Examples of smooth compact toric varieties that are not quasitoric manifolds

YUSUKE SUYAMA

We construct smooth compact toric varieties of complex dimension ≥ 4 whose orbit spaces by the action of the compact torus are not homeomorphic to simple polytopes (as manifolds with corners). These provide the first known examples of smooth compact toric varieties that are not quasitoric manifolds.

1 Introduction

A *toric variety* X of complex dimension n is a normal algebraic variety over $\mathbb{C}$ containing the algebraic torus $(\mathbb{C}^*)^n$ as an open dense subset, such that the natural action of $(\mathbb{C}^*)^n$ on itself extends to X, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. In this paper, a smooth compact toric variety is called a *toric manifold*. The orbit space $X/(S^1)^n$ of a toric manifold X by the restricted action of the compact torus $(S^1)^n \subset (\mathbb{C}^*)^n$ is a manifold with corners such that all faces are contractible and any non-empty intersection of faces is connected. If X is projective, then a moment map identifies the orbit space $X/(S^1)^n$ with a simple polytope, so the orbit space is homeomorphic to a simple polytope as a manifold with corners. Any toric manifold of complex dimension ≤ 2 is projective. Although there are many non-projective toric manifolds in complex dimension 3, their orbit spaces are all homeomorphic to simple polytopes as manifolds with corners; this follows from Steinitz's theorem on planar graphs. It has so far been unknown whether this is still the case in higher dimensions.

The purpose of this paper is to prove the following theorem:

Theorem 1 *For any integer $n \geq 4$, there are infinitely many smooth compact toric varieties X of complex dimension n whose orbit spaces $X/(S^1)^n$ are not homeomorphic to any simple polytope as manifolds with corners.*

A *quasitoric manifold* X of (real) dimension $2n$ over a simple polytope P is a closed smooth manifold with a smooth action of $(S^1)^n$ such that the action is locally standard and the orbit space $X/(S^1)^n$ is the simple polytope P [4]. The restricted action

of $(S^1)^n$ on a toric manifold of complex dimension n is always locally standard. Therefore, if a toric manifold is projective or of dimension $n \le 3$, then it is a quasitoric manifold [2]. Our theorem implies that if $n \ge 4$, then there are infinitely many toric manifolds of complex dimension n that are not quasitoric manifolds. This solves a problem posed by Buchstaber and Panov [2, Problem 5.20]. In a 2003 preprint, Y Civan [3] claimed the existence of a toric manifold as in Theorem 1, but it has been recognized that his explanation of non-polytopality was insufficient. Our construction is based on ideas found in [3] but is more explicit.

A *simplicial n–sphere* is a simplicial complex that is homeomorphic to S^n. A simplicial sphere is *polytopal* if it is combinatorially equivalent to the boundary complex of a simplicial polytope. The orbit space $X/(S^1)^n$ of a toric manifold X is homeomorphic to a simple convex polytope if and only if the underlying simplicial complex of the fan of X is polytopal [2]. The Barnette sphere is a simplicial 3–sphere which is not polytopal [1]. However, the Barnette sphere cannot be the underlying simplicial complex of a non-singular fan [6, Theorem 9.1]. So we first find a simplicial singular fan whose underlying simplicial complex is the Barnette sphere and change it into a non-singular fan by subdivision while keeping the non-polytopality of the underlying simplicial complex. Our proof of non-polytopality is similar to that of [5, Theorem 5.3, Chapter III]. Thus we obtain a desired toric manifold X of complex dimension 4. In fact, we can produce infinitely many such toric manifolds by performing subdivisions and suspensions on the fan of X.

The structure of the paper is as follows: In Section 2, we give a fan whose underlying simplicial complex is the Barnette sphere. In Section 3, we prove Theorem 1.

2 The Barnette sphere

The Barnette sphere is a simplicial 3–sphere consisting of the 8 vertices e_1, e_2, e_3, e_4, d_1, d_2, d_3, d_4 described in Figure 1; see [1] for details.

