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Low-dimensional linear representations of the mapping class group of a nonorientable surface

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Suppose that f is a homomorphism from the mapping class group $\mathcal{M}(N_{g,n})$ of a nonorientable surface of genus g with n boundary components to $\mathrm{GL}(m,\mathbb{C})$. We prove that if $g\geq 5$, $n\leq 1$ and $m\leq g-2$, then f factors through the abelianization of $\mathcal{M}(N_{g,n})$, which is $\mathbb{Z}_2\times\mathbb{Z}_2$ for $g\in\{5,6\}$ and $\mathbb{Z}_2$ for $g\geq 7$. If $g\geq 7$, $n=0$ and $m=g-1$, then either f has finite image (of order at most two if $g\neq 8$), or it is conjugate to one of four "homological representations". As an application we prove that for $g\geq 5$ and $h<g$, every homomorphism $\mathcal{M}(N_{g,0})\to\mathcal{M}(N_{h,0})$ factors through the abelianization of $\mathcal{M}(N_{g,0})$.

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1 Introduction

For a compact surface F, its *mapping class group* $\mathcal{M}(F)$ is the group of isotopy classes of all, orientation-preserving if F is orientable, homeomorphisms $F\to F$ equal to the identity on the boundary of F. A compact surface of genus g with n boundary components will be denoted by $S_{g,n}$ if it is orientable, and by $N_{g,n}$ if it is nonorientable. If $n=0$ then we drop it in the notation and write simply S_g or N_g. The first integral homology group of F will be denoted by $H_1(F)$.

After fixing a basis of $H_1(S_g)$, the action of $\mathcal{M}(S_g)$ on $H_1(S_g)$ gives rise to a homomorphism $\mathcal{M}(S_g)\to\mathrm{Sp}(2g,\mathbb{Z})$, which is well-known to be surjective, and whose kernel is known as the Torelli group. Gluing a disc along each boundary component of $S_{g,n}$ induces an epimorphism $\mathcal{M}(S_{g,n})\to\mathcal{M}(S_g)$, and by composing it with $\mathcal{M}(S_g)\to\mathrm{Sp}(2g,\mathbb{Z})$, and then with the inclusion $\mathrm{Sp}(2g,\mathbb{Z})\hookrightarrow\mathrm{GL}(2g,\mathbb{C})$, we obtain the map $\Phi\colon\mathcal{M}(S_{g,n})\to\mathrm{GL}(2g,\mathbb{C})$. Recently, the following two results were proved by J Franks, M Handel and M Korkmaz.

Theorem 1.1 (Franks and Handel [8], and Korkmaz [15]) *Let $g\geq 2$, $m\leq 2g-1$ and let $f\colon\mathcal{M}(S_{g,n})\to\mathrm{GL}(m,\mathbb{C})$ be a homomorphism. Then f is trivial if $g\geq 3$, and $\mathrm{Im}(f)$ is a quotient of $\mathbb{Z}_{10}$ if $g=2$.*

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We say that two homomorphisms f_1, f_2 from a group G to a group H are *conjugate* if there exits $h \in H$ such that $f_2(x) = hf_1(x)h^{-1}$ for $x \in G$.

Theorem 1.2 (Korkmaz [16]) *For $g \geq 3$, every nontrivial homomorphism*

$$f\colon \mathcal{M}(S_{g,n}) \to \mathrm{GL}(2g, \mathbb{C})$$

is conjugate to the map Φ.

In this paper we prove analogous results for $\mathcal{M}(N_g)$. Fix $g \geq 3$. Let R_g denote the quotient of $H_1(N_g)$ by its torsion. Hence R_g is a free $\mathbb{Z}$–module of rank $g-1$. There is a covering $P\colon S_{g-1} \to N_g$ of degree two. By a theorem of Birman and Chillingworth [3], $\mathcal{M}(N_g)$ is isomorphic to the subgroup of $\mathcal{M}(S_{g-1})$ consisting of the isotopy classes of orientation-preserving lifts of homeomorphisms of N_g, which gives an action of N_g on $H_1(S_{g-1})$. Let $K_g \subset H_1(S_{g-1})$ be the kernel of the composition of the induced map $P_*\colon H_1(S_{g-1}) \to H_1(N_g)$ with the canonical projection $H_1(N_g) \to R_g$. Then K_g is a $\mathcal{M}(N_g)$–invariant subgroup of rank $g-1$ and we have two homomorphisms

$$\Psi_1\colon \mathcal{M}(N_g) \to \mathrm{GL}(K_g) \quad \text{and} \quad \Psi_2\colon \mathcal{M}(N_g) \to \mathrm{GL}(H_1(S_{g-1})/K_g),$$

which after fixing bases will be treated as representations of $\mathcal{M}(N_g)$ in $\mathrm{GL}(g-1, \mathbb{C})$. One may see Ψ_1 and Ψ_2 as stemming from the actions of $\mathcal{M}(N_g)$ on homology groups of N_g with (local) coefficients in $\mathbb{Z}$ with non-trivial and trivial $\mathbb{Z}[\pi_1(N_g)]$–module structure respectively (see Remark 4.4). We will see that these representations are not conjugate, although $\ker\Psi_1 = \ker\Psi_2$.

Our first result is the following.

Theorem 1.3 *Suppose that $n \leq 1$, $g \geq 5$, $m \leq g-2$ and $f\colon \mathcal{M}(N_{g,n}) \to \mathrm{GL}(m, \mathbb{C})$ is a nontrivial homomorphism. Then $\mathrm{Im}(f)$ is either $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, the latter case being possible only for $g = 5$ or 6.*

Theorem 1.3 was proved in [15], in the more general setting of punctured surfaces, under the additional assumption that $m \leq g-3$ if g is even. Therefore the only novelty of our result is that it also covers the case $m = g-2$ for even g. As an application of Theorem 1.3, we prove the following result, which solves Problem 3.3 in [19].

Theorem 1.4 *Suppose that $g \geq 5$, $h < g$ and $f\colon \mathcal{M}(N_g) \to \mathcal{M}(N_h)$ is a nontrivial homomorphism. Then $\mathrm{Im}(f)$ is as in Theorem 1.3.*

The analogous theorem for mapping class groups of orientable surfaces was proved by Harvey and Korkmaz [13]. In the orientable case the two surfaces need not be closed, as shown in recent theorems of Castel [5], and Aramayona and Souto [1].

We will prove that both Theorem 1.3 and Theorem 1.4 fail for $g=4$, by showing that there is a homomorphism from $\mathcal{M}(N_4)$ to $\mathcal{M}(N_3)\cong \mathrm{GL}(2,\mathbb{Z})$ whose image is isomorphic to the infinite dihedral group.

Suppose that $g\geq 7$. Then the abelianization of $\mathcal{M}(N_g)$ is $\mathbb{Z}_2$ and we denote by $\mathrm{ab}\colon \mathcal{M}(N_g)\to\mathbb{Z}_2$ the canonical projection. For $i=1,2$ we set $\Psi_i'=(-1)^{\mathrm{ab}}\Psi_i$. Our next result is the following.

Theorem 1.5 *Suppose that $g\geq 7$, $g\neq 8$ and $f\colon \mathcal{M}(N_g)\to \mathrm{GL}(g-1,\mathbb{C})$ is a nontrivial homomorphism. Then either $\mathrm{Im}(f)\cong\mathbb{Z}_2$, or f is conjugate to one of Ψ_1, Ψ_1', Ψ_2, Ψ_2'.*

For $g=8$, other representations of $\mathcal{M}(N_8)$ in $\mathrm{GL}(7,\mathbb{C})$ occur, related to the fact that there is an epimorphism $\epsilon\colon \mathcal{M}(N_8)\to \mathrm{Sp}(6,\mathbb{Z}_2)$ and the last group admits irreducible representations in $\mathrm{GL}(7,\mathbb{C})$ (see [4]). We prove the following result.

Theorem 1.6 *Suppose that $f\colon \mathcal{M}(N_8)\to \mathrm{GL}(7,\mathbb{C})$ is a nontrivial homomorphism. Then one of the following holds.*

(1) $\mathrm{Im}(f)\cong\mathbb{Z}_2$

(2) *f or $(-1)^{\mathrm{ab}}f$ factors through $\epsilon\colon \mathcal{M}(N_8)\to \mathrm{Sp}(6,\mathbb{Z}_2)$.*

(3) *f is conjugate to one of Ψ_1, Ψ_1', Ψ_2, Ψ_2'.*

To prove our theorems we use the ideas and results from [8; 15; 16] with necessary modifications. While the case of odd genus is relatively easy, the case of even genus requires much more effort. This phenomenon is typical for the mapping class group of a nonorientable surface.

Throughout this paper we will often have to solve an equation of the form $L=R$, where L and R are products of matrices from $\mathrm{GL}(m,\mathbb{C})$ with some unknown coefficients. Although the dimension m is variable, the calculations of L and R always reduce to multiplication of blocks of size at most 7×7. With some patience, such calculations could be done by hand, but it is definitely easier to use a computer. We used GAP, but of course, any program that performs symbolic operations on matrices could be used as well.

2 Notation and algebraic preliminaries

Suppose that $m\geq 2$ is fixed. We denote by I_m the identity matrix of dimension m. We will sometimes write simply I, if m is clear from the context. We denote by

E_{ij} the elementary matrix with 1 on the position (i,j) and 0 elsewhere. Suppose that $M_1,\dots,M_k$ are nonsingular square matrices of dimensions $m_1,\dots,m_k$, where $m_1+\cdots+m_k=m$. Then we denote by $\mathrm{diag}(M_1,\dots,M_k)$ the $m\times m$ matrix with $M_1,\dots,M_k$ on the main diagonal and zeros elsewhere. Set

$$V=\begin{pmatrix}1&1\\0&1\end{pmatrix},\quad \widehat{V}=\begin{pmatrix}1&0\\-1&1\end{pmatrix},\quad W=\begin{pmatrix}1&1&0&-1\\0&1&0&0\\0&-1&1&1\\0&0&0&1\end{pmatrix}.$$

For $2\le 2i\le m$ we define

$$A_i=\mathrm{diag}(I_{2i-2},V,I_{m-2i}),\quad B_i=\mathrm{diag}(I_{2i-2},\widehat{V},I_{m-2i}),$$

and for $2\le 2j\le m-2$,

$$C_j=\mathrm{diag}(I_{2j-2},W,I_{m-2-2j}).$$

The proof of the following lemma is straightforward and we leave it as an exercise (cf [16, Lemma 2.2]).

Lemma 2.1 *Suppose that $1\le k\le l\le m/2$ and $M\in\mathrm{GL}(m,\mathbb{C})$ satisfies $A_iM=MA_i$, $B_iM=MB_i$ and $C_jM=MC_j$ for all i,j such that $k\le i\le l$, $k\le j\le l-1$. Then M has the form*

$$\begin{pmatrix}*&0&*\\0&\lambda I_{2(l-k+1)}&0\\ *&0&*\end{pmatrix}$$

for some $\lambda\in\mathbb{C}^*$, where the top left λ of the block $\lambda I_{2(l-k+1)}$ is at the position $(2k-1,2k-1)$.

Suppose that $L\in\mathrm{GL}(m,\mathbb{C})$ and λ is an eigenvalue of L. Then we denote by $\#\lambda$ the multiplicity of λ. For $k\ge 1$, we denote by $E^k(L,\lambda)$ the space $\ker(E-\lambda I)^k$. Thus $E^1(L,\lambda)$ is the eigenspace of L with respect to λ, and it will be also denoted by $E(L,\lambda)$. Note that if $L'\in\mathrm{GL}(m,\mathbb{C})$ commutes with L, then the spaces $E^k(L,\lambda)$ are L'–invariant for $k\ge 1$.

For $k\ge 2$ we denote by $\mathfrak{S}_k$ the full symmetric group of the set $\{1,\dots,k\}$. It is generated by the transpositions $\sigma_i=(i,i+1)$ for $1\le i\le k-1$. We will need the following result from the representation theory of the symmetric group; see for example Fulton and Harris [9, Exercise 4.14].

Lemma 2.2 *For $k\ge 5$, $\mathfrak{S}_k$ has no irreducible representation (over $\mathbb{C}$) of dimension $1<m<k-1$. If $k\ge 7$, then $\mathfrak{S}_k$ has two irreducible representations of dimension $k-1$: the standard one and the tensor product of the standard and sign representations.*

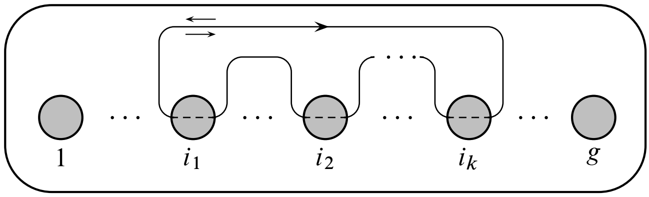

Figure 1: The surface $N_{g,n}$ and the curve ξ_I for $I = \{i_1, i_2, \ldots, i_k\}$

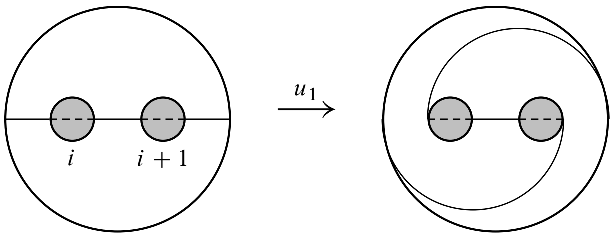

Figure 2: The crosscap transposition u_i

3 Mapping class group of a nonorientable surface

Let $n \in \{0, 1\}$ and $g \geq 2$. Let us represent $N_{g,n}$ as a sphere (if $n = 0$) or a disc (if $n = 1$) with g crosscaps. This means that the interiors of g small pairwise disjoint discs should be removed from the sphere/disc, and then antipodal points in each of the resulting boundary components should be identified. Let us arrange the crosscaps as shown on Figure 1 and number them from 1 to g. For each nonempty subset $I \subseteq \{1, \ldots, g\}$ let ξ_I be the simple closed curve shown on Figure 1. Note that ξ_I is two-sided if and only if I has even number of elements. In this case t_{ξ_I} will be the Dehn twist about γ_I in the direction indicated by arrows on Figure 1.

