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Eulerian cube complexes and reciprocity

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Let G be the fundamental group of a compact nonpositively curved cube complex Y. With respect to a basepoint x, one obtains an integer-valued length function on G by counting the number of edges in a minimal length edge-path representing each group element. The *growth series of G with respect to x* is then defined to be the power series $G_x(t) = \sum_g t^{\ell(g)}$, where $\ell(g)$ denotes the length of g. Using the fact that G admits a suitable automatic structure, $G_x(t)$ can be shown to be a rational function. We prove that if Y is a manifold of dimension n, then this rational function satisfies the reciprocity formula $G_x(t^{-1}) = (-1)^n G_x(t)$. We prove the formula in a more general setting, replacing the group with the fundamental groupoid, replacing the growth series with the characteristic series for a suitable regular language, and only assuming Y is Eulerian.

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1 Introduction

Given a group G and a length function $\ell\colon G \mapsto \mathbb{Z}$, the *growth series* of G with respect to ℓ is the formal power series

$$G_\ell(t) = \sum_{g \in G} t^{\ell(g)}.$$

The literature has several instances of known reciprocity formulas for growth series when the length function is derived from some action of the group on a manifold or nonsingular cell complex. Serre [13] observed that $G(t^{-1}) = \pm G(t)$ for affine Coxeter groups and the standard word length. Floyd and Plotnick [6] noted other examples for Fuchsian groups. Charney and Davis [1] generalized the formulas for affine Coxeter groups to multivariate growth series for Coxeter groups whose associated Davis complex was an Eulerian complex, and Dymara [4] and Davis, Dymara, Januszkiewicz and Okun [2] later gave a geometric interpretation in terms of "weighted L^2–cohomology" and Poincaré duality. The author [12] described a multivariate noncommuative version of the formula for right-angled Coxeter groups with Eulerian Davis complex, and with Okun [8] gave a partial interpretation of it in terms of weighted L^2–cohomology.

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Right-angled Coxeter groups fall into the more general class of groups acting on CAT(0) cube complexes. This paper arose from the observation that an extension of the author's methods in [12] establishes the reciprocity formula for many more groups in this class. In particular, reciprocity also holds for the fundamental group of any compact, Eulerian, nonpositively curved cubical complex. We stick to this case for the remainder of the introduction (and most of the paper) since the argument is easier to follow. A generalization to orbihedra (which includes the case of right-angled Coxeter groups) is mentioned in Section 5.3 at the end of the paper.

Let X be a connected CAT(0) cube complex and let G be a group acting freely, cellularly, and cocompactly on X. Then the quotient space $Y = X/G$ is a nonpositively curved cube complex with universal cover X and fundamental group isomorphic to G. We let V denote the vertices of X, and V/G denote the vertices of Y. Then the set of homotopy classes of paths in Y that start and end at vertices in V/G forms a groupoid which we denote by $\mathcal{G}$. We let $\mathcal{G}_{x,y}$ denote the morphisms in $\mathcal{G}$ that start at x and end at y. The vertex group $G_x = \mathcal{G}_{x,x}$ is precisely the fundmental group $\pi_1(Y, x)$ and, hence, isomorphic to G.

The complex X comes equipped with a family of *hyperplanes,* obtained by extending the perpendicular bisectors of the 1–dimensional cubes. We denote the set of hyperplanes in X by $\mathcal{H}$, and their images in Y by $\mathcal{H}/G$. For each element of $\mathcal{G}$, we define a multivariable "length" by counting the number of times a representative path crosses each hyperplane in $\mathcal{H}/G$. More precisely, we let $\mathbb{Q}((\boldsymbol{t}))$ denote the multivariate ring of formal Laurent series with indeterminates $\boldsymbol{t} = (t_h)$ indexed by elements $h \in \mathcal{H}/G$. For any homotopy class $[\gamma] \in \mathcal{G}_{x,y}$, we choose a representative γ that lies in the 1–skeleton of Y (ie an "edge path" in Y) and we assume that it is a minimal length representative (uses as few edges as possible). We then define the weight of $[\gamma]$ to be

$$\tau([\gamma]) = \prod_{h \in \mathcal{H}/G} t_h^{m_h},$$

where m_h counts the number of times that the edge path γ crosses the hyperplane h. It can be shown (Section 4) that the weight on $[\gamma]$ is independent of the choice of representative, so we obtain a well-defined *(multivariate) growth series* $\mathcal{G}_{x,y}(\boldsymbol{t}) \in \mathbb{Q}((\boldsymbol{t}))$ by summing over $\mathcal{G}_{x,y}$:

$$\mathcal{G}_{x,y}(\boldsymbol{t}) = \sum_{[\gamma] \in \mathcal{G}_{x,y}} \tau([\gamma]).$$

The *(ordinary) growth series* is the single-variable power series $\mathcal{G}_{x,y}(t)$ obtained by substituting t for each indeterminate t_h in the multivariate growth series. Using a result of Niblo and Reeves [7], it can be shown that these growth series are, in fact, rational functions.

In order to state our theorem, we briefly recall the definition of an Eulerian complex. Let K be a cell complex such that the link of every (nonempty) cell is a simplicial complex. We say that K is *Eulerian of dimension* n if every maximal cell is n–dimensional and for every (nonempty) cell σ in K, the Euler characteristic of the link is equal to the Euler characteristic of the sphere of the same dimension, ie

$$\chi(\mathrm{Lk}(\sigma)) = 1 + (-1)^{\dim \mathrm{Lk}(\sigma)}.$$

If, in addition, the Euler characteristic of K is the same as the Euler characteristic of the n–sphere, then K is called an *Eulerian* n*–sphere*. In particular, if K is an n–dimensional homology manifold, then it is Eulerian, and if in addition n is odd, K is an Eulerian n sphere (by Poincaré duality). The main result of this paper is the following theorem.

Theorem *Let G be a group acting freely, cellularly, and cocompactly on a connected* CAT(0) *cube complex X, and let V be the set of vertices of X. If X/G is Eulerian of dimension n, then for any $x, y \in V/G$, the multivariate growth series $\mathcal{G}_{x,y}(\boldsymbol{t})$ (regarded as a rational function) satisfies*

$$\mathcal{G}_{x,y}(\boldsymbol{t}^{-1}) = (-1)^n \mathcal{G}_{x,y}(\boldsymbol{t}).$$

The proof of the theorem actually establishes a more general formula that holds for a certain regular language encoding the fundamental groupoid. In Section 2, we collect and discuss requisite facts from formal language theory. In Section 3, we discuss groups acting on CAT(0) cube complexes. In Section 4, we discuss growth series and give some sample computations. In Section 5, we give the proof of the main theorem.

