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Nongeneric J–holomorphic curves and singular inflation

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This paper investigates the geometry of a symplectic 4–manifold (M,ω) relative to a J–holomorphic normal crossing divisor $\mathcal{S}$. Extending work by Biran, we give conditions under which a homology class $A \in H_2(M;\mathbb{Z})$ with nontrivial Gromov invariant has an embedded J–holomorphic representative for some $\mathcal{S}$–compatible J. This holds for example if the class A can be represented by an embedded sphere, or if the components of $\mathcal{S}$ are spheres with self-intersection -2. We also show that inflation relative to $\mathcal{S}$ is always possible, a result that allows one to calculate the relative symplectic cone. It also has important applications to various embedding problems, for example of ellipsoids or Lagrangian submanifolds.

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1 Introduction

1.1 Overview

Inflation is an important tool for understanding symplectic embeddings in dimension 4. Combined with Taubes–Seiberg–Witten theory, it provides a powerful method to study these embedding problems, especially in so-called rational or ruled symplectic manifolds. Nonexhaustive references for ball packings are McDuff and Polterovich [23], Biran [1] and McDuff [19]. In recent years, these results have been extended in several directions by McDuff and Schlenk [26], Buşe and Hind [4] and McDuff [21], building on a work of the first author on ellipsoid embeddings [20]. Unfortunately, that paper contains a gap which we now describe briefly. The classical inflation method requires that one finds an embedded symplectic curve in a given homology class A, that intersects some fixed divisor transversally and positively. When this divisor is regular in the sense of J–holomorphic curve theory — as in the case of ball packings, where it is an exceptional divisor — this embedded representative of A is found via Taubes' work on pseudo-holomorphic curves in dimension 4. For ellipsoid embeddings however, these divisors are not regular, so the relevant almost complex structures are not generic, and the theory must be adapted, which was not done in [20]. This discussion raises the following general question:

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Question 1.1.1 Given a homology class $A \in H_2(M)$ in a symplectic 4–manifold, with embedded J–representatives for a generic set of J, are there natural conditions that ensures that A also has an embedded J–representative, where J is now prescribed on some fixed divisor $\mathcal{S}$?

In fact, as realized by Li and Usher [14], a complete answer to this question is not needed for inflation: nonembedded representatives can also be used to inflate and, as was shown by the first author in [22], this suffices to deal with the main gap in [20].

Question 1.1.2 To which extent can nodal curves replace embedded ones as far as inflation is concerned?

The present paper is concerned with these two questions. The main results are Theorem 1.2.7 which gives conditions under which a class A has an embedded J–holomorphic representative for $\mathcal{S}$–adapted J and Theorem 1.2.12 which explains that nodal curves can be used for inflation in 1–parameter families relative to $\mathcal{S}$ (leading to a relative version of "deformation implies isotopy"; see McDuff [19]).

1.2 Main results

We assume throughout that (M,ω) is a closed symplectic 4–manifold. We first discuss the kind of singular sets $\mathcal{S}$ we consider, and give a local model for their neighborhoods. A neighborhood $\mathcal{N}(C)$ of a (2–dimensional) symplectic submanifold C can always be identified with a neighborhood of the zero section in a holomorphic line bundle $\mathcal{L}$ over C with Chern class $[C]\cdot[C]$. For a union $\mathcal{S} = \bigcup C^{S_i}$ of submanifolds that intersect positively and ω–orthogonally the local model is a plumbing: we identify the standard neighborhoods $\mathcal{N}(C^{S_i})$ with $\mathcal{N}(C^{S_j})$ at an intersection point $q \in C^{S_i} \cap C^{S_j}$ by preserving the local product structure but interchanging fiber and base. Thus each such q has a product neighborhood $\mathcal{N}_q$, and by a local isotopy we can always arrange that this product structure is compatible with ω, ie $\omega|_{\mathcal{N}_q}$ is the sum of the pullbacks of its restrictions to C^{S_i} and C^{S_j}. We call the resulting plumbed structure on the neighborhood $\mathcal{N}(\mathcal{S}) = \bigcup_i \mathcal{N}(C^{S_i})$ the *local fibered structure*.

Definition 1.2.1 A *singular set* $\mathcal{S} := C^{S_1} \cup \cdots \cup C^{S_s}$ of (M,ω) is a union of symplectically embedded curves of genus $g(S_i)$ in classes $S_1,\dots,S_s$ respectively whose pairwise intersections are transverse and ω–orthogonal. A component C^{S_i} is called *negative* if $(S_i)^2 < 0$ and *nonnegative* otherwise, and is called *regular* if $(S_i)^2 \geq g-1$. We write $\mathcal{S}_{sing}$ (resp. $\mathcal{S}_{irreg}$) for the collection of components that are negative and not regular (resp. not regular), and define $\mathcal{I}_{sing} := \{i : C^{S_i} \in \mathcal{S}_{sing}\}$ and $\mathcal{I}_{irreg} := \{i : C^{S_i} \in \mathcal{S}_{irreg}\}$.

We say that the *symplectic form ω is adapted to $\mathcal{S}$* if the conditions above are satisfied and if ω is compatible with the local fiber structure on some neighborhood $\mathcal{N}(\mathcal{S})$.

Given a closed fibered neighborhood $\overline{\mathcal{N}}$ of $\mathcal{S}$ we say that an ω–tame *almost complex structure J is $(\mathcal{S},\overline{\mathcal{N}})$–adapted* if it is integrable in $\overline{\mathcal{N}}$ and if each C^{S_i} as well as each local projection $\overline{\mathcal{N}}(C^{S_i}) \to C^{S_i}$ is J–holomorphic. We define $\mathcal{J}(\mathcal{S},\overline{\mathcal{N}}) := \mathcal{J}(\mathcal{S},\overline{\mathcal{N}},\omega)$ to be the space of all such almost complex structures J. The space of *$\mathcal{S}$–adapted* almost complex structures is the union $\mathcal{J}(\mathcal{S}) := \bigcup_{\overline{\mathcal{N}}} \mathcal{J}(\mathcal{S},\overline{\mathcal{N}})$ with the direct limit topology.

We suppose throughout that $\mathcal{S}$ satisfies the conditions of Definition 1.2.1, and will call it the singular set, even though some of its components may not be in any way singular.

Remark 1.2.2 (i) The regularity condition can also be written as $d(S_i) \geq 0$, where $d(S_i) := c_1(S_i) + (S_i)^2$ is the Seiberg–Witten degree. By (2.1.4), any regular component C^{S_i} can be given a J–holomorphic parametrization for some $J \in \mathcal{J}(\mathcal{S})$ such that the linearized Cauchy–Riemann operator is surjective. In other words, the parametrization is regular in the usual sense for J–holomorphic curves; see McDuff–Salamon [25, Chapter 3]. On the other hand, if C^{S_i} is not regular, this is impossible. Further, by Remark 2.1.9(ii), if C^{S_i} is regular but negative then it is an exceptional sphere. Therefore $\mathcal{S}_{sing}$ consists of spheres with self-intersection ≤ -2 and higher-genus curves with negative self-intersection.

(ii) The orthogonality condition (ii) in Definition 1.2.1 is purely technical. If all intersections are transverse and positively oriented we can always isotop the curves in $\mathcal{S}$ so that they intersect orthogonally; cf Proposition 3.1.3.

Example 1.2.3 Suppose that (M,ω,J) is a toric manifold whose moment polytope has a connected chain of edges $\epsilon_i, i = 1,\dots,s$, with Chern numbers $-k_i \leq -2$. Then the inverse image $\mathcal{S}$ of this chain of edges under the moment map is a chain of spheres with respect to the natural complex structure on M. Moreover the toric symplectic form is adapted to $\mathcal{S}$: in particular the spheres C^{S_i} do intersect orthogonally. Another example of $\mathcal{S}$ is a disjoint union of embedded spheres each with self-intersection ≤ -2.

Write $\mathcal{E} \subset H_2(M;\mathbb{Z})$ for the set of classes that can be represented by exceptional spheres, ie symplectically embedded spheres with self-intersection -1.

Definition 1.2.4 A nonzero class $A \in H_2(M;\mathbb{Z})$ is said to be *$\mathcal{S}$–good* if:

(i) $\mathrm{Gr}(A) \neq 0$.

(ii) If $A^2 = 0$ then A is a primitive class.

(iii) $A \cdot E \geq 0$ for every $E \in \mathcal{E} \setminus \{A\}$.

(iv) $A \cdot S_i \geq 0$ for $1 \leq i \leq s$.

Example 1.2.5 As we explain in more detail in Section 2.1, when M is rational (ie $S^2 \times S^2$ or a blow up of $\mathbb{CP}^2$) the Gromov invariant $\mathrm{Gr}(A)$ is nonzero whenever $A^2 > 0$, $\omega(A) > 0$ and the Seiberg–Witten degree $d(A) := A^2 + c_1(A)$ is ≥ 0. Thus condition (i) above is easy to satisfy. Further, if $A \notin \mathcal{E}$ satisfies (i) and (iii) then $A^2 \geq 0$.

Here is a more precise version of Question 1.1.1.

Question 1.2.6 Suppose that A is $\mathcal{S}$–good. When is there an embedded connected curve C^A in class A that is J–holomorphic for some $J \in \mathcal{J}(\mathcal{S})$?

If $\mathcal{S}_{sing} = \varnothing$, then the answer is "always". Therefore the interesting case is when at least one component of $\mathcal{S}$ is not regular.[1] So far we have not managed to answer this question by trying to construct C^A geometrically.[2] The difficulties with such a direct approach are explained in Section 4.1. Nevertheless, in various situations one can obtain a positive answer by using numerical arguments. In cases (iii) and (iv) below the class A has genus $g(A) = 0$, where, by the adjunction formula (2.1.1), $g(A) := 1 + \frac{1}{2}(A^2 - c_1(A))$ is the genus of any embedded and connected J–holomorphic representative of A. Our proof of (iv) adapts arguments from Li and Zhang [15], while that in (v) generalizes Biran [1, Lemma 2.2B]. Finally (ii) follows by an easy special case of the geometric construction that works because $\mathcal{S}$ is not very singular.

Theorem 1.2.7 *Let (M, ω) be a symplectic 4–manifold with a singular set $\mathcal{S}$, and suppose that $A \in H_2(M; \mathbb{Z})$ is $\mathcal{S}$–good.*

- *In the following cases there is $J \in \mathcal{J}(\mathcal{S})$ such that A has an embedded J–holomorphic representative:*
  - (i) *$\mathcal{S}_{sing} = \varnothing$, ie the only components of $\mathcal{S}$ with negative square are exceptional spheres.*
  - (ii) *$\mathcal{S}_{sing}$ consists of a single sphere with $S^2 = -k$, where $2 \leq k \leq 4$.*
- *In the following cases there is a residual subset $\mathcal{J}_{emb}(\mathcal{S}, A)$ of $\mathcal{J}(\mathcal{S})$ such that A is represented by an embedded J–holomorphic curve C^A for all $J \in \mathcal{J}_{emb}(\mathcal{S}, A)$:*
  - (iii) $A \in \mathcal{E}$.
  - (iv) $g(A) := 1 + \frac{1}{2}(A^2 - c_1(A)) = 0$.
  - (v) *The components of $\mathcal{S}_{irreg}$ have $c_1(S_i) = 0$ and A cannot be written as*
  $$\sum_{i \in \mathcal{I}_{irreg}} \ell_i S_i,$$
  where $\ell_i \geq 0$.

[1] See Remark 2.1.9(i) for an explanation of the problem in analytic terms.

[2] In a previous version of this paper, the first author claimed to carry out such a construction. However, the second author pointed out first that the complicated inductive argument had a flaw and, more seriously, that some of the geometric constructions were incomplete.

Moreover, any two elements $J_0, J_1 \in \mathcal{J}_{emb}(\mathcal{S}, A)$ can be joined by a path $J_t, t \in [0,1]$, in $\mathcal{J}(\mathcal{S})$ for which there is a smooth family of embedded J_t–holomorphic A–curves.

Remark 1.2.8 (i) Although we do not assume initially that $b_2^+ = 1$, it is well known that any 4–manifold M that has a class A with $\mathrm{Gr}(A) \neq 0$ and $d(A) > 0$ must have $b_2^+ = 1$; cf Fact 2.1.8. The same holds if $d(A) = g(A) = 0$ and $A \notin \mathcal{E}$; cf Lemma 2.1.7. Therefore in almost all cases covered by (iv) and (v) we must have $b_2^+ = 1$.

(ii) If $c_1(S_i) = 0$ and $g(S_i) > 0$ then $(S_i)^2 \geq 0$ so that $d(S_i) \geq 0$, in other words, C^{S_i} is regular. Therefore in (v) the components of $\mathcal{S}_{irreg}$ must be spheres.

(iii) As noted in Remark 4.1.6 below, the condition on k in part (ii) above can almost surely be improved. We restrict to $k \leq 4$ to simplify the proof, and because these are the only cases that have been applied; see Borman, Li and Wu [2] and Weiwei [30].

In general, the issues involved in constructing a single embedded representative of a class A are rather different from those involved in constructing a 1–parameter family of embedded J_t–holomorphic curves for a generic path $J_t \in \mathcal{J}(\mathcal{S})$. In particular, as we see in Lemma 3.1.5, the presence of positive but nonregular components of $\mathcal{S}$ can complicate matters. Further, in cases (i) and (ii) of Theorem 1.2.7 we have no independent characterization (eg via Fredholm theory) of those $J \in \mathcal{J}(\mathcal{S})$ that admit embedded A–curves, and also cannot guarantee that there is a 1–parameter family of embedded curves connecting any given pair of embedded curves. Even if we managed to include them as part of the boundary of a 1–manifold of curves, they may well not lie in the same connected component. Hence, without extra hypotheses, it makes very little sense to try to construct 1–parameter families of such curves for fixed symplectic form ω. However, if we add an extra hypothesis (such as (ii) below) then we can construct such families. We will prove a slightly more general result that applies when we are given a family $\omega_t, t \in [0,1]$, of $\mathcal{S}$–adapted symplectic forms.

Proposition 1.2.9 *Let (M, ω) be a blowup of a rational or ruled manifold, and let $\omega_t, t \in [0,1]$, be a smooth family of $\mathcal{S}$–adapted symplectic forms. Suppose that*

(i) *$\mathcal{S}_{sing}$ is either empty or contains one sphere of self-intersection $-k$ with $2 \leq k \leq 4$,*

(ii) *$d(A) > 0$ if M is rational, and $d(A) > g + \frac{n}{4}$ if M is the n–point blow up of a ruled surface of genus g.*

Then, possibly after reparametrization with respect to t, any pair $J_\alpha \in \mathcal{J}(\mathcal{S}, \omega_\alpha, A), \alpha = 0, 1$, for which A has an embedded holomorphic representative can be joined by a path $J_t \in \mathcal{J}(\mathcal{S}, \omega_t, A)$ for which there is a smooth family of embedded J_t–holomorphic A–curves.

Remark 1.2.10 The proof of parts (iii), (iv) and (v) of Theorem 1.2.7 easily extends to prove a similar statement in these cases, but without hypothesis (ii) on A.

The gap in [20] precisely consisted in the claim that every $\mathcal{S}$–good class A does have an $\mathcal{S}$–adapted embedded representative, and as explained already, this was used to justify certain inflations and hence the existence of certain embeddings. Even though we still have not found an answer to Question 1.2.6, as far as inflation goes one can avoid it: as explained in [22], one can in fact inflate along suitable nodal curves. Thus the following holds.

Lemma 1.2.11 *If A is $\mathcal{S}$–good, $A^2 \geq 0$, and $A \cdot S_i \geq 0$ for all components S_i of $\mathcal{S}$, then there is a family of symplectic forms $\omega_{\kappa,A}$ in class $[\omega] + \kappa\,\mathrm{PD}(A)$, $\kappa \geq 0$, that are nondegenerate on $\mathcal{S}$ and have $\omega_{0,A} = \omega$.*

This result (which is reproved in Lemma 5.2.1 below) suffices to establish the existence of the desired embeddings. However, to prove their uniqueness up to isotopy, one needs to inflate in 1–parameter families, in other words, we need the following relative version of the " deformation implies isotopy" result of [19].

Theorem 1.2.12 *Let (M, ω) be a blow up of a rational or ruled 4–manifold, and let $\mathcal{S} \subset M$ satisfy the conditions of Definition 1.2.1. Let ω' be any symplectic form on M such that the following conditions hold:*

(a) $[\omega'] = [\omega] \in H^2(M)$.

(b) *There is a family of possibly noncohomologous symplectic forms ω_t, $t \in [0, 1]$, on M that are nondegenerate on $\mathcal{S}$ and are such that $\omega_0 = \omega$ and $\omega_1 = \omega'$.*

Then there is a family ω_{st}, $s, t \in [0, 1]$, of symplectic forms such that

- *$\omega_{0t} = \omega_t$ for all t and $[\omega_{1t}]$, $t \in [0, 1]$ is constant,*
- *$\omega_{s0} = \omega$ and $\omega_{s1} = \omega'$ for all s,*
- *ω_{st} is nondegenerate on each component of $\mathcal{S}$.*

Moreover, if $\omega = \omega'$ near $\mathcal{S}$, we can arrange that all the forms ω_{1t}, $t \in [0, 1]$, equal ω near $\mathcal{S}$.

Corollary 1.2.13 *Under the assumptions of Theorem 1.2.12, there is an isotopy ϕ_t, $t \in [0, 1]$, of M such that $\phi_0 = \mathrm{id}$, $\phi_1^*(\omega') = \omega$ and $\phi_t(\mathcal{S}) = \mathcal{S}$ for all t. Moreover, if $\omega = \omega'$ near $\mathcal{S}$ we may choose this isotopy to be compactly supported in $M \setminus \mathcal{S}$.*

Remark 1.2.14 Li and Liu show in [13, Theorems 2,3] that every manifold with $b_2^+ = 1$ has enough nonvanishing Seiberg–Witten invariants to convert any family ω_t with cohomologous endpoints to an isotopy. It is likely that Proposition 1.2.9 and Theorem 1.2.12 also extends to this case since the rational/ruled hypothesis is needed only via Lemma 2.1.5, which guarantees conditions that ensure $\mathrm{Gr}(B) \neq 0$ in Propositions 3.2.3 and 5.1.6.

The results on inflation can be rephrased in terms of the *relative symplectic cone* $\mathrm{Cone}_\omega(M, \mathcal{S})$. Denote by $\Omega_\omega(M)$ the connected component containing ω of the space of symplectic forms on M, and by $\Omega_\omega(M, \mathcal{S})$ its subset consisting of forms that are nondegenerate on $\mathcal{S}$. Further given $a \in H^2(M)$ let $\Omega_\omega(M, \mathcal{S}, a)$ be the subset of $\Omega_\omega(M, \mathcal{S})$ consisting of forms in class a. Define

$$\mathrm{Cone}_\omega(M) := \bigl\{[\sigma] \bigm| \sigma \in \Omega_\omega(M)\bigr\} \subset H^2(M; \mathbb{R}),$$
$$\mathrm{Cone}_\omega(M, \mathcal{S}) := \bigl\{[\sigma] \bigm| \sigma \in \Omega_\omega(M, \mathcal{S})\bigr\} \subset H^2(M; \mathbb{R}).$$

Note that these cones are connected by definition. If (M, ω) is a blowup of a rational or ruled manifold, it is well known that

$$\mathrm{Cone}_\omega(M) = \{a \in H^2(M; \mathbb{R}) \bigm| a^2 > 0,\ a(E) > 0\ \forall E \in \mathcal{E}\};$$

see Li and Li [10] and the proof of Proposition 1.2.15 given below.[3] In this language, Lemma 1.2.11 and Theorem 1.2.12 can be restated as follows. Note that the case when $\mathcal{S}$ has a single component (and M has $b_2^+ = 1$) was proved in by Dorfmeister and Li in [6, Theorem 2.7].