We will write ξ_i instead of $\xi_{\{i\}}$. The following curves will play a special role and so we give them different names:

- $\delta_i = \xi_{\{i,i+1\}}$ for $1 \leq i \leq g-1$
- $\varepsilon_j = \xi_{\{1,2,\ldots,2j\}}$ for $2 \leq 2j \leq g$

Note that $\varepsilon_1 = \delta_1$.

For $1 \leq i \leq g-1$ we define the *crosscap transposition* u_i to be the isotopy class of the homeomorphism interchanging the i^{th} and the $(i+1)^{\text{st}}$ crosscaps as shown on Figure 2, and equal to the identity outside a disc containing these crosscaps.

The groups $\mathcal{M}(N_{1,n})$ are trivial for $n \le 1$ by Epstein [7, Theorem 3.4], and $\mathcal{M}(N_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by Lickorish [21]. Birman and Chillingworth obtained in [3, Theorem 3] a finite presentation for $\mathcal{M}(N_3)$, from which it is easy to deduce that this group is isomorphic to $\mathrm{GL}(2,\mathbb{Z})$. A direct geometric proof of this fact is given in González-Acuña and Márquez-Bobadilla [12]. For $g \ge 3$, a finite generating set for $\mathcal{M}(N_{g,n})$ was given in Chillingworth [6] for $n = 0$ and in Stukow [25] for $n > 0$. For $n \le 1$ this set can be reduced to the one given in the following theorem, which can be deduced from the main result of Paris and Szepietowski [23].

Theorem 3.1 *For $g \ge 4$ and $n \in \{0,1\}$, $\mathcal{M}(N_{g,n})$ is generated by u_{g-1}, t_{ε_2} and t_{δ_i} for $1 \le i \le g-1$.*

If $n > 1$, then we consider $N_{g,n}$ as the result of gluing $S_{0,n+1}$ to $N_{g,1}$ along the boundary component. We will need the following relations, satisfied in $\mathcal{M}(N_{g,n})$. Those between Dehn twists are the well-known disjointness and braid relations.

(R1) $t_{\delta_i} t_{\delta_j} = t_{\delta_j} t_{\delta_i}$ for $|i-j| > 1$

(R2) $t_{\varepsilon_i} t_{\varepsilon_j} = t_{\varepsilon_j} t_{\varepsilon_i}$ for all i, j

(R3) $t_{\varepsilon_i} t_{\delta_j} = t_{\delta_j} t_{\varepsilon_i}$ for $j \ne 2i$

(R4) $t_{\delta_i} t_{\delta_{i+1}} t_{\delta_i} = t_{\delta_{i+1}} t_{\delta_i} t_{\delta_{i+1}}$ for $1 \le i \le g-2$

(R5) $t_{\varepsilon_i} t_{\delta_{2i}} t_{\varepsilon_i} = t_{\delta_{2i}} t_{\varepsilon_i} t_{\delta_{2i}}$ for $2i < g$

The relations involving crosscap transpositions are not so well known and we refer the reader to Paris and Szepietowski [23], and Szepietowski [28], for their proofs.

(R6) $t_{\delta_i} u_j = u_j t_{\delta_i}$ for $|i-j| > 1$

(R7) $u_i u_j = u_j u_i$ for $|i-j| > 1$

(R8) $t_{\varepsilon_i} u_j = u_j t_{\varepsilon_i}$ for $j > 2i$

(R9) $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ for $1 \le i \le g-2$

(R10) $t_{\delta_i} u_{i+1} u_i = u_{i+1} u_i t_{\delta_{i+1}}$ for $1 \le i \le g-2$

(R11) $u_{i+1} t_{\delta_i} t_{\delta_{i+1}} u_i = t_{\delta_i} t_{\delta_{i+1}}$ for $1 \le i \le g-2$

(R12) $t_{\delta_i} u_i t_{\delta_i} = u_i$ for $1 \le i \le g-1$

If follows from (R4) that all t_{δ_i} are conjugate for $1 \le i \le g-1$, from (R5) that t_{ε_j} is conjugate to $t_{\delta_{2j}}$ for $2j < g$, and from (R12) that t_{δ_i} is conjugate to $t_{\delta_i}^{-1}$. Similarly, by (R9) all u_i are conjugate for $1 \le i \le g-1$, and by (R11) u_i is conjugate to u_i^{-1}.

For a group G we denote the abelianization $G/[G,G]$ by G^{ab}. The following theorem is proved in Korkmaz [17] for $n = 0$ and generalised to $n > 0$ in Stukow [25].

Theorem 3.2 *For $n \leq 1$ and $g \geq 3$, $\mathcal{M}(N_{g,n})^{\mathrm{ab}}$ has the following presentation as a $\mathbb{Z}$–module:*

$$\begin{array}{ll} \langle [t_{\delta_1}], [t_{\varepsilon_2}], [u_1] \mid 2[t_{\delta_1}] = 2[t_{\varepsilon_2}] = 2[u_1] = 0 \rangle & \text{if } g = 4 \\ \langle [t_{\delta_1}], [u_1] \mid 2[t_{\delta_1}] = 2[u_1] = 0 \rangle & \text{if } g \in \{3, 5, 6\} \\ \langle [u_1] \mid 2[u_1] = 0 \rangle & \text{if } g \geq 7 \end{array}$$

In particular, for $g \geq 7$ we have $[t_{\delta_1}] = 0$.

Lemma 3.3 *For $g \geq 5$ and $n \leq 1$, let α, β be two-sided curves on $N_{g,n}$, intersecting transversally in one point. If $f\colon \mathcal{M}(N_{g,n}) \to G$ is a homomorphism, such that $f(t_\alpha)$ commutes with $f(t_\beta)$, then $\mathrm{Im}(f)$ is abelian.*

Proof Let $N = N_{g,n}$ and $\mathcal{M} = \mathcal{M}(N_{g,n})$. Fix a regular neighbourhood A of $\alpha \cup \beta$. Note that A is homeomorphic to $S_{1,1}$ and $N \backslash A$ is homeomorphic to $N_{g-2,1}$. It follows that for each $i \leq g-2$ there is a homeomorphism $h\colon N \to N$ such that $h(\alpha) = \delta_i$ and $h(\beta) = \delta_{i+1}$. It follows that $ht_\alpha h^{-1} = t_{\delta_i}^{\varepsilon_1}$ and $ht_\beta h^{-1} = t_{\delta_{i+1}}^{\varepsilon_2}$, where $\varepsilon_j \in \{-1, 1\}$ for $j = 1, 2$. Hence $f(t_{\delta_i})$ commutes with $f(t_{\delta_{i+1}})$, and by the braid relation (R4), $f(t_{\delta_i}) = f(t_{\delta_{i+1}})$. Analogously, $f(t_{\varepsilon_2}) = f(t_{\delta_4})$. By Theorem 3.1, $\mathrm{Im}(f)$ is generated by $f(t_{\delta_1})$ and $f(u_{g-1})$, and since u_{g-1} commutes with t_{δ_1}, $\mathrm{Im}(f)$ is abelian. □

Lemma 3.4 *Suppose that $g \geq 4$ and $f\colon \mathcal{M}(N_{g,n}) \to G$ is a homomorphism. If $f(t_{\varepsilon_i}) = f(t_{\delta_j})$ for some $2i+1 \leq j \leq g-1$, then $f(t_{\delta_1}^2) = 1$.*

Proof Set $x = f(t_{\varepsilon_i}) = f(t_{\delta_j})$ and $y = f(u_j)$. By the relation (R8) we have $xy = yx$, and by (R12), $xyx = y$. Hence $x^2 = 1$, which finishes the proof, because t_{δ_j} is conjugate to t_{δ_1}. □

Let $g = 2r + s$, where $r \geq 1$, $s \in \{1, 2\}$ and $S = S_{g-1}$. Consider S as being embedded in $\mathbb{R}^3$ in such a way that it is invariant under the reflections about the xy, xz and yz planes, as shown on Figure 3. We define a homeomorphism $j\colon S \to S$ as $j(x, y, z) = (-x, -y, -z)$. The quotient space S/j is a nonorientable surface of genus g and the projection $p\colon S \to S/j$ is a covering map of degree 2. Let S' be the subsurface of S consisting of points $(x, y, z) \in S$ with $x \leq -\varepsilon$, where ε is a positive constant, so small that S' is homeomorphic to $S_{r,s}$. If g is even, then one of the boundary components of S' is isotopic to α_{r+1}. In this paper we identify isotopic curves, and therefore we will treat α_{r+1} as a curve on S'. Note that the restriction of p to S' is an embedding. For odd g we define γ' to be the arc of γ_r consisting of points with $x \leq 0$. For even g we define β' to be the arc of β_{r+1} consisting of points with $x \leq 0$. Note that $p(\gamma')$ and $p(\beta')$ are one-sided simple closed curves on S/j.

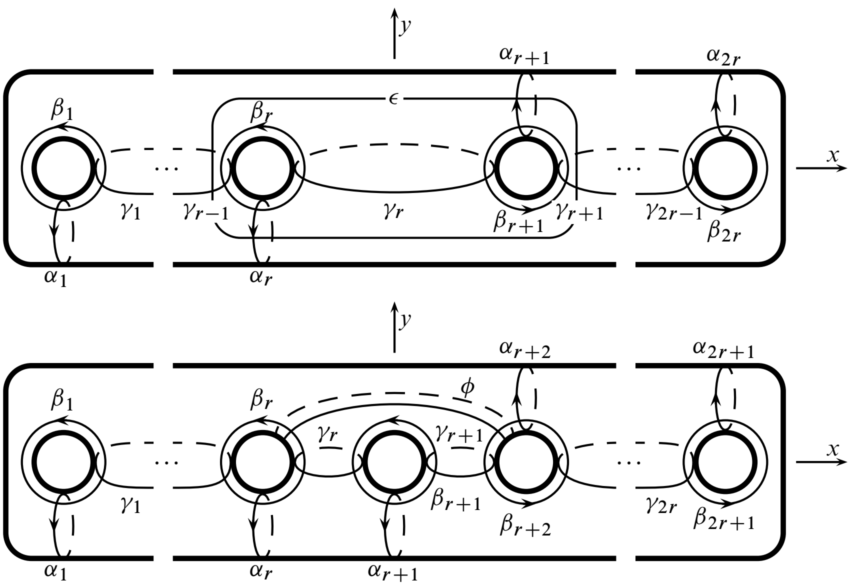

Figure 3: The surface S_{g-1} for $g = 2r + 1$ (top) and $g = 2r + 2$ (bottom)

Proposition 3.5 *There is a homeomorphism $\varphi\colon S_{g-1}/j \to N_g$ such that, for $P = \varphi \circ p$, up to isotopy:*

(1) $P(\beta_i) = \delta_{2i}$ *for* $1 \leq i \leq r$

(2) $P(\alpha_i) = \varepsilon_i$ *for* $2 \leq 2i \leq g$

(3) $P(\gamma_i) = \delta_{2i+1}$ *for* $2 \leq 2i \leq g-2$

(4) $P(\gamma') = \xi_g$ *if g is odd*

(5) $P(\beta') = \xi_g$ *if g is even*

Proof By altering the curves δ_i, ε_j and ξ_g by a small isotopy, we may assume that they intersect each other minimally. The curves δ_i for $1 \leq i \leq g-1$ form a chain of two-sided curves, which means that δ_i and δ_j intersect at one point if $|i-j| = 1$, and they are disjoint otherwise. The one-sided curve ξ_g intersects δ_{g-1} at one point and is disjoint from δ_i for $i < g-1$. Let Σ be a regular neighbourhood of the union of δ_i for $1 \leq i \leq g-1$, and let M be the union of Σ with a regular neighbourhood of ξ_g. Observe that Σ and M are homeomorphic to $S_{r,s}$ and $N_{g,1}$ respectively. We may choose Σ big enough to contain the curves ε_i for $2 \leq 2i \leq g$ (if g is even, then one of the boundary components of Σ is isotopic to ε_{r+1}). Let $M' \subset S_{g-1}/j$ be the union of $p(S')$ and a regular neighbourhood of the one-sided curve $p(\gamma')$ if g is odd, or $p(\beta')$ if g is even. There is a homeomorphism $\varphi\colon M' \to M$ such that for $P = \varphi \circ p$ we have $P(S') = \Sigma$ and

P maps the chain $(\alpha_1, \beta_1, \gamma_1, \beta_2, \gamma_2, \dots, \beta_r, \gamma')$ (or $(\alpha_1, \beta_1, \gamma_1, \beta_2, \gamma_2, \dots, \gamma_r, \beta')$) on $(\delta_1, \dots, \delta_{g-1}, \xi_g)$. Such a P satisfies conditions (1), (3), (4), (5) of the lemma. Since ε_i, ε_{i+1} and δ_{2i+1} bound a pair of pants in Σ for $2 \le 2i \le g-2$, φ may be taken to also satisfy (2). Finally, since $(S_{g-1}/j)\backslash M'$ and $N_g \backslash M$ are discs, φ may be extended to $\varphi\colon S_{g-1}/j \to N_g$. □

Corollary 3.6 *There is a homomorphism $\iota\colon \mathcal{M}(S') \to \mathcal{M}(N_{g,n})$ such that:*

- $\iota(t_{\beta_i}) = t_{\delta_{2i}}$ *for* $1 \le i \le r$
- $\iota(t_{\alpha_i}) = t_{\varepsilon_i}$ *for* $2 \le 2i \le g$
- $\iota(t_{\gamma_i}) = t_{\delta_{2i+1}}$ *for* $2 \le 2i \le g-2$

Here the Dehn twists about the curves on S' are right-handed with respect to the standard orientation.

