corollary 1.2 *For any integers $n \geq 2$ and $g \geq 2$, there exists a genus-g Heegaard splitting of a closed 3–manifold with distance exactly n.*

Remark 1.3 In [4], Guo, Qiu and Zou prove a statement that includes Corollary 1.2. In fact, they show in [4, Theorem 1] that for each pair of integers $n \geq 1$ and $g \geq 2$ with $(n, g) \neq (1, 2)$, there is a genus-g Heegaard splitting of a closed 3–manifold with distance n. We note that the pair $(n, g) = (1, 2)$ is not realizable (see Thompson [15, Proposition 1]).

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2 Preliminaries

Let S be a compact connected orientable surface with genus g and p boundary components. A simple closed curve in S is *essential* if it does not bound a disk in S and is not parallel to a component of ∂S. An arc properly embedded in S is *essential* if it does not cobound a disk in S together with an arc on ∂S. We say that S is *sporadic* if $g = 0$, $p \leq 4$ or $g = 1$, $p \leq 1$. We say that S is *simple* if S contains no essential simple closed curves. We note that S is simple if and only if S is a 2–sphere with at most three boundary components. A subsurface X in S is *essential* if each component of ∂X is contained in ∂S or is essential in S.

Heegaard splittings A connected 3–manifold V is a *compression body* if there exists a closed (possibly empty) surface F and a 0–handle B such that V is obtained from $F \times [0, 1] \cup B$ by adding 1–handles to $F \times \{1\} \cup \partial B$. The subsurface of ∂V corresponding to $F \times \{0\}$ is denoted by $\partial_- V$. Then $\partial_+ V$ denotes the subsurface $\partial V \setminus \partial_- V$ of ∂V. The genus of $\partial_+ V$ is the *genus* of the compression body V. A compression body V is called a *handlebody* if $\partial_- V = \varnothing$. A disk D properly embedded in V is called an *essential disk* if ∂D is an essential simple closed curve in $\partial_+ V$.

Let M be a compact orientable 3–manifold. We say that $C_1 \cup_P C_2$ is a *genus-g Heegaard splitting* of M if C_1 and C_2 are compression bodies of genus g in M such that $C_1 \cup C_2 = M$ and $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = P$. Alternatively, given a Heegaard splitting $C_1 \cup_P C_2$ of M, we may regard that there is a homeomorphism $f: \partial_+ C_1 \to \partial_+ C_2$ such that M is obtained from C_1 and C_2 by identifying $\partial_+ C_1$ and $\partial_+ C_2$ via f. When we take this viewpoint, we will denote the Heegaard splitting by the expression $C_1 \cup_f C_2$.

Curve complexes Except in sporadic cases, the *curve complex* $\mathcal{C}(S)$ is defined as follows: each vertex of $\mathcal{C}(S)$ is the isotopy class of an essential simple closed curve on S, and a collection of $k + 1$ vertices forms a k–simplex of $\mathcal{C}(S)$ if they can be realized by mutually disjoint curves in S. In sporadic cases, we need to modify the definition of the curve complex slightly, as follows. Note that the surface S is simple unless S is a torus, a torus with one boundary component, or a sphere with 4 boundary components. When S is a torus or a torus with one boundary component (resp. a sphere with 4 boundary components), a collection of $k + 1$ vertices forms a

k–simplex of $\mathcal{C}(S)$ if they can be realized by curves in S which mutually intersect exactly once (resp. twice). The *arc-and-curve complex* $\mathcal{AC}(S)$ is defined similarly, as follows: each vertex of $\mathcal{AC}(S)$ is the isotopy class of an essential properly embedded arc or an essential simple closed curve on S, and a collection of $k+1$ vertices forms a k–simplex of $\mathcal{AC}(S)$ if they can be realized by mutually disjoint arcs or simple closed curves in S. Throughout this paper, for a vertex $x \in \mathcal{C}(S)$ we often abuse notation and use x to represent (the isotopy class of) a geometric representative of x. The symbol $\mathcal{C}^0(S)$ (resp. $\mathcal{AC}^0(S)$) denotes the 0–skeleton of $\mathcal{C}(S)$ (resp. $\mathcal{AC}(S)$).

For two vertices a, b of $\mathcal{C}(S)$, we define the *distance* $d_{\mathcal{C}(S)}(a, b)$ between a and b, which will be denoted by $d_S(a, b)$ in brief, as the minimal number of 1–simplexes of a simplicial path in $\mathcal{C}(S)$ joining a and b. For two subsets A, B of $\mathcal{C}^0(S)$, we define $\operatorname{diam}_S(A, B) :=$ the diameter of $A \cup B$. Similarly, we can define $d_{\mathcal{AC}(S)}(a, b)$ for $a, b \in \mathcal{AC}^0(S)$ and $\operatorname{diam}_{\mathcal{AC}(S)}(A, B)$ for $A, B \subset \mathcal{AC}^0(S)$.

For a sequence $a_0, a_1, \ldots, a_n$ of vertices in $\mathcal{C}(S)$ with $a_i \cap a_{i+1} = \varnothing$, $i = 0, 1, \ldots, n-1$, we denote by $[a_0, a_1, \ldots, a_n]$ the path in $\mathcal{C}(S)$ with vertices $a_0, a_1, \ldots, a_n$ in this order. We say that a path $[a_0, a_1, \ldots, a_n]$ is a *geodesic* if $n = d_S(a_0, a_n)$.

Let V be a compression body. Then the *disk complex* $\mathcal{D}(V)$ is the subset of $\mathcal{C}^0(\partial_+ V)$ consisting of the vertices with representatives bounding essential disks of V. For a genus-$g (\geq 2)$ Heegaard splitting $C_1 \cup_P C_2$, the (Hempel) *distance* of $C_1 \cup_P C_2$ is $d_P(\mathcal{D}(C_1), \mathcal{D}(C_2)) = \min\{d_P(x, y) \mid x \in \mathcal{D}(C_1), y \in \mathcal{D}(C_2)\}$.