Proposition 1.2.15 *Let (M, ω) be a blowup of a rational or ruled manifold and $\mathcal{S}$ a singular set. Then the following hold:*

(i) $\mathrm{Cone}_\omega(M, \mathcal{S}) = \bigl\{a \in \mathrm{Cone}_\omega(M) \bigm| a(S_i) > 0,\ 1 \le i \le s\bigr\}$.

(ii) $\Omega_\omega(M, \mathcal{S}, a)$ *is path connected.*

Proof In (i) the left-hand side is clearly contained in the right-hand side. To prove the reverse inclusion, first notice that the set of classes represented by symplectic forms that evaluate positively on the S_i is open in $H^2(M, \mathbb{R})$. Hence, if $a \in \mathrm{Cone}(M, \omega)$, satisfies $a(S_i) > 0$ for all i, so does $a' = a - \varepsilon[\omega]$ for $\varepsilon > 0$ sufficiently small. Further, by perturbing ω, we may choose ε so that $a' \in H^2(M; \mathbb{Q})$. Since M is rational or ruled, the class $q\,\mathrm{PD}(a')$ is $\mathcal{S}$–good for q sufficiently large (see Corollary 2.1.6). Thus,

[3] It follows easily from Gromov–Witten theory that the set $\mathcal{E} = \mathcal{E}_{\omega'}$ of all classes represented by ω'–symplectically embedded -1 spheres is the same for all $\omega' \in \Omega_\omega(M)$. Therefore, this description of $\mathrm{Cone}_\omega(M)$ makes sense.

by Lemma 1.2.11, the class $[\omega]+\kappa q a'$ is represented by a symplectic form ω_κ for all $\kappa>0$. Taking $\kappa = 1/(q\varepsilon)$, we therefore obtain a symplectic form $\varepsilon\omega_\kappa$ in class a. This proves (i). Finally, (ii) holds because, by definition of $\Omega_\omega(M,\mathcal{S})$, any two symplectic forms in $\Omega_\omega(M,\mathcal{S},a)$ are deformation equivalent, thus isotopic by Theorem 1.2.12. □

Finally, we show that these singular inflation procedures combine with the Donaldson construction to provide approximate asymptotic answers to Question 1.2.6.

Theorem 1.2.16 *Let (M^4,ω) be a blow up of a rational or ruled manifold with a singular set $\mathcal{S}$ and an $\mathcal{S}$–good class $A\in H_2(M)$. Then:*

(i) *There is a union $\mathcal{T}$ of transversally and positively intersecting symplectic submanifolds $C^{T_1},\dots,C^{T_r}$, orthogonal to $\mathcal{S}$ and such that $\mathrm{PD}(\omega)=\sum_{j=1}^r \beta_j T_j$, where $\beta_j>0$. Further, we may take $r\le \operatorname{rank} H^2(M)$ and if $[\omega]$ is rational, we may take $r=1$.*

(ii) *For all positive $\varepsilon_1,\dots,\varepsilon_r\in\mathbb{Q}$, there are integers $N_0,k_0\ge 1$ such that $N_0(A+\sum\varepsilon_i T_i)$ is integral and each class $kN_0(A+\sum\varepsilon_i T_i), k\ge k_0$, is represented by an embedded J–curve for some $J\in\mathcal{J}(\mathcal{S}\cup\mathcal{T})$.*

Corollary 1.2.17 *If k closed balls of size $a_1,\dots,a_k\in\mathbb{Q}$ embed into $\mathbb{CP}^2$, and if $\mathcal{S}$ is any singular set in the k–fold blow-up of $\mathbb{CP}^2$ with $S_j\cdot(L-\sum a_iE_i)\ge 0$ $\forall j$, then there is N such that the class $N(L-\sum a_iE_i)$ has an embedded J–representative for some $J\in\mathcal{J}(\mathcal{S})$.*

1.3 Plan of the paper

Because this paper deals with nongeneric J, we must rework standard J–holomorphic curve theory, adding quite a few rather fussy technical details. For the convenience of the reader, we begin in Section 2 by surveying relevant aspects of Taubes–Seiberg–Witten theory, explaining in particular why Question 1.2.6 has a positive answer when $\mathcal{S}=\varnothing$. We then describe the modifications needed when J is $\mathcal{S}$–adapted.

Section 3 proves most cases of Theorem 1.2.7. The basic strategy of the proof is to represent the class A by an embedded J_ε–representative for sufficiently generic J_ε and let J_ε tend to some $J\in\mathcal{J}(\mathcal{S})$. By Gromov compactness, we get a nodal J–representative for A, whose properties are investigated in Lemmas 3.1.1 and 3.1.2. To prove part (i) of Theorem 1.2.7 it then suffices to amalgamate these components into a single curve, which is always possible for components with nonnegative self-intersection; see Corollary 3.1.4. Since this geometric approach gets considerably more complicated when $\mathcal{S}$ has negative components, the proof of part (ii) of Theorem 1.2.7

is deferred to Section 4. Proposition 1.2.9, which is a 1–parameter version of (i) and (ii), is proved in Corollaries 3.2.5 and 4.1.7. The other parts of Theorem 1.2.7 concern 1–parameter families, and their proof mixes geometric arguments with J–holomorphic curve theory. The main idea is to show that for generic families J_t one can find corresponding 1–parameter families of embedded A–curves. When $A \in \mathcal{E}$ (case (iii) of the theorem), generic families of A–curves are embedded by positivity of intersections. For more general A, we formulate hypotheses that guarantee the existence of suitable embedded families in Proposition 3.2.3. In Section 3.3 we then check that these hypotheses hold in cases (iv) and (v).

Section 4 is essentially independent of the rest of the paper. In Section 4.1 we explain how one might attempt a geometric construction of embedded A–curves. We give an extended example (Example 4.1.2), and prove part (ii) of Theorem 1.2.7 in Proposition 4.1.5. The asymptotic result Theorem 1.2.16 is explained in Section 4.2. The idea is that using Donaldson's construction of curves instead of Seiberg–Witten invariants and degenerations provides a much better control on the position of the curve relative to $\mathcal{S}$. The smoothing process is then very elementary. However, one pays for this by having less control over the class that has the embedded representatives. Note also that the proof of Theorem 1.2.16 depends on the existence of the symplectic forms constructed in Section 5.

Finally Section 5 deals with inflation, especially its 1–parameter version that is also called "deformation implies isotopy". This section provides explicit formulas for the inflation process along singular curves, and gives complete proofs of Lemma 1.2.11 and Theorem 1.2.12 in the absolute and relative cases. It relies on the results in Section 3.1 and Section 3.2, but is independent of the rest of Section 3 and of Section 4.

2 Consequences of Taubes–Seiberg–Witten theory

This section first recalls various well known results on J–holomorphic curve theory in dimension 4, and then explains the modifications necessary in the presence of a singular set $\mathcal{S}$. Throughout, unless specific mention is made to the contrary,[4] by a *curve* we mean the image of a smooth map $u\colon \Sigma \to M$, where Σ is a connected smooth Riemann surface. Thus an *immersed J–holomorphic curve* is the image of a smooth J–holomorphic immersion $u\colon \Sigma \to M$. In particular, all its double points have positive intersection number. A curve is called *simple* (or *somewhere injective*) if it is not multiply covered; see [25, Chapter 2].

[4] Occasionally we allow a curve to be disconnected, but it never has nodes unless the adjective "nodal" or "stable" is used.

2.1 Review of J–holomorphic curve theory

We begin this section by a brief review of Taubes' work relating Seiberg–Witten theory to J–holomorphic curves in order to explain the condition that $\mathrm{Gr}(A)\neq 0$. Here, $\mathrm{Gr}(A)$ is Taubes' version of the Gromov invariant of A, that to a first approximation counts embedded J–holomorphic curves in (M,ω) through $\frac{1}{2}d(A)$ generic points, where $d(A):=c_1(A)+A^2$ is the index of the appropriate Fredholm problem; see [29]. Thus $\mathrm{Gr}(A)\neq 0$ implies both that $d(A)\geq 0$ and that $\omega'(A)>0$ for all symplectic forms ω' that can be joined to ω by a deformation (ie a path of possibly noncohomologous symplectic forms). For 4–manifolds with $b_2^+=1$ (such as blow ups of rational and ruled manifolds), one shows that $\mathrm{Gr}(A)\neq 0$ by using the wall crossing formulas of Kronheimer and Mrowka [8] in the rational case and Li and Liu [11] in the ruled case.

When the intersection form on $H^2(M;\mathbb{R})$ has type $(1,N)$, the cone $\mathcal{P}:=\{a\in H^2(M)\mid a^2>0\}$ has two components; let $\mathcal{P}^+$ be the component containing $[\omega]$. Then we have the following useful fact.

Fact 2.1.1 Suppose that $b_2^+(M)=1$. If $a,b\in\overline{\mathcal{P}^+}\setminus\{0\}$ then $a\cdot b\geq 0$ with equality only if $a^2=0$ and b is a multiple of a.

Taubes' Gromov invariant $\mathrm{Gr}(A)$ in [29] counts holomorphic submanifolds and hence is somewhat different from the "usual" invariant due to Ruan and Tian that counts (perturbed) J–holomorphic maps $u\colon(\Sigma,j)\to(M,J)$ with a connected domain of fixed topological type modulo reparametrization. To explain the relation, we first make the following definition.

Definition 2.1.2 A class $A\in H_2(M;\mathbb{Z})$ is said to be *reduced* if $A\cdot E\geq 0$ for all $E\in\mathcal{E}\setminus\{A\}$.

For example, as noted at the beginning of Section 1.2, every $E\in\mathcal{E}$ is reduced. Now recall that the adjunction formula for a somewhere embedded J–curve $u\colon(\Sigma,j)\to(M,J)$ in class A with connected smooth domain of genus g_Σ states that

$$g_\Sigma\leq g(A):=1+\tfrac{1}{2}(A^2-c_1(A)), \tag{2.1.1}$$

with equality exactly if u is an embedding; see [25, Appendix E]. Using this, one can check that the only reduced classes A with $A^2<0$ and $d(A)\geq 0$ are those of the exceptional spheres $A\in\mathcal{E}$ (see Remark 2.1.9 (ii)). Taubes showed that if A is reduced and has $\mathrm{Gr}(A)\neq 0$ then A is represented by a holomorphic submanifold. Moreover, when $b_2^+(M)=1$ and $A^2>0$, it follows from Fact 2.1.1 that this manifold is connected, while if $A^2=0$ each component is a sphere or torus; see Lemma 2.2.4.

In fact, except in the case of tori of zero self-intersection (where double covers affect the count in a very delicate way), the following holds.

Fact 2.1.3 Assume that A is reduced and, if $g(A)=1$ and $A^2=0$, also primitive. Then for generic J, the invariant $\mathrm{Gr}(A)$ simply counts (with appropriate signs) the number of possibly disconnected, embedded J–holomorphic curves through $\frac{1}{2}d(A)$ generic points. Moreover if $b_2^+(M)=1$ this curve is connected with genus $g(A)$. Thus $\mathrm{Gr}(A)$ equals the standard J–holomorphic curve invariant that counts connected curves with genus $g(A)$ through $\frac{1}{2}d(A)$ generic points.

For example, $\mathrm{Gr}(E)=1$ for all $E\in\mathcal{E}$.

Now let us consider a general, not necessarily reduced class, A with $\mathrm{Gr}(A)\neq 0$. Then it is shown in McDuff [18, Proposition 3.1] that if we decompose A as

$$A = A' + \sum_{E\in\mathcal{E}(A)} |A\cdot E|\, E,\qquad \mathcal{E}(A)=\{E \mid E\cdot A<0\}, \tag{2.1.2}$$

then $E\cdot E'=0$ for $E, E'\in\mathcal{E}(A)$ and A' is reduced with $\omega(A')\geq 0$, $d(A')\geq d(A)\geq 0$ and $A'\cdot E=0$ for all $E\in\mathcal{E}(A)$. Further, for generic J the class A is represented by a main (possibly empty) embedded component $C^{A'}$ in class A', together with a finite number of disjoint curves C^E each with multiplicity $|A\cdot E|$ in the classes $E\in\mathcal{E}(A)$. This is proved by considering the structure of a J–holomorphic A–curve (where J is generic) through $\frac{1}{2}d'(A)$ points, where

$$d'(A) := c_1(A) + A^2 + \sum_{E\in\mathcal{E}} \big(|A\cdot E|^2 - |A\cdot E|\big). \tag{2.1.3}$$

It follows that $d'(A)=d(A')$. Moreover, Li and Liu show in [12] that the equivalence between Seiberg–Witten and Gromov invariants, previously established for reduced classes, extends to show that the class A has the same invariant as does its reduction A'. Thus we have the following fact:

Fact 2.1.4 Let A' be the reduction of A, and assume that A' is primitive if $g(A')=0$, $(A')^2=0$. Then $\mathrm{Gr}(A)=\mathrm{Gr}(A')$ counts the number of embedded A'–curves through $\frac{1}{2}d'(A)=\frac{1}{2}d(A')$ generic points. In particular, if $A'=0$ then $d'(A)=d(A')=0$.

We next discuss conditions that imply $\mathrm{Gr}(A)\neq 0$. The following is a sharper version of Li and Liu [13, Proposition 4.3]. (Their result applies to more general manifolds.)

Lemma 2.1.5 (i) *Let (M,ω) be $S^2\times S^2$ or a blowup of $\mathbb{CP}^2$. If $A\in H_2(M)$ satisfies $A^2\geq 0$, $\omega(A)>0$, and $d(A)\geq 0$, then* $\mathrm{Gr}(A)\neq 0$.

(ii) *Let M be the k–point blowup of a ruled surface with base of genus $g(M) \ge 1$. Then a sufficient (but not necessary) condition for* $\mathrm{Gr}(A)$ *to be nonzero is that $A \in \mathcal{P}^+$ and $d(A) > g(M) + \frac{1}{4}k$.*

(iii) *Let M be as in* (ii) *and $A \in H_2(M)$ be in the image of the Hurewicz map $\pi_2(M) \to H_2(M)$. Then $d(A) \ge 0$ implies that* $\mathrm{Gr}(A) \ne 0$.

Proof We prove (i). Since this can be proved by direct calculation when $M = S^2 \times S^2$, we suppose that (M, ω) is obtained from the standard $\mathbb{CP}^2$ by blowing up $N \ge 0$ points. Let $E_i \in H_2(M), i = 1, \dots, N$, be the classes of the corresponding exceptional divisors. Then the anticanonical class $K = -c_1(M)$ is standard, namely $K = -3L + \sum_{i=1}^N E_i$, where $L = [\mathbb{CP}^1]$. (As usual we identify $H_2(M;\mathbb{Z})$ with $H^2(M;\mathbb{Z})$ via Poincaré duality.) Because $d(A) \ge 0$, it follows from the wall crossing formula in [8] that exactly one of $\mathrm{Gr}(A), \mathrm{Gr}(K - A)$ is nonzero. Since $A^2 \ge 0$ and $\omega(A) > 0$, the Poincaré dual of A lies in $\overline{\mathcal{P}^+}$. Hence Fact 2.1.1 implies that $\omega'(A) > 0$ for all forms ω' obtained from ω by deformation. On the other hand if $\omega'(E_i)$ is sufficiently small for all i, $\omega'(K - A) < 0$. Therefore $K - A$ has no J–holomorphic representative for ω'–tame J, so that $\mathrm{Gr}(K - A)$ must be zero. Hence $\mathrm{Gr}(A) \ne 0$. This proves (i).

To prove (ii), we use [13, Lemma 3.4] which states that $\mathrm{Gr}(A) \ne 0$ if $d(A) \ge 0$ and $2A - K \in \mathcal{P}^+$. Therefore (ii) will hold provided that $2A - K \in \mathcal{P}^+$. Suppose that $M = (S^2 \times \Sigma_g) \# k\overline{\mathbb{CP}}^2$ is the k–fold blow up of the trivial bundle, where the exceptional divisors are $E_1, \dots, E_k$. Then

$$K = -2[\Sigma_g] + 2(g-1)[S^2] + \sum_{i=1}^{k} E_i$$

so that $K^2 = -8(g-1) - k \le 0$. Hence the nontrivial ruled surface over Σ_g also has $K^2 = -4(g-1)$, since its one point blowup is the same as the one point blowup of the trivial bundle and blowing up reduces K^2 by 1. Then

$$(2A - K)^2 = 4A^2 - 4A \cdot K + K^2 = 4d(A) - 4(g-1) - k > 0,$$

by our assumption. Therefore either $2A - K \in \mathcal{P}^+$ or $-(2A - K) \in \mathcal{P}^+$. But the displayed inequality also shows that $A \cdot K \le A^2 + \frac{1}{4}K^2 \le A^2$ so that $A \cdot (2A - K) = 2A^2 - A \cdot K > 0$. Hence, because $A \in \mathcal{P}^+$, Fact 2.1.1 implies that $2A - K \in \mathcal{P}^+$ as required.

To prove (iii), let us first consider the case when (M, ω) is minimal. Then $A = k[S^2]$, where $[S^2]$ is the class of the fiber. Further $k \ge 1$ since $d(A) = c_1(A) \ge 0$. Hence $\mathrm{Gr}(A) := \mathrm{Gr}(M, A) \ne 0$ by direct calculation. Note that this class takes values in $\mathbb{Z} \equiv \Lambda^0 H^1(M;\mathbb{Z})$. We can now use the blow-down formula of [13, Lemma 2.8]. This says that if (X, τ) is obtained from (X', τ') by blowing down the single exceptional class E, and if $\mathrm{Gr}(X, B)$ takes values in $\mathbb{Z} \equiv \Lambda^0 H^1(X; \mathbb{Z})$, then for all $\ell \geq 0$

$$d(B - \ell E) \geq 0 \Longrightarrow \mathrm{Gr}(X', B - \ell E) = \mathrm{Gr}(X, B) \in \mathbb{Z}.$$

Note also that $d(B - \ell E) = (B - \ell E)^2 + c_1(B) + \ell = d(B) - \ell(\ell - 1) \leq d(B)$. Therefore if we start with a class in the k–fold blow up with $d(A) \geq 0$, as we blow it down the degree $d(A)$ increases and we end up with a class $k[S^2], k > 0$, in the underlying minimal ruled surface. Hence $\mathrm{Gr}(A) \neq 0$. □

Corollary 2.1.6 *If M satisfies any of the hypotheses in Lemma 2.1.5 and $A \in \mathcal{P}^+$, then there is an integer q_0 such that $\mathrm{Gr}(qA) \neq 0$ for all $q \geq q_0$.*

Proof Since $(qA)^2 > 0$ grows quadratically with q while $c_1(qA)$ grows linearly, the sequence $d(qA)$ is eventually increasing with limit infinity. The result then follows from Lemma 2.1.5. □

The following recognition principle will be useful. It is taken from [24, Corollary 1.5], but here we explain some extra details in the proof.

Lemma 2.1.7 (i) *Suppose that (M^4, ω) admits a symplectically embedded submanifold Z with $c_1(Z) > 0$ that is not an exceptional sphere. Then (M, ω) is the blow up of a rational or ruled manifold.*

(ii) *The same conclusion holds if there is a J–holomorphic curve $u : (\Sigma, j) \to (M, J)$ in a class B with $c_1(B) > 0$, where $B \neq kE$ for some $E \in \mathcal{E}, k \geq 1$.*

Proof Since $0 < c_1(Z) = 2 - 2g + Z^2$, where g is the genus of the submanifold Z, we must have $Z^2 \geq 0$, since otherwise $Z^2 = -1$ and $g = 0$ so that Z is an exceptional sphere. But when $Z^2 \geq 0$ we can use the method of symplectic inflation from [9; 19] to deform ω to a symplectic form in the class $[\omega_\kappa] := [\omega] + \kappa\, \mathrm{PD}(Z)$ for any $\kappa \geq 0$. Therefore if K is Poincaré dual to $-c_1(M)$, then $K \cdot Z < 0$ so that for large κ we have $\omega_\kappa(K) < 0$. But by Taubes' structure theorems in [28], this is impossible when $b_2^+ > 1$. Thus $b_2^+ = 1$. The rest of the proof of (i) now follows the arguments given in [24]. The crucial ingredient is Liu's result that a minimal manifold with $K^2 < 0$ is ruled.