Proof By the proof of Proposition 3.5, the restriction of P to S' is a homeomorphism onto Σ satisfying conditions (1), (2), (3). There is an induced isomorphism $\mathcal{M}(S') \to \mathcal{M}(\Sigma)$, which may be composed with the homomorphism $\mathcal{M}(\Sigma) \to \mathcal{M}(N_{g,n})$ induced by the inclusion $\Sigma \hookrightarrow N_{g,n}$, for any $n \ge 0$, to obtain ι. □

For any homeomorphism $h\colon N_g \to N_g$ there is a unique orientation preserving lift $\widetilde{h}\colon S_{g-1} \to S_{g-1}$ such that $h \circ P = P \circ \widetilde{h}$. By [3], the mapping $h \mapsto \widetilde{h}$ induces a monomorphism $\theta\colon \mathcal{M}(N_g) \to \mathcal{M}(S_{g-1})$. The following proposition follows from [3] and [28, Theorem 10], where the lift of a crosscap transposition is determined.

Proposition 3.7 *There is a monomorphism $\theta\colon \mathcal{M}(N_g) \to \mathcal{M}(S_{g-1})$ such that*

$$\theta(t_{\varepsilon_i}) = t_{\alpha_i} t_{\alpha_{g-i}}^{-1}, \quad \theta(t_{\delta_{2i}}) = t_{\beta_i} t_{\beta_{g-i}}^{-1}, \quad \theta(t_{\delta_{2j+1}}) = t_{\gamma_j} t_{\gamma_{g-1-j}}^{-1},$$

for $1 \le i \le r$, $2 \le 2j \le g-2$ *and*

$$\theta(u_{g-1}) = \begin{cases} t_{\beta_r}^{-1} t_{\beta_{r+1}} (t_{\gamma_r} t_{\beta_r} t_{\beta_{r+1}})^2 t_{\epsilon}^{-1} & \text{if } g = 2r+1, \\ t_{\gamma_r}^{-1} t_{\gamma_{r+1}} (t_{\beta_{r+1}} t_{\gamma_r} t_{\gamma_{r+1}})^2 t_{\phi}^{-1} & \text{if } g = 2r+2. \end{cases}$$

4 Homological representations

Fix $g \ge 3$ and let $S = S_{g-1}$, $N = N_g$ and $P\colon S \to N$ be as in the previous section. The group $H_1(S)$ is a free $\mathbb{Z}$–module of rank $2(g-1)$ and the homology classes $a_i = [\alpha_i]$, $b_i = [\beta_i]$ for $1 \le i \le g-1$ form its basis, which is a symplectic basis with respect to the algebraic intersection form:

$$\langle a_i, a_j \rangle = 0, \quad \langle b_i, b_j \rangle = 0, \quad \langle a_i, b_j \rangle = \delta_{ij}$$

Let $\Phi\colon \mathcal{M}(S) \to \mathrm{Sp}(H_1(S))$ be the homomorphism induced by the action of $\mathcal{M}(S)$ on $H_1(S)$. If γ is an oriented simple closed curve on S, $[\gamma] \in H_1(S)$ is its homology class, and t_γ is the right Dehn twist, then $\Phi(t_\gamma)$ is the transvection

$$\Phi(t_\gamma)(h) = h + \langle [\gamma], h\rangle [\gamma] \quad \text{for } h \in H_1(S). \tag{1}$$

From (1) we immediately obtain that, with respect to the basis $(a_1, b_1, \dots, a_{g-1}, b_{g-1})$, we have

$$\Phi(t_{\alpha_i}) = A_i, \quad \Phi(t_{\beta_i}) = B_i, \quad \Phi(t_{\gamma_j}) = C_j,$$

for $1 \le i \le g-1$, $1 \le j \le g-2$, where A_i, B_i and C_j are the matrices defined in Section 2.

The group $H_1(N)$ has the following presentation, as a $\mathbb{Z}$–module:

$$H_1(N) = \langle x_1, \dots, x_g \mid 2(x_1 + \dots + x_g) = 0\rangle,$$

where $x_i = [\xi_i]$. Set $k = x_1 + \dots + x_g$ and $R = H_1(N)/\langle k\rangle$. Observe that k is the unique element of order two in $H_1(N)$ and R is a free $\mathbb{Z}$–module of rank $g-1$.

The map $P\colon S \to N$ induces $P_*\colon H_1(S) \to H_1(N)$, such that, for $1 \le i \le r$,

$$P_*(a_i) = x_1 + \dots + x_{2i} = -P_*(a_{g-i}),$$
$$P_*(b_i) = x_{2i} + x_{2i+1} = P_*(b_{g-i}),$$

and if $g = 2r+2$, then

$$P_*(a_{r+1}) = x_1 + \dots + x_g = k, \quad P_*(b_{r+1}) = 2x_g.$$

Let $q\colon H_1(S) \to R$ be the composition of P_* with the canonical projection $H_1(N) \to R$, and set $K = \ker q$. Observe that K has rank $g-1$ and the following elements form its basis:

$$e_i = a_i + a_{g-i}, \quad e_{r+i} = b_i - b_{g-i} \quad \text{for } 1 \le i \le r$$
$$e_{2r+1} = a_{r+1} \quad \text{for } g = 2r+2$$

We also set

$$f_i = b_i, \quad f_{r+i} = a_{g-i} \quad \text{for } 1 \le i \le r,$$
$$f_{2r+1} = b_{r+1} \quad \text{for } g = 2r+2.$$

The elements e_i, f_i for $1 \le i \le g-1$ form a symplectic basis of $H_1(S)$. It follows that $H_1(S)/K$ is a free $\mathbb{Z}$–module of rank $g-1$ that is canonically isomorphic to R if g is odd, or to an index-two subgroup of R if g is even. The group $\mathcal{M}(N)$

acts on $H_1(S)$ by the composition $\Phi\circ\theta\colon \mathcal{M}(N)\to \mathrm{Sp}(H_1(S))$. Observe that K is $M(N)$–invariant and hence we have two $(g-1)$–dimensional representations

$$\Psi_1\colon \mathcal{M}(N)\to \mathrm{GL}(K),\quad \Psi_2\colon \mathcal{M}(N)\to \mathrm{GL}(H_1(S)/K).$$

Lemma 4.1 $\ker\Psi_1=\ker\Psi_2$ *and* $\theta(\ker\Psi_1)\subset\ker\Phi$.

Proof Fix the basis $(e_1,\dots,e_{g-1},f_1,\dots,f_{g-1})$ of $H_1(S)$. For any $x\in\mathcal{M}(N)$ let X be the matrix of $\Phi(\theta(x))$. Since $\Phi(\theta(x))$ preserves K, we have

$$X=\begin{pmatrix}X_1 & Y\\ 0 & X_2\end{pmatrix},$$

where X_1,X_2,Y are $(g-1)\times(g-1)$ matrices. Furthermore, X_1 is the matrix of $\Psi_1(x)$ with respect to the basis $(e_i)_{1\le i\le g-1}$ of K, and X_2 is the matrix of $\Psi_2(x)$ with respect to the basis $(f_i+K)_{1\le i\le g-1}$ of $H_1(S)/K$. The algebraic intersection form on $H_1(S)$ has the matrix

$$\Omega=\begin{pmatrix}0 & I_{g-1}\\ -I_{g-1} & 0\end{pmatrix}.$$

Since X is symplectic, we have $X^t\Omega X=\Omega$, which gives $X_1^tX_2=I$. Therefore $X_1=I\Leftrightarrow X_2=I$, which proves $\ker\Psi_1=\ker\Psi_2$.

To prove the second assertion, suppose that $x\in\ker\Psi_1$. Then

$$X=\begin{pmatrix}I_{g-1} & Y\\ 0 & I_{g-1}\end{pmatrix},$$

and we have to show $Y=0$. Let $j_*\colon H_1(S)\to H_1(S)$ be the map induced by the covering involution j. We have $j_*(a_i)=-a_{g-i}$ and $j_*(b_i)=b_{g-i}$ for $1\le i\le g-1$. It follows that the matrix of j_* with respect to the basis $(e_1,\dots,e_{g-1},f_1,\dots,f_{g-1})$ has the form

$$J=\begin{pmatrix}-I_{g-1} & T\\ 0 & I_{g-1}\end{pmatrix}$$

for some T. Since $\theta(x)$ commutes with j, we have $XJ=JX$, which gives $Y+T=-Y+T$, hence $Y=0$. □

Note that $\ker\Phi$ is the Torelli group, which is well known to be torsion-free, and since θ is a monomorphism, we immediately obtain the following.

Corollary 4.2 $\ker\Psi_1$ *is torsion-free.* □

Remark 4.3 Let H denote the subgroup of $\mathcal{M}(N)$ consisting of the elements inducing the identity on $H_1(N)$. It was proved in Gastesi [11] that $\theta(H) \subset \ker \Phi$. We leave it as an exercise to check that if g is odd, then $H = \ker \Psi_2$, whereas if g is even, then H is an index-two subgroup of $\ker \Psi_2$. In the latter case, if $g = 2r + 2$, then we have $\ker \Psi_2 = H \cup t_{\varepsilon_{r+1}} H$.

Remark 4.4 There is a nontrivial action of $\pi_1(N)$ on $\mathbb{Z}$ defined as follows: $\gamma \in \pi_1(N)$ acts by multiplication by 1 or -1 according to whether γ preserves or reverses local orientations of N. This action gives rise to homology groups with local coefficients $H_*(N, \widetilde{\mathbb{Z}})$, where $\widetilde{\mathbb{Z}}$ is $\mathbb{Z}$ with the nontrivial $\mathbb{Z}[\pi_1(N)]$–module structure. By Hatcher [14, Example 3H.3], we have the exact sequence

$$H_2(N) \to H_1(N, \widetilde{\mathbb{Z}}) \to H_1(S) \xrightarrow{P_*} H_1(N),$$

which is a part of a long exact sequence of homology groups. Since $H_2(N) = 0$, we have a $\mathcal{M}(N)$–equivariant isomorphism $H_1(N, \widetilde{\mathbb{Z}}) \cong \ker P_*$. If g is odd, then $\ker P_* = K$, whereas if g is even, then $\ker P_*$ is an index-two subgroup of K. Therefore the representations Ψ_1 and Ψ_2 may be seen as coming from the actions of $\mathcal{M}(N)$ on $H_1(N, \widetilde{\mathbb{Z}})$ and $H_1(N)$ respectively.