Subsurface projection maps Let $\mathcal{P}(Y)$ denote the power set of a set Y. Suppose that X is an essential subsurface of S, where X is not a simple surface or a torus. We call the composition $\pi_0 \circ \pi_A$ of maps $\pi_A \colon \mathcal{C}^0(S) \to \mathcal{P}(\mathcal{AC}^0(X))$ and $\pi_0 \colon \mathcal{P}(\mathcal{AC}^0(X)) \to \mathcal{P}(\mathcal{C}^0(X))$ a *subsurface projection* if they satisfy the following: for a vertex α, take a representative of α so that $|\alpha \cap X|$ is minimal, where $|\cdot|$ is the number of connected components. Then

- $\pi_A(\alpha)$ is the set of all isotopy classes of the components of $\alpha \cap X$,
- $\pi_0(\{\alpha_1, \ldots, \alpha_n\})$ is the union, for all $i = 1, \ldots, n$, of the set of all isotopy classes of the components of $\partial N(\alpha_i \cup \partial X)$ which are essential in X, where $N(\alpha_i \cup \partial X)$ is a regular neighborhood of $\alpha_i \cup \partial X$ in X.

We denote the subsurface projection $\pi_0 \circ \pi_A$ by π_X. We say that α *misses* X (resp. α *cuts* X) if $\alpha \cap X = \varnothing$ (resp. $\alpha \cap X \neq \varnothing$).

Lemma 2.1 *Let X be as above. Let $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ be a path in $\mathcal{C}(S)$ such that every α_i cuts X. Then $\operatorname{diam}_X(\pi_X(\alpha_0), \pi_X(\alpha_n)) \leq 2n$.*

Proof Since $d_S(\alpha_i, \alpha_{i+1}) = 1$ and every α_i cuts X, we have

$$\operatorname{diam}_{AC(X)}(\pi_A(\alpha_i), \pi_A(\alpha_{i+1})) \le 1$$

for every $i = 0, 1, \ldots, n-1$. This together with [13, Lemma 2.2] implies

$$\operatorname{diam}_X(\pi_0(\pi_A(\alpha_i)), \pi_0(\pi_A(\alpha_{i+1})))(= \operatorname{diam}_X(\pi_X(\alpha_i), \pi_X(\alpha_{i+1}))) \le 2.$$

Since $\operatorname{diam}_X(\pi_X(\alpha_0), \pi_X(\alpha_n)) \le \sum_{i=0}^{n-1} \operatorname{diam}_X(\pi_X(\alpha_i), \pi_X(\alpha_{i+1}))$, this implies

$$\operatorname{diam}_X(\pi_X(\alpha_0), \pi_X(\alpha_n)) \le 2n. \qquad \square$$

Remark 2.2 Let X be an essential subsurface of S. Suppose that X is disconnected, and at least two components of X are nonsimple. Then for any pair of curves α, α' on S we have $\operatorname{diam}_X(\pi_X(\alpha), \pi_X(\alpha')) \le 2$. To be precise, let X_1 be one of the nonsimple components of X, and X_2 the union of the others. Let a and a' be elements of $\pi_X(\alpha)$ and $\pi_X(\alpha')$, respectively. If both a and a' are contained in X_i for some $i = 1, 2$, say X_1, then we can find a curve on X_2 that is disjoint from $a \cup a'$, which implies $d_X(a, a') \le 2$. If $a \subset X_1$ and $a' \subset X_2$ (or $a \subset X_2$ and $a' \subset X_1$), we have $d_X(a, a') \le 1$. Thus $d_X(a, a') \le 2$ for any pair of elements $a \in \pi_X(\alpha)$ and $a' \in \pi_X(\alpha')$, and hence we have $\operatorname{diam}_X(\pi_X(\alpha), \pi_X(\alpha')) \le 2$.

3 Disk complexes

Let $\mathcal{D}(V) (\subset \mathcal{C}^0(\partial_+ V))$ be the disk complex of a compression body V. We have a decomposition $\mathcal{D}(V) = \mathcal{D}_{\text{nonsep}}(V) \sqcup \mathcal{D}_{\text{sep}}(V)$, where $\mathcal{D}_{\text{nonsep}}(V)$ (resp. $\mathcal{D}_{\text{sep}}(V)$) denotes the subset of $\mathcal{D}(V)$ consisting of the vertices with representatives bounding nonseparating (resp. separating) essential disks of V. In this section, we prove the following proposition.

Proposition 3.1 Let V be a compression body obtained by adding a 1–handle to $F \times [0, 1]$, where F is a genus-$(g-1)$ closed orientable surface ($g \ge 2$). Then we have the following:

(1) $\mathcal{D}_{\text{nonsep}}(V)$ consists of a single vertex, say c_0.

(2) For each element c_α of $\mathcal{D}_{\text{sep}}(V)$, there is a 1–simplex in $\mathcal{C}(\partial_+ V)$ joining c_0 and c_α.

Remark 3.2 In fact, we can see that $\mathcal{D}_{\text{sep}}(V)$ is a countable, infinite set and that there is no 1–simplex between c_α and $c_{\alpha'}$ for each pair $c_\alpha, c_{\alpha'} \in \mathcal{D}_{\text{sep}}(V)$.

In the remainder of this section, V denotes a compression body obtained by adding a 1–handle to $F \times [0, 1]$, where F is a genus-$(g-1)$ closed orientable surface ($g \geq 2$). Then D denotes the essential disk of V corresponding to the cocore of the 1–handle. Proposition 3.1 follows from Lemmas 3.3 and 3.4 below.

Lemma 3.3 *Any nonseparating disk properly embedded in V is ambient isotopic to D.*

Proof Let D' be a nonseparating disk in V. Assume that D and D' intersect transversely, and $|D \cap D'|$ is minimized up to ambient isotopy class of D'.

Suppose $|D \cap D'| = 0$, ie, $D \cap D' = \varnothing$. Then D' is properly embedded disk in the manifold obtained from V by cutting along D, that is, $F \times [0, 1]$. Since any disk properly embedded in $F \times [0, 1]$ is boundary parallel and D' is nonseparating in V, we see that $D \cup D'$ bounds a product region, and hence D' is ambient isotopic to D.