This proves (i). To prove (ii), note first that by replacing u by its underlying simple curve we may assume that the map u is somewhere injective. Since this replaces the class B by $B' := \frac{1}{k} B$ for some $k > 1$, we still have $c_1(B') > 0, B' \notin \mathcal{E}$. Then perturb the image of u as in Proposition 3.1.3 below until it is symplectically embedded, and apply (i). □

We also recall from [28] that for general 4–dimensional symplectic manifolds, the classes with nonvanishing Gromov invariant are rigid:

Fact 2.1.8 If $b_2^+ > 1$ and $\mathrm{Gr}(A) \neq 0$, then $d(A) = 0$.

Finally we remind the reader of the standard theory of J–holomorphic curves as developed in [25], for example. An almost complex structure J is said to be *regular* for a J–holomorphic map $u : (\Sigma, j) \to (M, J)$ if the linearized Cauchy–Riemann operator $D_{u,J}$ is surjective. We will say that J is *semiregular* for u if $\dim \mathrm{Coker}\, D_{u,J} \le 1$. Here (Σ_g, j_Σ) is a smooth connected Riemann surface, and when $g := \mathrm{genus}(\Sigma) > 0$ we allow the complex structure j_Σ on Σ to vary, so that the tangent space $T_{j_\Sigma}\mathcal{T}$ at j_Σ to Teichmüller space $\mathcal{T}$ is part of the domain of $D_{u,J}$; see [18; 29]. Therefore, if u is a somewhere injective curve in class B the (adjusted) Fredholm index of the problem in dimension $2n = 4$ is

$$\mathrm{ind}(D_{u,J}) = 2n(1-g) + 6(g-1) + 2c_1(B) = 2(g + c_1(B) - 1). \tag{2.1.4}$$

This is the virtual dimension of the quotient space of J–holomorphic maps modulo the action of the reparametrization group, where we adjust by quotienting out by the reparametrization group (for genus $g_\Sigma = 0, 1$) and adding in the $(6g-6)$–dimensional tangent space to Teichmüller space when $g_\Sigma > 1$. Thus, if J is regular, the space $\mathcal{M}_{g,k}(M, B, J)$ of J–holomorphic maps $u : (\Sigma_g, j) \to (M, J)$ with k marked points modulo reparametrizations and with j varying in Teichmüller space is a manifold of dimension $\mathrm{ind}(D_{u,J}) + 2k$. Hence the evaluation map

$$\mathcal{M}_{g,k}(M, B, J) \to M^k \tag{2.1.5}$$

can be locally surjective only if $\mathrm{ind}(D_{u,J}) + 2k \ge 4k$, ie $\frac{1}{2}(\mathrm{ind}(D_{u,J})) \ge k$.

Now recall that the adjunction inequality (2.1.1) states that the genus $g(u)$ of the (connected) domain of any J–holomorphic curve in class B satisfies $g(u) \le g(B)$, where the algebraic genus $g(B) = 1 + \frac{1}{2}(B^2 - c_1(B))$ is the genus of an embedded representative of B. Therefore, (2.1.4) gives

$$\mathrm{ind}\, D_{u,J} = 2(c_1(B) + g(u) - 1) = c_1(B) + B^2 + 2(g(u) - g(B)).$$

In other words,

$$\mathrm{ind}\, D_{u,J} = d(B) + 2(g(u) - g(B)) \le d(B). \tag{2.1.6}$$

Remark 2.1.9 (i) The above inequality (2.1.6) implies that when J is regular for all B–curves the evaluation map $\mathcal{M}_{g,k}(M, B, J) \to M^k$ can be surjective only if $\frac{1}{2}d(B) \ge k$. Informally, we may say that a connected B–curve can go through at most

$\frac{1}{2}d(B)$ generic points of M. Note that nodal *regular* curves do worse. If Σ^B is a J–holomorphic nodal curve in class B with components in classes B_j, $j = 1, \dots, n$ then positivity of intersections implies that $B_i \cdot B_j \ge 0$ for all $i \neq j$ so that $\sum d(B_j) \le d(B)$, with strict inequality if any $B_i \cdot B_j > 0$. Hence if all components of Σ^B are regular and some $B_i \cdot B_j > 0$ (which always happens when Σ^B arises as a Gromov limit of connected curves), then such a nodal curve goes through at most $\frac{1}{2}\sum_j d(B_j) < \frac{1}{2}d(B)$ points. However, if some of the components of Σ^B are not regular (eg they lie in the singular set $\mathcal{S}$, or they are multiply covered exceptional spheres), their Taubes index might be negative, so others may have larger index, and could go through more points. The arguments that follow show how to deal with this problem in certain special cases.

(ii) If $A^2 < 0$, the condition $d(A) \ge 0$, combined with the formula

$$d(A) = 2(A^2 - g(A) + 1),$$

shows that $A^2 = -1$, $g(A) = 0$. Hence $g(u) = g(A) = 0$, so u is an embedded exceptional sphere.

2.2 The case $J \in \mathcal{J}(\mathcal{S})$

We now suppose that J belongs to the set $\mathcal{J}(\mathcal{S})$ of Definition 1.2.1, where this is given the direct limit topology. When we consider J–holomorphic representatives for a reduced class A for such J, the situation is rather different from before since the curves in $\mathcal{S}$ are not regular. Thus A could decompose as $A = \sum_i \ell_i S_i + A'$, where $\ell_i \ge 0$, and we need to consider generic representations of the class A'. But A' need not be reduced, and hence could have a disconnected representative as above with some multiply covered exceptional spheres. We will consider two subsets of $\mathcal{J}(\mathcal{S})$, first a set (defined carefully below) of regular J, that we call $\mathcal{J}_{reg}(\mathcal{S})$, and secondly a larger path connected set $\mathcal{J}_{semi}(\mathcal{S})$ whose elements retain some of the good properties of regular J. Specially important will be certain special paths in $\mathcal{J}_{semi}$ called regular homotopies.

Definition 2.2.1 If $\overline{\mathcal{N}}$ is a closed fibered neighborhood of $\mathcal{S}$, then $\mathcal{J}_{reg}(\mathcal{S}, \overline{\mathcal{N}}, \omega, \kappa)$, the space of *regular* $(\mathcal{S}, \overline{\mathcal{N}})$*–adapted* J, is the set of almost complex structures $J \in \mathcal{J}(\mathcal{S})$ satisfying the following conditions:

(i) J is $\mathcal{S}$–adapted on $\overline{\mathcal{N}}$.

(ii) J is regular for all somewhere injective elements $u : (\Sigma, j_\Sigma) \to (M, J)$ in class B with $\omega(B) \le \kappa$ and $\operatorname{im} u \cap (M \setminus \overline{\mathcal{N}}) \neq \varnothing$.

The space $\mathcal{J}_{semi}(\mathcal{S},\overline{\mathcal{N}},\omega,\kappa)$ of *semiregular $\mathcal{S}$–adapted J* consists of all $J \in \mathcal{J}(\mathcal{S})$ that are semiregular for all maps u satisfying the above conditions. We then define

$$\mathcal{J}_{reg}(\mathcal{S},\omega,\kappa) := \bigcup_{\overline{\mathcal{N}}} \mathcal{J}_{reg}(\mathcal{S},\overline{\mathcal{N}},\omega,\kappa),$$
$$\mathcal{J}_{semi}(\mathcal{S},\omega,\kappa) := \bigcup_{\overline{\mathcal{N}}} \mathcal{J}_{semi}(\mathcal{S},\overline{\mathcal{N}},\omega,\kappa),$$

and give these spaces the direct limit topology.

Remark 2.2.2 (i) In the case of spheres there is a close connection between the value of the Chern class $c_1(B)$ and the (semi-) regularity of a somewhere injective J–holomorphic sphere $u:(S^2,j)\to(M^4,J)$ in class B. Indeed, if $J\in\mathcal{J}_{semi}(\mathcal{S},\omega,\kappa)$ for some $\kappa\geq\omega(B)$ and B is represented by a somewhere injective curve that meets $M\setminus\overline{\mathcal{N}}$, then $c_1(B)>0$ because $\operatorname{ind} D_{u,J}=2c_1(B)-2$. Conversely, if u is immersed, then the condition $c_1(B)>0$ implies the surjectivity of $D_{u,J}$ by automatic regularity [7].

(ii) If $\mathcal{A}=\{A_1,\dots,A_k\}$ is a finite set of reduced classes A_j, we define the space $\mathcal{J}_{reg/semi}(\mathcal{S},\omega,\mathcal{A}):=\mathcal{J}_{reg/semi}(\mathcal{S},\omega,\kappa(\mathcal{A}))$ of almost complex structures, where $\kappa(\mathcal{A})=\max_j\omega(A_j)$. In practice, these complex structures are (semi-)regular at each component not in $\mathcal{S}$ of the stable maps that represent the A_j.

Lemma 2.2.3 *The subset $\mathcal{J}_{reg}(\mathcal{S},\kappa)$ of $\mathcal{J}(\mathcal{S})$ is residual in the sense of Baire. Further, $\mathcal{J}_{reg}(\mathcal{S},\kappa)\subset\mathcal{J}_{semi}(\mathcal{S},\kappa)$.*

Proof Let $\mathcal{J}(\mathcal{S},\overline{\mathcal{N}},\kappa)$ denote the subset of $\mathcal{S}$–adapted J satisfying Definition 2.2.1(i) for the given $\overline{\mathcal{N}}$. Because $\mathcal{J}_{reg}(\mathcal{S},\kappa)$ is a (countable) direct limit, it suffices to check that $\mathcal{J}_{reg}(\mathcal{S},\overline{\mathcal{N}},\kappa)$ is residual in $\mathcal{J}(\mathcal{S},\overline{\mathcal{N}},\kappa)$ for each $\overline{\mathcal{N}}$. When the domain Σ of u has genus zero this follows immediately from standard theory as developed in [25, Chapter 3.2], since we can vary J freely somewhere on $\operatorname{im} u$. The argument applies equally in the higher-genus case. One main technical ingredient is the version of the Riemann–Roch theorem in [25, Theorem C.1.10]. Since this theorem is stated for arbitrary genus, one can easily adapt the above proof to higher-genus curves as in [29; 18]. This proves the first statement. The rest of (i) is then immediate since the elements in $\mathcal{J}_{semi}(\mathcal{S},\kappa)$ satisfy fewer conditions than those in $\mathcal{J}_{reg}(\mathcal{S},\kappa)$. □

Lemma 2.2.4 *Let $J\in\mathcal{J}_{semi}(\mathcal{S},\overline{\mathcal{N}},\mathcal{A})$. The following statements hold for somewhere injective J–holomorphic curves u in a class B with $\omega(B)\leq\kappa(\mathcal{A})$:*

(i) *If $B\neq\sum\ell_iS_i$ with $\ell_i\geq0$, then $\operatorname{im} u\cap(M\setminus\overline{\mathcal{N}})\neq\varnothing$.*

(ii) *If $\operatorname{im} u\cap(M\setminus\overline{\mathcal{N}})\neq\varnothing$ then $d(B)\geq0$. Moreover, $B^2\geq0$ unless $B\in\mathcal{E}$, and if $B^2=0$ then B is represented by an embedded J–holomorphic sphere or torus.*

Proof Let $J \in \mathcal{J}(\mathcal{S})$ be $\mathcal{S}$–adapted on some fibered neighborhood $\mathcal{N}(\mathcal{S})$. If $u : (\Sigma, j) \to (\overline{\mathcal{N}}, J)$ is J–holomorphic, then $B = \sum \ell_i S_i$ for some ℓ_i, because there is a projection $\overline{\mathcal{N}} \to \mathcal{S}$. Moreover $\ell_i \geq 0$ because we can choose this projection to be J–holomorphic over some nonempty open subset of each curve C^{S_i} in $\mathcal{S}$. This proves (i).

To prove (ii), notice that since the index of a somewhere injective J–holomorphic curve with domain of genus g is even and $\dim \mathrm{Coker}(D_{u,J}) \leq 1$ when $J \in \mathcal{J}_{semi}(\mathcal{S}, \mathcal{A})$, we must have $\mathrm{ind}(D_{u,J}) \geq 0$. Hence $d(B) \geq \mathrm{ind}(D_{u,J}) \geq 0$ by Equation (2.1.6). Further, the only simple curves in a class B with $B^2 < 0$ and $d(B) \geq 0$ are embedded exceptional spheres (Remark 2.1.9 (ii)). Similarly, if $B^2 = 0$ we again have equality in the adjunction formula, so that the curve is embedded and $g(B) = 0$ or 1, as claimed. □

Remark 2.2.5 A path $J_t \in \mathcal{J}(\mathcal{S}), t \in [0, 1]$, is called an $(\mathcal{S}, \overline{\mathcal{N}})$*–regular homotopy* if the derivative $\partial_t J_t$ covers the cokernel of D_{u,J_t} for every map $u : (\Sigma, j_\Sigma) \to (M, J)$ that satisfies condition (ii) in Definition 2.2.1. Thus (J_t) is a path in $\mathcal{J}_{semi}(\mathcal{S}, \overline{\mathcal{N}}, \omega, \kappa)$ with the special property that for each t all the relevant cokernels are covered by the (restriction of the) single element $\partial_t J_t$. The proof of [25, Theorem 3.1.7] shows that any two elements $J_0, J_1 \in \mathcal{J}_{reg}(\mathcal{S}, \overline{\mathcal{N}}, \omega, \kappa)$ may be joined by a regular homotopy of this kind.

Let us denote by $\mathcal{M}_{g,k}(M \setminus \overline{\mathcal{N}}; B, J_t)$ the moduli space of all k–pointed curves as in (2.1.5) whose image meets $M \setminus \overline{\mathcal{N}}$. Then [25, Theorem 3.1.7] also shows that, for each B with $\omega(B) \leq \kappa$, the moduli space $\bigcup_{t \in [0,1]} \mathcal{M}_{g,k}(M \setminus \overline{\mathcal{N}}; B, J_t)$ is a smooth manifold of the "correct" dimension $\mathrm{ind}\, D_{u,J} + 2k + 1$ with boundary at $t = 0, 1$. Hence the corresponding evaluation map goes through at most $\frac{1}{2} d(B)$ generic points in $M \setminus \overline{\mathcal{N}}$; see Remark 2.1.9. Note also that if $B \neq \sum_i m_i S_i, m_i \geq 0$, then every B–curve meets $M \setminus \overline{\mathcal{N}}$ by Lemma 2.2.4(ii). Therefore, in this case

$$\mathcal{M}_{g,k}(M \setminus \overline{\mathcal{N}}; B, J_t) = \mathcal{M}_{g,k}(M; B, J_t).$$

3 The proof of Theorem 1.2.7

We first explain the structure of nodal representatives of A, and then in Proposition 3.1.3 show how to build embedded curves from components in classes B with $B^2 \geq 0$. As we see in Corollary 3.1.4 and Proposition 3.1.6, these arguments suffice to prove Theorem 1.2.7 in cases (i) and (iii). Section 3.2 explains how to construct 1–parameter families of embedded curves, while Section 3.3 proves Theorem 1.2.7 in cases (iv) and (v).

3.1 The structure of nodal curves

Throughout this section we assume that the class A is $\mathcal{S}$–good in the sense of Definition 1.2.4. For such A, as explained in Section 2.1 there is for each generic ω–tame J and each sufficiently generic set of $\frac{1}{2}d(A)$ points in M an embedded J–holomorphic curve $u:(\Sigma,j)\to(M,J)$ of genus $g(A):=1+\frac{1}{2}(A^2-c_1(A))$ through these points. Hence by Gromov compactness, for every ω–tame J and every set of $\frac{1}{2}d(A)$ points, there is a connected but possibly nodal representative of the class A through these points that is the limit of these embedded curves. We denote such nodal curves as Σ^A, reserving the notation C^A for a (smooth, often immersed) curve. This section explains the structure of these nodal curves. Recall from Definition 1.2.1 that $\mathcal{J}(\mathcal{S},\overline{\mathcal{N}})$ consists of ω–tame J that are fibered on the neighborhood $\overline{\mathcal{N}}$ of $\mathcal{S}$.

Lemma 3.1.1 *For each $J\in\mathcal{J}(\mathcal{S},\overline{\mathcal{N}})$ and $\mathcal{S}$–good class A, there is a connected J–holomorphic nodal curve Σ^A in class A whose components are either multiple covers of the components of $\mathcal{S}$ or lie in classes $B_j\neq\sum_i m_iS_i, m_i\ge 0$. The homology classes of these components provide a decomposition*

$$A=\sum_{i=1}^{s}\ell_iS_i+\sum_{j=1}^{k}n_jB_j \tag{3.1.1}$$

satisfying:

(i) $\ell_i\ge 0$ *and* $n_j>0$ *for all* j.

(ii) $B_j\cdot S_i\ge 0$ *for all* i,j.

(iii) *Each class* B_j *may be represented by a connected simple* J*–holomorphic curve* C^{B_j} *that intersects* $M\smallsetminus\overline{\mathcal{N}}$.

Further every J–holomorphic nodal curve Σ^A that is the Gromov limit of embedded J_n–holomorphic A–curves for some convergent sequence J_n has this structure.

Proof Let Σ^A be any J–holomorphic nodal curve. As explained above, these exist because $\mathrm{Gr}(A)\neq 0$. Then since we may replace every component in some class $\sum_j m_iS_i, m_i\ge 0$, by a union of copies of the C^{S_i}, we can suppose that no B_j has this form. Therefore A does decompose as in (3.1.1), and (i) and (ii) hold by positivity of intersections. To prove (iii), note first that we may take B_j to be the class of a simple curve underlying a possibly multiply covered component of Σ. The curve C^{B_j} must intersect $M\smallsetminus\overline{\mathcal{N}}$ by Lemma 2.2.4(i). □

The following sharpening of this result is useful in proving Theorem 1.2.7. Order the classes B_j (assumed distinct) so that $B_j\in\mathcal{E}$ for $j\le p$ and $B_j\notin\mathcal{E}$ otherwise, and write $E_j:=B_j$ for $1\le j\le p$, and $B:=\sum_{j>p}n_jB_j$. We then have

$$A = \sum_i \ell_i S_i + \sum_{j=1}^{p} m_j E_j + \sum_{j>p} n_j B_j = \sum_i \ell_i S_i + \sum_{j=1}^{p} m_j E_j + B, \tag{3.1.2}$$

where $B \cdot (A - B) > 0$ if $B \neq 0$ because Σ^A is connected.

Lemma 3.1.2 *Suppose that $J \in \mathcal{J}_{semi}(\mathcal{S}, A)$ and that the J–holomorphic nodal curve Σ^A is the Gromov limit of embedded curves. Then the components B_j, $j > p$, in its decomposition* (3.1.2) *also satisfy:*

- $d(B) \geq \sum_{j>p} d(B_j) \geq 0$.
- $B_j^2 \geq 0$ *for all* $j > p$.