For K we fix the basis

$$\begin{aligned} &(e_1, e_{r+1}, \dots, e_r, e_{2r}) \quad \text{if } g = 2r + 1, \\ &(e_1, e_{r+1}, \dots, e_r, e_{2r}, e_{2r+1}) \quad \text{if } g = 2r + 2. \end{aligned}$$

For $H_1(S)/K$ we fix the basis

$$\begin{aligned} &(a_1 + K, b_1 + K, \dots, a_r + K, b_r + K) \quad \text{if } g = 2r + 1, \\ &(a_1 + K, b_1 + K, \dots, a_r + K, b_r + K, b_{r+1} + K) \quad \text{if } g = 2r + 2. \end{aligned}$$

Having fixed bases for K and $H_1(S)/K$ we can now compute, for Ψ_1 and Ψ_2, the images of the generators of $\mathcal{M}(N)$. This is done by a straightforward calculation, using Proposition 3.7 and the formula (1). For $k = 1, 2$ and $1 \le i \le r$, $1 \le j \le r - 1$, we have

$$\Psi_k(t_{\varepsilon_i}) = A_i, \quad \Psi_k(t_{\delta_{2i}}) = B_i, \quad \Psi_k(t_{\delta_{2j+1}}) = C_j.$$

If $g = 2r + 1$, then

$$\Psi_1(u_{g-1}) = \begin{pmatrix} I_{g-3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad \Psi_2(u_{g-1}) = \begin{pmatrix} I_{g-3} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

If $g = 2r + 2$, then

$$\Psi_1(t_{\delta_{g-1}}) = \begin{pmatrix} I_{g-4} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}, \quad \Psi_2(t_{\delta_{g-1}}) = \begin{pmatrix} I_{g-4} & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Psi_1(u_{g-1}) = \begin{pmatrix} I_{g-4} & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}, \quad \Psi_2(u_{g-1}) = \begin{pmatrix} I_{g-4} & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

It is easy to see that Ψ_1 and Ψ_2 are not conjugate as homomorphisms to $\mathrm{GL}(g-1,\mathbb{C})$. For, suppose that there is $M \in \mathrm{GL}(g-1,\mathbb{C})$ such that $\Psi_1(x) = M\Psi_2(x)M^{-1}$ for all $x \in \mathcal{M}(N)$. Then M commutes with A_i, B_i, C_j for $1 \le i \le r$, $1 \le j \le r-1$, and by Lemma 2.1, $M = \alpha I_{2r}$ if $g = 2r+1$, or $M = \mathrm{diag}(\alpha I_{2r}, \beta)$ if $g = 2r+2$, for $\alpha, \beta \in \mathbb{C}$. In either case it is impossible that $\Psi_1(u_{g-1}) = M\Psi_2(u_{g-1})M^{-1}$.

5 Homomorphisms from $\mathcal{M}(N_{g,n})$ to $\mathrm{GL}(m,\mathbb{C})$ for $m < g-1$

The aim of this section is to prove Theorem 1.3. The proof is divided in two parts.

Proof of Theorem 1.3 for $(g,m) \neq (6,4)$ Suppose that $n \in \{0,1\}$, $g = 2r+s$ for $r \ge 2$, $s \in \{1,2\}$, $m \le g-2$ and $f\colon \mathcal{M}(N_{g,n}) \to \mathrm{GL}(m,\mathbb{C})$ is a homomorphism. By Theorem 3.2, it suffices to prove that $\mathrm{Im}(f)$ is abelian. Let $S' = S_{r,s}$ and $\iota\colon \mathcal{M}(S') \to \mathcal{M}(N_{g,n})$ be the homomorphism from Corollary 3.6. Set $f' = f \circ \iota$ and observe that if $\mathrm{Im}(f')$ is abelian, then so is $\mathrm{Im}(f)$, by Lemma 3.3.

Suppose that $m \le 2r-1$. Then $\mathrm{Im}(f')$ is either trivial or cyclic by Theorem 1.1 and we are done. This finishes the proof for odd g.

Suppose that $g = 2r+2$ for $r \ge 3$ and $m = 2r$. By Theorem 1.2, f' is either trivial or conjugate to the homological representation Φ. In the former case we are done. In the latter case, by the definition of Φ we have $\Phi(t_{\gamma_r}) = \Phi(t_{\alpha_r})$ because the curves γ_r and α_r become isotopic after gluing discs to the boundary of S'. It follows that $f(t_{\delta_{2r+1}}) = f(t_{\varepsilon_r})$ and, by Lemma 3.4, $f(t_{\delta_1}^2) = 1$. This is a contradiction because $\Phi(t_{\alpha_1})$ has infinite order. □

In order to prove Theorem 1.3 for $(g,m) = (6,4)$, we first prove some lemmas.

Lemma 5.1 *Suppose that $n \le 1$ and $f\colon \mathcal{M}(N_{6,n}) \to \mathrm{GL}(4,\mathbb{C})$ is a homomorphism such that $f(t_{\delta_1}^2) = 1$. Then $\mathrm{Im}(f)$ is abelian.*

Proof Let H be the normal closure of $t_{\delta_1}^2$ in $\mathcal{M}(N_{6,n})$ and set $G=\mathcal{M}(N_{6,n})/H$. We have an induced homomorphism $f'\colon G\to \mathrm{GL}(4,\mathbb{C})$ such that $f=f'\circ\pi$, where $\pi\colon \mathcal{M}(N_{6,n})\to G$ is the canonical projection. By the relations (R1), (R4), the mapping $\rho(\sigma_i)=\pi(t_{\delta_i})$, where σ_i is the transposition $(i,i+1)$ for $1\le i\le 5$, defines a homomorphism $\rho\colon \mathfrak{S}_6\to G$. Let $\phi\colon \mathfrak{S}_6\to \mathrm{GL}(4,\mathbb{C})$ be the composition $f'\circ\rho$. By Lemma 2.2, ϕ is the direct sum of one-dimensional representations. In particular the image of ϕ is abelian, and so is $\mathrm{Im}(f)$ by Lemma 3.3. □

Let R be the subsurface obtained by removing from $N_{6,n}$ a regular neighbourhood of $\delta_1\cup\delta_2$. Note that R is homeomorphic to $N_{4,n+1}$. The homomorphism $\mathcal{M}(R)\to\mathcal{M}(N_{6,n})$ induced by the inclusion of R in $N_{6,n}$ is injective, and we will treat $\mathcal{M}(R)$ as a subgroup of $\mathcal{M}(N_{6,n})$.

Lemma 5.2 *Suppose that $h\colon \mathcal{M}(R)\to \mathrm{GL}(2,\mathbb{C})$ is a homomorphism. Then, with respect to some basis, one of the following cases holds:*

(a) $h(t_{\delta_4})=h(t_{\delta_5})=h(t_{\varepsilon_2})=\lambda I,\quad \lambda\in\{-1,1\}$

(b) $h(t_{\delta_4})=h(t_{\delta_5})=h(t_{\varepsilon_2})=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

(c) $h(t_{\delta_5})=h(t_{\varepsilon_2})=\begin{pmatrix}1&1\\0&-1\end{pmatrix},\ h(t_{\delta_4})=\begin{pmatrix}-1&0\\1&1\end{pmatrix}$

In particular, $h(t_{\delta_4}^2)=1$.

Proof For $i=4,5$, let $L_i=h(t_{\delta_i})$, $M=h(t_{\varepsilon_2})$ and $U=h(u_5)$. Recall that the twists t_{δ_i} and t_{ε_2} are pairwise conjugate, and each of them is conjugate to its inverse (by (R12)).

Case 1 $\boldsymbol{M}$ **has only one eigenvalue** $\boldsymbol{\lambda}$ Since M is conjugate to M^{-1} we have $\lambda\in\{-1,1\}$. If $\dim E(M,\lambda)=2$, then we have case (a). Suppose that $\dim E(M,\lambda)=1$. Then with respect to some basis we have $M=\begin{pmatrix}\lambda&1\\0&\lambda\end{pmatrix}$, and since L_5 commutes with M, $L_5=\begin{pmatrix}\lambda&x\\0&\lambda\end{pmatrix}$ for some x.

If $E(M,\lambda)\ne E(L_4,\lambda)$, then we may arrange that the second vector of the basis is from $E(L_4,\lambda)$; thus, $L_4=\begin{pmatrix}\lambda&0\\y&\lambda\end{pmatrix}$ for some y. From $ML_4M=L_4ML_4$ we obtain $y=-1$, and from $L_4L_5L_4=L_5L_4L_5$ we have $x=1$, hence $M=L_5$. By Lemma 3.4 (for $i=2,\ j=5$) we have $M^2=I$, which is a contradiction.

If $E(M,\lambda)=E(L_4,\lambda)$, then $L_4=\begin{pmatrix}\lambda&y\\0&\lambda\end{pmatrix}$ for some y. From $ML_4M=L_4ML_4$ and $L_4L_5L_4=L_5L_4L_5$ we obtain $x=y=1$, hence $M=L_5$, which leads to a contradiction as above.

Case 2 $\boldsymbol{M}$ **has two eigenvalues** $\boldsymbol{\lambda},\boldsymbol{\mu}$ With respect to some basis we have $M=\begin{pmatrix}\lambda&0\\0&\mu\end{pmatrix}$, and since L_5 and U commute with M, they are also diagonal. In particular we have $UL_5 = L_5U$ and $L_5UL_5 = U$ (R12) gives $L_5^2 = 1$, which implies $\{\lambda, \mu\} = \{-1, 1\}$. Either $L_5 = M$ or $L_5 = -M$. In the latter case the braid relations $L_5L_4L_5 = L_4L_5L_4$ and $ML_4M = L_4ML_4$ imply $L_4ML_4 = 0$, a contradiction, hence $M = L_5$.

If $E(M, 1) \neq E(L_4, 1)$, then with respect to some basis we have $M = \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$, $L_4 = \left(\begin{smallmatrix} -1 & 0 \\ x & 1 \end{smallmatrix}\right)$. From $ML_4M = L_4ML_4$ we have $x = 1$ and we are in case (c). Analogously, if $E(M, -1) \neq E(L_4, -1)$, then with respect to some basis we have $M = \left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $L_4 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & -1 \end{smallmatrix}\right)$, and since $E(M, 1) \neq E(L_4, 1)$, we are in case (c) again. Finally, if $E(M, 1) = E(L_4, 1)$ and $E(M, -1) = E(L_4, -1)$, then with respect to some basis we have $M = L_4 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and we are in case (b). □

Lemma 5.3 *Suppose that $n \leq 1$, $f\colon \mathcal{M}(N_{6,n}) \to \mathrm{GL}(4, \mathbb{C})$ is a homomorphism and there exists a splitting $\mathbb{C}^4 = V_1 \oplus V_2$ such that V_i is a 2–dimensional $\mathcal{M}(R)$–invariant subspace for $i = 1, 2$. Then $\mathrm{Im}(f)$ is abelian.*

Proof Let f' be the restriction of f to $\mathcal{M}(R)$. With respect to the splitting $\mathbb{C}^4 = V_1 \oplus V_2$ we have $f' = f_1 \oplus f_2$ for some $f_i\colon \mathcal{M}(R) \to \mathrm{GL}(2, \mathbb{C})$, $i = 1, 2$. By Lemma 5.2 we have $f_i(t_{\delta_4}^2) = 1$ for $i = 1, 2$, hence $f(t_{\delta_4}^2) = 1$ and we are done by Lemma 5.1. □

Lemma 5.4 *Suppose that $n \leq 1$, $f\colon \mathcal{M}(N_{6,n}) \to \mathrm{GL}(4, \mathbb{C})$ is a homomorphism, $f(t_{\delta_1})$ has only one eigenvalue and there exists a 2–dimensional $\mathcal{M}(R)$–invariant subspace. Then $\mathrm{Im} f$ is abelian.*

Proof Fix a basis of $\mathbb{C}^4$ whose first two vectors span the $\mathcal{M}(R)$–invariant subspace. For $x \in \mathcal{M}(R)$ we have

$$f(x) = \begin{pmatrix} h_1(x) & * \\ 0 & h_2(x) \end{pmatrix},$$

where $h_1(x)$ and $h_2(x)$ are 2–dimensional matrices. We apply Lemma 5.2 to the homomorphisms h_1 and h_2. Because $f(t_{\delta_4})$ has only one eigenvalue, case (a) holds. It follows that $f(t_{\delta_4}) = \left(\begin{smallmatrix} \lambda I & X \\ 0 & \lambda I \end{smallmatrix}\right)$, $f(t_{\delta_5}) = \left(\begin{smallmatrix} \lambda I & Y \\ 0 & \lambda I \end{smallmatrix}\right)$, for some 2×2 matrices X, Y and $\lambda \in \{-1, 1\}$. In particular $f(t_{\delta_4})$ and $f(t_{\delta_5})$ commute and we are done by Lemma 3.3. □

Proof of Theorem 1.3 for $g = 6$, $m = 4$ Suppose that $n \in \{0, 1\}$ and $f\colon \mathcal{M}(N_{6,n}) \to \mathrm{GL}(4, \mathbb{C})$ is a homomorphism. For $1 \leq i \leq 5$, we set $L_i = f(t_{\delta_i})$ and $M = f(t_{\varepsilon_2})$, $U_5 = f(u_5)$. We consider the following cases.

(1) L_1 has 4 eigenvalues.

(2) L_1 has 3 eigenvalues.

(3) L_1 has 2 eigenvalues with equal multiplicities.

(4) L_1 has 2 eigenvalues with different multiplicities.

(5) L_1 has 1 eigenvalue.

In cases (1), (2), (3) it is easy to find a splitting $\mathbb{C}^4 = V_1 \oplus V_2$ such that V_i is a 2–dimensional $\mathcal{M}(R)$–invariant subspace for $i = 1, 2$. For example, suppose that L_1 has 3 eigenvalues $\lambda_1, \lambda_2, \lambda_3$ such that $\#\lambda_1 = \#\lambda_2 = 1$ and $\#\lambda_3 = 2$. Then we take $V_1 = E(L_1, \lambda_1) \oplus E(L_1, \lambda_2)$ and $V_2 = E(L_1, \lambda_3)$ if $\dim E(L_1, \lambda_3) = 2$ or $V_2 = E^2(L_1, \lambda_3)$ if $\dim E(L_1, \lambda_3) = 1$. Therefore in cases (1), (2), (3), we are done by Lemma 5.3.