Suppose $|D \cap D'| > 0$. By standard innermost disk arguments, we can see that $D \cap D'$ has no loop components. Note that there are at least two components of $D' \setminus D$ which are outermost in D'. Take a pair of such outermost components, say Δ_1 and Δ_2, which are next to each other, ie, there is a subarc $\beta \subset \partial D'$ such that $\beta \cap \Delta_1$ is an endpoint of β and $\beta \cap \Delta_2$ is the other endpoint of β, and β does not intersect any other outermost disk of $D' \setminus D$. Note that we can retrieve $F \times [0, 1]$ by cutting V along D. Let D^+, D^- be the copies of D in $F \times \{1\}$, and let $\overline{\Delta}_1$ (resp. $\overline{\Delta}_2$) be the closure of the image of Δ_1 (resp. Δ_2) in $F \times [0, 1]$. Note that $\overline{\Delta}_1$ and $\overline{\Delta}_2$ are disks properly embedded in $F \times [0, 1]$, and $\overline{\Delta}_i \cap (D^+ \cup D^-)$ consists of an arc properly embedded in $D^+ \cup D^-$. Let Γ_i ($i = 1, 2$) be the disk in $F \times \{1\}$ such that $\partial \Gamma_i = \partial \overline{\Delta}_i$. Without loss of generality, we may suppose $\overline{\Delta}_1 \cap (D^+ \cup D^-) = \overline{\Delta}_1 \cap D^+$. Note that if D^- is not contained in Γ_1, we can isotope D' in V via the product region between $\overline{\Delta}_1$ and Γ_1 to reduce $|D \cap D'|$, a contradiction. Hence, D^- is contained in Γ_1. Let β be the arc in $\partial D'$ as above. Then $\beta \cap D$ consists of finite number of points, say $p_0, p_1, \ldots, p_n$, where $\partial \beta = \{p_0, p_n\}$, $p_0 \in \partial \overline{\Delta}_1$, $p_n \in \partial \overline{\Delta}_2$, and $p_0, p_1, \ldots, p_n$ are arrayed on β in this order. Then a small neighborhood of p_0 in β is contained in a small neighborhood of D^- in $F \times [0, 1]$. If the other endpoint of the subarc $\overline{p_0 p_1}$ of β is contained in ∂D^-, then we see that the subarc $\overline{p_0 p_1}$ is an inessential arc in $\mathrm{Cl}(F \times \{1\} \setminus (D^+ \cup D^-))$. This shows that we can reduce $|D \cap D'|$ by an isotopy on D', a contradiction. By applying the same argument successively, we see that each subarc $\overline{p_{i-1} p_i}$ ($i = 1, 2, \ldots, n$) joins D^+ and D^-, and particularly, a small neighborhood of p_n in β is contained in a small neighborhood of D^+. This shows that $\overline{\Delta}_2 \cap (D^+ \cup D^-) = \overline{\Delta}_2 \cap D^-$. Then we see that D^+ is not contained in Γ_2, hence we have a contradiction by using the argument as above. $\square$

Let D' be a separating essential disk properly embedded in V. By an argument similar to that in the proof of Lemma 3.3, we can see that D' is ambient isotopic to a disk disjoint from D. Hence, we have the following lemma.

Lemma 3.4 *Any separating essential disk properly embedded in V can be isotoped to be disjoint from the nonseparating disk D.*

4 A pair of curves with distance exactly *n*

In this section, for each integer $n \geq 3$, we construct pairs of curves with distance exactly n. Let S be a closed connected orientable surface with genus greater than or equal to 2. We first prove Propositions 4.1 and 4.4. Then we describe the constructions of paths in $\mathcal{C}(S)$ of length n and show that they are geodesics in $\mathcal{C}(S)$.

Proposition 4.1 *For an integer $n(\geq 4)$, let $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ be a path in $\mathcal{C}(S)$ satisfying the following:*

(H1) $[\alpha_0, \ldots, \alpha_{n-2}]$ *and* $[\alpha_{n-2}, \alpha_{n-1}, \alpha_n]$ *are geodesics in $\mathcal{C}(S)$.*

(H2) $\operatorname{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_{n-4}), \pi_{X_{n-2}}(\alpha_n)) \geq 4n$, *where $X_{n-2} = \operatorname{Cl}(S \setminus N(\alpha_{n-2}))$.*

Then $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ is a geodesic in $\mathcal{C}(S)$.

Remark 4.2 In Proposition 4.1, we note that X_{n-2} is connected, ie, α_{n-2} is nonseparating in S. This can be shown by using Remark 2.2 together with the condition (H2).

Proof of Proposition 4.1 Let $[\beta_0, \beta_1, \ldots, \beta_m]$ be a geodesic in $\mathcal{C}(S)$ such that $\beta_0 = \alpha_0$, $\beta_m = \alpha_n$. Then note that $m \leq n$.

Claim 4.3 $\beta_j = \alpha_{n-2}$ *for some $j \in \{0, 1, \ldots, m\}$.*

Proof Assume on the contrary that $\beta_j \neq \alpha_{n-2}$ for any j. Then every β_j cuts X_{n-2} (ie, $\beta_j \cap X_{n-2} \neq \varnothing$) since $X_{n-2} = \operatorname{Cl}(S \setminus N(\alpha_{n-2}))$. By Lemma 2.1, we have $\operatorname{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\beta_0), \pi_{X_{n-2}}(\beta_m)) \leq 2m$. On the other hand, since $[\alpha_0, \alpha_1, \ldots, \alpha_{n-2}]$ is a geodesic, no α_i $(0 \leq i \leq n-3)$ is isotopic to α_{n-2}. Hence each α_i $(0 \leq i \leq n-3)$ cuts X_{n-2}. By Lemma 2.1, $\operatorname{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_0), \pi_{X_{n-2}}(\alpha_{n-4})) \leq 2(n-4) < 2n$. These imply

$$\operatorname{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_{n-4}), \pi_{X_{n-2}}(\alpha_n)) \leq \operatorname{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_{n-4}), \pi_{X_{n-2}}(\alpha_0))$$
$$+ \operatorname{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_0), \pi_{X_{n-2}}(\alpha_n))$$
$$< 2n + \operatorname{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\beta_0), \pi_{X_{n-2}}(\beta_m))$$
$$\leq 2n + 2m \leq 4n.$$

This contradicts the hypothesis (H2). $\qquad\qquad\square$

By Claim 4.3 and the hypothesis (H1), we have the equalities

$$j = d_S(\beta_0, \beta_j) = d_S(\alpha_0, \alpha_{n-2}) = n - 2,$$
$$m - j = d_S(\beta_j, \beta_m) = d_S(\alpha_{n-2}, \alpha_n) = 2.$$

By combining the above equalities, we have $m = n$. Recall that $[\beta_0, \beta_1, \ldots, \beta_m]$ is a geodesic in $\mathcal{C}(S)$ with $\beta_0 = \alpha_0$ and $\beta_m = \alpha_n$. Hence, $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ is a geodesic in $\mathcal{C}(S)$. □

Proposition 4.4 *For an integer $n(\geq 3)$, let $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ be a path in $\mathcal{C}(S)$ satisfying the following:*

(H1′) *$[\alpha_0, \ldots, \alpha_{n-1}]$ and $[\alpha_{n-2}, \alpha_{n-1}, \alpha_n]$ are geodesics in $\mathcal{C}(S)$.*

(H2′) *$\alpha_{n-2} \cup \alpha_{n-1}$ is nonseparating in S, and $\mathrm{diam}_{S'}(\pi_{S'}(\alpha_0), \pi_{S'}(\alpha_n)) > 2n$, where $S' = \mathrm{Cl}(S \setminus N(\alpha_{n-2} \cup \alpha_{n-1}))$.*

Then $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ is a geodesic in $\mathcal{C}(S)$.