2 Regular languages

Given a set $\mathcal{A}$, we let $\mathcal{A}^*$ denote the free monoid on $\mathcal{A}$. We refer to $\mathcal{A}$ as an *alphabet* and elements of $\mathcal{A}^*$ as *words* over $\mathcal{A}$. Any subset $\mathcal{L} \subseteq \mathcal{A}^*$ is called a *language* over $\mathcal{A}$. A language is called *regular* if it is the language accepted by some finite state automaton. For our purposes, a finite state automaton will consist of a finite directed graph with vertex set $\mathcal{S}$ (the *state set*) and two designated subsets $\mathcal{B}, \mathcal{E} \subseteq \mathcal{S}$ (the *initial states* and the *accept states*, respectively). The directed edges (called *transitions*) of the graph are labeled by elements of some alphabet $\mathcal{A}$, and the labeling is further assumed to have the property that for each $a \in \mathcal{A}$ and each $i \in \mathcal{S}$ there is at most one edge labeled a emanating from i. Any directed path in the graph then determines a word in $\mathcal{A}^*$ by writing down (left to right) the labels on the consecutive edges of the path. The language *accepted* by the automaton is the set of all words corresponding to paths that start at one of the initial or "begin" states $i \in \mathcal{B}$ and end at one of the accept or "end" states $i \in \mathcal{E}$.

Remark 2.0.1 Our definition of a finite state automaton is less restrictive than a *deterministic* one (we allow more than one start state), but more restrictive than a *nondeterministic* one (we don't include a padding symbol). Since deterministic and nondeterministic automata define the same language class (ie regular languages), so does ours.

2.1 Characteristic series

Formal languages are often considered in an algebraic context by identifying them with their characteristic series, a power series whose terms are the words in the language. For convenience, we shall work in the ring $\mathbb{Q}((\mathcal{A}))$ consisting of formal Laurent series over $\mathbb{Q}$ in noncommuting indeterminates indexed by $\mathcal{A}$. To avoid extra notation, we use the same symbols for these indeterminates as for the elements of $\mathcal{A}$, and for $a \in \mathcal{A}$, we let a^{-1} denote the formal inverse in $\mathbb{Q}((\mathcal{A}))$. Given a language $\mathcal{L}$ over $\mathcal{A}$, we define its *characteristic series* to be

$$\lambda = \sum_{\alpha \in \mathcal{L}} \alpha,$$

which we regard as an element of $\mathbb{Q}((\mathcal{A}))$. If $\mathcal{L}$ is a regular language, then its characteristic series λ has the rational algebraic representation (see [10, Theorem 5.1])

$$\lambda = B(I-Q)^{-1}E = B(I+Q+Q^2+Q^3+\cdots)E, \tag{1}$$

where B is a row vector with entries in $\mathbb{Q}$, E is a column vector with entries in $\mathbb{Q}$, and Q is a square matrix whose entries are formal sums of elements in $\mathcal{A}$. In fact, given a finite state automaton that accepts $\mathcal{L}$, the matrices B, E and Q can be given explicitly as follows.

- Q is the $\mathcal{S} \times \mathcal{S}$ matrix whose (i,j)–entry is the sum of all $a \in \mathcal{A}$ that label transitions from i to j.
- B is the $1 \times \mathcal{S}$ matrix with all zeros except for 1 in the entries corresponding to initial states.
- E is the $\mathcal{S} \times 1$ matrix with all zeros except for 1 in the entries corresponding to accept states.

Since Q is determined by the transitions in the automaton, we shall refer to Q as a *transition matrix* for λ.

2.2 Reciprocity

If $\mathcal{L}$ is a regular language, then the entries of any transition matrix Q will be a sum of elements in $\mathcal{A}$. By replacing each such element with its reciprocal (ie its formal inverse), we obtain an $\mathcal{S} \times \mathcal{S}$ matrix $\overline{Q}$ defined over $\mathbb{Q}((\mathcal{A}))$. We then define the *reciprocal* of the characteristic series λ by

$$\overline{\lambda} = B(I - \overline{Q})^{-1} E,$$

provided that $I - \overline{Q}$ is invertible over $\mathbb{Q}((\mathcal{A}))$ (note that the formal expansion $I + \overline{Q} + \overline{Q}^2 + \overline{Q}^3 + \cdots$ is not defined when $\overline{Q}$ has terms with negative exponents, so invertibility of $I - \overline{Q}$ is not automatic). It can be shown that if the reciprocal of the characteristic series for a regular language exists, then it is independent of the choice of automaton [12, Proposition 4.1].

2.3 Specialization

Given any ring homomorphism $\phi\colon \mathbb{Q}((\mathcal{A})) \to R$, we refer to the image of any element $\lambda \in \mathbb{Q}((\mathcal{A}))$ as the *specialization of* λ *(with respect to* ϕ*)*. The most common examples are specializations to commutative Laurent series rings induced by monomial substitutions. More precisely, given an arbitrary indexing set I, we let $\mathbb{Q}_I((\boldsymbol{t}))$ (or simply $\mathbb{Q}((\boldsymbol{t}))$ if I is clear from the context) denote the ring of Laurent series in commuting indeterminates $\boldsymbol{t} = (t_i)_{i \in I}$. Then any assignment $a \mapsto \boldsymbol{t}_a$ from elements in $\mathcal{A}$ to nontrivial monomials in $\mathbb{Q}((\boldsymbol{t}))$ induces a specialization homomorphism $\phi\colon \mathbb{Q}((\mathcal{A})) \to \mathbb{Q}((\boldsymbol{t}))$. For such a specialization, we denote the image of $\lambda \in \mathbb{Q}((\mathcal{A}))$ by $\lambda(\boldsymbol{t})$, suppressing the index set I and the homomorphism ϕ. If λ is the characterstic series for a regular language $\mathcal{L}$, then the series $\lambda(\boldsymbol{t})$ is the power series expansion of a rational function. (This follows from the fact that the specialization of $I - Q$ in the representation (1) is invertible over the ring of rational functions in $\boldsymbol{t}$.) For such a characteristic series λ, if the reciprocal $\overline{\lambda}$ exists, then its specialization is given by $\overline{\lambda}(\boldsymbol{t}) = \lambda(\boldsymbol{t}^{-1})$, where $\boldsymbol{t}^{-1}$ denotes the tuple $(t_i^{-1})_{i \in I}$.

3 Cube complexes and cube paths

A *cube complex* is a piecewise-Euclidean metric cell complex obtained by gluing Euclidean cubes together via isometries along their faces. A cube complex is CAT(0) if it is simply-connected, and the link of every vertex is a flag complex (a *flag* complex is a simplicial complex such that every set of pairwise-adjacent vertices spans a simplex). If X is CAT(0) and v is a vertex, we let $\mathrm{Lk}(v)$ denote the link of v.