Assume (5). Let λ be the unique eigenvalue of L_1 and $k = \dim E(L_1, \lambda)$. If $k = 4$ then $L_1 = \lambda I$ and the image of f is cyclic. If $k = 2$ or $k = 1$, then, respectively, $E(L_1, \lambda)$ or $E^2(L_1, \lambda)$ is a 2–dimensional $\mathcal{M}(R)$–invariant subspace, and we are done by Lemma 5.4. Suppose that $k = 3$. If $E(L_1, \lambda) \neq E(L_2, \lambda)$ then $E(L_1, \lambda) \cap E(L_2, \lambda)$ is a 2–dimensional $\mathcal{M}(R)$–invariant subspace, and we are done by Lemma 5.4. If $E(L_1, \lambda) = E(L_2, \lambda)$, then with respect to some basis we have

$$L_1 = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad L_2 = \begin{pmatrix} \lambda & 0 & 0 & x \\ 0 & \lambda & 0 & y \\ 0 & 0 & \lambda & z \\ 0 & 0 & 0 & \lambda \end{pmatrix}.$$

In particular L_1 and L_2 commute and we are done by Lemma 3.3.

It remains to consider case (4). Suppose that L_1 has eigenvalues μ, λ, with $\#\mu = 1$ and $\#\lambda = 3$. Since L_1 is conjugate to L_1^{-1}, we have $\{\mu, \lambda\} = \{-1, 1\}$. It follows from Theorem 3.2 that there is a homomorphism $\tau(\mathcal{M}(N_6)) \to \{-1, 1\}$ such that $\tau(t_{\delta_1}) = -1$. By multiplying f by τ if necessary, we may assume $\mu = -1$, $\lambda = 1$. The Jordan form of L_1 is one of the three matrices

$$\text{(i)} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{(ii)} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{(iii)} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In case (i) we have $L_1^2 = I$ and we are done by Lemma 5.1.

In case (ii) the following subspaces are $\mathcal{M}(R)$–invariant: $E(L_1, -1)$, $E(L_1, 1)$, $E^2(L_1, 1)$, $E^3(L_1, 1)$. It follows that

$$M = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & v_1 & v_2 \\ 0 & 0 & x_3 & v_3 \\ 0 & 0 & 0 & x_4 \end{pmatrix}, \quad L_4 = \begin{pmatrix} y_1 & 0 & 0 & 0 \\ 0 & y_2 & w_1 & w_2 \\ 0 & 0 & y_3 & w_3 \\ 0 & 0 & 0 & y_4 \end{pmatrix}.$$

The braid relation $ML_4M = L_4ML_4$ implies $x_i = y_i$ for $1 \leq i \leq 4$. Since the first two vectors of the basis are eigenvectors of M, they have to correspond to different eigenvalues of M. Therefore $x_2 = -x_1$, $x_3 = x_4 = 1$ and $x_1 = 1$ or $x_1 = -1$. In either case $ML_4M = L_4ML_4$ holds only if $M = L_4$. We are done by Lemma 3.3.

In case (iii) the following subspaces are $\mathcal{M}(R)$–invariant: $E(L_1,-1)$, $E(L_1,1)$, $E^2(L_1,1)$. We have $\dim E(L_1,1) = 2$. For $x \in \mathcal{M}(R)$ let $h(x)$ be the restriction of $f(x)$ to $E(L_1,1)$. By applying Lemma 5.2 to h we obtain three sub-cases.

Case (iii a) h satisfies (a) of Lemma 5.2. We have

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & x_1 \\ 2 & 0 & 1 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As in case (ii), the braid relation implies $M = L_4$ and we are done by Lemma 3.3.

Case (iii b) h satisfies (b) of Lemma 5.2. By changing the basis of $E(L_1,1)$, we may assume that

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & x_1 \\ 0 & 0 & 1 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As in case (ii), the braid relation implies $M = L_4$ and we are done by Lemma 3.3.

Case (iii c) h satisfies (c) of Lemma 5.2. By changing the basis of $E(L_1,1)$ we may assume that

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & x_1 \\ 0 & 0 & -1 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & y_1 \\ 0 & 1 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & z_1 \\ 0 & 0 & -1 & z_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By solving the equations $ML_4M = L_4ML_4$ and $L_5L_4L_5 = L_4L_5L_4$, we obtain $x_2 = -(2x_1 + y_1 + 2y_2)$, $z_2 = -(2z_1 + y_1 + 2y_2)$, and from $ML_5 = L_5M$ we obtain $x_2 = z_2$. Thus $M = L_5$, and, by Lemma 3.4, $L_1^2 = 1$. We are done by Lemma 5.1. □

6 Homomorphisms between mapping class groups

The aim of this section is to prove Theorem 1.4. Fix $g \geq 5$ and set $\mathcal{M} = \mathcal{M}(N_g)$. We are going to use the fact that $s = t_{\delta_1} \cdots t_{\delta_{g-1}}$ has finite order in $\mathcal{M}$ (equal to g if it is even, or $2g$ otherwise; see [23, Proposition 3.2]). By the relations (R1), (R4), we have

$$t_{\delta_{i+1}} s = s t_{\delta_i} \quad \text{for } 1 \le i \le g-2. \tag{2}$$

By Theorem 3.2 we have $s \in [\mathcal{M}, \mathcal{M}]$ for $g \ge 7$ and $g = 5$, $s^2 \in [\mathcal{M}, \mathcal{M}]$ for $g = 6$.

Proof of Theorem 1.4 Suppose that $g \ge 5$, $h < g$ and $f\colon \mathcal{M}(N_g) \to \mathcal{M}(N_h)$ is a homomorphism. Since $M(N_h)$ is abelian for $h \le 2$, we are assuming $h \ge 3$.

Let $f'\colon \mathcal{M}(N_g) \to \mathrm{GL}(h-1, \mathbb{C})$ be the composition $\Psi_1 \circ f$ and $K = \ker \Psi_1$. By Theorem 1.3, $\mathrm{Im}(f')$ is abelian, hence $f([\mathcal{M}(N_g), \mathcal{M}(N_g)]) \subseteq K$. Suppose that $g \ge 7$ or $g = 5$. Then $f(s) \in K$, and since K is torsion-free by Corollary 4.2, $f(s) = 1$. This gives, by (2), $f(t_{\delta_1}) = f(t_{\delta_2})$ and we are done by Lemma 3.3. If $g = 6$ then $f(s^2) \in K$, which gives $f(s^2) = 1$ and $f(t_{\delta_2}) = f(t_{\delta_4})$. Since t_{δ_1} commutes with t_{δ_4}, $f(t_{\delta_1})$ commutes with $f(t_{\delta_2})$ and we are done by Lemma 3.3. □

Note that Theorems 1.3 and 1.4 are trivially true for $g \le 3$ because $\mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$, $\mathcal{M}(N_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $M(N_1) = 1$ are abelian groups. On the other hand, Corollary 6.2 below shows that both theorems are false for $g = 4$ (recall that $\mathcal{M}(N_3) \cong \mathrm{GL}(2, \mathbb{Z})$). Let D_∞ denote the infinite dihedral group, defined by the presentation

$$D_\infty = \langle x, y \mid x^2 = y^2 = 1 \rangle.$$

Lemma 6.1 *There is an epimorphism* $\phi\colon \mathcal{M}(N_4) \to D_\infty$.

Proof According to the main result of Szepietowski [27], simplified in [23], $\mathcal{M}(N_4)$ admits a presentation with generators t_{ε_2}, t_{δ_i}, u_i for $i = 1, 2, 3$ and relations (R1), (R3), (R4), (R6), (R7), (R9), (R10), (R11), (R12) and

$$t_{\delta_{i+1}} u_i u_{i+1} = u_i u_{i+1} t_{\delta_i} \quad \text{for } i = 1, 2,$$
$$(t_{\varepsilon_2} u_3)^2 = 1, \quad t_{\delta_1} (t_{\delta_2} t_{\delta_3} u_3 u_2) t_{\delta_1} = (t_{\delta_2} t_{\delta_3} u_3 u_2).$$

It is easy to check that the mapping $\phi(t_{\varepsilon_2}) = xy$, $\phi(t_{\delta_i}) = 1$, $\phi(u_i) = y$ for $i = 1, 2, 3$ respects the defining relations of $\mathcal{M}(N_4)$, hence it defines a homomorphism onto D_∞. □

Corollary 6.2 *For $h \ge 3$, there is a homomorphism* $f\colon \mathcal{M}(N_4) \to \mathcal{M}(N_h)$ *such that* $\mathrm{Im}(f)$ *is isomorphic to* D_∞.

Proof Fix $h \ge 3$. By the proof of Szepietowski [26, Theorem 3], t_{δ_1} can be written in $\mathcal{M}(N_h)$ as a product of two involutions σ, τ. Since t_{δ_1} has infinite order in $\mathcal{M}(N_h)$, the mapping $x \mapsto \sigma$, $y \mapsto \tau$ defines an embedding $D_\infty \to \mathcal{M}(N_h)$. By precomposing this embedding with the epimorphism ϕ from Lemma 6.1, we obtain f. □

The following two theorems can be proved by the same method as Theorem 1.4. We leave the details to the reader.

Theorem 6.3 *Suppose that* $g \geq 5$, $g \geq 2h + 2$ *and* $f\colon \mathcal{M}(N_g) \to \mathcal{M}(S_h)$ *is a homomorphism. Then* $\mathrm{Im}(f)$ *is abelian.*

Theorem 6.4 *Suppose that* $g \geq 3$ *and* $h \leq 2g$. *Then the only homomorphism from* $\mathcal{M}(S_g)$ *to* $\mathcal{M}(N_h)$ *is the trivial one.*

7 Homomorphisms from $\mathcal{M}(N_g)$ to $\mathrm{GL}(g-1,\mathbb{C})$

The aim of this section is to prove Theorem 1.5. The proof is divided into two cases, according to the parity of the genus.

Let $g = 2r + s$, $s \in \{1, 2\}$, $S' = S_{r,s}$ and $\iota\colon \mathcal{M}(S') \to \mathcal{M}(N_{g,n})$ be the homomorphism from Corollary 3.6. If $f\colon \mathcal{M}(N_{g,n}) \to \mathrm{GL}(m,\mathbb{C})$ is a homomorphism, then we set $f' = f \circ \iota$.

Proof of Theorem 1.5 for odd g Suppose that $N = N_{2r+1}$, $r \geq 3$ and $f\colon \mathcal{M}(N) \to \mathrm{GL}(2r,\mathbb{C})$ is a homomorphism such that $\mathrm{Im}(f)$ is not abelian. By Theorem 1.2, f' is conjugate to the homological representation Φ, and thus there exists a basis such that $f(t_{\varepsilon_i}) = f'(t_{\alpha_i}) = A_i$, $f(t_{\delta_{2i}}) = f'(t_{\beta_i}) = B_i$ for $1 \leq i \leq r$ and $f(t_{\delta_{2j+1}}) = f'(t_{\gamma_j}) = C_j$ for $1 \leq j \leq r-1$. Set $U_k = f(u_k)$ for $1 \leq k \leq 2r$.

Since U_{2r} commutes with A_i and B_i for $1 \leq i \leq r$, and with C_j for $j = 1, \dots, r-2$ (R6), (R8), by Lemma 2.1 we have

$$U_{2r} = \begin{pmatrix} \lambda I_{2r-2} & 0 \\ 0 & X \end{pmatrix}$$

for some 2×2 matrix X. Since U_{2r} is conjugate to U_{2r}^{-1} we have $\lambda \in \{-1, 1\}$ and by multiplying f by $(-1)^{\mathrm{ab}}$ if necessary, we may assume $\lambda = 1$. The relation $B_r U_{2r} B_r = U_{2r}$ (R12) implies

$$X = \begin{pmatrix} x & 0 \\ y & -x \end{pmatrix}.$$

From (R11) and (R7), we have

$$\begin{gathered} U_{2r-2} = (C_{r-1} B_r B_{r-1} C_{r-1})^{-1} U_{2r} (C_{r-1} B_r B_{r-1} C_{r-1}), \\ U_{2r} U_{2r-2} - U_{2r-2} U_{2r} = 0, \end{gathered}$$

and since the left hand side of the last equation is equal to

$$(1-x^2)(E_{2r,2r-3}+E_{2r-2,2r-1}),$$

where $E_{i,j}$ is the elementary matrix defined in Section 2, $x^2=1$. We have $U_{2r}^{-1}=U_{2r}$, and from (R11) and (R9),

$$U_{2r-1}=(C_{r-1}B_r)^{-1}U_{2r}(C_{r-1}B_r),$$
$$U_{2r}U_{2r-1}U_{2r}-U_{2r-1}U_{2r}U_{2r-1}=0.$$

By considering the cases $x=1$ and $x=-1$ separately, we find that the left hand side of the last equation is of the form $(y-x)^2Z$, where $Z\neq 0$. Hence $x=y$ and $U_{2r}=\Psi_1(u_{2r})$ if $x=1$, or $U_{2r}=\Psi_2(u_{2r})$ if $x=-1$. By Theorem 3.1, f is equal to Ψ_1 or Ψ_2 on the generators of $\mathcal{M}(N)$. □

Now we will borrow some arguments from [16] to prove Lemma 7.3 below, which will be a starting point for the proof of Theorem 1.5 for even genus.