We assign to the vertices of the Barnette sphere certain points in $\mathbb{R}^4$ as follows: e_1, e_2, e_3, e_4 are the standard basis of $\mathbb{Z}^4 \subset \mathbb{R}^4$ and

$$d_1 = (-1, 0, -2, 1), \qquad d_2 = (-2, -1, 0, 1),$$
$$d_3 = (0, -2, -1, 1), \qquad d_4 = (1, 0, 1, -1).$$

Let Δ be the set consisting of 19 4–dimensional cones as in Table 1 and their faces. The determinants of the matrices formed by the four edge vectors of each 4–dimensional cone in Δ are also noted in Table 1.
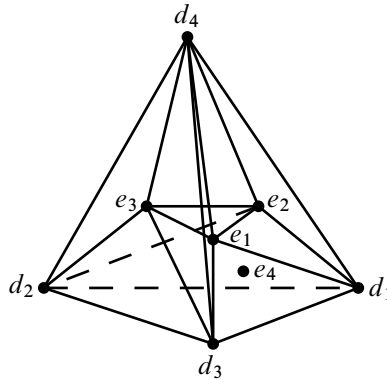
Figure 1: Diagram of the Barnette sphere (some edges are omitted)

cone	edges	det	cone	edges	det	cone	edges	det
σ_1	$e_1e_2e_3e_4$	1	σ_8	$d_1e_2d_3e_4$	1	σ_{15}	$e_1d_1d_3d_4$	2
σ_2	$d_1e_2e_3e_4$	-1	σ_9	$e_1d_2e_3d_3$	1	σ_{16}	$d_1e_2d_2d_4$	1
σ_3	$e_1d_2e_3e_4$	-1	σ_{10}	$e_1e_2d_3d_1$	1	σ_{17}	$d_3d_2e_3d_4$	3
σ_4	$e_1e_2d_3e_4$	-1	σ_{11}	$d_1e_2e_3d_2$	1	σ_{18}	$d_1d_2d_3e_4$	-9
σ_5	$e_1e_2e_3d_4$	-1	σ_{12}	$e_1e_2d_1d_4$	1	σ_{19}	$d_1d_2d_3d_4$	3
σ_6	$d_1d_2e_3e_4$	1	σ_{13}	$e_1d_3e_3d_4$	2			
σ_7	$e_1d_2d_3e_4$	1	σ_{14}	$d_2e_2e_3d_4$	1			

Table 1: 4–dimensional cones in Δ

Lemma 2 Δ *is a simplicial complete fan.*

Proof By construction, any k–dimensional cone of Δ is generated by linearly independent k vectors in $\mathbb{Z}^4$. So any cone in Δ is simplicial. One can check that for each 3–dimensional cone τ in Δ, the two 4–dimensional cones containing it as a common face have no intersection except τ. For example, let τ be the 3–dimensional cone generated by d_1, e_2, d_3. The cones containing τ are σ_8 and σ_{10}. Since $e_1+e_4=-d_1+4e_2+2d_3$, σ_8 and σ_{10} are on opposite sides of the 3–subspace containing τ. This implies that $\mathbb{R}^4$ is covered by the 4–dimensional cones uniformly. Hence if some cones overlap, then every cone is covered by the union of some other cones. However, one can check by elementary calculations that, for example, each of $\sigma_2, \dots, \sigma_{19}$ has no points whose coordinates are all positive (that is, interior points of σ_1). For example, if $a_1e_1+a_2e_2+a_3d_1+a_4d_4 \in \sigma_{12}$ ($a_i \geq 0$) is an interior point of σ_1, then we must have $a_1-a_3+a_4>0, a_2>0, -2a_3+a_4>0, a_3-a_4>0$. The latter two inequalities imply $-a_3>0$, which is a contradiction. So there are no overlaps among the 4–dimensional cones in Δ, which means that Δ is a complete fan. This completes the proof. $\square$

Remark   A computer calculation shows that the $81^4 = 43{,}046{,}721$ lattice points in

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid x_i \in \mathbb{Z}, -40 \le x_i \le 40\}$$

are classified into the five types in Table 2. The 260 relative interior points of the common 1–face of more than two cones are

$$\{me_1, me_2, me_3, me_4, nd_1, nd_2, nd_3, md_4 \in \mathbb{Z}^4 \mid m, n \in \mathbb{Z}, 1 \le m \le 40, 1 \le n \le 20\}.$$

The relative interior point of the common 0–face of more than two cones is the origin. The sum of the numbers in Table 2 is 81^4, which supports the completeness of the fan Δ.