Moreover, if B_j is represented by a J–sphere, we have $\mathrm{Gr}(B_j) \neq 0$. *(This case occurs only if M is the blow up of a rational or ruled manifold.)*

Proof Apply Lemma 3.1.1 to Σ^A. By Lemma 3.1.1(iii), we can apply Lemma 2.2.4(ii) to curves in class B_j to find $d(B_j) \geq 0$ for all j. Since $B_j \cdot B_k$ for all $j \neq k$,

$$d(B) = \Big(\sum_{j>p} B_j\Big)^2 + c_1\Big(\sum_{j>p} B_j\Big) \geq \sum_{j>p} B_j^2 + c_1(B_j) = \sum_{j>p} d(B_j) \geq 0,$$

which proves the first claim. Since $d(B_j) \geq 0$, Remark 2.1.9(ii) shows that either $B_j^2 \geq 0$ or B_j is represented by a J–holomorphic (-1)–sphere. The latter is ruled out by definition, so B_j^2 is indeed nonnegative for all $j > p$. When B_j is represented by a J–sphere

$$u : (S^2, j) \longrightarrow (M, J),$$

then we saw in Remark 2.2.2(i) that $c_1(B_j) > 0$. Therefore, because $B_j^2 \geq 0$ we also have $d(B_j) > 0$. Therefore Lemma 2.1.7(ii) implies that M is the blow up of a rational or ruled manifold. Finally because the class B_j is the J–holomorphic image of a sphere, we conclude from Lemma 2.1.5(i) and (iii) that $\mathrm{Gr}(B_j) \neq 0$. □

These lemmas give enough preparation for the proof of part (iii) of Theorem 1.2.7 (the case $A \in \mathcal{E}$). We next prove a general position result that allows us to "clean up" a nodal representation of the class A. The result when $\mathcal{S} = \varnothing$ is well known. Besides being the key to the handling of the components in $\mathcal{S} \smallsetminus \mathcal{S}_{sing}$, this lemma will be very useful when discussing inflation in Section 5. Note that in distinction to the decomposition $B = \sum n_j B_j$ considered above where by definition $B_j \neq S_i$ for any i, j, we now allow $T_j = S_i$ in some cases.

Proposition 3.1.3 *Let $T = \sum_{j=1}^{N} n_j T_j \in H_2(M)$ be such that the following hold:*

(i) *$T_j \neq T_k$ for each $j \neq k$, and $n_j \geq 1$.*

(ii) *For some $J_0 \in \mathcal{J}(\mathcal{S})$, each T_j can be represented by a simple connected J_0–holomorphic curve C^{T_j}.*

(iii) *$T_j^2 \geq 0$ unless C^{T_j} is an exceptional sphere.*

(iv) *$T \cdot S_i \geq 0$ for all i and $T \cdot T_j \geq 0$ for all j. Further, $T_j \cdot S_i \geq 0$ for all i, j unless $T_j = S_i$, where C^{S_i} is an exceptional sphere.*

Then T can also be represented by a (possibly disconnected) embedded curve that is orthogonal to $\mathcal{S}$ and J–holomorphic for some $J \in \mathcal{J}(\mathcal{S})$.

Proof **Case 1** We assume $N = n_1 = 1$. If $T = S_i$ for some i, then there is the required embedded representative, namely C^{S_i}. Therefore, assume $T \neq S_i$ for any i. By hypothesis there is a connected simple J_0–holomorphic curve C^T, and our task is to resolve its singularities to make it embedded. By general theory (see for example [25, Appendix E]), C^T has at most a finite number of singular points $q_i = u(z_i)$. Suppose first that none lie on $\mathcal{S}$. At each of these it is possible to perturb C^T locally to an immersed J_0–holomorphic curve by [17, Theorem 4.1.1], and then patch this new piece of curve to the rest of C^T by the technique of [16, Lemma 4.3] to obtain a positively immersed symplectic curve C'. The curve C' is J_0–holomorphic except close to $C^T \cap \mathit{Shell}$, where Shell is the union of spherical shells $\mathit{Shell}(q) := T_{r_1}(q) \smallsetminus T_{r_2}(q)$ centered at the finite number of singular points q. Thus we can make it J–holomorphic for some J near J_0 that equals J_0 away from $C' \cap \mathit{Shell}$. Hence even if some singular point q is in some C^{S_i} we can assume $J \in \mathcal{J}(\mathcal{S})$.

Then C' is immersed, and can be homotoped (keeping it symplectic) so that it has at most transverse double points that are disjoint from its intersections with the curves C^{S_i} in $\mathcal{S}$. Then we deform C' so that it is vertical near its intersections p with each C^{S_i}, in the sense that it coincides with the fiber of the normal bundle to $\mathcal{S}$ at p. (A parametric version of this maneuver is carried out in more detail in Lemma 3.2.1 below.) Then C' meets each component C^{S_i} of $\mathcal{S}$ orthogonally in distinct points. Moreover, by resolving all its double points (which lie away from C^{S_i}), we can assume that C' is embedded and still J–holomorphic for some $J \in \mathcal{J}(\mathcal{S})$. This completes the proof when $N = n_1 = 1$. Notice also that C' is connected since we assumed that the initial curve C^T is connected.

Case 2 We assume $T = nT_0$, where $n > 1$ and $T_0 \neq S_i$ for any i. By the above we can suppose that C^{T_0} is embedded, orthogonal to $\mathcal{S}$ and J–holomorphic for some $J \in \mathcal{J}(\mathcal{S})$. Then, for suitable $J \in \mathcal{J}(\mathcal{S})$, a neighborhood $\mathcal{N}(C^{T_0}, J)$ of C^{T_0} can be identified with a neighborhood of the zero section in a holomorphic line bundle over C^{T_0} with nonnegative Chern class. (Since $n > 0$, condition (iv) implies that $(T_0)^2 \geq 0$.) Moreover, since the condition $J \in \mathcal{J}(\mathcal{S})$ only affects the complex structure on $\mathcal{N}(C^{T_0})$ near a finite set of points, we may choose J so that this bundle has nonzero holomorphic sections. Hence we may represent the class nT_0 by the union of n generic J–holomorphic sections of this bundle that intersect transversally. If $T_0^2 > 0$ each pair of these sections intersect, and by choosing generic sections we can assume that the intersection points do not lie on $\mathcal{S}$. Hence after resolving these intersections as before, we get an embedded (possibly disconnected) representative of nT_0 that we finally perturb to be orthogonal to $\mathcal{S}$.

Case 3 We assume $T = nT_0$, where $n > 1$ and $T_0 = S_i$ for some i. This is much as Case 2: we just need to pick $J \in \mathcal{J}(\mathcal{S})$ so that the normal bundle to $C^{T_0} = C^{S_i}$ has holomorphic sections that intersect the zero set transversally in a finite number of points. This is possible because by condition (iv) we have $T \cdot S_i = n(T_0)^2 \geq 0$.

Case 4 We assume $N > 1$ and $T_j^2 \geq 0$ for all j. We first resolve all singularities, so that each simple curve C^{T_j} is embedded and meets all the other curves C^{T_k} and C^{S_i} transversally and positively in double points. Because $T_j^2 \geq 0$, even if $T_j = S_i$ we may replace C^{T_j} by a suitable section of its normal bundle that is transverse to C^{S_i}. Next, we perturb all double points to be orthogonal. Since $(T_j)^2 \geq 0$ by assumption, we may represent every class $n_j T_j$ by embedded curves as in Cases 2 and 3 above. Finally, we patch all double points to get an embedded curve in class T.

Case 5 This is the general case. Because $T \cdot T_j \geq 0$ for all j, each exceptional class T_j must intersect some other component in T. If two different exceptional spheres C^{T_k}, C^{T_ℓ} intersect, then we may form a symplectically embedded curve C' that is transverse to $\mathcal{S}$ by patching together two meromorphic and nonvanishing sections of their normal bundles each with their pole at one of the intersection points with the other curve. (See Lemma 4.1.1 below for further discussion of patching meromorphic sections.) Then by perturbing C' further we can suppose that it is J–holomorphic for some $J \in \mathcal{J}(\mathcal{S})$. Therefore we can replace these two components T_k, T_ℓ of T with the single component $T' := T_k + T_\ell$. If $T' \neq T_{j_0}$ for any j_0, then the decomposition $T = T' + \sum n'_j T_j$, where $n'_j = n_j - 1$ for $j = k, \ell$ and $= n_j$ otherwise, satisfies all the conditions (i) through (iv). In particular, by construction $T_j \cdot T' = 0 = T_k \cdot T'$. Otherwise we can write $T = \sum n'_j T_j$ where $n'_j = n_j - 1$ for $j = k, \ell$, and $n'_{j_0} = n_{j_0} + 1$, which clearly also satisfies the required conditions. Because the meromorphic sections do not vanish, this procedure works equally well if one or both spheres C^{T_k}, C^{T_ℓ} are in $\mathcal{S}$. It also works if an exceptional sphere C^{T_k} intersects some nonnegative component of T. Therefore, after a finite number of steps of this kind, we arrive at a decomposition $T = \sum n'_j T'_j$ with no exceptional spheres, hence the conclusion follows by Case 4. □

Corollary 3.1.4 *Part (i) of Theorem 1.2.7 holds.*

Proof Suppose that $\mathcal{S} = \mathcal{S}_{reg} \cup \mathcal{S}_{nonneg}$, let $J \in \mathcal{J}_{semi}(\mathcal{S})$, and choose a J–holomorphic nodal representative Σ^A of A as in Lemma 3.1.2. Then write $A = \sum_{j=1}^N n_j T_j$, where T_j is one of the classes S_i, E_j, B_j occurring in (3.1.2). By assumption, the classes S_i either have $(S_i)^2 \geq 0$ or are represented by an embedded curve C^{S_i} that is Fredholm regular and hence must be an exceptional sphere by Remark 2.1.9(ii). If $A \in \mathcal{E}$ then we might be in the case $N = 1 = n_1$ with $A = E_1$, in which case A has the embedded representative C^{E_1}. Otherwise, because A is reduced we must have $A \cdot E_j \geq 0$ for all j, and $A \cdot S_i \geq 0$ for all i because A is $\mathcal{S}$–good. Therefore, in this case the result follows from Proposition 3.1.3. □

For the next result, denote by $\mathcal{I}_{neg}$, respectively $\mathcal{I}_{nonneg}$, the classes with $(S_i)^2 < 0$, respectively $(S_i)^2 \geq 0$. Recall from Remark 1.2.2 that the elements in $\mathcal{I}_{neg}$ are either represented by exceptional spheres or are in $\mathcal{I}_{sing}$.

Lemma 3.1.5 *Suppose that A is $\mathcal{S}$–good. Then we may write*

$$A = \sum_{i \in \mathcal{I}_{neg}} \ell_i S_i + \sum_{k=1}^{q} m_k E_k + B, \quad \ell_i \geq 0, \ m_k > 0, \tag{3.1.3}$$

where:

- (i) *If $\ell_i > 0$ and C^{S_i} is an exceptional sphere, then $S_i \cdot B = 0$.*
- (ii) *Each E_k for $k \leq q$ satisfies $E_k \cdot E_j = 0, j \neq k$, $E_k \cdot S_j \geq 0$ for $1 \leq j \leq s$ and $E_k \cdot B = 0$.*
- (iii) *B has an embedded representative C^B that intersects $M \setminus \mathcal{S}$ and is J–holomorphic for some $J \in \mathcal{J}(\mathcal{S})$.*
- (iv) *If all S_i with $S_i^2 \geq 0$ are regular, then $d(B) \geq 0$.*

Proof Consider a decomposition of $A = \sum_i \ell_i S_i + \sum_{j=1}^p m_j E_j + B$ as in (3.1.2) given by a J–holomorphic nodal curve, where $J \in \mathcal{J}_{semi}(\mathcal{S})$. As in the proof of Proposition 3.1.3 we may incorporate all nonnegative components $\ell_i S_i$ into B.[5] If $E_j \cdot E_k > 0$, then, as in the proof of Case 5 of Proposition 3.1.3, we may reduce each of m_j, m_k by 1 and add a component in class $E_j + E_k$ to B. Similarly, if $E_j \cdot B_k > 0$, or if $E_j \cdot S_i > 0$ or $B_j \cdot S_i > 0$ for some i for which C^{S_i} is an exceptional sphere, we may incorporate one copy of the exceptional class S_i or E_j into the B_j. Repeating

[5] Since these components need not be Fredholm regular and could have $d(S_i) < 0$, we may lose control of $d(B)$ at this step.

this process, we arrive at a situation in which (i) and (ii) hold, and B (if nonzero) has an embedded representative C^B that intersects $M \setminus \mathcal{S}$ and is J–holomorphic for suitable $J \in \mathcal{J}(\mathcal{S})$.

To prove (iv), notice that if there are no irregular nonnegative components, $d(B)$ cannot decrease as we incorporate the various components C^{S_i} and C^{E_j} into the B–curve. Because we begin with $d(B) \geq 0$ by Lemma 3.1.2, this proves (iv). □

We end this section by proving case (iii) of Theorem 1.2.7.

Proposition 3.1.6 *Theorem 1.2.7 holds when $A \in \mathcal{E}$. Moreover, we may choose $\mathcal{J}_{emb}(\mathcal{S}, A) \supset \mathcal{J}_{semi}(\mathcal{S}, A)$.*

Proof We first show that when $J \in \mathcal{J}_{semi}(\mathcal{S}, A)$ each $A \in \mathcal{E}$ has an embedded J–holomorphic representative. Suppose, to the contrary, that this does not hold for some $\mathcal{S}$–good $A \in \mathcal{E}$ and some $J \in \mathcal{J}_{semi}(\mathcal{S}, A)$. Consider the J–holomorphic nodal representative Σ^A with decomposition

$$A = \sum_i \ell_i S_i + \sum_{j=1}^{p} m_j E_j + \sum_{j>p} n_j B_j$$

as in (3.1.2). Since Σ^A is the Gromov limit of spheres, each component of Σ^A is represented by a sphere. If there is just one component, this must be somewhere injective since the class A is primitive, and hence by the adjunction formula (2.1.1) must be embedded. Therefore, we can assume that Σ^A has several components. Because $A \cdot A = -1$ and $A \cdot S_i \geq 0$ for all i, the class A must have negative intersection with one of the E_j or B_j. But because this decomposition is nontrivial, $\omega(E_j) < \omega(A)$ for each j, so that $A \neq E_j$. Hence, because $A, E_j \in \mathcal{E}$, we must have $A \cdot E_j \geq 0$ for all j. Therefore, there is $j > p$ such that $A \cdot B_j < 0$. Next, notice that $\mathrm{Gr}(B_j) \neq 0$ by Lemma 3.1.2. Therefore, by Fact 2.1.3, for generic $J' \in \mathcal{J}(M)$ the class A has an embedded J'–holomorphic representative while, by the discussion after (2.1.2), B_j (which need not be reduced) can be represented by an embedded J'–holomorphic curve in some reduced class B_j' together with possibly multiply covered exceptional spheres in classes E_α'. But $E_\alpha' \neq A$ since $\omega(E_\alpha') \leq \omega(B_j) < \omega(A)$. Hence $A \cdot E_\alpha' \geq 0$, and also $A \cdot B_j' \geq 0$. Therefore $A \cdot B_j \geq 0$, which contradicts the choice of B_j. We conclude that the class A must have an embedded J–holomorphic representative for each $J \in \mathcal{J}_{semi}(\mathcal{S}, A)$. Further, A can have no other nodal J–holomorphic representative, since if it did A would have nonnegative intersection with each of its components, and hence with A itself, which is impossible because $A \in \mathcal{E}$.

Next define $\mathcal{J}_{emb}(\mathcal{S}, A)$ to be the set of $J \in \mathcal{J}(\mathcal{S})$ for which A has an embedded representative. This set is residual in $\mathcal{J}(\mathcal{S})$, because it contains $\mathcal{J}_{semi}(\mathcal{S}, A)$, which is residual by Lemma 2.2.3. Further, it is open since embedded curves in class A are regular by automatic regularity (see Remark 2.2.2) and hence deform to nearby embedded curves when J deforms. It remains to check that $\mathcal{J}_{emb}(\mathcal{S}, A)$ is path connected. But this holds because any two elements $J_0, J_1 \in \mathcal{J}_{emb}(\mathcal{S}, A)$ can be slightly perturbed to $J_0', J_1' \in \mathcal{J}_{emb}(\mathcal{S}, A) \cap \mathcal{J}_{reg}(\mathcal{S}, A)$, and then by Remark 2.2.5, joined by a regular homotopy in $\mathcal{J}_{semi}(\mathcal{S}, A) \subset \mathcal{J}_{emb}(\mathcal{S}, A)$. □

Remark 3.1.7 Of course, classes $E \in \mathcal{E}$ do degenerate, for example as $(E - E') + E'$, where $E' \in \mathcal{E}$. But such degenerations (a) happen for J in a set of codimension at least 2, and (b) have the property that the intersection of E with the class of the nonregular component(s) (in this case $E - E'$) is *negative*. The argument above shows the presence of nonregular components in classes S_i with $E \cdot S_i \geq 0$ does not affect the situation.

3.2 One-parameter families

We begin with a useful geometric result.

Lemma 3.2.1 *Let $J_t, t \in [0, 1]$, be a path in $\mathcal{J}(\mathcal{S})$ and suppose given a smooth family Σ_t^A of J_t–holomorphic representatives of A all with the same decomposition*

$$A = \sum_{i=1}^{s} \ell_i S_i + \sum_{j=1}^{p} m_j E_j + \sum_{j>p} n_j B_j$$

as in (3.1.2). *Suppose further that the components of Σ_t^A in classes E_j and B_j are embedded (though possibly disconnected). Then, after perturbing J_t in $\mathcal{J}(\mathcal{S})$, we can assume in addition that for each t that all intersections of these components with each other as well as with $\mathcal{S}$ are ω_t–orthogonal.*

Proof We first arrange that all intersections are transverse which is possible because such tangencies happen in codimension at least 2. Then these intersections occur at a finite number of points $p_{t,i}$ that vary smoothly with the parameter t. Fix i, and denote by C_1^t, C_2^t the two branches of $\mathcal{S} \cup \Sigma_t^A$ that meet at $p_{i,t}$, labelled putting the branch that lies in $\mathcal{S}$ first. Thus C_1^t, C_2^t are smoothly varying (local) curves, and, using a 1–parameter version of Darboux's theorem, we may choose smoothly varying Darboux charts

$$p_{t,i} \in U_t \xrightarrow{\varphi_t} B^4(\varepsilon)$$

such that

$$\varphi_t(U_t \cap C_1^t) = B^4(\varepsilon) \cap \{z_1 = 0\}, \quad (\varphi_t(0))_*(J_t) = J_0,$$

where J_0 is the standard complex structure on $B^4(\varepsilon) \subset \mathbb{C}^2$. Moreover, if $C_1^t \subset \mathcal{S}$, we may arrange that φ_t takes the fiber at $p_{t,i}$ of the normal bundle to $\mathcal{S}$ to the axis $z_2 = 0$. By shrinking $\varepsilon > 0$ (which we assume small, but fixed) we can also assume that the image $\varphi_t(C_2^t) \cap B^4(\varepsilon)$ is the graph $z_2 = f_t(z_1)$ of some function such that $f_t(0) = 0$ and $df_t(0)$ is complex linear. If $df_t(0) = 0$, the proof is complete. So we suppose below that $df_t(0) \neq 0$.

An obvious 1–parameter perturbation of C_2^t near p_t provides us with curves C_t' which coincide with C_2^t outside of some small ball and with the graph of $df_t(0)$ near the origin (in the coordinates given by ϕ_t). Since this perturbation can be made $\mathcal{C}^1$–small, C_t' remains symplectic. In other words, we can assume that there is $0 < \delta < \varepsilon$ such that

$$\varphi_t(C_2^t) \cap B^4(\delta) = \operatorname{graph} df_t(0) \cap B^4(\delta) = \{(z, a_t \cdot z),\ z \in \mathbb{C}\} \cap B^4(\delta),$$

where $a_t \cdot$ denotes the multiplication by the nonvanishing complex number $a_t \approx df_t(0)$.

Let now $\rho : [0, \delta] \to [0, 1]$ be a nondecreasing cut-off function that equals 0 near 0 and 1 near δ, and consider the curves $C_t'' := \{(z, \rho(|z|) a_t z)\} \cap B^4(\delta)$. These curves are embedded, coincide with $\{z_2 = 0\}$ near 0, with $\varphi_t(C_2^t) = \{(z, a_t z)\}$ near $\partial B^4(\delta)$, and they are symplectic because $\mathrm{Jac}(z \mapsto \rho(|z|) a_t z) = \rho'(|z|)|a_t||z| \geq 0$ (in polar coordinates, $\rho(|z|) a_t z$ is the map $(r, \theta) \mapsto (\rho(r)|a_t|, \theta + \arg a_t)$). We may therefore replace $C_2^t \cap \varphi_t^{-1}(B^4(\delta))$ by $\varphi_t^{-1}(C_t'')$. This is symplectically embedded (and hence J–holomorphic for some ω–tame J), and ω–orthogonal to C_1^t at $p_{t,i}$. Finally, if $C_1^t \subset \mathcal{S}$ we need to check that the new C_2^t is J–holomorphic for some $J \in \mathcal{J}(\mathcal{S})$. But this holds because we constructed C_2^t to coincide with the normal fiber to $\mathcal{S}$ at $p_{t,i}$. □

A family of nodal curves Σ_t that satisfies the conclusions of the above lemma for a fixed ω will be called *$\mathcal{S}$–adapted.* In particular this means that the corresponding homological decomposition of A is fixed, as is the intersection pattern of its components. The next result gives conditions under which A is represented by a family of embedded curves.