Lemma 7.1 *Suppose that $n\leq 1$, $g\geq 5$ and $f\colon \mathcal{M}(N_{g,n})\to \mathrm{GL}(m,\mathbb{C})$ is a homomorphism. If there is a flag $0=W_0\subset W_1\subset\cdots\subset W_k=\mathbb{C}^m$ of $\mathcal{M}(N_{g,n})$–invariant subspaces such that $\dim(W_i/W_{i-1})<g-1$ for $i=1,\dots,k$, then $\mathrm{Im}(f)$ is abelian.*

Proof We use a similar argument as in the proof of [16, Lemma 4.8]. For $i=1,\dots,k$, set $m_i=\dim(W_i/W_{i-1})$. Fix a basis $(v_1,\dots,v_m)$ of $\mathbb{C}^m$, such that the vectors v_j for $1\leq j\leq m_1+\cdots+m_i$ form a basis of W_i. For $x\in\mathcal{M}(N_{g,n})$, the matrix of $f(x)$ with respect to this basis is

$$\begin{pmatrix} X_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & X_k \end{pmatrix},$$

where X_i is a square matrix of dimension m_i for $i=1,\dots,k$. Thus we have k homomorphisms $f_i\colon \mathcal{M}(N_{g,n})\to\mathrm{GL}(m_i,\mathbb{C})$ defined by $f_i(x)=X_i$. Since $m_i<g-1$, the image of each f_i is abelian by Theorem 1.3. It follows that $f[\mathcal{M}(N_{g,n}),\mathcal{M}(N_{g,n})]$ is contained in the subgroup of upper triangular matrices with 1 on the diagonal. Since this subgroup is nilpotent and $[\mathcal{M}(S'),\mathcal{M}(S')]$ is perfect [20, Theorem 4.2], it follows that $f'[\mathcal{M}(S'),\mathcal{M}(S')]$ is trivial, which means that $\mathrm{Im}(f')$ is abelian, hence so is $\mathrm{Im}(f)$. □

Lemma 7.2 *Suppose that $N=N_{2r+2}$, $r\geq 3$ and $f\colon\mathcal{M}(N)\to\mathrm{GL}(2r+1,\mathbb{C})$ is a homomorphism, such that $\mathrm{Im}(f)$ is not abelian. Then $L_1=f(t_{\delta_1})$ has an eigenvalue λ such that $\dim E(L_1,\lambda)=2r$.*

Proof By [16, Corollary 4.6] applied to f', L_1 has at most two eigenvalues. It follows that there is an eigenvalue λ with $\#\lambda \geq r+1 \geq 4$. Set $m = \dim E(L_1, \lambda)$. Since $\mathrm{Im}(f)$ is not abelian, $m \leq 2r$. We are going to show that $m = 2r$.

Let R be the subsurface obtained by removing from N a regular neighbourhood of $\delta_1 \cup \delta_2$. We have $R \approx N_{2r,1}$. We treat $\mathcal{M}(R)$ as a subgroup of $\mathcal{M}(N)$.

Suppose $m \leq 2r-2$. Let $W = E^k(L_1, \lambda)$, where $k = \max\{4-m, 1\}$. Observe that W is a $\mathcal{M}(R)$–invariant subspace with $3 \leq \dim W \leq 2r-2$. By Lemma 7.1, $f(\mathcal{M}(R))$ is abelian, hence $f(t_{\delta_4})$ and $f(t_{\delta_5})$ commute. By Lemma 3.3, $\mathrm{Im}(f)$ is abelian, a contradiction.

Suppose that $m = 2r-1$ and set $L_2 = f(t_{\delta_2})$. If $E(L_1, \lambda) \neq E(L_2, \lambda)$, then $E(L_1, \lambda) \cap E(L_2, \lambda)$ is a $\mathcal{M}(R)$–invariant subspace of dimension $2r-3$ or $2r-2$ and we can use the same argument as above to obtain a contradiction. If $E(L_1, \lambda) = E(L_2, \lambda)$, then by [16, Lemma 4.3] applied to f', $E(L_1, \lambda)$ is a $\mathcal{M}(S')$–invariant subspace of dimension $2r-1$, and by [16, Lemma 4.8] f' is trivial. It follows that $\mathrm{Im} f$ is abelian, a contradiction. □

Lemma 7.3 *Suppose that $N = N_{2r+2}$, $r \geq 3$ and $f\colon \mathcal{M}(N) \to \mathrm{GL}(2r+1, \mathbb{C})$ is a homomorphism. If $r = 3$ then assume that 1 is the unique eigenvalue of $f(t_{\delta_1})$. Then either $\mathrm{Im}(f)$ is abelian, or with respect to some basis $f(t_{\varepsilon_i}) = A_i$, $f(t_{\delta_{2i}}) = B_i$ for $i = 1, \dots, r$.*

Proof Suppose that $\mathrm{Im}(f)$ is not abelian. The result will follow from [16, Lemma 4.7] applied to $f'\colon \mathcal{M}(S') \to \mathrm{GL}(2r+1, \mathbb{C})$. Therefore it suffices to show that f' satisfies the hypothesis of [16, Lemma 4.7], namely: (1) the Jordan form of $f'(t_{\alpha_1})$ is $\left(\begin{smallmatrix} V & 0 \\ 0 & I_{2r-1} \end{smallmatrix}\right)$, and (2) $E(f'(t_{\alpha_1}), 1) \neq E(f'(t_{\beta_1}), 1)$.

By Lemma 7.2, $L_1 = f'(t_{\alpha_1}) = f(t_{\delta_1})$ has an eigenvalue λ with $\dim E(L_1, \lambda) = 2r$. If $r = 3$, then $\lambda = 1$ by assumption. For $r \geq 4$ we will prove $\lambda = 1$ by using the argument from the proof of [16, Lemma 5.2]. Set $t_1 = t_{\alpha_1}$ and choose 6 Dehn twits $t_2, \dots, t_7$ about nonseparating simple closed curves on S' such that the lantern relation $t_1 t_2 t_3 t_4 = t_5 t_6 t_7$ holds in $\mathcal{M}(S')$. By applying f' to both sides we obtain $L_1 L_2 L_3 L_4 = L_5 L_6 L_7$, where $L_i = f'(t_i)$. Since the L_i are conjugate, we have $\dim E(L_i, \lambda) = 2r$ for $i = 1, \dots, 7$. Set $W = \bigcap_{i=1}^7 E(L_i, \lambda)$ and observe that $\dim W > 0$. For a nonzero vector $v \in W$, we have $\lambda^4 v = L_1 L_2 L_3 L_4(v) = L_5 L_6 L_7(v) = \lambda^3 v$, hence $\lambda = 1$. Since $\mathcal{M}(S')$ is perfect (Korkmaz [18]), $\det L_1 = 1$ and $\lambda = 1$ is the unique eigenvalue. It follows that L_1 has the desired Jordan form.

Set $M = f'(t_{\beta_1}) = f(t_{\delta_2})$ and suppose $E(L_1, 1) = E(M, 1)$. Then L_1 and M commute by the same argument as in case (5) of the proof of Theorem 1.3 for $(g, m) = (6, 4)$. By Lemma 3.3, $\mathrm{Im}(f)$ is abelian, a contradiction. Thus $E(L_1, 1) \neq E(M, 1)$. $\square$

Proof of Theorem 1.5 for even *g* Suppose that $N = N_{2r+2}$, $r \geq 4$, and

$$f\colon \mathcal{M}(N) \to \mathrm{GL}(2r+1, \mathbb{C})$$

is a homomorphism such that $\mathrm{Im}(f)$ is not abelian. By Lemma 7.3, there is a basis such that $f(t_{\varepsilon_i}) = A_i$ and $f(t_{\delta_{2i}}) = B_i$ for $1 \leq i \leq r$. Set $D_i = f(t_{\delta_{2i+1}})$ for $1 \leq i \leq r$ and $U_j = f(u_j)$ for $1 \leq j \leq 2r+1$.

Fix $i \in \{1, \dots, r-1\}$. For $j \notin \{i, i+1\}$ we have $D_i A_j = A_j D_i$ and $D_i B_j = B_j D_i$. Setting $M = D_i$ and $k = l = j$ in Lemma 2.1, we obtain

$$D_i = \begin{pmatrix} * & 0 & * \\ 0 & \lambda I_2 & 0 \\ * & 0 & * \end{pmatrix},$$

where λ is at the positions $(2j-1, 2j-1)$ and $(2j, 2j)$. Since D_i is conjugate to A_1, 1 is its unique eigenvalue, hence $\lambda = 1$. It follows that D_i has the form

$$D_i = \begin{pmatrix} I_{2(i-1)} & 0 & 0 & 0 & 0 \\ 0 & F_{11} & F_{12} & 0 & X_1 \\ 0 & F_{21} & F_{22} & 0 & X_2 \\ 0 & 0 & 0 & I_{2(r-i-1)} & 0 \\ 0 & Y_1 & Y_2 & 0 & z \end{pmatrix},$$

where F_{kl} are 2×2 matrices, X_k are 2×1 vectors, Y_k are 1×2 vectors and z is a complex number. The relations $D_i A_i = A_i D_i$ and $D_i A_{i+1} = A_{i+1} D_i$ give: $V F_{kk} = F_{kk} V$, $V X_k = X_k$, $Y_k V = Y_k$ for $k = 1, 2$, and $V F_{kl} = F_{kl} = F_{kl} V$ for $k \neq l$. It follows that

$$F_{11} = \begin{pmatrix} s_1 & t_1 \\ 0 & s_1 \end{pmatrix}, \quad F_{12} = \begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix},$$
$$F_{21} = \begin{pmatrix} 0 & v_2 \\ 0 & 0 \end{pmatrix}, \quad F_{22} = \begin{pmatrix} s_2 & t_2 \\ 0 & s_2 \end{pmatrix}, \quad X_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix},$$
$$Y_1 = \begin{pmatrix} 0 & y_1 \end{pmatrix}, \qquad Y_2 = \begin{pmatrix} 0 & y_2 \end{pmatrix}.$$

Since s_1, s_2 are eigenvalues, we have $s_1 = s_2 = 1$ and $\det D_i = z$, which gives $z = 1$. Now, by solving the equations $B_i D_i B_i - D_i B_i D_i = 0$ and $B_{i+1} D_i B_{i+1} - D_i B_{i+1} D_i = 0$, we obtain $t_1 = t_2 = 1$, $v_1 v_2 = 1$, $y_2 = y_1 v_1$, $x_2 = x_1 v_2$, $x_1 y_1 = 0$.

Thus, for $i = 1, \dots, r-1$ we have

$$D_i = \begin{pmatrix} I_{2(i-1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \alpha_i & 0 & \alpha_i x_i \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_i^{-1} & 1 & 1 & 0 & x_i \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{2(r-i-1)} & 0 \\ 0 & 0 & y_i & 0 & \alpha_i y_i & 0 & 1 \end{pmatrix}, \quad x_i y_i = 0.$$

Analogous calculations, using the relations $A_i D_r = D_r A_i$ and $B_i D_r = D_r B_i$ for $1 \le i \le r$, $A_r D_r = D_r A_r$ and $D_r B_r D_r = B_r D_r B_r$, lead to the following form of D_r:

$$D_r = \begin{pmatrix} I_{2r-2} & 0 & 0 & 0 \\ 0 & 1 & 1 & x_r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & y_r & 1 \end{pmatrix}, \quad x_r y_r = 0$$

It is not possible that $x_r = y_r = 0$, because then $D_r = A_r$ and Lemma 3.4 would give a contradiction. For $1 \le i \le r-1$, by solving the equation $D_i D_r - D_r D_i = 0$ we obtain $x_i y_r = 0$ and $x_r y_i = 0$. It follows that either $x_i = 0$ for all $i = 1, \dots, r$, or $y_i = 0$ for all $i = 1, \dots, r$. We are going to show that it is possible to change the basis so that $\alpha_i = -1$ for $i = 1, \dots, r-1$ and $x_r + y_r = -2$. Suppose that the old basis is $\beta_1 = (v_1, w_1, \dots, v_r, w_r, v_{r+1})$. We consider two cases.

Case 1 $x_r = 0$. Then $y_r \neq 0$ and the new basis is

$$\begin{aligned} &v'_i = (-1)^{r-i}\alpha_i \cdots \alpha_{r-1} v_i, \quad w'_i = (-1)^{r-i}\alpha_i \cdots \alpha_{r-1} w_i, \quad i = 1, \dots, r-1, \\ &v'_r = v_r, \quad w'_r = w_r, \quad v'_{r+1} = -\frac{y_r}{2}\, v_{r+1}. \end{aligned}$$

In the new basis, we have $D_r = \Psi_1(t_{\delta_{2r+1}})$, $D_i = C_i + x'_i(E_{2r+1,2i} - E_{2r+1,2i+2})$ for $i = 1, \dots, r-1$.