Classification	#
Interior points of a cone	41,315,292
Rel. int. points of the common facet of two cones	1,696,978
Rel. int. points of the common 2–face of more than two cones	34,190
Rel. int. points of the common 1–face of more than two cones	260
Rel. int. point of the common 0–face of more than two cones	1

Table 2: Classification of lattice points

3   Proof of Theorem 1

According to Table 1, the singular 4–dimensional cones of Δ are $\sigma_{13}, \sigma_{15}, \sigma_{17}, \sigma_{18}$ and σ_{19}. We shall subdivide them so that the resulting 4–dimensional cones are all non-singular. We denote a cone by arranged edge vectors in $\mathbb{R}^4$ (eg $\sigma_{13} = e_1 d_3 e_3 d_4$).

Subdivision of σ_{13} and σ_{15}   We introduce a point

$$c_1 = \tfrac{1}{2}e_1 + \tfrac{1}{2}d_3 + \tfrac{1}{2}d_4 = (1, -1, 0, 0).$$

Note that c_1 is on the 3–dimensional cone $e_1 d_3 d_4$. We subdivide the cones σ_{13} and σ_{15} as (see Figure 2)

$$\sigma_{13} = e_1 d_3 e_3 d_4 \text{ into } c_1 d_3 e_3 d_4, \ e_1 c_1 e_3 d_4, \ e_1 d_3 e_3 c_1,$$
$$\sigma_{15} = e_1 d_1 d_3 d_4 \text{ into } c_1 d_1 d_3 d_4, \ e_1 d_1 c_1 d_4, \ e_1 d_1 d_3 c_1.$$

All the determinants of the resulting cones are 1.

Subdivision of σ_{17}   We introduce a point

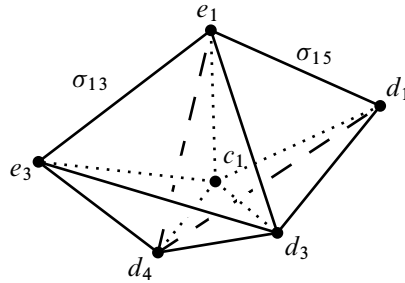$$c_2 = \tfrac{1}{3}d_3 + \tfrac{1}{3}d_2 + \tfrac{2}{3}e_3 + \tfrac{2}{3}d_4 = (0, -1, 1, 0)$$

Figure 2: Subdivision of σ_{13} and σ_{15}

and subdivide $\sigma_{17} = d_3 d_2 e_3 d_4$ into

$$c_2 d_2 e_3 d_4, \quad d_3 c_2 e_3 d_4, \quad d_3 d_2 c_2 d_4, \quad d_3 d_2 e_3 c_2.$$

The determinants of the cones $d_3 d_2 c_2 d_4, d_3 d_2 e_3 c_2$ are 2. So we further introduce a point

$$c_3 = \tfrac{1}{2} d_3 + \tfrac{1}{2} d_2 + \tfrac{1}{2} c_2 = (-1, -2, 0, 1).$$

Note that c_3 is on the 3–dimensional cone $d_3 d_2 c_2$. We subdivide the cones $d_3 d_2 c_2 d_4$, $d_3 d_2 e_3 c_2$ as (see Figure 3)

$$d_3 d_2 c_2 d_4 \quad \text{into} \quad c_3 d_2 c_2 d_4, \quad d_3 c_3 c_2 d_4, \quad d_3 d_2 c_3 d_4,$$
$$d_3 d_2 e_3 c_2 \quad \text{into} \quad c_3 d_2 e_3 c_2, \quad d_3 c_3 e_3 c_2, \quad d_3 d_2 e_3 c_3.$$
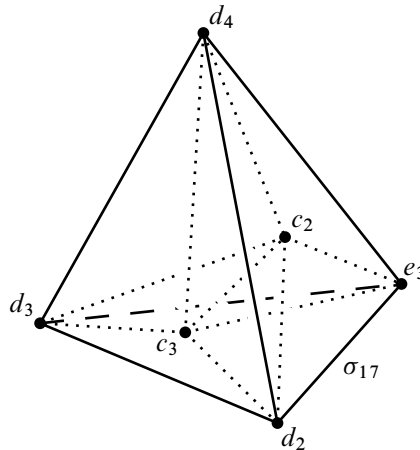
All the determinants of the resulting cones are 1.