3.1 Diagonals and cube paths

Let X be a CAT(0) cube complex. Every cell in X is an isometrically embedded cube. A segment that starts at one vertex of an embedded cube and ends at the opposite vertex will be called a *(directed) diagonal* in X. Note that we include the set of *trivial diagonals*, which start and end at the same vertex. For a diagonal d, we adopt the following notation:

- d^* is the oppositely oriented diagonal ($d^*=d$ if and only if d is a trivial diagonal).
- $\alpha(d)$ is the initial vertex.
- $\omega(d)$ is the terminal vertex.
- $C(d)$ is the cube spanned by d.
- $\dim d$ or $|d|$ is the dimension of the cube $C(d)$.
- $\sigma(d)$ is the image of $C(d)$ in $\mathrm{Lk}(\alpha(d))$.

We define a *cube path* in X to be a sequence $\boldsymbol{d} = (d_1, \dots, d_n)$ of diagonals such that $\omega(d_i) = \alpha(d_{i+1})$ for $i = 1, \dots, n-1$. Note that this condition means that for each i, $\sigma(d_i^*)$ and $\sigma(d_{i+1})$ are both simplices in the link of the vertex $v_i = \omega(d_i)$. The *length* of a cube path $\boldsymbol{d}$, which we denote by $|\boldsymbol{d}|$ is the sum of the dimensions of the corresponding cubes, ie

$$|\boldsymbol{d}| = |d_1| + \cdots + |d_n|.$$

A cube path is *reduced* if it contains no trivial diagonals. A reduced cube path is called *normal* if $\mathrm{St}(\sigma(d_i^*)) \cap \sigma(d_{i+1}) = \varnothing$. Here $\mathrm{St}(\sigma)$, the *star* of σ, denotes the union of all simplices in the link that contain σ as a face.

Remark 3.1.1 The defining condition for normal cube paths are often given in terms of cubes in X rather than simplices in the link. In this case, the requirement would be $\mathrm{St}(C(d_i)) \cap C(d_{i+1}) = \{v_i\}$ (where $\mathrm{St}(C)$ now denotes the union of all *cubes* in X that contain C as a face).

Proposition 3.1.2 [7, Proposition 3.3] *If X is a connected* CAT(0) *cube complex and u and v are any two vertices in X, then there exists a unique normal cube path from u to v.*

3.2 The fundamental groupoid induced by a G–action

Let G be a group acting freely, cellularly, and cocompactly on a connected CAT(0) cube complex X. Then X is the universal cover of X/G and we let $p\colon X \to X/G$ denote the projection map. Let V denote the vertex set of X. Since G acts on V, we can define a groupoid $\mathcal{G}$ whose objects are the orbits V/G and whose morphisms are homotopy classes of paths in X/G that start and end in V/G. In other words, $\mathcal{G}$ is the subgroupoid of the fundamental groupoid $\pi_1(X/G)$ obtained by restricting the objects to the subset $V/G \subseteq X/G$. Given $x, y \in V/G$, we let $\mathcal{G}_{x,y}$ denote the set of morphisms from x to y, and we let $\mathcal{G}_x$ denote the vertex group $\mathcal{G}_{x,x}$. Since the action on X is free, $\mathcal{G}_x$ coincides with the fundamental group $\pi_1(X/G, x)$, which is in turn isomorphic to G.

Given a cube C or diagonal d in X, we let $\overline{C}$ and $\overline{d}$ denote the respective projections in X/G. We call $\overline{C}$ (resp., $\overline{d}$) a *cube in X/G* (resp., *diagonal in X/G*). Any diagonal $\overline{d}$ has a well-defined initial and final vertex in V/G defined by $\alpha(\overline{d}) = \overline{\alpha(d)}$ and $\omega(\overline{d}) = \overline{\omega(d)}$. Thus, we can define a *cube path in X/G* to be any sequence of diagonals $\overline{\boldsymbol{d}} = (\overline{d}_1, \overline{d}_2, \dots, \overline{d}_n)$ satisfying $\omega(\overline{d}_i) = \alpha(\overline{d}_{i+1})$. Given a lift v for the initial vertex $\alpha(\overline{d}_1)$, any such path has a unique lift to a cube path in X starting at v. Thus, cube paths in X/G correspond to G–orbits of cube paths in X. A cube path in X/G will be called *normal* if any (hence every) lift is a normal cube path in X.

Given $x, y \in V/G$ and respective lifts $\tilde{x}, \tilde{y} \in X$, any (continuous) path γ from x to y in X/G has a unique lift to a path $\tilde{\gamma}$ from $\tilde{x}$ to $g\tilde{y}$ for some (unique) $g \in G$. Since X is the universal cover for X/G, any path homotopic to $\tilde{\gamma}$ in X projects to a path homotopic to γ in X/G. Since any path in X with endpoints in V is homotopic to a cube path, any element of the groupoid $\mathcal{G}$ is represented by a cube path in X/G. Moreover, since there is a unique normal representative for any such cube path (by Proposition 3.1.2), the elements (morphisms) of $\mathcal{G}$ correspond bijectively to normal cube paths in X/G or, equivalently, to G–orbits of normal cube paths in X.

3.3 Automata for the groupoid

Let $\mathcal{A}$ denote the set of all nontrivial diagonals in X/G. Given any cube path in X/G, we obtain a word in $\mathcal{A}^*$ by reading off the nontrivial diagonals in the cube path. We let $\mathcal{L} \subseteq \mathcal{A}^*$ denote the set of words corresponding to normal cube paths. By the previous paragraph, we have a bijections $\mathcal{L} \to \mathcal{G}$, hence $\mathcal{L}$ defines a normal form for the groupoid $\mathcal{G}$.

Proposition 3.3.1 [7, Propositions 5.1 and 5.2] *The normal form $\mathcal{L}$ provides a biautomatic structure for $\mathcal{G}$ (in the sense of [5, Chapter 11]). In particular, $\mathcal{L}$ is a regular language.*

Given $x, y \in V/G$, we let $\mathcal{L}_{x,y} \subseteq \mathcal{L}$ denote the sublanguages corresponding to $\mathcal{G}_{x,y}$. A non-deterministic automaton for $\mathcal{L}$ is given in [7] using $\mathcal{A}$ as both the alphabet and the state set. Since we will be interested in the characteristic series for the sublanguages $\mathcal{L}_{x,y}$, we modify the automaton in [7] by enlarging the state set to $\mathcal{S} = \mathcal{A} \cup (V/G)$. Thus, elements of $\mathcal{S}$ are precisely the diagonals (both nontrivial and trivial) in X/G. Before defining the transitions, we first note that because the G–action is free, we have the following:

- The map $j \mapsto j^*\colon \mathcal{S} \to \mathcal{S}$ given by $j^* = \overline{d^*}$, where d is any lift of j is a well-defined involution on $\mathcal{S}$.
- For any vertex $v \in V$, the projection $\sigma \mapsto \overline{\sigma}$ defines an isomorphism from the link of v in X to the link of $\overline{v}$ in X/G. For any diagonal d with initial vertex v, we let $\sigma(\overline{d})$ denote the image of $\sigma(d)$ in $\mathrm{Lk}(\overline{v})$.