Case 2 $y_r = 0$. Then $x_r \neq 0$ and the new basis is

$$\begin{aligned} &v'_i = (-1)^{r-i+1}\alpha_i \cdots \alpha_{r-1} \frac{x_r}{2}\, v_i, \quad w'_i = (-1)^{r-i+1}\alpha_i \cdots \alpha_{r-1} \frac{x_r}{2}\, w_i, \\ &i = 1, \dots, r-1, \quad v'_r = -\frac{x_r}{2}\, v_r, \quad w'_r = -\frac{x_r}{2}\, w_r, \quad v'_{r+1} = v_{r+1}. \end{aligned}$$

In the new basis we have: $D_r = \Psi_2(t_{\delta_{2r+1}})$, $D_i = C_i + x'_i(E_{2i-1,2r+1} - E_{2i+1,2r+1})$ for $i = 1, \dots, r-1$.

Since U_{2r+1} commutes with A_i and B_i for $1 \le i \le r-1$, by Lemma 2.1 we have

$$U_{2r+1} = \operatorname{diag}(\lambda_1 I_2, \lambda_2 I_2, \dots, \lambda_{r-1} I_2, X)$$

for some 3×3 matrix X. The relations $A_r U_{2r+1} = U_{2r+1} A_r$ (R8) and $D_r U_{2r+1} D_r = U_{2r+1}$ (R12) imply that X has the form

$$X = \begin{pmatrix} \lambda_r & \alpha & \lambda_r \\ 0 & \lambda_r & 0 \\ 0 & \beta & -\lambda_r \end{pmatrix} \quad \text{or} \quad X = \begin{pmatrix} \lambda_r & \alpha & \beta \\ 0 & \lambda_r & 0 \\ 0 & \lambda_r & -\lambda_r \end{pmatrix},$$

respectively, in Case 1 and Case 2. For $1 \le i \le r-1$, by the relation (R6) we have $D_i U_{2r+1} - U_{2r+1} D_i = 0$. By solving this equation we obtain $\lambda_i = \lambda_{i+1}$ and $x_i' = 0$, hence $D_i = C_i$. We also see that U_{2r+1} has two eigenvalues $\lambda_r, -\lambda_r$ with $\#\lambda_r = 2r$. Since U_{2r+1} is conjugate to U_{2r+1}^{-1}, we have $\lambda_r \in \{-1, 1\}$ and by multiplying f by $(-1)^{\mathrm{ab}}$ if necessary, we may assume $\lambda_r = 1$.

By the relation (R11), we have

$$\begin{aligned} U_{2r} &= (B_r C_r)^{-1} U_{2r+1}^{-1} (B_r C_r), \\ U_{2r-1} &= (B_r C_r C_{r-1} B_r)^{-1} U_{2r+1} (B_r C_r C_{r-1} B_r). \end{aligned}$$

Similarly as in the proof for odd g, by solving $U_{2r+1}U_{2r-1} - U_{2r-1}U_{2r+1} = 0$, we obtain $\beta = -2\alpha$, and then by solving $U_{2r+1}U_{2r}U_{2r+1} - U_{2r}U_{2r+1}U_{2r} = 0$, we obtain $\alpha = -1$ in Case 1, or $\alpha = 1$ in Case 2. Hence $U_{2r+1} = \Psi_1(u_{2r+1})$ in Case 1, or $U_{2r+1} = \Psi_2(u_{2r+1})$ in Case 2. By Theorem 3.1, f is equal to Ψ_1 in Case 1, and equal to Ψ_2 in Case 2, on generators of $\mathcal{M}(N)$. □

8 Homomorphisms from $\mathcal{M}(N_8)$ to $\mathrm{GL}(7, \mathbb{C})$

The aim of this section is to prove Theorem 1.6. First we have to define the epimorphism $\epsilon\colon \mathcal{M}(N_{2r+2}) \to \mathrm{Sp}(2r, \mathbb{Z}_2)$.

Fix $r \ge 1$ and set $V = H_1(N_{2r+2}, \mathbb{Z}_2)$. V is a vector space over $\mathbb{Z}_2$ of dimension $2r+2$ with basis $\overline{x_i} = [\xi_i]_2$ for $1 \le i \le 2r+2$, where $[\xi_i]_2$ denotes the mod 2 homology class of the curve ξ_i. The mod 2 intersection pairing is the symmetric bilinear form on V satisfying $\langle \overline{x_i}, \overline{x_j} \rangle = \delta_{ij}$. We define another basis for V. For $1 \le i \le r$ we set

$$\begin{gathered} v_i = [\varepsilon_i]_2 = \overline{x_1} + \dots + \overline{x_{2i}}, \quad w_i = [\delta_{2i}]_2 = \overline{x_{2i}} + \overline{x_{2i+1}}, \\ c = \overline{x_{2r+2}}, \quad d = \overline{x_1} + \dots + \overline{x_{2r+2}}. \end{gathered}$$

For $i, j \in \{1, \dots, r\}$ we have $\langle v_i, v_j \rangle = \langle w_i, w_j \rangle = 0$, $\langle v_i, w_j \rangle = \delta_{ij}$ and $\langle v_i, c \rangle = \langle w_i, d \rangle = 0$. We also have $\langle c, c \rangle = \langle c, d \rangle = 1$ and $\langle d, d \rangle = 0$.

Lemma 8.1 *Let* $\mathrm{Iso}(V)$ *be the group of automorphisms of* V *preserving the form* $\langle\cdot,\cdot\rangle$. *Then* $\mathrm{Iso}(V)$ *is isomorphic to a semi-direct product* $\mathrm{Sp}(2r,\mathbb{Z}_2)\ltimes\mathbb{Z}_2^{2r+1}$.

Proof Note that d is the unique vector of V such that $\langle\overline{x_i},d\rangle=\langle\overline{x_i},\overline{x_i}\rangle$ for $i=1,\dots,r$. It follows that d is the unique vector satisfying $\langle x,d\rangle=\langle x,x\rangle$ for all $x\in V$, hence d is fixed by all elements of $\mathrm{Iso}(V)$.

Let $W=\mathrm{span}\{v_i,w_i\mid i=1,\dots,r\}$ and observe that the restriction of $\langle\cdot,\cdot\rangle$ to W is nondegenerate and $\langle x,x\rangle=0$ for $x\in W$, hence it is a symplectic form on W. For $R\in\mathrm{Sp}(W)$, we define $A_R\in\mathrm{Iso}(V)$ as

$$A_R(d)=d,\quad A_R(c)=c,\quad A_R(x)=R(x)\quad\text{for } x\in W.$$

Any $x\in V$ can be written as $x=\gamma c+\delta d+w$, where $w\in W$ and $\gamma,\delta\in\mathbb{Z}_2$. We have $\langle x,c\rangle=\gamma+\delta$ and $\langle x,d\rangle=\gamma$. It follows that $W=\{x\in V\mid\langle x,d\rangle=\langle x,c\rangle=0\}$. Suppose $L\in\mathrm{Iso}(V)$ fixes c. Then, since $L(d)=d$, L preserves W. Hence $L=A_R$ for some $R\in\mathrm{Sp}(W)$. It follows that the mapping $R\mapsto A_R$ defines an isomorphism $\mathrm{Sp}(W)\to\mathrm{Stab}_{\mathrm{Iso}(V)}(c)$.

For $x\in\mathbb{Z}_2$ and $z\in W$, we define $B_{x,z}\in\mathrm{Iso}(V)$ as

$$B_{x,z}(d)=d,\quad B_{x,z}(c)=c+xd+z,\quad B_{x,z}(w)=w+\langle w,z\rangle d\quad\text{for } w\in W.$$

For arbitrary $v\in V$, we have

$$B_{x,z}(v)=v+\langle d,v\rangle z+\langle z+xd,v\rangle d.$$

Set $N=\{B_{x,z}\mid x\in\mathbb{Z}_2,z\in W\}$. This is a subgroup of $\mathrm{Iso}(V)$ with the group law

$$B_{x_1,z_1}B_{x_2,z_2}=B_{x_1+x_2+\langle z_1,z_2\rangle,z_1+z_2}.$$

It follows that N is abelian and $B_{x,z}^2=1$ for all x,z. Thus N is isomorphic to $\mathbb{Z}_2^{2r+1}$.

Let $L\in\mathrm{Iso}(V)$ be arbitrary. Since $\langle L(c),d\rangle=\langle L(c),L(d)\rangle=\langle c,d\rangle=1$, $L(c)=c+xd+z$ for some $x\in\mathbb{Z}_2$, $z\in W$. It follows that $B_{x,z}^{-1}L\in\mathrm{Stab}_{\mathrm{Iso}(V)}(c)$ and hence $L=B_{x,z}A_R$ for some $R\in\mathrm{Sp}(W)$. This decomposition is clearly unique, and since $A_RB_{x,z}A_R^{-1}=B_{x,R(z)}$, N is normal in $\mathrm{Iso}(V)$ and $\mathrm{Iso}(V)=N\rtimes\mathrm{Stab}_{\mathrm{Iso}(V)}(c)$. □

Lemma 8.2 *For* $r\geq 2$, *there is an epimorphism*

$$\epsilon\colon\mathcal{M}(N_{2r+2})\to\mathrm{Sp}(2r,\mathbb{Z}_2),$$

whose kernel is normally generated by $t_{\delta_{2r+1}}u_{2r+1}$ *and* $t_{\delta_{2r+1}}t_{\varepsilon_r}^{-1}$.

Proof Let $\mathcal{M}=\mathcal{M}(N_{2r+2})$. The action of $\mathcal{M}$ on $V=H_1(N_{2r+2},\mathbb{Z}_2)$ induces a homomorphism $\rho\colon \mathcal{M}\to \mathrm{Iso}(V)$, which was proved to be surjective in Gadgil and Pancholi [10], and McCarthy and Pinkall [22], and whose kernel is the normal closure of $t_{\delta_{2r+1}}u_{2r+1}$ by Szepietowski [29]. By Lemma 8.1, there exists a normal subgroup N of $\mathrm{Iso}(V)$ such that $\mathrm{Iso}(V)/N$ is isomorphic to $\mathrm{Sp}(2r,\mathbb{Z}_2)$. We define ϵ to be the composition of ρ with the canonical projection $\mathrm{Iso}(V)\to\mathrm{Iso}(V)/N$.

Let K be the normal closure of $t_{\delta_{2r+1}}u_{2r+1}$ and $t_{\delta_{2r+1}}t_{\varepsilon_r}^{-1}$ in $\mathcal{M}$. We claim that $K\subseteq\ker\epsilon$. We have $t_{\delta_{2r+1}}u_{2r+1}\in\ker\rho\subset\ker\epsilon$. Since $[\varepsilon_r]_2=v_r$ and $[\delta_{2r+1}]_2=v_r+d$, for $x\in V$ we have $\rho(t_{\varepsilon_r})(x)=x+\langle v_r,x\rangle v_r$ and $\rho(t_{\delta_{2r+1}})(x)=x+\langle v_r+d,x\rangle(v_r+d)=B_{1,v_r}(\rho(t_{\varepsilon_r})(x))$. Thus $\rho(t_{\delta_{2r+1}}t_{\varepsilon_r}^{-1})=B_{1,v_r}\in N$ and $t_{\delta_{2r+1}}t_{\varepsilon_r}^{-1}\in\ker\epsilon$. It follows that there is an induced epimorphism

$$\epsilon'\colon \mathcal{M}/K\to\mathrm{Iso}(V)/N\cong\mathrm{Sp}(2r,\mathbb{Z}_2).$$

To prove that ϵ' is an isomorphism, it suffices to show $[\mathcal{M}:K]\leq|\mathrm{Sp}(2r,\mathbb{Z}_2)|$. We are going to prove the last inequality by exhibiting an epimorphism $\mathrm{Sp}(2r,\mathbb{Z}_2)\to\mathcal{M}/K$.