Lemma 3.2.2 *Let $\mathcal{S}$ be any singular set and A be $\mathcal{S}$–good. Suppose that for every $J \in \mathcal{J}_{semi}(\mathcal{S}, A)$ and every decomposition* (3.1.2) *given by a J–holomorphic stable map Σ^A that is a limit of embedded curves we have $d(B) \leq d(A)$ with equality only if $B = A$ (so that the decomposition is trivial). Then:*

(i) *For each $J \in \mathcal{J}_{semi}(\mathcal{S}, A)$ there is an embedded J–holomorphic A–curve of genus $g(A)$ through a generic $\frac{1}{2}d(A)$–tuple of points in M.*

(ii) *Any two elements $J_0, J_1 \in \mathcal{J}_{reg}(\mathcal{S}, A)$ can be joined by a path $J_t, t \in [0, 1]$, in $\mathcal{J}_{semi}(\mathcal{S}, A)$ for which there is a smooth family of embedded J_t–holomorphic A–curves.*

Proof Let us first suppose that $d(A) > 0$. Then by Fact 2.1.8 we are in the case $b_2^+ = 1$. By definition of $\mathrm{Gr}(A)$ (see Fact 2.1.3), there is for each generic ω–tame J and each sufficiently generic set $\boldsymbol{x}$ of $\frac{1}{2}d(A) \ge 1$ points in M an embedded J–holomorphic curve $u : (\Sigma, j) \to (M, J)$ that goes through these points, where (Σ, j) is some smooth Riemann surface of genus $g(A)$. Hence by Gromov compactness, for every ω–tame J and every set of $\frac{1}{2}d(A)$ points, there is a possibly nodal representative of the class A through these points. We show below that when $J \in \mathcal{J}_{semi}(\mathcal{S}, A)$ a generic set $\boldsymbol{x}$ does not lie on a nonsmooth nodal J–holomorphic representative for A. Hence, as above, it must lie on an embedded representative.

Consider a J–holomorphic representative Σ^A of A with nontrivial decomposition (3.1.2). If we remove the rigid components in the classes $\ell_i S_i$ and $m_j E_j$ from Σ^A we are left with a stable map Σ in the class $B = \sum_j n_j B_j$. Since $B_j^2 \ge 0$ by Lemma 3.1.2, we can resolve all singularities and double points of the components of B as in Case 4 for the proof of Proposition 3.1.3, obtaining an embedded representative C^B of the class B. Moreover because $b_2^+ = 1$ the curve C^B is connected unless $B^2 = 0$, and in which case it has m components in class B_0, where B_0 is primitive and $B = mB_0$ (Fact 2.1.1). In the latter case $d(B) = c_1(B) = md(B_0)$, and in either case $d(B) < d(A)$ by hypothesis. But if C^B is connected, then we conclude from Equation (2.1.6) that the Fredholm index of a simple connected curve in class B is at most $d(B)$. Because J is semiregular, each B–curve is an element in a moduli space of dimension at most $d(B) + 1$. Hence it cannot go through more than $\frac{1}{2}d(B) < \frac{1}{2}d(A)$ generic points. Similarly, if $B = mB_0$, then $d(B) = md(B_0) < d(A)$. As above, a B_0–curve can go through at most $\frac{1}{2}d(B_0)$ points, so that a B–curve goes through at most $\frac{m}{2}d(B_0) = \frac{1}{2}d(B)$ points. This shows that no simple representative of B goes through a generic set $\boldsymbol{x}$. However, as explained in Remark 2.1.9(i), the nodal representatives involved by the decomposition (3.1.2) are even more constrained, because their components satisfy $B_j^2 \ge 0$ and $C^{B_j} \cap (M \smallsetminus \overline{\mathcal{N}}) \neq 0$ by Lemma 3.1.1. Hence there is no J–holomorphic representative of B through $\boldsymbol{x}$. This completes the proof of (i).

To prove (ii), given $J_0, J_1 \in \mathcal{J}_{reg}(\mathcal{S}, A)$, join them by a regular homotopy $J_t \in \mathcal{J}_{semi}(\mathcal{S}, A)$ as in Remark 2.2.5. Then the space of B–curves that are J_t–holomorphic for some t and intersect $M \smallsetminus \overline{\mathcal{N}}$ forms a manifold of dimension $d(B) + 1$. Hence again we may choose tuple $\boldsymbol{x}$ of $\frac{1}{2}d(A)$ points in $M \smallsetminus \overline{\mathcal{N}}$ that does not lie on any such B–curve. Therefore the space of embedded A–curves through $\boldsymbol{x}$ is a compact 1–manifold with boundary at $\alpha = 0, 1$. But because A is $\mathcal{S}$–good, $\mathrm{Gr}(A) \neq 0$. Hence there is at least one component of this manifold with one boundary at $\alpha = 0$ and the other at $\alpha = 1$. Thus for some continuous function $\phi : [0, 1] \to [0, 1]$ with $\phi(0) = 0$ and $\phi(1) = 1$ there is a family of embedded $J_{\phi(t)}$–holomorphic A–curves. This proves (ii).

When $d(A)=0$ the argument is similar. In this case the hypothesis means that unless $A=B$ we have $d(B)<0$. Since $d(B)$ is even, this means that $d(B)\le -2$. But then $\dim\operatorname{Coker} D_{u,J}\ge 2$ for every B–curve u. Hence given any regular homotopy $J_t\in\mathcal{J}(\mathcal{S})$, the class B has no J_t–holomorphic representatives for any t, so that all representatives of A must be embedded. Therefore the previous argument applies. □

The next proposition applies in the situation of Proposition 1.2.9 where the manifold is rational or ruled and we have a smooth family ω_t of $\mathcal{S}$–adapted symplectic forms.

Proposition 3.2.3 *Let M be a rational or ruled symplectic manifold, and let A an $\mathcal{S}$–good class with $d(A)>0$. Suppose further that $d(A)>g+\frac{1}{4}k$ if M is the k–point blow up of a ruled surface of genus g. Let $J_t\in\mathcal{J}(\mathcal{S},\omega_t,A)$, $t\in[0,1]$, be a smooth path with endpoints in $\mathcal{J}_{reg}(\mathcal{S},A)$ Then, possibly after reparametrization with respect to t, the path $(J_t)_{t\in[0,1]}$ can be perturbed to a smooth $\mathcal{S}$–adapted path $(J'_t\in\mathcal{J}_{semi}(\mathcal{S},\omega_t,A))_{t\in[0,1]}$ such that there is a smooth family Σ^A_t, $t\in[0,1]$, of J'_t–holomorphic and $\mathcal{S}$–adapted nodal curves in class A. Moreover the corresponding decomposition*

$$A=\sum_{i=1}^{s}\ell_i S_i+\sum_j m_j E_j+B,\quad E_j^2=-1,$$

of (3.1.2) *has* $\mathrm{Gr}(B)\neq 0$.

Proof **Step 1: Preliminaries** Because we are in dimension 4, (M,ω_t) is semi-positive in the sense of [25]. Hence by the results of [25, Chapter 6] we may join J_0,J_1 by a regular homotopy $J_t\in\mathcal{J}(\mathcal{S},\omega_t,A)$. As in Remark 2.2.5, this means that $\partial_t J_t$ covers the cokernel of D_{u,J_t} for every relevant map u, so that the moduli spaces $\bigcup_{t\in[0,1]}\mathcal{M}(M\smallsetminus\overline{\mathcal{N}},B,J_t)$ are smooth manifolds with boundary of the "correct" dimension. In particular each $J_t\in\mathcal{J}_{semi}(\mathcal{S},\omega_t,A)$.

Given such J_t, consider the compact space of stable maps

$$X:=\bigcup_{t\in[0,1]}\overline{\mathcal{M}}(A,J_t).$$

This space is stratified according to the topological type $\mathcal{T}$ of the domains of the stable maps, where $\mathcal{T}$ keeps track both of the structure of the domain and the homology classes of the corresponding curves. These strata $X_{\mathcal{T}}$ are ordered by the relation that $\mathcal{T}'\le\mathcal{T}$ if a stable curve with domain of type $\mathcal{T}$ can degenerate into one of type $\mathcal{T}'$. Since J_t ranges in a compact set there are a finite number of such decompositions $A=\sum_i\ell_i S_i+\sum_j m_j E_j+B$ as in (3.1.2). Let $d_{\max}$ be the maximum of the numbers $d(B)$, where B occurs in such a decomposition for some $t\in[0,1]$. We claim that

$d_{\max} \geq d(A)$. For otherwise Lemma 3.2.2 implies that for each t there are embedded J_t–holomorphic A–curves. Since this is one of the decompositions considered in the definition of $d_{\max}$, we must have $d_{\max} \geq d(A)$.

Next, consider a decomposition

$$A = \sum_{i=1}^{s} \ell_i S_i + \sum_{j=1}^{e} m_j E_j + B \tag{3.2.1}$$

of the given type with $d(B) = d_{\max}$ and with maximal multiplicities (ℓ_i), in the sense that there is no other representative of A with decomposition

$$\sum_{i=1}^{s} \ell_i' S_i + \sum_{j=1}^{e'} m_j' E_j' + B',$$

where $d(B') = d_{\max}$, $\ell_i' \geq \ell_i$ for all i and $\ell_i' > \ell_i$ for some i.

Step 2: In this situation, we have Gr(B) $\neq 0$ If M is rational, this follows from Lemma 2.1.5(i) since $B^2 \geq 0$, $\omega(B) > 0$ by construction, and $d(B) \geq d(A) \geq 0$. So suppose that M is the blowup of a ruled surface. If $B^2 = 0$ then we may write $B = mB_0$, where $m \geq 1$ and B_0 is represented by an embedded curve. This must be a sphere or torus, since in all other cases the Fredholm index of the class is ≤ -2, so that by definition of $\mathcal{J}_{semi}$ they are not represented. In the case of a sphere we have $\mathrm{Gr}(B) = 1$, since for generic J there is a unique B–curve through each set of m generic points. On the other hand, in the case of a torus $d(B_0) = d(B) = 0$. Since $d(B) = d_{\max} \geq d(A) > g + \frac{1}{4}k \gg 0$ by hypothesis this case does not occur. Therefore, it remains to consider the case when $B^2 > 0$. Since $\omega(B) > 0$ by construction, this means that $B \in \mathcal{P}^+$. Therefore $\mathrm{Gr}(B) \neq 0$ by Lemma 2.1.5(ii) applied to the class B.

Step 3: Completion of the proof Since the classes E_j, B in (3.2.1) have nontrivial Gromov invariant, they are always represented in some form for each J_t. By Proposition 3.1.6 the classes E_j are in fact always represented by embedded curves $C_t^{E_j}$ when $J \in \mathcal{J}_{semi}(\mathcal{S}, \omega_t, A)$ since $J \in \mathcal{J}_{semi}(\mathcal{S}, \omega_t, A) \subset \mathcal{J}_{semi}(\mathcal{S}, \omega_t, E_j)$ because $\omega(A) \geq \omega(E_j)$. We next check that we can choose the regular homotopy $J_t' \in \mathcal{J}_{semi}(\mathcal{S}, \omega_t, A)$ so that the class B does not decompose. As in the proof of Lemma 3.2.2, this will follow if we can show that for each decomposition $B = \sum_j B_j'$ of the B–curve, the sum of the Fredholm indices of its nonrigid components is strictly less than the Fredholm index $d(B)$ of the class B. If the components of the B curve are all transverse to $\mathcal{S}$, then this calculation is standard; see Remark 2.1.9(i). On the other hand, if for some J_t' the decomposition is a stable map $(\Sigma_B)'$ that involves some components of $\mathcal{S}$ with others in class B', then the maximality of the pair $d(B) = d_{\max}$

and (ℓ_i) implies that $d(B') < d(B)$, and since $d(B')$ is always even, we actually have $d(B') \leq d(B) - 2$. Therefore in a regular path J'_t, the dimension of the moduli space of these stable maps is at most $d(B) - 1$, and hence those curves cannot go through $k := \frac{1}{2}d(B)$ generic points.[6] Thus, the space of embedded B–curves that are J'_t–holomorphic for some t and go through k generic points is a compact 1–manifold with boundary. Moreover, because $\mathrm{Gr}(B) \neq 0$ there is at least one connected component of this 1–manifold with one end at $t = 0$ and the other at $t = 1$. Taking such a component, and reparametrizing with respect to t as necessary, we therefore have a family C_t^B of embedded J'_t–holomorphic curves in class B. □

Remark 3.2.4 The above proposition constructs 1–parameter families of nodal curves whose components are covers of embedded curves. Lemma 3.2.1 shows that if we start with a family of nodal curves whose components are embedded (or immersed) we can perturb them so that they intersect ω_t–orthogonally. The patching arguments in Proposition 3.1.3 that resolve double points and amalgamate transversally intersecting components in classes B, B' with $B^2, (B')^2 \geq -1$ also work for 1–parameter families. Therefore, we can apply Proposition 3.1.3 to these 1–parameter families of nodal curves. (The only part of this proposition that might fail in a 1–parameter family is the initial resolution of singularities.)

Corollary 3.2.5 *Proposition 1.2.9 holds when $\mathcal{S}_{sing} = \varnothing$.*

Proof This holds by applying the 1–parameter version of Proposition 3.1.3 as in Remark 3.2.4 to obtain the required family of embedded curves. □

3.3 Numerical arguments

This section proves Theorem 1.2.7 under hypotheses (iv) and (v) by showing in both cases that the hypotheses in Lemma 3.2.2 are satisfied. First we discuss the genus-zero situation, using an argument adapted from Li and Zhang [15, Lemma 4.9].[7]

Lemma 3.3.1 *Let $\mathcal{S}$ be any singular set. Then the hypothesis of Lemma 3.2.2 holds for every $\mathcal{S}$–good A such that*

$$g(A) := 1 + \tfrac{1}{2}(A^2 - c_1(A)) = 0, \quad d(A) := A^2 + c_1(A) > 0,$$

and every $J \in \mathcal{J}_{semi}(\mathcal{S}, A)$.

[6] Strictly speaking, we can only control the dimension of the family of curves that go through some point in $M \setminus \overline{\mathcal{N}}$. However, since $d(A) > 0$ we only need consider curves that go through at least one fixed point that we can choose far from $\mathcal{S}$.

[7] Instead of requiring J to be in some way generic, they use the hypothesis that A is J–NEF, which also implies that $A \cdot B_j \geq 0$ for all j.

Proof Note first that we must be in the situation $b_2^+ = 1$, since by Fact 2.1.8, $\mathrm{Gr}(A) \neq 0$ can only be consistent with $d(A) > 0$ in this case. Consider a nontrivial decomposition $A = \sum \ell_i S_i + \sum m_j E_j + B$ as in Lemma 3.1.2, given by a J–holomorphic stable map Σ^A with $J \in \mathcal{J}_{semi}(\mathcal{S}, A)$ that, by construction, is the limit of embedded A–curves. We must check that $d(B) < d(A)$.

Let us suppose first that B has a connected, smooth and somewhere injective J–holomorphic representative $u : (\Sigma, j) \to (M, J)$. Then the adjunction formula (2.1.1) implies that $g(B) \geq g_\Sigma \geq 0$,[8] so that

$$\tfrac{1}{2} d(B) = 1 + B^2 - g(B) \leq 1 + B^2.$$

Thus $\frac{1}{2} d(B) \leq 1 + B^2$ while our hypotheses imply that $\frac{1}{2} d(A) = 1 + A^2$. Thus it suffices to show that $B^2 < A^2$. But

$$\begin{aligned} A^2 - B^2 = (A + B) \cdot (A - B) = A \cdot \Big(\sum \ell_i S_i + \sum n_j E_j \Big) + B \cdot (A - B) \\ \geq B \cdot (A - B) > 0, \end{aligned}$$

where the first inequality holds because A is $\mathcal{S}$–good, and the second (strict) inequality holds because, as we noted above, Σ^A is connected and $A \neq B$.

By Fact 2.1.3, this completes the proof unless $B = nB_0$ where B_0 is a primitive class with $B_0^2 = 0$. Since $g(B_0) = 0$ by construction, Lemma 2.2.4(ii) implies that each B_0–curve is an embedded sphere. Hence $\frac{1}{2} d(nB_0) = n$. Thus we need to show that $\frac{1}{2} d(A) = 1 + A^2 > n$. But this holds because, by the above calculation

$$A^2 = A^2 - B^2 \geq B \cdot (A - B) \geq n,$$

where the last inequality holds because B has n disjoint components. □

We next extend an argument from Biran [1]. Recall that $\mathcal{S}_{sing}$ consists of all the negative components of $\mathcal{S}$ that are not exceptional spheres.

Lemma 3.3.2 *Let $\mathcal{S}$ be any singular set such that $c_1(S_i) = 0$ for all i with $C^{S_i} \in \mathcal{S}_{irreg}$. Then the hypothesis of Lemma 3.2.2 holds for every $\mathcal{S}$–good class $A \notin \mathcal{E}$ such that $A \neq \sum_{i \in \mathcal{S}_{sing}} \ell_i S_i$ and every $J \in \mathcal{J}_{semi}(\mathcal{S}, A)$.*

Proof Starting with a nodal curve Σ^A with decomposition with $d(B) \geq 0$ as in Lemma 3.1.2, add to B all regular components C^{S_j} and all exceptional spheres that intersect B as in Lemma 3.1.5. As we remarked in the proof of Lemma 3.1.5(iv), $d(B)$

[8] Although we know each $g(B_j) = 0$, it is a priori possible that $g(B) > 0$. Li and Zhang's argument shows that in fact this does not happen. However, we do not need to use this.

does not decrease when we do this. By hypothesis all irregular components of $\mathcal{S}$ are negative because they have $d(S) = S^2 + c_1(S) < 0$. Therefore it suffices to show that in any decomposition

$$A = \sum_{i \in \mathcal{S}_{neg}} \ell_i S_i + \sum_k m_k E_k + B$$

we have $d(B) < d(A)$. Rewrite this decomposition as

$$A = \sum_{i \in \mathcal{S}_{sing}} \ell_i S_i + \sum_j n_j E'_j + B,$$

where we have grouped the sums $\sum_{i \in \mathcal{S}_{neg} \smallsetminus \mathcal{S}_{irreg}} \ell_i S_i$ and $\sum_k m_k E_k$ into a single sum over classes $E'_j \in \mathcal{E}$. We write $Z := \sum_{i \in \mathcal{S}_{sing}} \ell_i S_i$, and note that by hypothesis $c_1(Z) = 0$.

First suppose that $B = 0$ so that $A = Z + \sum_j n_j E'_j$. We must show that $d(A) > 0 = d(B)$. By assumption $A \neq Z$. Further,

$$d(A) = A^2 + c_1\Big(Z + \sum_j n_j E'_j\Big) = A^2 + \sum n_j > 0$$

unless $A \in \mathcal{E}$ and $\sum n_j = 1$. But we excluded the case $A \in \mathcal{E}$. Hence when $B = 0$ we have $d(A) > 0$ as required.