Figure 3: Subdivision of σ_{17}

Subdivision of σ_{18} We introduce a point

$$c_4 = \tfrac{1}{3} d_1 + \tfrac{1}{3} d_2 + \tfrac{1}{3} d_3 = (-1, -1, -1, 1).$$

Note that c_4 is on the 3–dimensional cone $d_1 d_2 d_3$. We subdivide $\sigma_{18} = d_1 d_2 e_3 d_4$ into $c_4 d_2 d_3 e_4, d_1 c_4 d_3 e_4, d_1 d_2 c_4 e_4$. All the determinants of the resulting cones are -3. So we further introduce points

$$c_5 = \tfrac{1}{3}c_4 + \tfrac{1}{3}d_2 + \tfrac{2}{3}d_3 + \tfrac{2}{3}e_4 = (-1, -2, -1, 2),$$
$$c_7 = \tfrac{2}{3}d_1 + \tfrac{1}{3}c_4 + \tfrac{1}{3}d_3 + \tfrac{2}{3}e_4 = (-1, -1, -2, 2),$$
$$c_9 = \tfrac{1}{3}d_1 + \tfrac{2}{3}d_2 + \tfrac{1}{3}c_4 + \tfrac{2}{3}e_4 = (-2, -1, -1, 2)$$

and we subdivide the cones $c_4 d_2 d_3 e_4, d_1 c_4 d_3 e_4, d_1 d_2 c_4 e_4$ as

$$c_4 d_2 d_3 e_4 \quad \text{into} \quad c_5 d_2 d_3 e_4, \quad c_4 c_5 d_3 e_4, \quad c_4 d_2 c_5 e_4, \quad c_4 d_2 d_3 c_5,$$
$$d_1 c_4 d_3 e_4 \quad \text{into} \quad c_7 c_4 d_3 e_4, \quad d_1 c_7 d_3 e_4, \quad d_1 c_4 c_7 e_4, \quad d_1 c_4 d_3 c_7,$$
$$d_1 d_2 c_4 e_4 \quad \text{into} \quad c_9 d_2 c_4 e_4, \quad d_1 c_9 c_4 e_4, \quad d_1 d_2 c_9 e_4, \quad d_1 d_2 c_4 c_9.$$

The determinants of the cones $c_4 d_2 c_5 e_4, c_4 d_2 d_3 c_5, c_7 c_4 d_3 e_4, d_1 c_4 d_3 c_7, d_1 c_9 c_4 e_4$ and $d_1 d_2 c_4 c_9$ are -2. So we further introduce points

$$c_6 = \tfrac{1}{2}c_4 + \tfrac{1}{2}d_2 + \tfrac{1}{2}c_5 = (-2, -2, -1, 2),$$
$$c_8 = \tfrac{1}{2}c_7 + \tfrac{1}{2}c_4 + \tfrac{1}{2}d_3 = (-1, -2, -2, 2),$$
$$c_{10} = \tfrac{1}{2}d_1 + \tfrac{1}{2}c_9 + \tfrac{1}{2}c_4 = (-2, -1, -2, 2)$$

and we subdivide the cones as (see Figure 4)