Observe that the map $\eta\colon\mathcal{M}(S')\to\mathcal{M}/K$ defined to be the composition of $\iota\colon\mathcal{M}(S')\to\mathcal{M}$ from Corollary 3.6 with the canonical projection $\pi\colon\mathcal{M}\to\mathcal{M}/K$ is surjective because $\mathcal{M}$ is generated by twists about curves on $P(S')$ and $t_{\delta_{2r+1}}u_{2r+1}$ by Theorem 3.1. Gluing a disc along the boundary component of S' bounding a pair of pants with α_r and γ_r induces an epimorphism $\mathcal{M}(S')\to\mathcal{M}(S_{r,1})$ whose kernel is normally generated by $t_{\gamma_r}t_{\alpha_r}^{-1}$ (see [16, Proposition 3.8]). Since $\iota(t_{\gamma_r}t_{\alpha_r}^{-1})=t_{\delta_{2r+1}}t_{\varepsilon_r}^{-1}\in K$, it follows that we have an induced epimorphism $\eta'\colon\mathcal{M}(S_{r,1})\to\mathcal{M}/K$. There is an epimorphism $\mathcal{M}(S_{r,1})\to\mathrm{Sp}(2r,\mathbb{Z}_2)$ induced by the action of $\mathcal{M}(S_{r,1})$ on $H_1(S_{r,1},\mathbb{Z}_2)$ whose kernel is normally generated by $t_{\alpha_1}^2$ (see Berrick, Gebhardt and Paris [2, Theorem 5.7]; here we are using the assumption $r\geq 2$). By applying Lemma 3.4 (with $i=r$, $j=2r+1$) to $\pi\colon\mathcal{M}\to\mathcal{M}/K$, we have $\eta'(t_{\alpha_1}^2)=\pi(t_{\delta_1}^2)=1$. It follows that there is an induced epimorphism $\eta''\colon\mathrm{Sp}(2r,\mathbb{Z}_2)\to\mathcal{M}/K$.

$$\begin{tikzcd}
\mathcal{M}(S') \arrow[r, "\iota"] \arrow[d] & \mathcal{M} \arrow[r, "\pi"] & \mathcal{M}/K \\
\mathcal{M}(S_{r,1}) \arrow[urr, "\eta'"] \arrow[d] & & \\
\mathrm{Sp}(2r,\mathbb{Z}_2) \arrow[uurr, "\eta''"] & &
\end{tikzcd}$$

The existence of η'' proves that ϵ' is an isomorphism and $K=\ker\epsilon$. □

Lemma 8.3 *Suppose that* $f\colon \mathcal{M}(N_8) \to \mathrm{GL}(7,\mathbb{C})$ *is a homomorphism such that* $f(t_{\delta_1})$ *has order* 2. *Then* f *or* $(-1)^{\mathrm{ab}} f$ *factors through the epimorphism* $\epsilon\colon \mathcal{M}(N_8) \to \mathrm{Sp}(6,\mathbb{Z}_2)$.

Proof Let H be the normal closure of $t_{\delta_1}^2$ in $\mathcal{M} = \mathcal{M}(N_8)$ and $G = \mathcal{M}/H$. Since $H \subseteq \ker f$, we have a homomorphism $f'\colon G \to \mathrm{GL}(7,\mathbb{C})$ such that $f = f' \circ \pi$, where $\pi\colon \mathcal{M} \to G$ is the canonical projection. There is a homomorphism $\rho\colon \mathfrak{S}_8 \to G$, defined as $\rho(\sigma_i) = \pi(t_{\delta_i})$, where $\sigma_i = (i, i+1)$, for $1 \le i \le 7$. Let $\phi\colon \mathfrak{S}_8 \to \mathrm{GL}(7,\mathbb{C})$ be the composition $\phi = f' \circ \rho$. If ϕ is reducible, then $\mathrm{Im}(\phi)$ is abelian by Lemma 2.2, $f(t_{\delta_1}) = \phi(\sigma_1) = \phi(\sigma_2) = f(t_{\delta_2})$, and $\mathrm{Im}(f)$ is also abelian by Lemma 3.3, which implies $f(t_{\delta_1}) = 1$ by Theorem 3.2, a contradiction. Hence ϕ is irreducible and since $\det f(t_{\delta_1}) = 1$ (by Theorem 3.2), ϕ is the tensor product of the standard and sign representations (by Lemma 2.2). For $1 \le i \le 7$, set $L_i = f(t_{\delta_i}) = \phi(\sigma_i)$. With respect to some basis $(v_1, \dots, v_7)$, we have

$$L_1 = \mathrm{diag}(A, -I_5), \quad L_7 = \mathrm{diag}(-I_5, B), \quad L_i = \mathrm{diag}(-I_{i-2}, C, -I_{6-i})$$

for $2 \le i \le 6$, where

$$A = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let M be the matrix of $f(\varepsilon_3)$. Since M commutes with L_i for $i \neq 6$ (R5), it preserves $E(L_i, 1) = \mathrm{span}\{v_i\}$. Hence $M(v_i) = x_i v_i$ for $i \neq 6$ and $M(v_6) = y_1 v_1 + \cdots + y_7 v_7$ for some complex numbers x_i, y_j. By solving the equations $ML_i = L_i M$ for $1 \le i \le 5$ and $i = 7$, we obtain

$$x_i = x_1, \quad y_i = i y_1 \quad \text{for } 1 \le i \le 5, \quad y_6 = x_1 + 6y_1, \quad x_7 = y_6 - 2y_7.$$

Since M and L_i are conjugate, they have the same eigenvalues, which gives $x_1 = -1$ and $y_6 = -x_7$. If $y_6 = 1$, then $y_1 = 1/3$ and $y_7 = 1$, which contradicts the braid relation $ML_6M = L_6ML_6$ (R5). Hence $y_6 = -1$, $y_1 = 0$ and $y_7 = -1$, which means $M = L_7$.

For $i = 1, \dots, 7$, let U_i be the matrix of $f(u_i)$. Since U_7 commutes with L_j for $1 \le j \le 5$ (R6) and with $M = L_7$ (R8), we obtain, as above, that

$$\begin{aligned} U_7(v_i) &= x v_i \quad \text{for } 1 \le i \le 5, \\ U_7(v_6) &= y(v_1 + 2v_2 + 3v_3 + 4v_4 + 5v_5) + (x + 6y)v_6 + z v_7, \\ U_7(v_7) &= (x + 6y - 2z)v_7, \end{aligned}$$

for some complex numbers x, y, z. Since U_7 is conjugate to its inverse, and x is an eigenvalue of multiplicity at least 5, we have $x = \pm 1$, and by multiplying f by $(-1)^{\mathrm{ab}}$ if necessary, we may assume $x = -1$. By (R11) we have $U_5 = (L_6 L_7 L_5 L_6)^{-1} U_7 (L_6 L_7 L_5 L_6)$, and by solving $U_5 U_7 = U_7 U_5$ we obtain $y = 0$. Since $\det U_7 = \pm 1$, either $-1 - 2z = 1$ or $-1 - 2z = -1$. In the latter case we have $U_7 = -I$, and since U_6 is conjugate to U_7, we have $U_6 = -I$, and the relation $L_6 U_7 U_6 = U_7 U_6 L_7$ (R10) gives $L_6 = L_7$, a contradiction. Hence $z = -1$ and $U_7 = L_7$.

We have $M = U_7 = L_7$, and since $L_7^2 = I$, we have $\{t_{\delta_7} t_{\varepsilon_3}^{-1}, t_{\delta_7} u_7\} \subset \ker f$, which implies, by Lemma 8.2, that f factors through ϵ. □

Proof of Theorem 1.6 Suppose that $f\colon \mathcal{M}(N_8) \to \mathrm{GL}(7, \mathbb{C})$ is a homomorphism, such that $\mathrm{Im}(f)$ is not abelian. By Lemma 7.2, $L = f(t_{\delta_1})$ has an eigenvalue λ such that $\dim E(L, \lambda) = 6$. Since L is conjugate to L^{-1}, we have $\lambda^2 = 1$. Suppose that $\lambda = -1$. Then since $\det L = 1$ we have $\#\lambda = 6$, and there is another eigenvalue $\mu = 1$. It follows that L has order 2 and the case (2) holds by Lemma 8.3. If $\lambda = 1$ then it must be the unique eigenvalue, and the case (3) holds by Lemma 7.3 and the proof of Theorem 1.5 for even g. □

Remark 8.4 Suppose that G is a finite quotient of $\mathcal{M}(N_g)$ for $g \geq 7$, $g \neq 8$, and $f\colon G \to \mathrm{GL}(g-1, \mathbb{C})$ is a homomorphism. Then, by Theorem 1.5, $\mathrm{Im}(f)$ is abelian. For example, by Lemma 8.2, for $r \geq 4$ the image of every homomorphism $\mathrm{Sp}(2r, \mathbb{Z}_2) \to \mathrm{GL}(2r+1, \mathbb{C})$ is abelian. It is a classical result that $\mathrm{Sp}(2r, \mathbb{Z}_d)$ is perfect for $r \geq 3$ and all d (note that the last group is a quotient of $\mathcal{M}(S_r)$, which is perfect for $r \geq 3$ [24]). It follows that for $r \geq 4$ the only homomorphism from $\mathrm{Sp}(2r, \mathbb{Z}_2)$ to $\mathrm{GL}(2r+1, \mathbb{C})$ is the trivial one.

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References

[1] **J Aramayona**, **J Souto**, *Homomorphisms between mapping class groups*, Geom. Topol. 16 (2012) 2285–2341 MR3033518

[2] **J A Berrick**, **V Gebhardt**, **L Paris**, *Finite index subgroups of mapping class groups*, to appear in Proc. London Math. Soc

[3] **J S Birman**, **D R J Chillingworth**, *On the homeotopy group of a non-orientable surface*, Proc. Cambridge Philos. Soc. 71 (1972) 437–448 MR0300288

[4] **J Bray**, **S Linton**, **S Norton**, **R Parker**, **S Rogers**, **I Suleiman**, **J Tripp**, **P Walsh**, **R A Wilson**, *Atlas of finite group representations* (2001) Available at `http://brauer.maths.qmul.ac.uk/Atlas/v3/`

[5] **F Castel**, *Geometric representations of the braid groups* `arXiv:1104.3698`

[6] **D R J Chillingworth**, *A finite set of generators for the homeotopy group of a non-orientable surface*, Proc. Cambridge Philos. Soc. 65 (1969) 409–430 MR0235583

[7] **D B A Epstein**, *Curves on 2–manifolds and isotopies*, Acta Math. 115 (1966) 83–107 MR0214087

[8] **J Franks**, **M Handel**, *Triviality of some representations of* $\mathrm{MCG}(S_g)$ *in* $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{Diff}(S^2)$ *and* $\mathrm{Homeo}(\mathbb{T}^2)$, Proc. AMS 141 (2013) 2951–2962 MR3068948

[9] **W Fulton**, **J Harris**, *Representation theory: A first course*, Graduate Texts in Mathematics 129, Springer, New York (1991) MR1153249

[10] **S Gadgil**, **D Pancholi**, *Homeomorphisms and the homology of non-orientable surfaces*, Proc. Indian Acad. Sci. Math. Sci. 115 (2005) 251–257 MR2161731

[11] **P A Gastesi**, *A note of the Torelli spaces of non-orientable compact Klein surfaces*, Ann. Acad. Sci. Fenn. Math. 24 (1999) 23–30 MR1677997

[12] **F J González-Acuña**, **J M Márquez-Bobadilla**, *On the homeotopy group of the non orientable surface of genus three*, Rev. Colombiana Mat. 40 (2006) 75–79 MR2321592

[13] **W J Harvey**, **M Korkmaz**, *Homomorphisms from mapping class groups*, Bull. London Math. Soc. 37 (2005) 275–284 MR2119027

[14] **A Hatcher**, *Algebraic topology*, Cambridge Univ. Press (2002) MR1867354

[15] **M Korkmaz**, *Low-dimensional linear representations of mapping class groups* `arXiv:1104.4816`

[16] **M Korkmaz**, *The symplectic representation of the mapping class group is unique* `arXiv:1108.3241`

[17] **M Korkmaz**, *First homology group of mapping class groups of nonorientable surfaces*, Math. Proc. Cambridge Philos. Soc. 123 (1998) 487–499 MR1607985

[18] **M Korkmaz**, *Low-dimensional homology groups of mapping class groups: A survey*, Turkish J. Math. 26 (2002) 101–114 MR1892804

[19] **M Korkmaz**, *Problems on homomorphisms of mapping class groups*, from: "Problems on mapping class groups and related topics", (B Farb, editor), Proc. Sympos. Pure Math. 74, Amer. Math. Soc. (2006) 81–89 MR2264533

[20] **M Korkmaz**, **J D McCarthy**, *Surface mapping class groups are ultrahopfian*, Math. Proc. Cambridge Philos. Soc. 129 (2000) 35–53 MR1757776

[21] **W B R Lickorish**, *Homeomorphisms of non-orientable two-manifolds*, Proc. Cambridge Philos. Soc. 59 (1963) 307–317 MR0145498

[22] **J D McCarthy**, **U Pinkall**, *Representing homology automorphisms of nonorientable surfaces*, preprint MPI/SFB 85–11, Max Planck Institute (2004) Available at `http://www.math.msu.edu/~mccarthy`

[23] **L Paris**, **B Szepietowski**, *A presentation for the mapping class group of a nonorientable surface* `arXiv:1308.5856`

[24] **J Powell**, *Two theorems on the mapping class group of a surface*, Proc. Amer. Math. Soc. 68 (1978) 347–350 MR0494115

[25] **M Stukow**, *Generating mapping class groups of nonorientable surfaces with boundary*, Adv. Geom. 10 (2010) 249–273 MR2629814

[26] **B Szepietowski**, *Mapping class group of a non-orientable surface and moduli space of Klein surfaces*, C. R. Math. Acad. Sci. Paris 335 (2002) 1053–1056 MR1955587

[27] **B Szepietowski**, *A presentation for the mapping class group of the closed non-orientable surface of genus* 4, J. Pure Appl. Algebra 213 (2009) 2001–2016 MR2533302

[28] **B Szepietowski**, *Embedding the braid group in mapping class groups*, Publ. Mat. 54 (2010) 359–368 MR2675928

[29] **B Szepietowski**, *Crosscap slides and the level* 2 *mapping class group of a nonorientable surface*, Geom. Dedicata 160 (2012) 169–183 MR2970047

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