The transitions for our automaton are then defined as follows. Given (nontrivial) states $i, j \in \mathcal{A}$, there is a transition from i to j labeled by i whenever $\omega(i) = \alpha(j)$ and $\mathrm{St}(\sigma(i^*)) \cap \sigma(j) = \varnothing$. We also have transitions from states in $\mathcal{A}$ to states in V/G. Namely, for each $i \in \mathcal{A}$, we add a transition from i to $y \in V/G$ labeled i whenever $\omega(i) = y$. Since the condition defining transitions is precisely the condition defining normal cube paths, we obtain an automaton that accepts $\mathcal{L}_{x,y}$ by taking the initial states to be $\mathcal{B}_\alpha = \{i \mid \alpha(i) = x\}$ and the accept states to be the singleton set $\mathcal{E} = \{y\}$.

Remark 3.3.2 The automaton above has transitions corresponding to pairs of composable diagonals such that the final vertex of the first diagonal is the initial vertex of the second. One obtains a different automaton (accepting the same language) by defining transitions for composable diagonals when the initial vertex of the first diagonal is the final vertex of the second. More precisely, given $i, j \in \mathcal{S}$, there is a transition from i to j labeled i^* whenever $\alpha(i) = \omega(j)$ and $\mathrm{St}(\sigma(i)) \cap \sigma(j^*) = \varnothing$. Since accept states will now correspond to initial vertices of diagonals, we have additional transitions from $i \in \mathcal{A}$ to $y \in V/G$ labeled i^* whenever $\alpha(i) = y$. The language $\mathcal{L}_{x,y}$ is then accepted by the automaton with initial states $\mathcal{B}_\omega = \{i \mid \omega(i) = x\}$, and with accept states the singleton $\mathcal{E} = \{y\}$.

Remark 3.3.3 The nondeterministic automaton for the full language $\mathcal{L}$ described in [7] has states corresponding only to the non-vertex diagonals $\mathcal{A}$, and every state is both an initial state and an accept state. The transitions (directed edges in the automaton) are the same as ours, but are labeled by the final state rather than the initial state. That is, the transition from i to j is labeled by j rather than by i. There is no *a priori* reason to prefer one convention over the other; we have chosen ours different from [7] only because it matches the setup in [12], and our proofs are based on those results. On the other hand, the fact that our automaton uses more states than just those in $\mathcal{A}$ is worth justifying. The key issue is that with only nontrivial diagonals as states, the map from paths in the automaton to words in $\mathcal{L}$ would not be injective. For example, if the transitions are labeled by their final states, as in [7], then *every* edge in the automaton that ends at the state j will correspond to the same word in $\mathcal{L}$, namely j. Likewise, if transitions are labeled by their initial states, then every edge in the automaton that starts at i will correspond to the same word i. Adding states corresponding to vertices in X/G, with the appropriate transitions, fixes the non-injectivity problem.

3.4 Characteristic series for the groupoid

Let $\lambda_{x,y}$ denote the characteristic series for $\mathcal{L}_{x,y}$. Then by discussion in Section 2.1 and the definition of the automaton above, we have the rational representation

$$\lambda_{x,y} = B_\alpha (I - Q_+)^{-1} E, \tag{2}$$

where B_α is the $1 \times \mathcal{S}$ row vector with all zeros except 1s in the entries corresponding to the diagonals with initial vertex x, E is the $\mathcal{S} \times 1$ column vector with all zeros except for a 1 in the entry corresponding to y, and Q_+ is the $\mathcal{S} \times \mathcal{S}$ matrix given by

$$(Q_+)_{i,j} = \begin{cases} i & \text{if } i, j \in \mathcal{A},\ \alpha(j) = \omega(i) \text{ and } \mathrm{St}(\sigma(i^*)) \cap \sigma(j) = \varnothing, \\ i & \text{if } i \in \mathcal{A},\ j \in V/G \text{ and } \omega(i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, using the other automaton described in Remark 3.3.2, we have

$$\lambda_{x,y} = B_\omega (I - Q_-)^{-1} E, \tag{3}$$

where B_ω is the $1 \times \mathcal{S}$ row vector with all zeros except 1 in the entries corresponding to the diagonals with *final* vertex x, E is as before, and Q_- is the $\mathcal{S} \times \mathcal{S}$ matrix given by

$$(Q_-)_{i,j} = \begin{cases} i^* & \text{if } i, j \in \mathcal{A},\ \alpha(i) = \omega(j) \text{ and } \mathrm{St}(\sigma(i)) \cap \sigma(j^*) = \varnothing \\ i^* & \text{if } i \in \mathcal{A},\ j \in V/G \text{ and } \alpha(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

The transition matrices Q_+ and Q_- can be related using the involution $j \mapsto j^*\colon \mathcal{A} \to \mathcal{A}$. For convenience, we extend this involution $\mathcal{S}$ by having it act trivially on the trivial diagonals V/G. The following fact follows from the explicit descriptions given above for the transition matrices.

Proposition 3.4.1 *Let $[*]$ denote the $\mathcal{S}\times\mathcal{S}$ permutation matrix induced by the involution $j \mapsto j^*\colon \mathcal{S}\to\mathcal{S}$. Then*

$$Q_- = [*]Q_+[*].$$

4 Growth series and examples

Throughout this section X is a CAT(0) cube complex, and G is a group acting freely, cellularly, and cocompactly on X.

4.1 Hyperplanes and weights on cube paths

Parallelism between (unoriented) edges in each cube extends to an equivalence relation on the edges of X as follows. Two edges e and e' are equivalent if there exists a sequence of edges $e = e_1, e_2, \dots, e_n = e'$ and a sequence of cubes $C_1, \dots, C_{n-1}$ such that e_i is parallel to e_{i+1} in the cube C_i. Given an edge e, we let $[e]$ denote its equivalence class in X, and we let $\mathcal{H}$ denote the set of all such equivalence classes.

Remark 4.1.1 Given a cube C in X, we define a *midplane* to be the intersection of C with any hyperplane passing through the geometric center of C which is parallel to a codimension-one face. Any midplane bisects the edges which are perpendicular to it. Given a parallel class of edges in X, we define the corresponding *hyperplane* or *wall* to be the union of all midplanes that bisect some edge in the parallel class. Any such hyperplane is an isometrically embedded CAT(0) cube complex and separates X into two parts called *halfspaces*. There is a bijection from parallel classes of edges to the set of hyperplanes given by mapping the parallel class of an edge e to the (unique) hyperplane spanned my any midplane that bisects e. (Details can be found in [9].) For this reason, we shall often refer to the elements of $\mathcal{H}$ as hyperplanes, and will say that an edge e *crosses* the hyperplane H if $[e] = H$.