Proof Let $[\beta_0, \beta_1, \ldots, \beta_m]$ be a geodesic in $\mathcal{C}(S)$ such that $\beta_0 = \alpha_0$, $\beta_m = \alpha_n$. Then note that $m \leq n$.

Claim 4.5 *There exists $j \in \{0, 1, \ldots, m\}$ such that $\beta_j = \alpha_{n-2}$ or $\beta_j = \alpha_{n-1}$.*

Proof Suppose that $\beta_j \neq \alpha_{n-2}$ and $\beta_j \neq \alpha_{n-1}$ for any j. Since $\alpha_{n-2} \cup \alpha_{n-1}$ is nonseparating in S, each β_j cuts S'. Hence, by Lemma 2.1, we have

$$\mathrm{diam}_{S'}(\pi_{S'}(\beta_0), \pi_{S'}(\beta_m)) \leq 2m \leq 2n.$$

On the other hand, by (H2′), $\mathrm{diam}_{S'}(\pi_{S'}(\beta_0), \pi_{S'}(\beta_m)) > 2n$, a contradiction. □

Suppose $\beta_j = \alpha_{n-2}$. Then we have the equalities

$$j = d_S(\beta_0, \beta_j) = d_S(\alpha_0, \alpha_{n-2}) = n - 2,$$
$$m - j = d_S(\beta_j, \beta_m) = d_S(\alpha_{n-2}, \alpha_n) = 2.$$

By combining the above equalities, we have $n = m$. Hence, $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ is a geodesic in $\mathcal{C}(S)$. We can use a similar argument for the case when $\beta_j = \alpha_{n-1}$. This completes the proof of Proposition 4.4. □

Construction 4.6 (The case when n is even) We first assume that n is an even integer with $n \geq 4$. Let α_0, α_2 be essential nonseparating simple closed curves on S which intersect transversely in one point, and let α_1 be an essential simple closed curve on S which is disjoint from $\alpha_0 \cup \alpha_2$. Let $X_2 = \mathrm{Cl}(S \setminus N(\alpha_2))$. Note that $[\alpha_0, \alpha_1, \alpha_2]$ is a geodesic of length two in $\mathcal{C}(S)$. Choose a homeomorphism $\varphi_2 \colon S \to S$ such that $\varphi_2(N(\alpha_2)) = N(\alpha_2)$ and that $\mathrm{diam}_{X_2}(\pi_{X_2}(\alpha_0), \pi_{X_2}(\varphi_2(\alpha_0))) \geq 4n$. This is possible by [12, Proposition 4.6]. Let $\alpha_3 = \varphi_2(\alpha_1)$ and $\alpha_4 = \varphi_2(\alpha_0)$. Note that $[\alpha_2, \alpha_3, \alpha_4]$ is a geodesic of length two in $\mathcal{C}(S)$, and $|\alpha_2 \cap \alpha_4| = 1$.
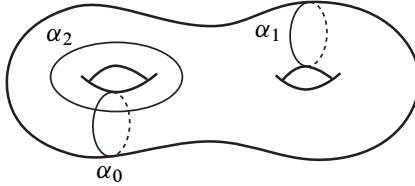

Figure 1

We repeat this process to construct a path $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ inductively as follows. Suppose we have constructed a path $[\alpha_0, \alpha_1, \ldots, \alpha_i]$ with $|\alpha_{i-2} \cap \alpha_i| = 1$ for an even integer $i \, (< n)$. Let $X_i = \mathrm{Cl}(S \setminus N(\alpha_i))$. Choose a homeomorphism $\varphi_i \colon S \to S$ such that $\varphi_i(N(\alpha_i)) = N(\alpha_i)$ and that

$$(4\text{-}1) \qquad \mathrm{diam}_{X_i}(\pi_{X_i}(\alpha_{i-2}), \pi_{X_i}(\varphi_i(\alpha_{i-2}))) \geq 4n.$$

Then we let $\alpha_{i+1} = \varphi_i(\alpha_{i-1})$ and $\alpha_{i+2} = \varphi_i(\alpha_{i-2})$. Note that $[\alpha_i, \alpha_{i+1}, \alpha_{i+2}]$ is a geodesic of length two in $\mathcal{C}(S)$, and we have obtained the path $[\alpha_0, \alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \alpha_{i+2}]$ with $|\alpha_i \cap \alpha_{i+2}| = 1$.

Assertion 4.7 *For each $k \in \{2, 4, \ldots, n\}$, the path $[\alpha_0, \alpha_1, \ldots, \alpha_k]$ in $\mathcal{C}(S)$ is a geodesic.*

Proof We prove the proposition by mathematical induction on k. It is clear that $[\alpha_0, \alpha_1, \alpha_2]$ is a geodesic in $\mathcal{C}(S)$. Hence, Assertion 4.7 holds for $k = 2$. Assume that $[\alpha_0, \alpha_1, \ldots, \alpha_k]$ is a geodesic in $\mathcal{C}(S)$ for some $k \in \{2, 4, \ldots, n-2\}$. We note that $[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]$ is a geodesic in $\mathcal{C}(S)$. Furthermore, by the inequality (4-1), we have $\mathrm{diam}_{X_k}(\pi_{X_k}(\alpha_{k-2}), \pi_{X_k}(\alpha_{k+2})) \geq 4n \geq 4(k+2)$. Hence, by Proposition 4.1, the path $[\alpha_0, \alpha_1, \ldots, \alpha_{k+2}]$ is a geodesic in $\mathcal{C}(S)$, which shows that Assertion 4.7 holds for $k+2$. This completes the proof of Assertion 4.7. $\qquad\square$