$$c_4 d_2 c_5 e_4 \quad \text{into} \quad c_6 d_2 c_5 e_4, \quad c_4 c_6 c_5 e_4, \quad c_4 d_2 c_6 e_4,$$
$$c_4 d_2 d_3 c_5 \quad \text{into} \quad c_6 d_2 d_3 c_5, \quad c_4 c_6 d_3 c_5, \quad c_4 d_2 d_3 c_6,$$
$$c_7 c_4 d_3 e_4 \quad \text{into} \quad c_8 c_4 d_3 e_4, \quad c_7 c_8 d_3 e_4, \quad c_7 c_4 c_8 e_4,$$
$$d_1 c_4 d_3 c_7 \quad \text{into} \quad d_1 c_8 d_3 c_7, \quad d_1 c_4 c_8 c_7, \quad d_1 c_4 d_3 c_8,$$
$$d_1 c_9 c_4 e_4 \quad \text{into} \quad c_{10} c_9 c_4 e_4, \quad d_1 c_{10} c_4 e_4, \quad d_1 c_9 c_{10} e_4,$$
$$d_1 d_2 c_4 c_9 \quad \text{into} \quad c_{10} d_2 c_4 c_9, \quad d_1 d_2 c_{10} c_9, \quad d_1 d_2 c_4 c_{10}.$$

All the determinants of the resulting cones are 1.

Subdivision of σ_{19} We subdivide $\sigma_{19} = d_1 d_2 d_3 d_4$ into $c_4 d_2 d_3 d_4, d_1 c_4 d_3 d_4,$ $d_1 d_2 c_4 d_4$. All the determinants of the resulting cones are 1.

Thus we replaced $\sigma_{13}, \sigma_{15}, \sigma_{17}, \sigma_{18}$ and σ_{19} by the cones in Table 3.

Now we have a refinement Δ' of Δ which has 18 edges and 55 4–dimensional cones. The determinant of each 4–dimensional cone of Δ' is 1 or -1. So Δ' is a non-singular complete fan and the corresponding toric variety $X(\Delta')$ is a toric manifold.
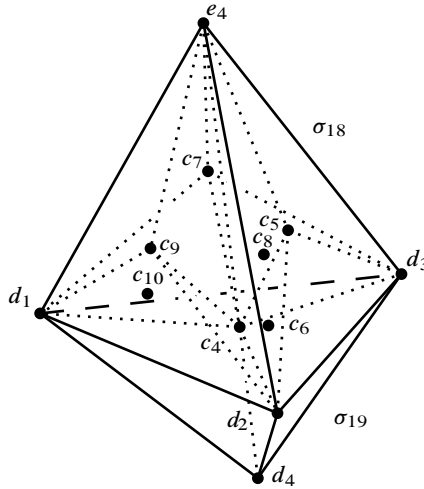
Figure 4: Subdivision of σ_{18} and σ_{19}

cone	sign	cone	sign	cone	sign	cone	sign
$c_1 d_3 e_3 d_4$	$+$	$c_3 d_2 e_3 c_2$	$+$	$c_8 c_4 d_3 e_4$	$-$	$d_1 c_9 c_{10} e_4$	$-$
$e_1 c_1 e_3 d_4$	$+$	$d_3 c_3 e_3 c_2$	$+$	$c_7 c_8 d_3 e_4$	$-$	$d_1 d_2 c_9 e_4$	$-$
$e_1 d_3 e_3 c_1$	$+$	$d_3 d_2 e_3 c_3$	$+$	$c_7 c_4 c_8 e_4$	$-$	$c_{10} d_2 c_4 c_9$	$-$
$c_1 d_1 d_3 d_4$	$+$	$c_5 d_2 d_3 e_4$	$-$	$d_1 c_7 d_3 e_4$	$-$	$d_1 d_2 c_{10} c_9$	$-$
$e_1 d_1 c_1 d_4$	$+$	$c_4 c_5 d_3 e_4$	$-$	$d_1 c_4 c_7 e_4$	$-$	$d_1 d_2 c_4 c_{10}$	$-$
$e_1 d_1 d_3 c_1$	$+$	$c_6 d_2 c_5 e_4$	$-$	$d_1 c_8 d_3 c_7$	$-$	$c_4 d_2 d_3 d_4$	$+$
$c_2 d_2 e_3 d_4$	$+$	$c_4 c_6 c_5 e_4$	$-$	$d_1 c_4 c_8 c_7$	$-$	$d_1 c_4 d_3 d_4$	$+$
$d_3 c_2 e_3 d_4$	$+$	$c_4 d_2 c_6 e_4$	$-$	$d_1 c_4 d_3 c_8$	$-$	$d_1 d_2 c_4 d_4$	$+$
$c_3 d_2 c_2 d_4$	$+$	$c_6 d_2 d_3 c_5$	$-$	$c_9 d_2 c_4 e_4$	$-$		
$d_3 c_3 c_2 d_4$	$+$	$c_4 c_6 d_3 c_5$	$-$	$c_{10} c_9 c_4 e_4$	$-$		
$d_3 d_2 c_3 d_4$	$+$	$c_4 d_2 d_3 c_6$	$-$	$d_1 c_{10} c_4 e_4$	$-$		