Now suppose that $B \neq 0$. By Lemma 3.1.5, we may assume that $B \cdot E'_j = 0 = E'_j \cdot E'_k$ for all $j \neq k$. Further $(A - B) \cdot B > 0$ since the classes $A - B$ and B are both represented by J–nodal curves with no common component, and their union in class A is connected. Hence

$$\begin{aligned} d(A) - d(B) &= \Big(Z + \sum n_j E'_j + B\Big) \cdot A - B^2 + c_1\Big(Z + \sum n_j E'_j\Big) \\ &= \Big(Z + \sum n_j E'_j\Big) \cdot A + B \cdot (A - B) + \sum n_j c_1(E'_j) + c_1(Z) \\ &> Z \cdot A + \sum n_j \geq 0, \end{aligned}$$

where the strict inequality uses the fact that $B \cdot (A - B) > 0$. This completes the proof. □

Corollary 3.3.3 *Parts (iv) and (v) of Theorem 1.2.7 hold.*

Proof If $A \in \mathcal{E}$ the result follows from Proposition 3.1.6 . Therefore we will assume $A \notin \mathcal{E}$. To prove Theorem 1.2.7(iv) notice that $d(A) \geq 0$ because A is $\mathcal{S}$–good. Moreover when $g(A) = 0$, the equality $d(A) = 0$ implies that $A^2 = -1$, so that $A \in \mathcal{E}$, contrary to hypothesis. Thus $d(A) > 0$. But then Lemma 3.3.1 combined with

Lemma 3.2.2 shows that A has an embedded J–representative for $J \in \mathcal{J}_{semi}(\mathcal{S}, A)$. Since $\mathcal{J}_{semi}(\mathcal{S}, A)$ is residual by Lemma 2.2.3(ii), this proves part (iii) of Theorem 1.2.7. Part (v), again with $\mathcal{J}_{emb}(\mathcal{S}, A) = \mathcal{J}_{semi}(\mathcal{S}, A)$, follows similarly using Lemmas 3.3.2 and 3.2.2. □

Remark 3.3.4 Biran actually assumed the weaker condition $A \cdot S_i + c_1(S_i) \geq 0$ for all i, but worked with disjoint curves C^{S_i}.

4 Constructions

In Section 4.1 we explain some geometric constructions for embedded curves, and then prove part (ii) of Theorem 1.2.7 and the second case of Proposition 1.2.9. The asymptotic result Theorem 1.2.16 is proved in Section 4.2.

4.1 Building embedded curves by hand

The naive strategy for answering Question 1.2.6 is to take the nodal curve Σ_A and try to piece its components together. A basic tool on which this strategy builds on is the following easy patching lemma.

Lemma 4.1.1 *Suppose that the integers $\ell, m > 0$ have no common divisor > 1. Given two nonvanishing and holomorphic functions h_1, h_2 in a neighborhood of $0 \in \mathbb{C}$ and $\varepsilon > 0$ small enough, there is an embedded symplectic submanifold*

$$C_f := \{f(z,w) = 0\} \subset \mathbb{C}^2 \smallsetminus \{zw = 0\}$$

which coincides with $\{w^\ell = \varepsilon h_1(z) z^{-m}\}$ on $|z| < \varepsilon_1$ and with $\{z^m = \varepsilon h_2(w) w^{-\ell}\}$ on $|z| > \varepsilon_2$, and is disjoint from the axes.

Note that when $\ell = m = 1$ we are patching the graph of a meromorphic section $w = az^{-1}$ over the z–axis to the graph of a meromorphic section $z = bw^{-1}$ over the w–axis via the cylinder C_f. Similarly, one can patch two transversally intersecting curves, and also a simple pole (the graph of $w = az^{-1}$) to the transverse axis $z = 0$. In the latter case, for example, C_f would coincide with the graph of $w = az^{-1}$ for $|z|$ large and with the axis $z = 0$ for $|w|$ large. We will not prove this lemma here (or state it very precisely) since we do not use it in any serious way in this paper. However, we describe some applications in Example 4.1.2 below. Note that Li and Usher [14] also use this idea of patching curves via meromorphic sections.

The way this lemma would ideally apply is the following. To fix ideas, consider the case where the S_i are spheres of self-intersection $-k_i \leq -2$. For the decomposition

$A = \sum \ell_i S_i + \sum m_i E_i + B$ associated to a nodal map Σ^A, the numerical condition $A \cdot S_i \geq 0$ implies that

$$(*) \qquad \ell_i k_i \leq \sum_{j \neq i} \ell_j S_i \cdot S_j + \sum m_j E_j \cdot S_i + B \cdot S_i.$$

We consider a holomorphic cover $\Sigma_i \xrightarrow{f_i} S_i$ of degree ℓ_i, totally ramified at the intersections between C^{S_i} and each C^{S_j} and C^{E_j}. We pull back the normal bundle $\mathcal{L}_i$ to C^{S_i} by f_i, and consider a smooth section σ_i of $f_i^*\mathcal{L}_i$ that is holomorphic near its zeros and poles, has poles of order ℓ_j, m_j at each (unique) preimage of the intersections of C^{S_i} with C^{S_j} and C^{E_j}, respectively, as well as one additional simple pole at some preimage of each intersection of C^{S_i} with C^B, and no other poles. Since the pullback bundle $f_i^*\mathcal{L}_i$ has degree $-\ell_i k_i$, the condition $(*)$ precisely means that the existence of such smooth sections is not homologically obstructed. We do the same for C^{E_i} and for C^B (for the latter we do not need to consider a covering). Now the pushforward of these sections to $\mathcal{L}_i$ provide multisections with singularities modeled on $w^{\ell_i} = z^{-\ell_j}$ (or $w^{\ell_i} = z^{-m_j}$ or $w^{\ell_i} = z^{-1}$) near each intersection. For example, at an intersection $q \in C^{S_i} \cap C^{S_j}$ let us use the coordinate z along C^{S_i} and w along C^{S_j}. Then the two branched covering maps are

$$(z', w) \mapsto \big((z')^{\ell_i} = z, w\big), \quad (z, w') \mapsto \big(z, (w')^{\ell_j} = w\big).$$

Hence the sections $w = a\,(z')^{-\ell_j}$, $z = b\,(w')^{-\ell_i}$ push forward to the curves

$$w^{\ell_i} = a^{\ell_i} z^{-\ell_j}, \quad z^{\ell_j} = b^{\ell_i} w^{-\ell_i}.$$

Thus Lemma 4.1.1 implies that for sufficiently small ε the sections $\varepsilon f_{i*}\sigma_i$ and $\varepsilon f_{j*}\sigma_j$ can be patched together to give a curve that does not meet $C^{S_i} \cup C^{S_j}$ near the intersection point q. More generally, all these (rescaled) multi-sections can be patched together in the neighborhood of the intersections to form a symplectic curve in class A that is transverse to $\mathcal{S}$.

Now this curve may have self-intersections coming from the folding of the section σ_i when we push it forward to $\mathcal{L}_i$. When σ_i is holomorphic, these self-intersections are positive, so they can be resolved and the procedure gives an embedded symplectic curve in class A that intersects the $\mathcal{S}$ transversally and positively. However, the criteria for the existence of such a holomorphic section is not of topological nature but of analytical one (it is given by the Riemann–Roch theorem). Hence there is no guarantee that one can find suitable sections σ_i. The next example illustrates these difficulties, which in this case arise from a multiply covered exceptional curve C^E. It also suggests some ways around them.

Example 4.1.2 Suppose that $\mathcal{S}$ consists of a single sphere C^S in class S with $S\cdot S=-k$, that E is the class of an exceptional divisor C^E with $E\cdot S=m$ and that B satisfies $B\cdot S=1, B\cdot E=0$. Then $A:=S+mE+B$ has

$$A\cdot E=0,\quad A\cdot S=m^2-k+1,$$
$$d(A)=d(S+mE)+d(B)+2S\cdot B=4-2k+m^2+m+d(B)\ge 0.$$

Because $d(B)$ can be arbitrarily large, the condition $d(A)\ge 0$ gives no information. Therefore, the only numerical information we have on k is that $k\le m^2+1$. Note also that if $k\le m^2$, then $A':=S+mE$ satisfies $A'\cdot S=0$, and we can try to form an embedded curve in class $A'=S+mE$ and then join it to the B curve to get the final embedded A curve. The virtue of this approach is that it gives us better understanding of the genus since $g(A')$ is a function of m only. In fact, because $A'\cdot B=1$, we have

$$g(A'+B)=g(A')+g(B)\quad\text{and}\quad g(A')=1+\tfrac12\big((A')^2-c_1(A')\big)=\tfrac12 m(m-1).$$

Therefore if $m=4$ and $k\le 16$, we should be able to construct an embedded curve in class $A'=S+4E$ of genus 6 and hence a curve in class A of genus $6+g(B)$. We show below that the embedded A'–curve exists when $k\le 13$, but may not exist when $14\le k\le 16$.

The case $k\le 4$ In this case it is very easy to construct such a curve. We may assume that C^E intersects C^S transversally at 4 distinct points $p_1,\dots,p_4$, and then choose a small meromorphic section σ_S of the normal bundle to C^S with simple poles at the four points $p_1,\dots,p_4$ and $4-k$ zeros. Then take 4 different small nonvanishing meromorphic sections $\rho_1,\dots,\rho_4$ of the normal bundle to C^E, where ρ_i has a simple pole at p_i. Note that these sections are inverse to holomorphic sections of the bundle over S^2 with Chern class 1 and so each pair intersects once transversally. Next patch ρ_i to σ_S at p_i. (This is possible because the graphs of σ_S and ρ_i satisfy an equation of the form $zw=$ constant near p_i and so we can cut out small discs from each of these graphs and replace it by a cylinder. One needs to check that this cylinder can be chosen to be disjoint from the other sections ρ_j; but this holds because ρ_i is relatively much larger than the ρ_j, $j\ne i$, near p_i since it has a pole there.) This process gives an immersed curve of genus 0 with 6 positive self-intersections, one for each (unordered) pair i,j, $i\ne j$.[9] Therefore we obtain the desired embedded curve of genus 6 by resolving these intersections.

[9] These intersection points occur at the places where the graphs of ρ_i,ρ_j intersect, far away from the poles. Note that although the graph of each ρ_i meets C^S at 3 points, one near each p_j, $j\ne i$, these intersections disappear after gluing since the part of C^S near p_j is cut out during the gluing with ρ_j.

The case $4 < k \le 10$ We can refine the above argument by choosing the sections $\rho_1, \dots, \rho_4$ to have different orders of magnitude, with $\rho_1 \gg \rho_2 \gg \rho_3 \gg \rho_4$. Thus ρ_1 has a simple pole at p_1 and its graph intersects C^S at points p_{1i}, $i = 2, 3, 4$, moderately near p_i. We match these zeroes and poles with 4 poles of σ_S. If ρ_2 is much smaller than ρ_1, then we can construct σ_S to have another pole at p_2 that matches with ρ_2, together with two more poles at p_{2i}, $i = 3, 4$, that are much closer to p_i. (Note that the point of intersection of the graph of ρ_2 with C^S that is near p_1 is cut out of C^S by the first patching process, and so we cannot put another pole there.) Similarly, we can choose ρ_3 so that it patches with 2 poles of σ_S and then can take $\rho_4 = 0$ to patch with one further pole. This procedure accommodates up to 10 poles. Moreover, it is not hard to check that the corresponding embedded curve has genus 6. For example if k is 10 we have patched five spheres together at $4 + 3 + 2 + 1 = 10$ points and so get a possibly immersed curve of genus 6. As before, the branch points would come from intersections of ρ_i with ρ_j for $i \neq j$. But these are all cut out during the patching process: for example, because $\rho_2 \ll \rho_1$ the intersection point of these sections lies near the pole on ρ_2 and so is cut out when this pole is patched to the pole of σ_S at p_2.

The case $10 < k \le 16$ It is possible to refine this argument by using branched coverings as suggested at the beginning of this section. Note that near a point where σ_S has a pole of order n its graph satisfies an equation of the form $wz^n = \text{constant}$, where z is the coordinate along C^S and w is the normal coordinate. It is not hard to check that this pole may be patched to the pushforward of a section ρ with a simple pole $\rho(w') = \varepsilon/w'$ by a branched covering map $w' \mapsto (w')^n := w$: indeed the graph of ρ satisfies $zw' = \varepsilon$, which gives $z^n (w')^n = \varepsilon^n$, so that its pushforward satisfies $z^n w = \varepsilon^n$. Since A contains E with multiplicity 4, we can in principle take any $n \le m = 4$ and hence accommodate up to 16 poles of σ_S. We now investigate this construction in more detail.

The case $k > 10$ Our initial strategy for constructing a curve in class $A' = S + 4E$ when $k > 4$ is the following:

(a) Take a meromorphic section σ_S of the normal bundle to C^S with poles of order 4 at each point $p_1, \dots, p_4$ and $16 - k$ zeros.

(b) Take a branched cover $f : \Sigma \to S^2$ of C^E of order 4 that is totally ramified at each of the points $q_i := f^{-1}(p_i)$, $i = 1, \dots, 4$ (and hence has local model $w' \mapsto (w')^4$).

(c) Choose a meromorphic section ρ_Σ over Σ of the pullback by f of the normal bundle to C^E with simple poles at the branch points $q_1, \dots, q_4$.

(d) Patch the multisection $f_*(\operatorname{graph} \rho_\Sigma)$ to the graph of σ_S obtaining an immersed curve with only positive intersections with itself and with $\mathcal{S}$.

Step (d) gives an immersed curve which is made by patching an immersed curve of genus $g(\Sigma)$ with 4 punctures to a sphere with 4 punctures. Hence it has genus $g(\Sigma)+3+a$, where a is the number of self-intersection points of $f_*(\text{graph}\,\rho_\Sigma)$. By the Riemann–Hurwitz formula, the Euler characteristic $\chi(\Sigma)$ equals $4\chi(S^2)-12=-4$, where $12=4\times 3$ is the number of "missing vertices". Therefore $g(\Sigma)=3$. Hence if this process worked we would have $a=0$. Thus the curve in (d) would actually be embedded. It is not hard to see that all the above steps can be achieved except (possibly) for (c). The problem here is that because Σ is not a sphere there is no guarantee that we can find a meromorphic section with poles at the given points. Here are some ways to try to get around this problem.

- Relax the condition on the section in (c), simply choosing any section with these poles. But then there is no guarantee that the pushforward multisection $f_*(\text{graph}\,\rho_\Sigma)$ has only positive self-intersections. In fact, in cases where we have tried this, we have managed only to construct sections with simple poles at the q_i that push forward to multisections with both positive and negative self-intersections; and it is not clear that these can be made to cancel.

- Change the cover in (b) so that $g(\Sigma)$ is smaller, since then we can prescribe the positions of $4-g(\Sigma)$ poles of ρ_Σ. Suppose for example that f has three branch points $q_1,\dots,q_3$ of orders $b_i=4$, $i=1,2$, and $b_3=3$. Then the Riemann–Hurwitz formula gives

$$2-2g(\Sigma)=\chi(\Sigma)=4\chi(S^2)-\sum_{i=1}^{3}(b_i-1)=0,$$

so that $g(\Sigma)=1$. Moreover there is a cover with this branching because there are three elements $\gamma_1,\dots,\gamma_3$ in the symmetric group S_4 on 4 letters such that

- γ_i has order b_i for all i,
- $\gamma_1\gamma_2\gamma_3=\mathrm{id}$.

(Take γ_1,γ_2 to be cycles of order 4 whose product fixes just one point and hence is a cycle of order 3.) Choose a meromorphic section ρ_Σ with simple poles at the branch points q_1,q_2,q_3 and at one other arbitrary point v_4. Then alter f by postcomposing with a diffeomorphism $\phi: S^2\to C^E$ so that $\phi\circ f:\Sigma\to C^E$ maps the four points $q_1,\dots,v_4$, where ρ_Σ has poles to the intersection points $\{p_1,\dots,p_4\}=C^S\cap C^E$. Then one can check that the pushforward of ρ_Σ by $\phi\circ f$ can be patched to a section σ_S with poles of order 4 at p_1,p_2, of order 3 at p_3 and of order 1 at p_4, a total of 12 poles. Since the other branch points of f just push forward to smooth points, the result is an immersed curve with genus $g(\Sigma)+3=6$ which in fact must be embedded.

It is not hard to check that this is best one can easily do with this approach: adding more branching increases $g(\Sigma)$ and hence decreases the number of points where one can allow σ_S to have higher-order poles. However, in this case it is possible to accommodate one more pole, because there happens to be a special 3–fold cover $f: T^2 \to S^2$ totally ramified over three points, say p_2, p_3, p_4; see Remark 4.1.3. We may therefore take a largish section ρ_1 of the normal bundle to E with a pole at p_1 whose graph intersects C^S at three points close to p_2, p_3, p_4, and a very small pushforward multisection $f_*(\sigma_T)$ that patches to poles of order 3 at p_2, p_3, p_4. This patches 13 poles. However, it is not clear how to deal with the cases $14 \le k \le 16$.

Remark 4.1.3 We now briefly describe the special 3–fold branched cover $f: T^2 \to S^2$. It has three totally ramified branch points q_1, q_2, q_3 such that the pullback bundle has a meromorphic section with its three poles precisely at q_1, q_2, q_3. Consider the torus $\mathbb{T}_0$ given by the Fermat curve $x^3 + y^3 + z^3 = 0$ in $\mathbb{CP}^2$, with deformations $\mathbb{T}_\varepsilon := x^3 + y^3 + z^3 = \varepsilon xyz$. There is a natural degree 9 cover

$$F: (\mathbb{CP}^2, \mathbb{T}_0) \to (\mathbb{CP}^2, \mathbb{CP}^1), \quad [x:y:z] \mapsto [x^3 : y^3 : z^3],$$

which quotients out by the action of the group $\mathbb{Z}_3 \times \mathbb{Z}_3$ on $\mathbb{T}_0$ by

$$[x:y:z] \mapsto [\tau^i x : \tau^j y : z], \quad i, j \in \mathbb{Z}_3.$$

The action of the subgroup $G_{\mathrm{free}} := \{(j, -j), j \in \mathbb{Z}_3\}$ has no fixed points, in fact acting on all the tori $\mathbb{T}_\varepsilon$ by a translation of order 3. Therefore the map $F: \mathbb{T}_0 \to \mathbb{CP}^1$ descends to

$$f: \Sigma := \mathbb{T}_0 / G_{\mathrm{free}} \to \mathbb{CP}^1.$$

On the other hand the group $G_{\mathrm{fix}} := (j, 0), j \in \mathbb{Z}_3$ fixes the three points

$$[0:1:-1], \quad [0:\tau:-\tau^2], \quad [0:\tau^2:-\tau],$$

acting on the tangent space of each by a rotation through $2\pi/3$. These points form one orbit under G_{free}. Hence this gives one totally ramified point of f in Σ. Similarly, G_{free} permutes the three points $[1:-1:0], [\tau:-\tau^2:0], [\tau^2:-\tau:0]$ and the corresponding set of points with 0 in the second place. Again, each of these gives rise to one totally ramified point in the quotient cover f. Note that F has 9 branch points, each of order 3, lying in three distinct fibers of the quotient map $\mathbb{T}_0 \to \Sigma := \mathbb{T}_0 / G_{\mathrm{free}}$.

One can see the section as follows. The normal bundle $\mathcal{L}_N$ to $\mathbb{T}_0$ in $\mathbb{CP}^2$ is the pullback by F of the normal bundle of the line $x + y + z = 0$. The 9 branch points of F lie on all the curves $\mathbb{T}_\varepsilon$. Define a section Y_ε of $\mathcal{L}_N$ by first embedding a neighbourhood of the zero section in the normal bundle of $\mathbb{T}_0$ into $\mathbb{CP}^2$ using the exponential map with respect to the standard Kähler metric, and then defining Y_ε so that $\exp_z(Y_\varepsilon(z)) \in \mathbb{T}_\varepsilon$

for all $z \in \mathbb{T}_0$. Then its derivative $\partial_\varepsilon Y_\varepsilon|_{\varepsilon=0}$ is a holomorphic section of the normal bundle. Thus this is a holomorphic section of $\mathcal{L}_N$ with precisely 9 simple zeros at the branch points of F. To get the bundle and section we are looking for, it remains to quotient out by G_{free}, which acts on the curves $\mathbb{T}_\varepsilon$ and also by isometries on $\mathbb{CP}^2$.