Construction 4.8 (The case when n is odd) Suppose that n is an odd integer with $n \geq 3$. Let $[\alpha_0, \alpha_1, \ldots, \alpha_{n-1}]$ be a geodesic in $\mathcal{C}(S)$ as in the previous subsection. Here,

in addition, we assume that each α_i is a nonseparating curve. (It is easy to see that this holds if we take a nonseparating curve in S for α_1 at the beginning of the construction of the geodesic.) Note that α_{n-3} intersects α_{n-1} transversely in one point and is disjoint from α_{n-2}. It is easy to see that these imply that $\alpha_{n-1} \cup \alpha_{n-2}$ is nonseparating. Choose a nonseparating essential simple closed curve γ on S such that $\gamma \cap \alpha_{n-1} = \varnothing$ and γ intersects α_{n-2} transversely in one point. Let $S' = \mathrm{Cl}(S \setminus N(\alpha_{n-2} \cup \alpha_{n-1}))$. By [12, Theorem 1.1], the diameter of $\mathcal{C}(S')$ is infinite. This shows that there exists $\gamma' \in \mathcal{C}^0(S')$ such that $d_{S'}(\gamma', \pi_{S'}(\alpha_0)) > 2n+2$. If $g > 2$, it is easy to find $\gamma'' \in \mathcal{C}^0(S')$ such that $d_{S'}(\gamma'', \gamma') \le 2$ and that γ'' cuts off a pair of pants P with $\partial N(\alpha_{n-2}) \subset \partial P$. If $g = 2$, then γ' separates S' into two pairs of pants P_1 and P_2. If $\partial N(\alpha_{n-2}) \subset \partial P_i$ ($i = 1$ or 2), then set $\gamma'' = \gamma'$, otherwise, take any essential simple closed curve γ'' in S' such that $\gamma'' \cap \gamma'$ consists of two points (ie, $d_{S'}(\gamma'', \gamma') = 1$) and that γ'' cuts off a pair of pants P with $\partial N(\alpha_{n-2}) \subset \partial P$. Since γ'' cuts off a pair of pants P with $\partial N(\alpha_{n-2}) \subset \partial P$ in either case, there is a simple closed curve $\alpha_n (\subset S)$ intersecting α_{n-2} in one point such that $\alpha_n \cap \alpha_{n-1} = \varnothing$ and that $\pi_{S'}(\alpha_n) = \gamma''$. Then we have

$$\mathrm{diam}_{S'}(\pi_{S'}(\alpha_0), \pi_{S'}(\alpha_n)) \ge \mathrm{diam}_{S'}(\pi_{S'}(\alpha_0), \gamma') - d_{S'}(\gamma'', \gamma')$$
$$> (2n + 2) - 2 = 2n.$$

On the other hand, since $\alpha_n \cap \alpha_{n-1} = \varnothing$ and α_n intersects α_{n-2} transversely in one point, $[\alpha_{n-2}, \alpha_{n-1}, \alpha_n]$ is a geodesic in $\mathcal{C}(S)$. Further, $[\alpha_0, \ldots, \alpha_{n-1}]$ is also a geodesic in $\mathcal{C}(S)$. Hence, by Proposition 4.4 together with the above inequality $\mathrm{diam}_{S'}(\pi_{S'}(\alpha_0), \pi_{S'}(\alpha_n)) > 2n$, we see that the path $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ is a geodesic in $\mathcal{C}(S)$.
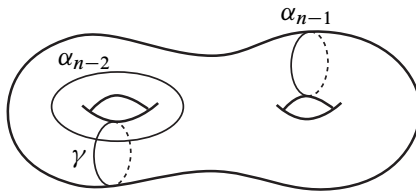

Figure 2

5　Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1 Let C_1 and C_2 be copies of the compression body obtained by adding a 1–handle to $F \times [0, 1]$, where F is a genus-$(g - 1)$ closed orientable surface ($g \ge 2$). Let α_0 be the boundary of the nonseparating essential disk D_1

properly embedded in C_1 and α_2 a simple closed curve on $\partial_+ C_1$ which intersects α_0 transversely in one point. Then we construct a geodesic $[\alpha_0, \alpha_1, \dots, \alpha_{n+2}]$ on $\partial_+ C_1$ as in Section 4. Note that α_{n+2} intersects α_n transversely in one point by the construction. Take any homeomorphism $f\colon \partial_+ C_1 \to \partial_+ C_2$ such that $f(\alpha_{n+2}) = \partial D_2$, where D_2 is the nonseparating essential disk properly embedded in C_2. We identify the boundary components $\partial_+ C_1$ and $\partial_+ C_2$ by f, and let $P = \partial_+ C_1 = f^{-1}(\partial_+ C_2)$. Then $C_1 \cup_P C_2$ is a genus-g Heegaard splitting of a compact orientable 3–manifold.

Let D_1' be a separating essential disk in C_1 disjoint from α_2 obtained as follows. Let D_1^+ and D_1^- be the components of $\mathrm{Cl}(\partial N(D_1) \setminus \partial_+ C_1)$, where $N(D_1)$ is a regular neighborhood of D_1 in C_1. Take the subarc of α_2 lying outside of the product region $N(D_1)$ between D_1^+ and D_1^-. Then D_1' is obtained from $D_1^+ \cup D_1^-$ by adding a band along the subarc of α_2. Similarly, we can obtain a separating essential disk D_2' in C_2 disjoint from α_n, by using D_2 and α_n. On the other hand, we have $d_P(\alpha_2, \alpha_n) = n-2$ since $[\alpha_0, \alpha_1, \dots, \alpha_{n+2}]$ is a geodesic in $\mathcal{C}(P)$. Hence

$$d_P(\partial D_1', \partial D_2') \le d_P(\partial D_1', \alpha_2) + d_P(\alpha_2, \alpha_n) + d_P(\alpha_n, \partial D_2')$$
$$= 1 + (n-2) + 1 = n.$$

Let $D_1'' \subset C_1$ and $D_2'' \subset C_2$ be any essential disks. By Proposition 3.1, we have $d_P(\partial D_i'', \partial D_i) \le 1$ for $i = 1, 2$. This implies

$$d_P(\partial D_1, \partial D_2) \le d_P(\partial D_1, \partial D_1'') + d_P(\partial D_1'', \partial D_2'') + d_P(\partial D_2'', \partial D_2)$$
$$\le 1 + d_P(\partial D_1'', \partial D_2'') + 1,$$

and hence

$$d_P(\partial D_1'', \partial D_2'') \ge d_P(\partial D_1, \partial D_2) - 2$$
$$= d_P(\alpha_0, \alpha_{n+2}) - 2 = (n+2) - 2 = n.$$