Table 3: Subdivided cones

Proposition 3 *The underlying simplicial complex $K_{\Delta'}$ is not polytopal. So the orbit space $X(\Delta')/(S^1)^n$ of the corresponding toric manifold $X(\Delta')$ is not homeomorphic to any simple polytope as a manifold with corners, that is, $X(\Delta')$ is not a quasitoric manifold.*

Proof Our proof is similar to the proof of [5, 5.3 Theorem, Chapter III]. Suppose that $K_{\Delta'}$ is polytopal. We denote the 3–simplex corresponding to σ_i by A_i and denote a 3–simplex by its arranged vertices. Take a Schlegel diagram of $K_{\Delta'}$ to the

3–simplex $A_{11} = d_1e_2e_3d_2$. The two 3–simplices A_2 and A_6 intersect along the common face $d_1e_3e_4$, and the point c_1 is not in $A_2 \cup A_6$. The star $\mathrm{st}(e_1d_3)$ of the edge e_1d_3 is the union of the six 3–simplices $A_9 = e_1d_2e_3d_3, e_1d_3e_3c_1, e_1d_1d_3c_1$, $A_{10} = e_1e_2d_3d_1$, $A_4 = e_1e_2d_3e_4$, $A_7 = e_1d_2d_3e_4$. Since the link $\mathrm{lk}(e_1d_3)$ of e_1d_3 consists of the simplices in $\mathrm{st}(e_1d_3)$ which do not intersect e_1d_3, it consists of six edges $d_2e_3, e_3c_1, c_1d_1, d_1e_2, e_2e_4, e_4d_2$. In the Schlegel diagram, c_1 is in the interior of A_{11} and is in the exterior of $A_2 = d_1e_2e_3e_4$ and $A_6 = d_1d_2e_3e_4$ since c_1 is not in A_2 nor in A_6. Hence either e_3c_1 or c_1d_1 passes through the triangle $e_2e_4d_2$. So the two triangles $e_2e_4d_2$ and $e_3c_1d_1$ are linked as links of a chain (see Figure 5).
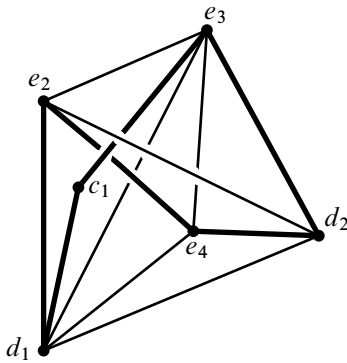

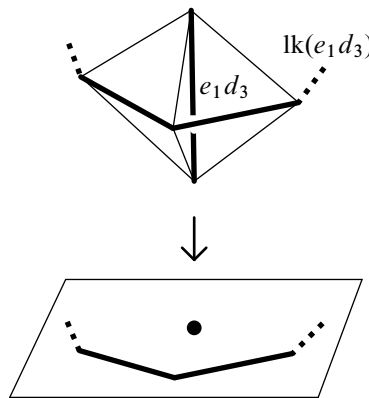Figure 5: Schlegel diagram of $K_{\Delta'}$ to the simplex A_{11}

Figure 6: Edge, link and their projections

Consider the projection of the diagram Figure 5 onto a plane. Since $e_2e_4d_2$ and $e_3c_1d_1$ are linked, their images on the plane intersect. Thus the image of the diagram in Figure 5 onto a plane falls into seven types in Figure 7 essentially. The former three diagrams in Figure 7 are the case where each point of d_1, e_2, e_3, d_2 is a boundary point of the image of A_{11}, and the latter four diagrams in Figure 7 are the case where one point of d_1, e_2, e_3, d_2 is an interior point of the image of A_{11}. The positions of the points e_4 and c_1 may differ from the graphs, but in any case, the image of $\mathrm{lk}(e_1d_3)$ has a self-intersection.