Remark 4.1.4 It is not clear how special the section in Remark 4.1.3 really is. Are there cases in which there are no meromorphic sections with the required zeros, but there are symplectic sections with these zeros whose pushforward has positive self-intersection? If so, the local structure of symplectic nodal curves would be significantly different from that of holomorphic ones.

Proposition 4.1.5 *Theorem 1.2.7(ii) holds.*

Proof By assumption $\mathcal{S}$ has one class S with $S^2 < -1$, some classes labelled by $i \in \mathcal{I}_{\mathcal{E}}$ with C^{S_i} an exceptional sphere, and classes S_i with $(S_i)^2 \geq 0$. By Lemma 3.1.5 we may assume that A has a connected nodal representative Σ^A with decomposition

$$A = \ell S + \sum_{i \in I_{\mathcal{E}}} \ell_i S_i + \sum_j m_j E_j + B, \tag{4.1.1}$$

as in Equation (3.1.3), where

(I) $A \cdot S \geq 0$, $A \cdot E_j \geq 0$, $A \cdot S_i \geq 0$ for all i, j with nonzero coefficients,

(II) $S_i \cdot E_j = S_i \cdot B = E_j \cdot B = 0$ for all i, j with nonzero coefficients,

(III) B (if nonzero) has an embedded representative C^B that is J–holomorphic for some $J \in \mathcal{J}(\mathcal{S})$.

Step 1 *If $\ell = 1$ in* (4.1.1) *then A has an embedded representative that intersects $\mathcal{S}$ and the C^{E_i} orthogonally.*

Proof We use the constructions and notations of Example 4.1.2. Let us first suppose that $B = 0$ and that there is a single curve E_i in class E so that $A = S + mE$. Then, with $a := S \cdot E$, and $2 \leq k := -S \cdot S \leq 4$, we must have

$$E \cdot A = a - m \geq 0, \quad S \cdot A = -k + ma \geq 0 \implies k \leq ma \leq a^2.$$

If $m = 1$ then we can construct the desired curve as in the case $k \leq 5$ in Example 4.1.2. If $m \geq 2$ then $a \geq 2$. We take $\Sigma = S^2$, and $f : \Sigma \to S^2 = C^E$ an m–fold cover branched at two of the intersection points of C^E with C^S. Because $g(\Sigma) = 0$ we can put the poles of ρ_Σ at the two branch points and hence can accommodate up to $2m \geq k$ poles.

Now suppose that all coefficients ℓ_i in (4.1.1) vanish. Because $E_i \cdot E_j = 0$, $i \neq j$, and $B \cdot E_i = 0$ for all i, we must have

$$a_i := E_i \cdot S \geq m_i, \quad B \neq 0 \Longrightarrow h := B \cdot S > 0,$$
$$k \leq \sum a_i m_i + h.$$

Since $k \leq 4$, if $\sum a_i + h \geq 4$, the claim holds because one can use sections of the normal bundle to the C^{E_i} and to C^B with simple poles at each intersection point with C^S to accommodate the four poles of a section of the normal bundle to C^S. The claim is also true when $A = S + mE$, as we saw above. Therefore, we need only consider the situation where $\sum a_i + h \leq 3$ and either $B \neq 0$ (so that $h \geq 1$) or there are at least two E_i. This is possible only if all $a_i \leq 2$. But because $m_i \leq a_i$ this means that again we only need consider two-fold covers. Therefore the argument proceeds as before.

The general case, in which some coefficients ℓ_i are nonzero, is similar. Indeed, since the construction yields a representative that is orthogonal to the exceptional curves it makes no difference whether these lie in $\mathcal{S}$ or are other curves C^{E_i}. □

Step 2: Completion of the proof Suppose inductively that the results holds for all $\ell < \ell_0$ and consider A with a decomposition (4.1.1) with $\ell = \ell_0 > 1$. We aim to show that there are nonnegative integers $\ell'_j \leq \ell_j, m'_i \leq m_i$ such that $A' := S + \sum_{j \in \mathcal{I}_\mathcal{E}} \ell'_j S_j + \sum m'_i E_i + B$ satisfies condition (I) in (4.1.1). Then, because (II), (III) are automatically true, we may apply Step 1 to conclude that A' has an embedded representative. Therefore, the decomposition

$$A = (\ell_0 - 1)S + \sum_{j \in \mathcal{I}_\mathcal{E}} (\ell_j - \ell'_j)S_j + \sum_i (m_i - m'_i)E_i + A',$$

also has the properties of (4.1.1) but with $\ell < \ell_0$. Hence it has an embedded representative by the inductive hypothesis.

Therefore it remains to find suitable ℓ'_j, m'_i. For simplicity, let us first suppose that $\ell_i = 0$ for all i. As in Step 1, define $a_i := E_i \cdot S$, and $h := B \cdot S$ so that

$$(*) \quad E_i \cdot A = \ell_0 a_i - m_i \geq 0, \quad (**) \quad S \cdot A = -\ell_0 k + \sum a_i m_i + h \geq 0.$$

Here are some situations in which we can check that there is a class $A' = S + \sum m'_i E_i + B$ that satisfies the numeric conditions (I).

(a) If $h + \sum_i a_i \geq 4 \geq k$, then we may take $A' = S + \sum E_i + B$.

(b) If all $m_i = 1$ we are in the previous case, and may take $A' = S + \sum E_i + B$.

(c) If $a_i = 1$ for all i then $m_i \leq \ell_0$ for all i so that $m_i/\ell_0 \leq 1 = a_i$ for all i, so that $(**)$ gives $k \leq \sum m_i/\ell_0 + h/\ell_0 \leq \sum_i a_i + h$ and we are again in case (a).

(d) If there is i with $m_i \geq 2$ and $a_i \geq 2$, then $k \leq 4 \leq 2a_i$ so that we can take $A' = S + 2E_i$.

(e) If there is only one curve E_i, then we may take $n := \lceil m/\ell_0 \rceil$, and $A' = S + nE + B$. Note that in this case $E \cdot A' = a - n \geq 0$ since a is an integer $\geq m/\ell_0$.

If none of these cases occur then there are at least two curves E_1, E_2 where $a_1 > 1$, $m_1 = 1$ and $m_2 > 1$, $a_2 = 1$. Further since $h + \sum_i a_i \leq k - 1$ we must have $k = 4$, $h = 0$, $a_1 = 2$, and no other E_i. But then $m_2 \leq \ell_0$ by $(*)$ and $4\ell_0 \leq 2 + m_2$ by $(**)$, which is impossible. Hence in all cases there is a suitable class A'.

Since the above argument is purely algebraic, it works equally well if some of the exceptional spheres in (4.1.1) lie in $\mathcal{S}$. This completes the inductive step and hence the proof. □

Remark 4.1.6 By using the special 3–fold cover in Remark 4.1.3 one should be able to extend this argument to larger values of k.

Corollary 4.1.7 *Proposition 1.2.9 holds for this $\mathcal{S}$.*

Proof Under the given assumptions for the class A, Proposition 3.2.3 constructs a 1–parameter family of $\mathcal{S}$–adapted nodal A–curves. The above proof that amalgamates these into a single embedded A–curve uses patching procedures that are only slightly more complicated than those in Proposition 3.1.3. Hence, as in Remark 3.2.4, they may be carried out for a 1–parameter family, giving the required family of embedded curves. □

4.2 The asymptotic problem

We now prove Theorem 1.2.16 and Corollary 1.2.17, using the patching procedures described in Section 4.1, as well as the inflation results explained in Section 5. Since the latter results are established only when M is rational/ruled we work under this hypothesis here, although the arguments below apply more widely.

Proof of Theorem 1.2.16 Here we assume that $(M, \omega, \mathcal{S}, J)$ is a rational/ruled manifold with singular set $\mathcal{S}$, $J \in \mathcal{J}(\mathcal{S})$ and Σ^A is a nodal J–representative of some class $A \in H_2(M)$. If ω is a rational class, a classical refinement of Donaldson's construction produces a symplectic curve C^T for $T = \mathrm{PD}(N\omega)$, $N \gg 1$, which intersects $\mathcal{S}, \Sigma^A$ transversally and positively [5]. The first statement of the theorem is a further refinement explained in [27]: Whatever ω, for some $r \le \operatorname{rank} H^2(M)$, there is a decomposition

$$[\omega] = \sum_{i=1}^{r} \beta_i \ \mathrm{PD}(T_i), \quad \beta_i > 0,$$

where T_i are represented by embedded symplectic curves C^{T_i}, which again intersect $\mathcal{S} \cup \Sigma^A$ positively and transversally. At this point, we therefore have a J–nodal curve $\mathcal{S} \cup \Sigma^A \cup \mathcal{T}$ (where $\mathcal{T} = \bigcup C^{T_i}$) for some $J \in \mathcal{J}(\omega)$. As will be explained in Section 5 (see Lemma 5.1.4), we can perform a small inflation along $\mathcal{T}$ in order to get a symplectic form $\omega' = \sum_i \beta_i' \ \mathrm{PD}(T_i)$ in a rational class, close to ω, still J–compatible.

On the other hand, Proposition 5.1.2 guarantees the existence of a J–compatible symplectic form ω_κ' in class $\mathrm{PD}(A) + [\omega']/\kappa$ for arbitrary large κ. Given the ε_i, choose $\kappa \in \mathbb{Q}$ so that $\varepsilon_i - \beta_i'/\kappa \ge 0$ and then choose N_0 so that $N_0\varepsilon_i$, $N_0\beta_i'/\kappa \in \mathbb{Z}$ for all i. Again by Donaldson's construction, for $k \gg 1$ there is an embedded curve Σ that is approximately J–holomorphic (hence ω–symplectic) and in class

$$[\Sigma] = kN_0\left(A + \frac{\mathrm{PD}[\omega']}{\kappa}\right) = kN_0\left(A + \sum \frac{\beta_i'}{\kappa} \ T_i\right),$$

As before, Σ can be required to intersect $\mathcal{S}, \mathcal{T}$ transversely and positively, meaning that Σ is J'–holomorphic for some $J' \in \mathcal{J}(\mathcal{S}, \omega)$. Then the given class $kN_0A_\varepsilon := kN_0(A + \sum \varepsilon_i T_i)$ is represented by the nodal curve

$$\Sigma \cup \bigcup kN_0\left(\varepsilon_i - \frac{\beta_i'}{\kappa}\right) C^{T_i}.$$

(Note that by construction each $N_0(\varepsilon_i - \beta_i'/\kappa)$ is a positive integer.) Since Σ has only transverse and positive intersections with $\mathcal{T}$, we can smooth this nodal curve to an embedded one as in Lemma 4.1.1 (with $\ell = m = 0$). □

Proof of Corollary 1.2.17 Consider the k–fold blow-up $\widehat{\mathbb{CP}}{}^2_k$ of $\mathbb{CP}^2$ endowed with a symplectic form ω, a singular set $\mathcal{S}$, and a class $A = L - \sum \mu_i E_i$. Slightly perturb ω if necessary so that $[\omega] = \ell - \sum \alpha_i e_i$ is rational. Since the union of closed balls $\bigsqcup \overline{B}(\mu_i)$ embeds into $\mathbb{CP}^2$, there is a symplectic form in class $\ell - \sum \mu_i e_i$, and hence in nearby classes $\ell - \sum(\mu_i + \delta_i)e_i$. It follows that for sufficiently small $|\delta_i|$, chosen so that $\mu_i + \delta_i$ is rational, every integral class of the form $A' = q(L - \sum(\mu_i + \delta_i)E_i) \in H_2(\widehat{\mathbb{CP}}{}^2_k)$, where $q > 0$, is reduced and has nonvanishing Gromov invariant. Applying Theorem 1.2.16 with $r = 1$ and to such a class A', we get an integral class $T = \mathrm{PD}(N_0\omega)$ and, for all positive $\varepsilon \in \mathbb{Q}$, a symplectically embedded curve positively transverse to $\mathcal{S}$ in a class of the form

$$\begin{aligned} N'(A'+\varepsilon T) &= N'\Big(L-\sum(\mu_i+\delta_i)E_i+\varepsilon N_0\Big(L-\sum\alpha_i E_i\Big)\Big) \\ &= N'\Big((1+\varepsilon N_0)L-\sum(\mu_i+\delta_i+\varepsilon N_0\alpha_i)E_i\Big) \\ &= N'(1+\varepsilon N_0)\Big(L-\sum\frac{\mu_i+\delta_i+\varepsilon N_0\alpha_i}{1+\varepsilon N_0}E_i\Big). \end{aligned}$$

Note that the choice of N_0 is independent of that of δ_i and ε, though N' depends on the latter choices. For sufficiently small (rational) $\varepsilon>0$ we may choose $\delta_i:=\varepsilon N_0(\mu_i-\alpha_i)$, so that the class $N'(A'+\varepsilon T)$ is a multiple of $A=L-\sum\mu_i E_i$. We conclude as claimed that for some N the class NA is represented by a J–curve for some $J\in\mathcal{J}(\mathcal{S},\omega)$. $\square$

5 Symplectic inflation

We assume throughout this section that (M,ω) is a blow up of a rational or ruled manifold so that the calculation of $\mathrm{Gr}(A)$ is given by Lemma 2.1.5. For short, we simply say that M is rational/ruled. We begin in Section 5.1 by explaining the inflation process and proving Theorem 1.2.12 modulo some technical results. Even in the absolute case, the details here are new: we explain a streamlined version of the construction that is easy to generalize to the relative case. The proofs of the technical results are deferred to Section 5.2. In particular, Lemma 5.2.1 is a more detailed version of Lemma 1.2.11.

5.1 The main construction

In this section we work relative to a collection $\mathcal{C}$ of surfaces C^{T_j}, $1\le j\le L$, that may contain some or all of the components of $\mathcal{S}$ and satisfies the following conditions:

Condition 5.1.1 (a) Each C^{T_j} is ω–symplectically embedded and lies in a class T_j with $T_j\cdot T_j=n_j\in\mathbb{Z}$.

(b) Each surface C^{T_j} is ω–orthogonal to all the components C^{S_i} of $\mathcal{S}\smallsetminus\mathcal{C}$ as well as to the other C^{T_k}, $k\ne j$.

In this situation we say that $\mathcal{C}$ is $(\mathcal{S},\omega)$*–adapted*, or simply $\mathcal{S}$*–adapted*, and that the form ω is $\mathcal{C}\cup\mathcal{S}$*–adapted.*[10] A component C^{T_i} is called *positive* (resp. *negative)* if $n_i\ge 0$ (resp. $n_i<0$). We say that $\mathcal{C}$ is J–holomorphic if the tangent space to each of its components is J–invariant. Similarly, we say that a nodal curve Σ^A is

[10] This amounts to requiring that ω satisfy the conditions in Definition 1.2.1 with respect to the collection $\mathcal{S}\cup\mathcal{C}$. For we always assume that ω is compatible with the given fibered structure near $\mathcal{S}$, and because of the orthogonality condition (b) we can always choose a compatible fibered structure near $\mathcal{C}$.

$(\mathcal{S}\cup\mathcal{C})$*–adapted* if the collection of its components satisfies the above conditions with respect to $\mathcal{S}\cup\mathcal{C}$.

In applications, we will represent the class A along which we want to inflate by a nodal curve Σ^A whose components give a decomposition $A=\sum \ell_i S_i + \sum n_j B_j$ as in (3.1.1), and then take $\mathcal{C}$ to contain the curves in the singular set $\mathcal{S}$ together with suitable embedded representatives of the classes B_j obtained via Proposition 3.1.3. Thus we can write $A=\sum m_j T_j$ for some integers $m_j\geq 0$, where T_j are the classes of the components of $\mathcal{C}$. Here, as always, we take the class A to be integral. However it is just as easy, and convenient specially in the relative case, to inflate along classes $Y\in H_2(M;\mathbb{R})$ of the form $Y:=\sum \lambda_i T_i$, where $\lambda_i\geq 0$ are real numbers. As will become clear, the important point is not whether Y is integral but that the classes T_i are represented by the submanifolds in $\mathcal{C}$.

We begin by stating a version of the basic inflation result. (A simpler version was proved in [22] using the pairwise sum as in [14].)

Proposition 5.1.2 *With $\mathcal{C}$ as above, let $Y:=\sum_{i=1}^L \lambda_i T_i$, where $\lambda_i\geq 0$, and define $\lambda_{\max}:=\max_i \lambda_i$. Then there are constants $\kappa^0,\kappa^1>0$, depending on Y,ω and $\mathcal{C}$ and a smooth family of symplectic forms $\omega_{\kappa,Y}$, $\kappa\in[-\kappa^0,\kappa^1]$, on M such that the following holds for all κ:*

- (i) $[\omega_{\kappa,Y}]=[\omega_0]+\kappa\,\mathrm{PD}(Y)$, *where* $\mathrm{PD}(Y)$ *denotes the Poincaré dual of* Y.
- (ii) $\omega_{\kappa,Y}$ *is* $(\mathcal{S}\cup\mathcal{C})$*–adapted.*
- (iii) *If* $Y\cdot T_j=0$ *for some* j *the restrictions of* $\omega_{\kappa,Y}$ *and* ω *to* C^{T_j} *are equal.*
- (iv) *The constant* κ^0 *depends on geometric information, namely* $\omega,\mathcal{C}$ *and* $\lambda_{\max}$, *while* κ^1 *depends only on* $[\omega]$, $\lambda_{\max}$, *and the homology classes* T_i. *Moreover, if* $Y\cdot T_i\geq 0$ *for all* i *then* κ^1 *can be arbitrarily large.*

For short we will say these forms $\omega_{\kappa,Y}$ are constructed by $\mathcal{C}$*–adapted inflation*. We will see in the proof (given in Section 5.2) that the curves along which we inflate are part of $\mathcal{C}$.

Note also that in this result we allow κ to be slightly negative. We will call a deformation from ω_0 to $\omega_{-\varepsilon}$ a *negative inflation*. However, just as "inflation" along a class S with $S^2<0$ decreases $\omega(S)$, negative inflation along such a class increases $\omega(S)$. The next example shows why we cannot always take κ^1 to be arbitrarily large.

Example 5.1.3 If $T=E$ is the class of an exceptional divisor C^E, then negative inflation along $Y=E$ by $-\kappa$ changes $[\omega]$ to $[\omega]-\kappa\,\mathrm{PD}(E)$, increasing the size of C^E to $\omega(E)+\kappa$. On the other hand, positive inflation by κ to $[\omega]+\kappa\,\mathrm{PD}(E)$ decreases it to $\omega(E)-\kappa$ and so is possible only if $\kappa<\omega(E)$.

The same argument works in 1–parameter families. More precisely, the following holds.

Lemma 5.1.4 *Let ω_t, $t\in[0,1]$, be a smooth family of symplectic forms on M and $\mathcal{C}_t$, $t\in[0,1]$, be a smooth family of $(\mathcal{S},\omega_t)$–adapted submanifolds in the classes T_i, $1\le i\le L$. Let $Y_t := \sum_{i=1}^{L}\lambda_i(t)T_i$ with $\lambda_i(t)\ge 0$. Then the following holds:*

- (i) *There are constants $\kappa^0,\kappa^1>0$ and a 2–parameter family of symplectic forms $\omega_{t,\kappa,Y}$, $t\in[0,1]$, $-\kappa^0\le\kappa\le\kappa^1$ that for each t satisfies the conditions (i)–(iv) of Proposition 5.1.2 with respect to $\mathcal{C}_t$ and Y_t. In particular, $[\omega_{t,\kappa,Y}] = [\omega_t]+\kappa\,\mathrm{PD}(Y_t)$ for all $t\in[0,1]$, $\kappa\in[-\kappa^0,\kappa^1]$.*
- (ii) *One can construct this family $\omega_{t,\kappa,Y}$, $t\in[0,1]$, $-\kappa^0\le\kappa\le\kappa^1$, so that it extends any given paths for $t=0,1$ that are constructed by $\mathcal{C}_0$– (or $\mathcal{C}_1$–) adapted inflation.*

In order to apply Lemma 5.1.4 to prove Theorem 1.2.12 we need first to find suitable classes A along which to inflate, and then construct the families $\mathcal{C}_t$. The following argument that deals with the case $\mathcal{S}=\varnothing$ is adapted from [19]. For simplicity, we explain it only when M is a blow up of $\mathbb{CP}^2$.