Therefore, $d_P(\partial D_1'', \partial D_2'') \ge n$ for any pair of essential disks $D_1'' \subset C_1$ and $D_2'' \subset C_2$, which implies $d_P(\mathcal{D}(C_1), \mathcal{D}(C_2)) \ge n$. Since $d_P(\partial D_1', \partial D_2') \le n$, we have that $d_P(\mathcal{D}(C_1), \mathcal{D}(C_2)) = n$. $\square$

In the remainder of the paper, we prove Corollary 1.2 by using the following proposition. (Throughout this paper, given an embedding $\varphi\colon X \to Y$ between compact surfaces X and Y, we abuse notation and use φ to denote the map $\mathcal{C}^0(X) \to \mathcal{C}^0(Y)$ or $\mathcal{P}(\mathcal{C}^0(X)) \to \mathcal{P}(\mathcal{C}^0(Y))$ induced by $\varphi\colon X \to Y$.)

Proposition 5.1 Let $V_1 \cup_f V_2$ be a genus-g (≥ 2) Heegaard splitting, where V_1 and V_2 are handlebodies. Let D_0 be a separating essential disk in V_1, and let $\mathcal{D}_2$ be either $\mathcal{D}(V_2)$ or a finite subset of $\mathcal{D}(V_2)$. Assume that $d_{\partial V_2}(f(\partial D_0), \mathcal{D}_2) = n \ge 3$. Then there exists a homeomorphism $g\colon \partial V_1 \to \partial V_1$ such that $d_{\partial V_2}(fg(\mathcal{D}(V_1)), \mathcal{D}_2) = n$.

Proof Let V_1^1 and V_1^2 be the closures of the two components of $V_1 \setminus D_0$. For $i = 1, 2$, let F_i be the subsurface $\partial V_1^i \cap \partial V_1$ of ∂V_1, and let $\pi_{F_i} = \pi_0 \circ \pi_A^i \colon \mathcal{C}^0(\partial V_1) \to \mathcal{P}(\mathcal{AC}^0(F_i)) \to \mathcal{P}(\mathcal{C}^0(F_i))$ be the subsurface projection introduced in Section 2. Let $P_i \colon F_i \to F_i \cup D_0$ be the inclusion map. Since D_0 is separating, the image of any essential simple closed curve in F_i by P_i is essential in $F_i \cup D_0$. This immediately implies the following.

Claim 5.2 *For any nonempty subset E of $\mathcal{C}^0(F_i)$, we have that*

- *$P_i(E)$ is nonempty,*
- *$\mathrm{diam}_{F_i \cup D_0}(P_i(E)) \leq \mathrm{diam}_{F_i}(E)$.*

We note that there exists a constant N such that

$$(5\text{-}1) \qquad\qquad \mathrm{diam}_{F_i}(\pi_{F_i} f^{-1}(\mathcal{D}_2)) \leq N \quad (i = 1, 2).$$

In fact, if $\mathcal{D}_2$ is a finite subset of $\mathcal{D}(V_2)$, this is clear. Thus assume $\mathcal{D}_2 = \mathcal{D}(V_2)$. We claim that the condition $d_{\partial V_2}(f(\partial D_0), \mathcal{D}_2) \geq 3$ implies the pair $(V_2, f(F_1) \cup f(F_2))$ is not homeomorphic to a ($[0, 1]$–bundle, the associated $\partial[0, 1]$–bundle). In fact, otherwise, we have a contradiction as follows. Take an essential arc α and an essential simple closed curve l in $f(F_1)$. If $(V_2, f(F_1) \cup f(F_2))$ is not homeomorphic to a $[0, 1]$–bundle, then $f(\partial D_0)$, l and $\alpha \times [0, 1]$ give a path of length 2. Further, $\alpha \times [0, 1]$ is an essential disk in V_2, a contradiction to $d_{\partial V_2}(f(\partial D_0), \mathcal{D}_2) \geq 3$. Hence, by [10, Theorem 1] together with the assumption $d_{\partial V_2}(f(\partial D_0), \mathcal{D}_2) \geq 3$, we see that $\mathrm{diam}_{f(F_i)}(\pi_{f(F_i)}(\mathcal{D}_2)) \leq 12$, which means $\mathrm{diam}_{F_i}(\pi_{F_i} f^{-1}(\mathcal{D}_2)) \leq 12$. By Claim 5.2, the inequality (5-1) implies

$$(5\text{-}2) \qquad\qquad \mathrm{diam}_{F_i \cup D_0}(P_i \pi_{F_i} f^{-1}(\mathcal{D}_2)) \leq N \quad (i = 1, 2).$$

Let $\mathcal{D}'(V_1^i)$ be the subset of $\mathcal{C}^0(F_i)$ consisting of simple closed curves that bound disks in V_1^i ($i = 1, 2$). By the inequality (5-2) and [7] (see also [1]), we see that there exists a homeomorphism $g \colon \partial V_1 \to \partial V_1$ such that $g(\partial D_0) = \partial D_0$ and

$$(5\text{-}3) \qquad\qquad d_{F_i \cup D_0}(P_i(\mathcal{D}'(V_1^i)), \widehat{g}_i^{-1}(P_i \pi_{F_i} f^{-1}(\mathcal{D}_2))) \geq 2n$$

for each $i = 1, 2$, where $\widehat{g}_i \colon F_i \cup D_0 \to F_i \cup D_0$ is a homeomorphism obtained by extending $g|_{F_i} \colon F_i \to F_i$. (We note that $g|_{F_i} \colon F_i \to F_i$ extends to a homeomorphism $\widehat{g}_i \colon F_i \cup D_0 \to F_i \cup D_0$ in a unique way up to isotopy in D_0 by Alexander's trick.) Since $g(\partial D_0) = \partial D_0$, it is easy to see that $\widehat{g}_i^{-1}(P_i \pi_{F_i} f^{-1}(\mathcal{D}_2)) = P_i(g|_{F_i})^{-1} \pi_{F_i} f^{-1}(\mathcal{D}_2) = P_i \pi_{F_i} g^{-1} f^{-1}(\mathcal{D}_2) = P_i \pi_{F_i}(fg)^{-1}(\mathcal{D}_2)$. We denote the

map $P_i \pi_{F_i}(fg)^{-1} (= \hat{g}_i^{-1} P_i \pi_{F_i} f^{-1}) \colon \mathcal{C}^0(\partial V_2) \to \mathcal{P}(\mathcal{C}^0(F_i \cup D_0))$ by Φ_i. Then, by the inequality (5-3), we have

$$(5\text{-}4) \qquad d_{F_i \cup D_0}(P_i(\mathcal{D}'(V_1^i)), \Phi_i(\mathcal{D}_2)) \geq 2n \quad (i = 1, 2).$$