Now let $\mathcal{H}/G$ denote the set of G–orbits of hyperplanes, and let $\mathbb{Q}((\boldsymbol{t}))$ denote the ring of Laurent series in commuting indeterminates $\boldsymbol{t} = (t_h)$ indexed by $h \in \mathcal{H}/G$. We then obtain a monomial "weighting" on the set of cube paths in X/G as follows. First we define this weighting on edge paths. By definition, an *edge path* is a cube path consisting only of 1–dimensional diagonals (ie consisting only of *oriented* edges). For an edge path $\boldsymbol{d} = (d_1, \dots, d_n)$ in X, we let $[d_1], \dots, [d_n]$ denote the corresponding sequence of parallel classes (forgetting orientations), and we let $h_1, \dots, h_n$ denote the corresponding G–orbits in $\mathcal{H}/G$. Then we define the *G–weight* of $\boldsymbol{d}$ to be the monomial in $\mathbb{Q}((\boldsymbol{t}))$ given by

$$\tau(\boldsymbol{d}) = t_{h_1} t_{h_2} \cdots t_{h_n}.$$

In other words, if we let m_h denote the number of times the edge path $\boldsymbol{d}$ crosses a hyperplane in the orbit h, then

$$\tau(\boldsymbol{d}) = \prod_h t_h^{m_h}.$$

More generally, we define the G–weight of *any* cube path. Given a diagonal d with $\dim d \geq 1$, we replace it with a minimum length edge path that starts at $\alpha(d)$ and ends at $\omega(d)$. If $\dim d = m$ then this edge path will cross m hyperplaners, and this set of hyperplanes depends only on d, not the choice of edge path. Given an arbitrary cube path $\boldsymbol{d}$, we then replace each diagonal d_i with a corresponding edge path and define $\tau(\boldsymbol{d})$ to be the G–weight of the resulting edge path. (Note that the length of a cube path $\boldsymbol{d}$ is precisely the length of a representative edge path; this was the reason for defining the length $|\boldsymbol{d}|$ as we did.)

Proposition 4.1.2 (Sageev [9]) *A minimum length edge path crosses a hyperplane at most once, and two minimum length edge paths with the same endpoints must cross the same set of hyperplanes.*

In particular, the G–weight of a minimum length cube path depends only on its endpoints.

4.2 Growth series

We can transfer the G–weighting on cube paths to a weighting on the groupoid $\mathcal{G}$ as follows. Given a homotopy class $[\gamma]$ in $\mathcal{G}$, we choose a representative path γ and choose a lift $\tilde{\gamma}$ to X. Since the path $\tilde{\gamma}$ starts and ends at a vertex of X, we can find a minimum length cube path $\boldsymbol{d}$ in the same homotopy class as $\tilde{\gamma}$. The G–weight of this cube path is independent of all choices (by Proposition 4.1.2), so gives a well-defined *G–weight* to the morphism $[\gamma]$. We denote this weight by $\tau([\gamma])$.

Definition 4.2.1 Let G be a group acting freely, cellularly, and cocompactly on a connected CAT(0) cube complex X. Then the *multivariate growth series for $\mathcal{G}$* is the power series $\mathcal{G}(\boldsymbol{t}) \in \mathbb{Q}((\boldsymbol{t}))$ defined by

$$\mathcal{G}(\boldsymbol{t}) = \sum_{[\gamma]\in\mathcal{G}} \tau([\gamma]).$$

The *(ordinary) growth series for $\mathcal{G}$* is the single-variable power series $\mathcal{G}(t) \in \mathbb{Q}((t))$ obtained by substituting t for each indeterminate t_h in the multivariate growth series.

Given $x, y \in V/G$, we define the growth series $\mathcal{G}_{x,y}(\boldsymbol{t})$ and $\mathcal{G}_{x,y}(t)$ by the same formulas, but only sum over homotopy paths from x to y. In particular, each $x \in V/G$ determines growth series for the group G, given by $G_x(\boldsymbol{t}) = \mathcal{G}_{x,x}(\boldsymbol{t})$ and $G_x(t) = \mathcal{G}_{x,x}(t)$. It follows from the bijection $\mathcal{L} \to \mathcal{G}$ and the discussions above, that the growth series $\mathcal{G}_{x,y}(\boldsymbol{t})$ is the specialization (discussed in Section 2.3) of the characteristic series $\lambda_{x,y}$ with respect to the substitution $\mathcal{A} \to \mathbb{Q}(\boldsymbol{t})$ that maps each diagonal $j = \overline{d}$ in $\mathcal{A}$ to its corresponding weight $\tau([j]) = \tau(d)$. Applying this same substitution to (2) and (3), gives the following.

Proposition 4.2.2 *Let $Q_+(\boldsymbol{t})$ (resp., $Q_+(t)$) denote the transition matrix Q_+ with each letter $j = \overline{d} \in \mathcal{A}$ replaced by the monomial $\tau(d)$ (resp., $t^{|d|}$) where d is any lift of $\overline{d}$. Define $Q_-(\boldsymbol{t})$ and $Q_-(t)$ similarly. Then we have the rational function representations*

$$\mathcal{G}_{x,y}(\boldsymbol{t}) = B_\alpha(I - Q_+(\boldsymbol{t}))^{-1}E = B_\omega(I - Q_-(\boldsymbol{t}))^{-1}E,$$
$$\mathcal{G}_{x,y}(t) = B_\alpha(I - Q_+(t))^{-1}E = B_\omega(I - Q_-(t))^{-1}E.$$

4.3 Examples

For a first example, let Y be the target graph in Figure 1, and let X be the universal cover. Then X is a connected CAT(0) cube complex and the group $G = \pi_1(Y) \cong \mathbb{Z}$ acts freely on X.