However, if the simplicial complex is polytopal, the link of any edge can be projected onto a plane perpendicular to the affine hull of the edge without self-intersection [5, Theorem 5.3, Chapter III]. Indeed, the 3–simplices of $\mathrm{st}(e_1d_3)$ are attached to e_1d_3 along their faces and $\mathrm{lk}(e_1d_3)$ is projected without self-intersection (see Figure 6). This is a contradiction. Thus we proved the proposition. □

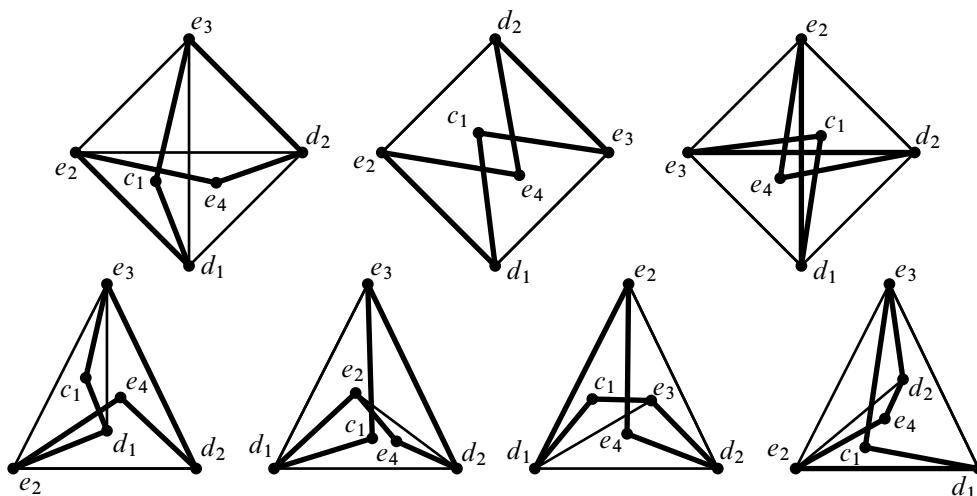The main theorem is deduced by using the fan Δ'.

Figure 7: Images of the Schlegel diagram (up to mirror images)

Proof of Theorem 1 Subdividing the cone σ_{16} in Δ' by its interior point $d_1 + e_2 + d_2 + d_4$, we have another toric manifold of complex dimension 4 whose orbit space by the compact torus is not homeomorphic to any simple polytope as a manifold with corners. Successive subdivisions produce infinitely many such toric manifolds.

If a simplicial sphere is non-polytopal, then its suspension is also non-polytopal. Because if its suspension were polytopal, the link of a new vertex would also be polytopal, which contradicts that the link is the original non-polytopal simplicial sphere. Thus for any $n \geq 4$, we have infinitely many toric manifolds of complex dimension n whose orbit spaces by the compact torus are not homeomorphic to any simple polytope as manifolds with corners. This completes the proof of Theorem 1. $\square$

References

[1] **D Barnette**, *Diagrams and Schlegel diagrams*, from: "Combinatorial Structures and their Applications", Gordon and Breach, New York (1970) 1–4 MR0270266

[2] **V M Buchstaber**, **T E Panov**, *Torus actions and their applications in topology and combinatorics*, University Lecture Series 24, Amer. Math. Soc. (2002) MR1897064

[3] **Y Civan**, *Some examples in toric geometry*, preprint `arXiv:math/0306029`

[4] **M W Davis**, **T Januszkiewicz**, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. 62 (1991) 417–451   MR1104531

[5] **G Ewald**, *Combinatorial convexity and algebraic geometry*, Graduate Texts in Mathematics 168, Springer, New York (1996)   MR1418400

[6] **H Ishida**, **Y Fukukawa**, **M Masuda**, *Topological toric manifolds*, Mosc. Math. J. 13 (2013) 57–98, 189–190   MR3112216

Department of Mathematics, Osaka City University
3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

`m13saU0r13@ex.media.osaka-cu.ac.jp`