Note that $d_{\partial V_2}(fg(\mathcal{D}(V_1)), \mathcal{D}_2) \leq n$ since $f(\partial D_0) = fg(\partial D_0) \in fg(\mathcal{D}(V_1))$ and $d_{\partial V_2}(f(\partial D_0), \mathcal{D}_2) = n$ by the assumption. To prove $d_{\partial V_2}(fg(\mathcal{D}(V_1)), \mathcal{D}_2) = n$, assume on the contrary that $d_{\partial V_2}(fg(\mathcal{D}(V_1)), \mathcal{D}_2) < n$, or equivalently, $d_{\partial V_1}(\mathcal{D}(V_1), (fg)^{-1}(\mathcal{D}_2)) < n$. Then there exist $a \in \mathcal{D}(V_1)$ and $b \in \mathcal{D}_2$ such that

$$(5\text{-}5) \qquad d_{\partial V_1}(a, (fg)^{-1}(b)) = m < n.$$

Let $[\gamma_0, \gamma_1, \ldots, \gamma_m]$ be a geodesic in $\mathcal{C}(\partial V_1)$ from a to $(fg)^{-1}(b)$.

Claim 5.3 *Every γ_j $(j = 1, 2, \ldots, m)$ cuts both F_1 and F_2.*

Proof Assume that γ_j does not cut F_i for some $j \in \{1, 2, \ldots, m\}$ and some $i \in \{1, 2\}$. Then γ_j is disjoint from $\partial D_0 (= \partial F_1 = \partial F_2)$, and hence we have

$$\begin{aligned} n &= d_{\partial V_1}(\partial D_0, (fg)^{-1}(\mathcal{D}_2)) \\ &\leq d_{\partial V_1}(\partial D_0, \gamma_j) + d_{\partial V_1}(\gamma_j, (fg)^{-1}(\mathcal{D}_2)) \\ &\leq d_{\partial V_1}(\partial D_0, \gamma_j) + d_{\partial V_1}(\gamma_j, \gamma_m) \leq 1 + (m - j) < 1 + n - j, \end{aligned}$$

a contradiction. $\qquad \square$

Let D_a be a disk in V_1 bounded by a. We may assume that $|D_a \cap D_0|$ is minimal. By using innermost disk arguments, we see that $D_a \cap D_0$ has no loop components.

Case 1: $|D_a \cap D_0| \neq 0$ Let Δ be a component of $D_a \setminus D_0$ that is outermost in D_a. Then $\Delta \subset V_1^i$ for some $i = 1, 2$. Without loss of generality, we may assume that $\Delta \subset V_1^1$, which implies $a(= \gamma_0)$ cuts F_1. This, together with Claim 5.3, shows that every γ_j $(j = 0, 1, \ldots, m)$ in the geodesic $[\gamma_0, \gamma_1, \ldots, \gamma_m]$ from a to $(fg)^{-1}(b)$ cuts F_1. Hence, by Lemma 2.1, we have

$$(5\text{-}6) \qquad \mathrm{diam}_{F_1}(\pi_{F_1}(a), \pi_{F_1}(fg)^{-1}(b)) \leq 2m < 2n,$$

which implies, by Claim 5.2,

$$(5\text{-}7) \qquad \mathrm{diam}_{F_1 \cup D_0}(P_1 \pi_{F_1}(a), \Phi_1(b)) < 2n.$$

Note that $\partial \Delta \cap F_1$ is an element of the image of a by $\pi_A^1 \colon \mathcal{C}^0(\partial V_1) \to \mathcal{P}(\mathcal{AC}^0(F_1))$. Further, by the minimality of $|D_a \cap D_0|$, the disk Δ is essential in V_1^1. Let D_0^1 and D_0^2 be the two components of $D_0 \setminus \Delta$, and let Δ' be one of the disks properly embedded in V_1^1 which is parallel to $D_0^1 \cup \Delta$ or $D_0^2 \cup \Delta$. Then we have $\partial \Delta' \in P_1(\mathcal{D}'(V_1^1))$,

and also $\partial\Delta' \in P_1\pi_0(\partial\Delta \cap F_1) \subset P_1\pi_0\pi_A^1(a) = P_1\pi_{F_1}(a)$. These, together with the inequality (5-7), imply

$$d_{F_1 \cup D_0}(P_1(\mathcal{D}'(V_1^1)), \Phi_1(\mathcal{D}_2)) \le d_{F_1 \cup D_0}(\partial\Delta', \Phi_1(b))$$
$$\le \operatorname{diam}_{F_1 \cup D_0}(P_1\pi_{F_1}(a), \Phi_1(b)) < 2n,$$

which contradicts the inequality (5-4).

Case 2: $|D_a \cap D_0| = 0$ In this case, the arguments in Case 1 work with regarding $D_a = \Delta'$ to have a contradiction.

The above contradictions give $d_{\partial V_2}(fg(\mathcal{D}(V_1)), \mathcal{D}_2) = n$. □

Remark 5.4 If we pose the assumption that the distance $d(V_1 \cup_f V_2)$ of the genus-g Heegaard splitting $V_1 \cup_f V_2$ is greater than or equal to 2 in Proposition 5.1, then the statement of the proposition can be strengthened as in the following form.

Let D_0 be a separating essential disk in V_1, and let $\mathcal{D}_2$ be any subset of $\mathcal{D}(V_2)$. If $d_{\partial V_2}(f(\partial D_0), \mathcal{D}_2) = n$, then there exists a homeomorphism $g: \partial V_1 \to \partial V_1$ such that $d_{\partial V_2}(fg(\mathcal{D}(V_1)), \mathcal{D}_2) = n$.

In fact, the statement can be proved basically by using the arguments of the proof of Proposition 5.1. The difference is the proof of inequality (5-1). We should replace the argument with the following.