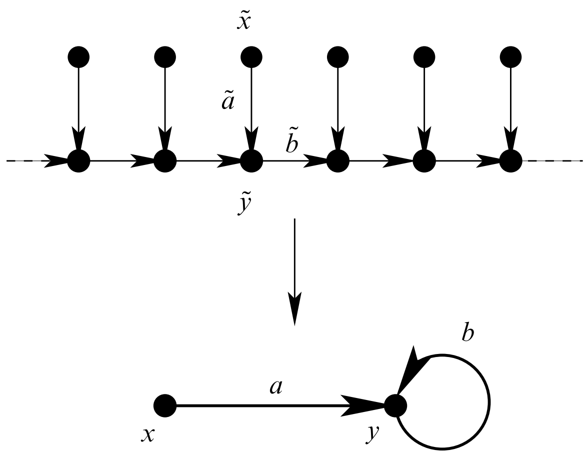

Figure 1: A 1–dimensional cube complex and its universal cover

There are 2 trivial diagonals x and y and 4 nontrivial diagonals a, a^*, b, b^* in $Y = X/G$. Thus $\mathcal{A} = \{a, a^*, b, b^*\}$ and $\mathcal{S} = \{a, a^*, b, b^*, x, y\}$. The corresponding transition matrix (with respect to the ordering (x, y, a, a^*, b, b^*)) is given by

$$Q_+ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & a & a \\ a^* & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & b & b & 0 \\ 0 & b^* & 0 & b^* & 0 & b^* \end{bmatrix}.$$

Substituting t_1, t_2, t_3, t_4 for the symbols $a, a^*, b, b^* \in \mathcal{A}$, respectively, and using Proposition 4.2.2, we obtain the multivariate growth series

$$\mathcal{G}_{x,x}(\boldsymbol{t}) = 1 - \frac{t_1 t_2(2t_3t_4 - t_3 - t_4)}{(1-t_3)(1-t_4)}, \qquad \mathcal{G}_{x,y}(\boldsymbol{t}) = \frac{t_1(1-t_3t_4)}{(1-t_3)(1-t_4)},$$
$$\mathcal{G}_{y,x}(\boldsymbol{t}) = \frac{t_2(1-t_3t_4)}{(1-t_3)(1-t_4)}, \qquad \mathcal{G}_{y,y}(\boldsymbol{t}) = \frac{1-t_3t_4}{(1-t_3)(1-t_4)}.$$

With the substitutions $t_i = t$, we obtain the ordinary growth series

$$\mathcal{G}_{x,x}(t) = \frac{2t^3 - t + 1}{1-t} = 1 + 2t^3 + 2t^4 + 2t^5 + \cdots,$$
$$\mathcal{G}_{x,y}(t) = \mathcal{G}_{y,x}(t) = \frac{t^2 + t}{1-t} = t + 2t^2 + 2t^3 + 2t^4 + \cdots,$$
$$\mathcal{G}_{y,y}(t) = \frac{1+t}{1-t} = 1 + 2t + 2t^2 + 2t^3 + \cdots.$$

For a more intricate example, and one that illustrates our reciprocity formula, we let X be the cell decomposition of the hyperbolic plane $\mathbb{H}^2$ that is dual to the tesselation by right-angled hexagons (Figure 2).

All of the 2–cells of X are squares and if one gives each of them the standard Euclidean metric, X is a CAT(0) square complex. If W is the right-angled Coxeter group generated by reflections across the sides of one of the hexagons, then W acts cellularly and cocompactly (but not freely) on X. If $s_1, \dots, s_6$ are the reflections across the lines numbered $1, 2, 3, 4, 5, 6$ in the figure, then there is a surjective homomorphism $W \to \mathbb{Z}_2 \times \mathbb{Z}_2$ defined by mapping s_1, s_3, s_5 to $(1, 0)$ and s_2, s_4, s_6 to $(0, 1)$. The kernel G of this homomorphism is an index-4 subgroup acting freely on X, and the quotient X/G is the surface of genus-2 shown in Figure 3.

The cell structure on X/G consists of 4 vertices (x, y, z, w), 12 1–cells, and 6 2–cells. It follows that the state set $\mathcal{S}$ will have 52 diagonals, 4 of which are trivial. There are 12 orbits of edge classes in $\mathcal{H}/G$, so the matrix $I - Q_+(\boldsymbol{t})$ is a 52×52

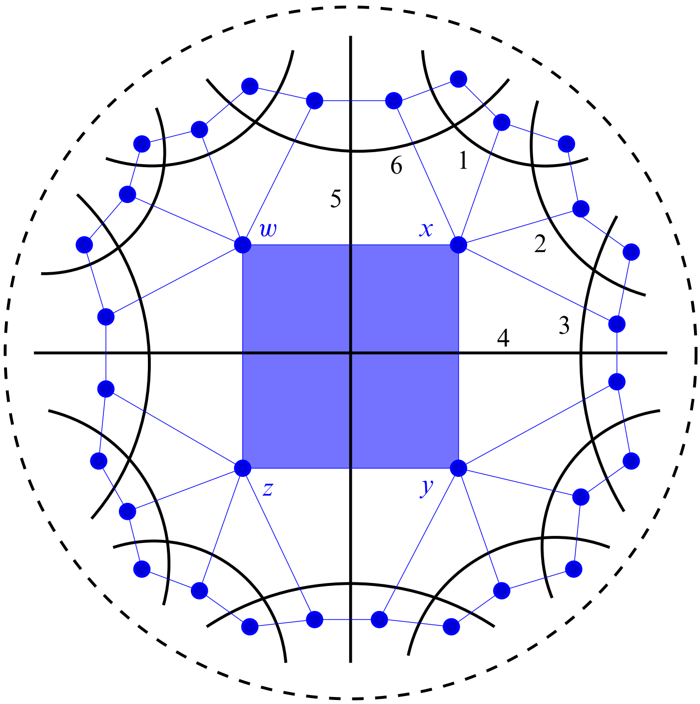

Figure 2: A right-angled hexagonal tiling of $\mathbb{H}^2$ and its dual tiling by squares

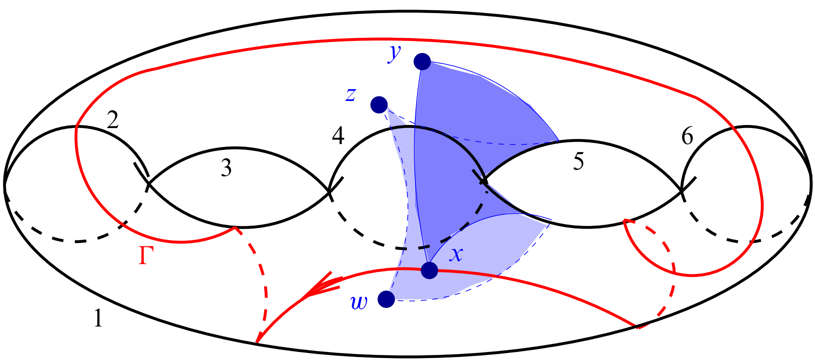

Figure 3: $\mathbb{H}^2/G$ tiled by 6 squares

matrix with 12 indeterminates. Inverting the single variable matrix $I - Q_+(t)$ is much more manageable and, with Proposition 4.2.2, yields

$$\mathcal{G}_{x,x}(t) = \frac{1-2t^2+t^4}{1-14t^2+t^4}, \quad \mathcal{G}_{x,y}(t) = \frac{3t+3t^3}{1-14t^2+t^4}, \quad \mathcal{G}_{x,z}(t) = \frac{12t^2}{1-14t^2+t^4}.$$

(By symmetry, each of the remaining series $\mathcal{G}_{i,j}(t)$ is equal to one of these.) Note that the weighting on edge paths corresponding to the single variable growth series has the interpretation of counting the number of times the path crosses one of the 6 numbered curves (these are the hyperplane orbits) in Figure 3. In particular, expanding the series for $\mathcal{G}_{x,x} = \pi_1(X/G, x)$ gives the growth series for the surface group G relative to the basepoint x,

$$G_x(t) = 1 + 12t^2 + 168t^4 + 2340t^6 + \cdots,$$

so there are 12 elements that cross 2 curves, 168 that cross 4 curves, 2340 that cross 6 curves, and so on. The indicated loop Γ, for example, is a minimal length representative in its homotopy class. It crosses the curves numbered 2, 3, 5, 6 each once and crosses the curve numbered 1 twice, thus it represents one of the 2340 group elements of length 6.

For this example, the surface X/G is an Eulerian cube complex (in fact, a manifold), so the hypotheses of our main theorem hold. In particular, all three of the growth series $\mathcal{G}_{x,x}(t)$, $\mathcal{G}_{x,y}(t)$ and $\mathcal{G}_{x,z}(t)$ satisfy the reciprocity formula $G(t^{-1}) = G(t)$.

5 Reciprocity for Eulerian cube complexes

In this section we shall prove the main theorem stated in the introduction, but in the more general setting of characteristic series rather than growth series. To state this theorem, we first extend the involution $\overline{d} \mapsto \overline{d}^*$ defined on $\mathcal{A}$ to the entire monoid $\mathcal{A}^*$. Given a word $\overline{\boldsymbol{d}} = \overline{d}_1\overline{d}_2\cdots\overline{d}_n$, we define $\overline{\boldsymbol{d}}^*$ to be $\overline{d}_1^*\overline{d}_2^*\cdots\overline{d}_n^*$. By linearity, this extends to an involution on $\mathbb{Q}((\mathcal{A}))$. Given a matrix M defined over $\mathbb{Q}((\mathcal{A}))$ we define M^* to be the matrix defined by $(M^*)_{i,j} = M_{i,j}^*$. It follows that if a series $\lambda \in \mathbb{Q}((\mathcal{A}))$ has rational expression of the form

$$\lambda = B(I - Q)^{-1}E$$

as in Section 2.1, then so does λ^* and we have

$$\lambda^* = B(I - Q^*)^{-1}E. \tag{4}$$

In terms of the characteristic series for normal cube paths, our reciprocity formula is the following.

Theorem 5.0.1 *Let G be a group acting freely, cellularly, and cocompactly on a* CAT(0) *cube complex X, and let V be the set of vertices of X. If X/G is Eulerian of dimension n, then for any $x, y \in V/G$, the reciprocal $\overline{\lambda}_{x,y}$ exists and is given by*

$$\overline{\lambda}_{x,y} = (-1)^n \lambda_{x,y}^*.$$

To obtain the theorem stated in the introduction, we simply specialize the series λ and λ^* to the commutative ring $\mathbb{Q}((\boldsymbol{t}))$. Any diagonal $\overline{d} \in \mathcal{A}$ is mapped to the product of indeterminates t_h where h is a hyperplane in X/G crossed by $\overline{d}$. Since $\overline{d}$ and $\overline{d}^*$ cross the same hyperplanes, both $\lambda_{x,y}$ and $\lambda^*_{x,y}$ specialize to the same growth series $\mathcal{G}_{x,y}(\boldsymbol{t})$. Since $\overline{\lambda}_{x,y}$ specializes to $\mathcal{G}_{x,y}(\boldsymbol{t}^{-1})$, we then obtain the formula

$$\mathcal{G}_{x,y}(\boldsymbol{t}^{-1}) = \mathcal{G}_{x,y}(\boldsymbol{t})$$

from the introduction. The remainder of this section will be devoted to the proof of Theorem 5.0.1.

5.1 Properties of the transition matrices

Let Q denote the transition matrix Q_+ for the automaton given in Section 3.4 for the language $\mathcal{L}$. Since the nonzero entries in a given row are always equal, we can factor out a diagonal matrix on the left. We define $\mathcal{S} \times \mathcal{S}$ matrices J_0 and D_0 by

$$(J_0)_{i,j} = \begin{cases} (-1)^{|i|-1} & \text{if } Q_{i,j} \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$
$$(D_0)_{i,j} = \begin{cases} (-1)^{|i|-1} i & \text{if } i = j \in \mathcal{A}, \\ 0 & \text{otherwise,} \end{cases}$$

where $|i|$ denotes the length $|d|$ for any of any lift d for i. Then we have the factorization $Q = D_0 J_0$.

To enable various matrix inversions, we define extensions of the matrices J_0 and D_0 by replacing the zero diagonal entries corresponding to states in V/G with ± 1. We define J and D by

$$J_{i,j} = \begin{cases} -1 & \text{if } i = j \in V/G, \\ (J_0)_{i,j} & \text{otherwise,} \end{cases} \tag{5}$$
$$D_{i,j} = \begin{cases} 1 & \text{if } i = j \in V/G, \\ (D_0)_{i,j} & \text{otherwise.} \end{cases}$$

We now collect various properties of these matrices for future reference. Recall that the reduced Euler characteristic of a simplicial complex K, denoted by $\overline{\chi}(K)$, is $\chi(K) - 1$ where $\chi(K)$ is the usual Euler characteristic.

Lemma 5.1.1 (1) $Q = D_0 J_0 = D J_0 = D_0 J$.

(2) $[*] D [*] = D^*$ *and* $[*] D_0 [*] = D_0^*$.

(3) *For any* $j \in \mathcal{S}$, *the sum of the entries in the* j^{th} *column of* $[*]J$ *is equal to* $\overline{\chi}(\mathrm{Lk}(\alpha(j)))$.

(4) *If* X/G *is Eulerian, then* J *is invertible and* $J^{-1} = [*] J [*]$.