Note that $f(\partial D_0)(= f(\partial F_1) = f(\partial F_2))$ intersects with every essential loop in $\mathcal{D}(V_2)$, since $d_{\partial V_2}(f(\partial D_0), \mathcal{D}(V_2)) \ge d(V_1 \cup_f V_2) \ge 2$. By [10, Theorem 1], either

$$(5\text{-}8) \qquad \operatorname{diam}_{F_i}(\pi_{F_i} f^{-1}(\mathcal{D}_2)) \le \operatorname{diam}_{F_i}(\pi_{F_i} f^{-1}(\mathcal{D}(V_2))) \le 12$$

or V_2 is a $[0,1]$–bundle over $f(F_1)$. In the latter case, it is easy to see that g must be even and that the union of V_2 and $N(D_0)$ is homeomorphic to a $[0,1]$–bundle over a closed surface, say S, of genus $g/2$. Note that the exterior of the union of V_2 and $N(D_0)$ is $\operatorname{Cl}(V_1 \setminus N(D_0))$ and consists of two handlebodies of genus $g/2$. Thus, S is a Heegaard surface of genus $g/2$, and $\partial V_2(= f(\partial V_1))$ is a stabilization of S. This implies $d(V_1 \cup_f V_2) = 0$, a contradiction. Hence, we have the inequality (5-8).

Proof of Corollary 1.2 We first note that the proof of the corollary for the case when $n = 2$ is exceptional, and we give it in the Appendix of this paper, and in this proof we show the corollary for the case $n \ge 3$. Let $C_1 \cup_P C_2 = C_1 \cup_f C_2$ be a genus-g Heegaard splitting with distance $n(\ge 3)$ obtained in Theorem 1.1. By the proof of Theorem 1.1, there are separating essential disks D_1 and D_2 in C_1 and C_2, respectively, such that $d_{\partial_+ C_2}(f(\partial D_1), \partial D_2) = n$. Let H_i $(i = 1, 2)$ be a handlebody of genus $(g - 1)$. Take

and fix any homeomorphism $h_i \colon \partial H_i \to \partial_- C_i$, and put $V_i := C_i \cup_{h_i} H_i$ (hence, V_i is a handlebody of genus g). Then $V_1 \cup_f V_2$ is a genus-g Heegaard splitting.

By Proposition 5.1, there exists a homeomorphism $g_1 \colon \partial V_1 \to \partial V_1$ such that

$$d_{\partial V_2}(fg_1(\mathcal{D}(V_1)), \partial D_2) = n.$$

By applying Proposition 5.1 again to $V_2 \cup_{(fg_1)^{-1}} V_1$, we see that there exists a homeomorphism $g_2 \colon \partial V_2 \to \partial V_2$ such that

$$d_{\partial V_1}((fg_1)^{-1} g_2(\mathcal{D}(V_2)), \mathcal{D}(V_1)) = n.$$

That is, the distance of the Heegaard splitting $V_1 \cup_{g_2^{-1} fg_1} V_2$ is exactly n. $\qquad \square$

Appendix: A construction of distance 2 examples

In this Appendix, we show for each $g \geq 2$, there is a genus-g Heegaard splitting of a closed 3–manifold with distance 2. The examples are given by using the construction of strongly irreducible Heegaard splittings by Kobayashi and Rieck in [9]. For the description of the construction we will use the notation $(H, A_1 \cup A_2)$, N, R etc from [9, Section 2.1].

For the case when $g = 2$, let F be an annulus, and let $R = F \times [0, 1]$. For the case when $g \geq 3$, let F be a genus-$(g - 2)$ nonorientable surface (connected sum of $g - 2$ copies of projective planes) with two holes, and let R be the orientable twisted $[0, 1]$–bundle over F. Note that F is homotopy equivalent to a bouquet of $g - 1$ circles, hence R is homeomorphic to the genus-$(g - 1)$ handlebody. Let R' be a copy of R. Then let $\mathcal{A}_1 \cup \mathcal{A}_2$ (resp. $\mathcal{A}'_1 \cup \mathcal{A}'_2$) be the union of annuli in ∂R (resp. $\partial R'$) corresponding to the $[0, 1]$–bundle over ∂F. Then let N be the manifold obtained from R and R' by identifying the subsurfaces of the boundaries corresponding to the associated $\partial [0, 1]$–bundle. It is easy to see that the manifolds N, R and R' satisfy [9, page 639, Conditions (1)–(3)].

Recall from [9] that H is a genus-2 handlebody, and $\{A_1, A_2\}$ is a pair of primitive annuli in ∂H. Let $(H', A'_1 \cup A'_2)$ be a copy of $(H, A_1 \cup A_2)$. Then it is observed in [9] that for any 2–bridge link L in S^3 there is a homeomorphism $h \colon \mathrm{Cl}(\partial H \setminus (A_1 \cup A_2)) \to \mathrm{Cl}(\partial H' \setminus (A'_1 \cup A'_2))$ such that the manifold obtained from H and H' by identifying $\mathrm{Cl}(\partial H \setminus (A_1 \cup A_2))$ and $\mathrm{Cl}(\partial H' \setminus (A'_1 \cup A'_2))$ by h is homeomorphic to the exterior $E(L)$ of L. Then let M be the 3–manifold obtained from $E(L)$ and N by identifying their boundaries by an orientation-reversing homeomorphism such that $\mathcal{A}_i$ (resp. $\mathcal{A}'_i$) is identified with A_i (resp. A'_i). Then it is shown

in [9, Section 2.1] that $H \cup R$ and $H' \cup R'$ are genus-g handlebodies, and these handlebodies give a Heegaard splitting of M.

Then we have the following.

Assertion *Suppose that the 2–bridge link L is not a trivial link or a Hopf link, then the distance of the Heegaard splitting $(H \cup R) \cup (H' \cup R')$ is exactly 2.*

Proof Since L is not a trivial link or a Hopf link, we see, by [9, Proposition 2.1], that $(H \cup R) \cup (H' \cup R')$ is strongly irreducible, ie the distance of the Heegaard splitting is greater than or equal to 2. On the other hand, since $\partial E(L) \, (= \partial N) \subset M$ is an essential torus, we see, by Hartshorn [5], that the distance of any Heegaard splitting of M is at most 2, and this together with the above shows that the distance of the Heegaard splitting $(H \cup R) \cup (H' \cup R')$ is exactly 2. $\qquad\square$

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