Proof The first two properties follow directly from the definitions. For the second two, we notice that the entries of the matrix $[*]J$ are given by

$$([*]J)_{i,j} = J_{i^*,j} = \begin{cases} -1 & \text{if } i = j \in V/G, \\ (-1)^{|i|-1} & \text{if } i \in \mathcal{A},\ j \in V/G \text{ and } \alpha(i) = j, \\ (-1)^{|i|-1} & \text{if } i, j \in \mathcal{A},\ \alpha(i) = \alpha(j), \text{ and } \mathrm{St}(\sigma(i)) \cap \sigma(j) = \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the (i, j)–entry is zero unless $\alpha(i) = \alpha(j)$, so the matrix breaks into diagonal blocks corresponding to the partition of $\mathcal{S} = \coprod_x \mathcal{S}_x$, where $\mathcal{S}_x$ is the set of diagonals with initial vertex x. That is

$$[*]J = \bigoplus_{x \in V/G} J_x,$$

where J_x is the $\mathcal{S}_x \times \mathcal{S}_x$ block of $[*]J$. For each x, the matrix J_x is the "anti-incidence matrix" introduced in [12, Definition 7.1] for the simplicial complex $\mathrm{Lk}(x)$. By [12, Theorem 7.13], the columns of J_x all have sum $\overline{\chi}(\mathrm{Lk}(x))$. By the block decomposition of $[*]J$, the j^{th} column of $[*]J$ has sum $\overline{\chi}(\mathrm{Lk}(\alpha(j)))$, proving (3). The second part of [12, Theorem 7.13] states that the anti-incidence matrix for $\mathrm{Lk}(x)$ is an involution if $\mathrm{Lk}(x)$ is an Eulerian sphere. It follows that J_x, and hence $[*]J$ is an involution, proving (4). $\square$

5.2 Proof of reciprocity

We now prove Theorem 5.0.1. As above, we let $Q = Q_+$. By definition, the reciprocal $\overline{\lambda}_{x,y}$ exists if the matrix $I - \overline{Q}$ (obtained by replacing every entry $j \in \mathcal{A}$ with j^{-1}) is invertible over $\mathbb{Q}((\mathcal{A}))$, in which case we have

$$\overline{\lambda}_{x,y} = B_\alpha (I - \overline{Q})^{-1} E. \tag{6}$$

Since $D^{-1} = \overline{D}$, we can rewrite $I - \overline{Q}$ as

$$I - \overline{Q} = I - \overline{D}_0 J = \overline{D} D - \overline{D}_0 J = \overline{D} D_0 - \overline{D} J,$$

where the last equation follows from the fact that $\overline{D} D$ is obtained from $\overline{D} D_0$ by putting 1 in entries i, j whenever $i = j \in V/G$, and $\overline{D} J$ is obtained from $\overline{D}_0 J$ by putting -1 in these same entries. Now using algebra and (1), (2) and (4) of Lemma 5.1.1, we have

$$\begin{aligned} I - \overline{Q} = -\overline{D}(I - D_0 J^{-1}) J &= -\overline{D}(I - D_0 [*] J [*]) J \\ &= -\overline{D}(I - [*] D_0^* J [*]) J = -\overline{D}(I - [*] Q^* [*]) J. \end{aligned}$$

By Proposition 3.4.1, this can be rewritten as

$$I - \overline{Q} = -\overline{D}(I - Q_-^*)J,$$

which is invertible with inverse given by

$$(I - \overline{Q})^{-1} = -[*]J[*](I - Q_-^*)^{-1}D.$$

Substituting into (6), we have

$$\overline{\lambda}_{x,y} = -B_\alpha[*]J[*](I - Q_-^*)^{-1}DE. \tag{7}$$

Since D is diagonal with (y,y)–entry equal to 1, we have $DE = E$. The vector B_α only has nonzero entries (which are all 1) for those states j whose initial vertex is x, hence by the block decomposition of $[*]J$ and Lemma 5.1.1(3), we have $B_\alpha[*]J = \overline{\chi}(\mathrm{Lk}(x))B_\alpha$. Substituting into (7), and using the fact that $B_\alpha[*] = B_\omega$, gives

$$\overline{\lambda}_{x,y} = -\overline{\chi}(\mathrm{Lk}(x))B_\omega(I - Q_-^*)^{-1}E.$$

Since the vertex links of X/G are Eulerian spheres of dimension $n-1$, this simplifies to

$$\overline{\lambda}_{x,y} = (-1)^n B_\omega(I - Q_-^*)^{-1}E.$$

Finally, using the rational expression for $\lambda_{x,y}$ in (3) and Equation (4), we then have

$$\overline{\lambda}_{x,y} = (-1)^n\lambda_{x,y}^*.$$

This completes the proof. □

5.3 Generalization to orbihedra

One can weaken the assumption slightly on the G–action and assume only that the action on the vertex set V is free. Since one then still has unique lifts from vertex links in X/G to vertex links in X, all of the above arguments go through verbatim. In particular, the main result of this paper then includes the case proved by the author in [12] where G is a right-angled Coxeter group and X is the corresponding Davis complex. In this case, there is a single free vertex orbit, but every (nonoriented) diagonal in X has an order-2 stabilizer that reverses its direction. It follows that the involution $*$ on the state set $\mathcal{S}$ is trivial (so J is its own inverse) and the action of $*$ on $\mathbb{Q}((\boldsymbol{t}))$ is trivial. The reciprocity formula therefore takes the form

$$\overline{\lambda}_{x,y} = (-1)^n\lambda_{x,y}.$$

Theorem 5.0.1 also applies more generally to the right-angled *mock reflection groups* studied in [3; 11]. By definition, G is a right-angled mock reflection group if it acts on a CAT(0) cube complex X with the property that the action is simply-transitive on the vertex set and the edge stabilizers are all nontrivial. (Any right-angled Coxeter group acting on its Davis complex is a special case.) Again one has a single free vertex orbit and for each edge, an involution in the group that reverses its orientation. It follows that the involution $*\colon \mathcal{S} \to \mathcal{S}$ on the state set is trivial on the 0 and 1–dimensional diagonals in X/G. However, for mock reflection groups, the hyperplane dual to an edge need not be stabilized pointwise by the edge stabilizer, so the involution $*$ need not be trivial on higher-dimensional diagonals. This time, the reciprocity formula for the characteristic series requires the more general form $\overline{\lambda}_{x,y} = (-1)^n \lambda^*_{x,y}$. However, since $\lambda_{x,y}$ and $\lambda^*_{x,y}$ specialize to the same commutative-variable series, one still obtains

$$\mathcal{G}_{x,y}(\boldsymbol{t}^{-1}) = (-1)^n \mathcal{G}_{x,y}(\boldsymbol{t}).$$

In light of this generalization to orbihedra, our theorem applies to finite index subgroups of right-angled Coxeter groups and, more generally, to finite index subgroups of right-angled mock reflection groups.

Corollary 5.3.1 *Let W be a right-angled Coxeter group or mock reflection group acting on its corresponding* CAT(0) *cube complex X, and assume that X is Eulerian of dimension n. If G is any finite index subgroup of W, and x is the G–orbit of any vertex, then the growth series of G with respect to x satisfies*

$$G_x(\boldsymbol{t}^{-1}) = (-1)^n G_x(\boldsymbol{t}).$$

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