Lemma 5.1.5 *Let M be a blow up of $\mathbb{CP}^2$, and suppose given a smooth family of symplectic forms ω_t, $t\in[0,1]$, on M with $[\omega_0]=[\omega_1]$. Then there is a family of symplectic forms ω_{st}, $0\le s,t\le 1$, such that*

$$\begin{aligned} \omega_{s0}&=\omega_0 \quad \textit{and} \quad \omega_{s1}=\omega_1 \quad \textit{for all } s,\\ \omega_{0t}&=\omega_t \quad \textit{and} \quad [\omega_{1t}]=[\omega_0] \quad \textit{for all } t. \end{aligned}$$

Proof Write L, E_j, $j=1,\dots,K$, for the homology classes of the line and the obvious exceptional divisors, and then define $\ell := \mathrm{PD}(L)$, $e_j=\mathrm{PD}(E_j)$ so that $e_j(E_j)=-1$.

Case 1: $[\omega_0]$ is rational. We claim that for sufficiently large integer N the following conditions hold, where $\mathcal{P}^+$ is the positive cone as in Fact 2.1.1:

- $N[\omega_0]\pm e_j\in\mathcal{P}^+$ for all j.
- The class $A_j^{\pm}=\mathrm{PD}(N[\omega_0]\pm e_j)$ is reduced, ie $A_j^{\pm}\cdot E\ge 0$ for all $E\in\mathcal{E}$.

By the openness of the space of symplectic forms, there is an $\varepsilon_0>0$ such that the classes $[\omega_0]+\varepsilon e_j$ have symplectic representatives ω_ε for all $|\varepsilon|<\varepsilon_0$. Taking $N<1/\varepsilon_0$, the first claim obviously holds. Moreover, $N[\omega_0]\pm e_j$ must evaluate positively on each exceptional class $E\in\mathcal{E}(\omega_{\pm 1/N})$. The claim follows by deformation invariance of $\mathcal{E}$.

Next, with $A_j^{\pm}=\mathrm{PD}(N[\omega_0]\pm e_j)$, Corollary 2.1.6 implies that $\mathrm{Gr}(qA_j^{\pm})\neq 0$ for sufficiently large q. It follows that for any deformation σ_t, given a generic 1–parameter family J_t of σ_t–tame almost complex structures, there is (after possible reparametrization with respect to t) a family of embedded connected J_t–holomorphic submanifolds $C^{\pm}_{t,j}$ in class $qA^{\pm}_j$. If we do this for each of the classes $A^+_1, A^-_1, A^+_2, \ldots$ in turn, possibly reparametrizing at each step, we may suppose that there is a family $\mathcal{C}^J_t$ of J_t–holomorphic submanifolds in these classes. We can finally perturb them to get a family $\mathcal{C}^{\mathcal{A}}_t, t \in [0,1]$, composed of ω_t–orthogonally intersecting curves for each t. Observe that the homological intersections $A^{\pm}_j \cdot A^{\pm}_i$ are all nonnegative when $i \neq j$ (as well as for $A^+_i \cdot A^-_i$) because the classes are J–represented; also $(A^{\pm}_i)^2 > 0$ by hypothesis ($A^{\pm}_i \in \mathcal{P}^+$). Hence every class $\sum \lambda_j A^{\pm}_j$, with $\lambda_j \geq 0$, intersects every component of $\mathcal{C}^{\mathcal{A}}_t$ positively for all t, and so can be used for arbitrary positive inflations by Lemma 5.1.4.

The family ω_{st} is constructed in three stages. The first stage for $s \in [0, s_1]$ implements the reparametrization. The second stage is the inflation.

Each class $[\omega_t]$ has a unique decomposition as

$$[\omega_t] = c(t)[\omega_0] + \sum_{j \in \mathcal{I}^+(t)} \lambda_j(t) e_j - \sum_{j \in \mathcal{I}^-(t)} \lambda_j(t) e_j, \quad c(t), \lambda_j(t) > 0,$$

where $\mathcal{I}^+(t), \mathcal{I}^-(t)$ are suitable disjoint subsets of $\{1, \ldots, K\}$ for each t, and the functions $c(t), \lambda_j(t)$ are smooth. Define the class

$$Y_t := \frac{1}{s_2 - s_1}\Bigg(\sum_{j \in \mathcal{I}^+(t)} \lambda_j(t) A^-_j + \sum_{j \in \mathcal{I}^-(t)} \lambda_j(t) A^+_j \Bigg).$$

Note that here we pair $j \in \mathcal{I}^+(t)$ with $A^-_j = \mathrm{PD}(N[\omega_0] - e_j)$. It follows that inflation along Y_t gives a smooth family of symplectic forms ω_{st}, $s \in [s_1, s_2]$, in class

$$\begin{aligned}
[\omega_{st}] &= [\omega_t] + \frac{s - s_1}{s_2 - s_1}\Bigg(\sum_{\mathcal{I}^+(t)} \lambda_j(t)\, \mathrm{PD}(A^-_j) + \sum_{\mathcal{I}^-(t)} \lambda_j(t)\, \mathrm{PD}(A^+_j) \Bigg) \\
&= \Big(1 + N \frac{s - s_1}{s_2 - s_1} \sum \lambda_j(t)\Big)[\omega_0] + \sum_{\mathcal{I}^+(t)} \Big(1 - \frac{s - s_1}{s_2 - s_1}\Big)\lambda_j(t) e_j \\
&\qquad - \sum_{\mathcal{I}^-(t)} \Big(1 - \frac{s - s_1}{s_2 - s_1}\Big)\lambda_j(t) e_j.
\end{aligned}$$

For $s = s_2$, the classes $[\omega_{s_2 t}]$ are proportional to $[\omega_0]$. The third stage consists of a rescaling, which gives $[\omega_{1t}] = [\omega_0]$. Observe that $Y_0 = Y_1 = 0$ so that this whole process does not modify ω_0 and ω_1.

Case 2: $[\omega_0]$ is irrational In this simple situation ($\mathcal{S} = \varnothing$), it is well-known that the "deformation to isotopy" statement is equivalent to the claim that the space of symplectic embeddings of disjoint closed balls of a fixed size into $\mathbb{CP}^2$ is path connected. But if this holds for balls of rational size, it is obviously also true for balls of irrational size since we can always extend an embedding of irrational balls to slightly larger rational balls, isotop this as required, and then restrict the isotopy to the original balls. We now give the formal proof that keeps track of this argument, because it will adapt to the situation $\mathcal{S} \neq \varnothing$.

Rescale so that $\omega_0(L) = 1$ and write $[\omega_0] = \mathrm{PD}(L) - \sum_j \lambda_j e_j$, where $\lambda_j > 0$.[11] For $t = 0, 1$ choose a generic ω_t–tame almost complex structure J_t. Then there are disjoint embedded J_t–holomorphic curves $C_t^{E_j}$ in the classes E_j, $1 \le j \le K$, for $t = 0, 1$. Choose $\kappa^0 > 0$ so that we can negatively inflate along these curves for $t = 0, 1$ and for $-\kappa^0 \le \kappa \le 0$, and then choose rational numbers $\mu_j = \lambda_j + \delta_j$ with $\delta_j < \kappa^0$. Then, by negatively inflating along the curves $C_0^{E_j}, C_1^{E_j}$, construct families of forms ω_t for $t \in [-1, 0]$ and $t \in [1, 2]$ so that $[\omega_{-1}] = [\omega_2]$ is rational:

$$[\omega_{-1}] = [\omega_2] = \mathrm{PD}(L) - \sum_j \mu_j e_j, \quad \mu_j \in \mathbb{Q}.$$

Because the endpoints of the path $\omega_t, -1 \le t \le 2$ are now equal and rational, as before we may homotop this deformation to an isotopy ρ_t, $t \in [-1, 2]$, with $\rho_t = \omega_t$ at $t = -1, 2$. Note that the set of classes $A_j^{\pm} = N[\omega_{-1}] \pm e_j$ along which we must now inflate depends on $[\omega_{-1}]$. Hence the family $\mathcal{C}_t^{\mathcal{A}}$, $t \in [-1, 2]$, does as well. Further, because the classes $E_j = \mathrm{PD}(e_j) \in \mathcal{E}$ are represented by unique J_t–holomorphic embedded spheres for all generic 1–parameter path J_t, we may simply add representatives of the classes E_j to the family $\mathcal{C}_t^J$, and then straighten out the components of the curves in $\mathcal{C}_t^J$ using Lemma 3.2.1 to obtain a family $(\mathcal{C}_t^{\mathcal{A}})'$, $t \in [-1, 2]$, of curves with pairwise orthogonal intersections, that contains embedded representatives $C_t^{E_j}$ of each class E_j as well as the components of $\mathcal{C}_t^{\mathcal{A}}$. Moreover, we can suppose at $t = -1, 2$ that these curves equal the previously chosen ones at $t = 0, 1$ respectively. Then by Lemma 5.1.4, the isotopy ρ_t, $t \in [-1, 2]$, consists of forms that are nondegenerate on the $C_t^{E_j}$.

More precisely, in the three stages defined above, we get for some $0 < s_3 < 1$ a 2–dimensional family ω_{st} of symplectic forms, $t \in [-1, 2]$, $s \in [0, s_3]$ that homotops ω_t (for $s = 0$) to $\rho_t = \omega_{s_3 t}$, where $[\omega_{s_3 t}] \equiv [\omega_{-1}] = [\omega_0] - \sum \delta_j e_j$. By construction, the curves $C_t^{E_j}$ are $\omega_{s_3 t}$–symplectic, with area larger than δ_j, so the last stage consists in performing a positive inflation of size δ_j along them, and a reparametrization in t, in order to straighten $\omega_{s_3 t}$, $t \in [-1, 2]$, to ω_{1t}, $t \in [0, 1]$, in class $[\omega_0]$. Note that at the endpoints this last step reverses the original negative inflation of ω_0 to ω_{-1} and

[11] This is the only step in the argument that fails when M is ruled. In this case, one should replace L by the class of some section of the ruling that has nontrivial Gromov invariant, and add the class of the fiber F (which is always represented) to the exceptional classes.

ω_1 to ω_2. Therefore the final isotopy ω_{1t}, $t \in [0,1]$, starts at ω_0 and ends at ω_1, as required. □

In order to carry out this proof in the case of isotopies relative to $\mathcal{S}$, one needs to find suitable representatives of all the classes involved in the above proof, the $A_j^{\pm}$ when $[\omega_0]$ is rational, and also suitable substitutes for the E_j in the general case. In order to deal with the latter we will need to work relative to a smooth $\mathcal{S}$–adapted family that for each t contains representatives of the classes corresponding to the E_j. Here is the main result about the existence of such representatives. Note also that the condition on $d(A_j)$ comes from Lemma 2.1.5, and is needed to ensure some Gromov invariant does not vanish.

Proposition 5.1.6 *Let ω_t, $t \in [0,1]$, be a path of symplectic forms as in Theorem 1.2.12 and $\mathcal{C}_t$ be a smooth $(\mathcal{S},\omega_t)$–adapted family of surfaces in the classes $T_1,\dots,T_L$. Suppose given a finite set $\mathcal{A} = \{A_1,\dots,A_K\}$ of $\mathcal{S}$–good classes such that:*

- *$A_i \cdot T_j \geq 0$ for all $1 \leq j \leq L$.*
- *$d(A_j) > 0$ for all j. Moreover, $d(A_j) \geq g + \frac{1}{4}k$, if M is the k–point blow up of a ruled surface of genus g.*

Choose a smooth path $J_t \in \mathcal{J}(\mathcal{S},\omega_t,\mathcal{A})$, $t \in [0,1]$ of $(\mathcal{C}_t,\omega_t)$–adapted almost complex structures. Then, possibly after reparametrization with respect to t, the path $(J_t)_{t\in[0,1]}$ can be perturbed to a smooth $(\mathcal{C}_t,\omega_t)$–adapted path $(J'_t \in \mathcal{J}_{semi}(\mathcal{S},\omega_t,\mathcal{A}))_{t\in[0,1]}$ such that for each $1 \leq j \leq K$ there is a smooth family $\Sigma_t^{A_j}$, $t \in [0,1]$, of J'_t–holomorphic and $(\mathcal{S}\cup\mathcal{C}_t,\omega_t)$–adapted nodal curves in class A_j. Moreover the corresponding decompositions

$$A_j = \sum \ell_{ji} S_i + \sum m_{ji} E_{ji} + B_j, \quad E_{ji}^2 = -1,$$

of (3.1.2) *have the property that* $\mathrm{Gr}(B_j) \neq 0$ *for all j.*

Proof The proof when there is only one class A and when $\mathcal{C}_t = \mathcal{S}$ is essentially the same as that of Proposition 3.2.3. The argument works just as well if $\mathcal{C}_t$ is strictly larger than $\mathcal{S}$. Since by hypothesis $d(B) \geq d(A) > 0$, we can always choose the set of $k := \frac{1}{2}d(B)$ points so that at least one does not lie in the three-dimensional set $\bigcup_t \mathcal{C}_t$. Hence we are free to perturb J'_t near some point on the B–curve which means that the genericity arguments work as before.

Finally, if $N > 1$ we argue by induction on N. Note that at each stage we may have to reparametrize. Further, to finish the i^{th} stage we should apply the straightening argument in Lemma 3.2.1 to make the components of the A_i–nodal curve orthogonal to $\mathcal{C}_t$ and all components for the previously constructed nodal curves $\Sigma_t^{A_j}$, $j < i$. Then at the $(i+1)^{\text{st}}$ stage, we repeat the argument with this enlarged family $\mathcal{C}'_t$. □

Proof of Theorem 1.2.12 Recall the statement: M is a blow-up of $\mathbb{CP}^2$ or a ruled surface, we have a family of symplectic forms ω_t, $t\in[0,1]$ with $[\omega_0]=[\omega_1]$ and, as in the previous lemma, we want to find a homotopy of symplectic forms ω_{st} between ω_t (for $s=0$) and an isotopy ω_{1t} (meaning that $[\omega_{1t}]$ is constant) with fixed ends: $\omega_{s0}=\omega_0$, $\omega_{s1}=\omega_1$ for all s. This time, the situation is *relative to* $\mathcal{S}$, meaning that we assume that the forms ω_t are nondegenerate on $\mathcal{S}$, and we want our homotopy ω_{st} to have the same property.

Case 1: $[\omega]$ is rational In order to adapt the proof of Lemma 5.1.5 we first choose an analog of the basis L, E_j for $H_2(M)$. We take L to be any class with nonzero Gromov invariant, so that $\omega_t(L)>0$ for all t and then choose integral classes $d_1,\dots,d_K$ that together with $\mathrm{PD}(L)$ form a basis of $H^2(M;\mathbb{Q})$. Define the classes $A_j^{\pm}:=\mathrm{PD}(N\omega_0\pm d_j)$ as before, using the openness of the space of symplectic forms to find a suitable value of N for which these classes are all $\mathcal{S}$–good and also satisfy the enhanced condition on $d(A_j)$ when M is ruled. This is possible by Corollary 2.1.6. We then use Proposition 5.1.6 with $\mathcal{C}_t=\mathcal{S}$ and $\mathcal{A}=\{A_j^{\pm}:1\le j\le K\}$ to get a smooth family of $(\mathcal{S},\omega_t)$–adapted nodal curves $\Sigma_{t,j}^{\pm}$ in classes $A_j^{\pm}$. Straighten out their components using Lemma 3.2.1 to obtain an $(\mathcal{S},\omega_t)$–adapted family $\mathcal{C}_t^{\mathcal{A}}$ that contains $\mathcal{S}$.

We next claim that each class $A_j^{\pm}$ has nonnegative intersection with the classes of the components of $\mathcal{C}_t^{\mathcal{A}}$. To see this, consider the decomposition

$$A_j^{\pm}=\sum \ell_{ji}^{\pm}\,S_i+\sum m_{ji}^{\pm}\,E_{ji}+B_j^{\pm},\quad E_{ji}^2=-1,\ \mathrm{Gr}(B_j^{\pm})\neq 0$$

associated via Proposition 5.1.6 to the nodal curves $\Sigma_{t,j}^{\pm}$. We chose the classes $A_j^{\pm}$ to be $\mathcal{S}$–good. Therefore they have nonnegative intersection with the components of $\mathcal{S}$ as well as all exceptional classes $E_j^{\pm}$. Further they have nonnegative intersection with the $B_j^{\pm}$ because both the $A_j^{\pm}$ and $B_k^{\pm}$ have nontrivial Gromov invariant and hence are represented by embedded curves for generic J. Hence Lemma 5.1.4 allows inflation along any nonnegative linear combination of the $A_j^{\pm}$, and these inflations provide symplectic forms which are nondegenerate on $\mathcal{S}$.

The family ω_{st} is then constructed in the same three stages as in the previous proof: reparametrization, inflation along the classes

$$Y_t=\sum_{\mathcal{I}^+(t)}\lambda_j(t)A_j^-+\sum_{\mathcal{I}^-(t)}\lambda_j(t)A_j^+$$

(where $[\omega_t]=c(t)[\omega_0]+\sum_{\mathcal{I}^+}\lambda_j(t)d_j-\sum_{\mathcal{I}^-}\lambda_j(t)d_j$), and rescaling. The result at $s=1$ is an isotopy ω_{1t}, $t\in[0,1]$, consisting of symplectic forms that restrict on

$\mathcal{S}$ to a possibly varying family of forms that are $\mathcal{S}$–adapted and all lie in the same cohomology class.[12]

Finally, if $\omega = \omega'$ near $\mathcal{S}$ then $\omega_{10} = \omega_{11} = \omega$ near $\mathcal{S}$ by construction, and we can arrange that the final isotopy ω_{1t} is constant near $\mathcal{S}$ by an easy application of a Moser's type argument. Details are left to the reader.

Case 2: $[\omega]$ is irrational When $[\omega]$ is irrational, we reduce to the rational case by first doing a small "negative inflation" along suitable classes, $F_1, \dots, F_K$, where $K = k+1$ or $k+2$ depending on whether M is a k–fold blow-up of $\mathbb{CP}^2$ or of a ruled surface. These classes are obtained as follows. Choose integral classes $a_1, \dots, a_K$ that are multiples of classes close to $[\omega] = [\omega']$, so that

$$[\omega] = \sum_{i=1}^{K} \mu_i a_i$$

for some $\mu_i \in \mathbb{R}^+$. By the openness of the space of symplectic forms, we may assume that the classes a_i have symplectic representatives and take positive values on the components S_i of $\mathcal{S}$. Then the classes $F_i := \mathrm{PD}(a_i)$ satisfy all the conditions needed to be $\mathcal{S}$–good except that $\mathrm{Gr}(F_i)$ could vanish. Therefore, by replacing the a_i by suitable multiples as in Corollary 2.1.6, we can assume that each $F_i := \mathrm{PD}(a_i)$ is $\mathcal{S}$–good, and, if relevant, has $d(F_i) \geq g + \frac{1}{4}k$ as in Proposition 5.1.6. By applying this proposition with $\mathcal{C}_t = \mathcal{S}$, we can find a smooth path $(J'_t \in \mathcal{J}_{semi}(\mathcal{S}, \omega_t, \mathcal{A}))_{t\in[0,1]}$ such that for each $1 \leq i \leq K$ there is a smooth family $\Sigma_t^{F_i}$, $t \in [0,1]$, of J'_t–holomorphic and $(\mathcal{S}, \omega_t)$–adapted nodal curves in class F_i. Straighten out their components using Lemma 3.2.1 to obtain an $(\mathcal{S}, \omega_t)$–adapted family $\mathcal{C}_t^{\mathcal{F}}$.

As in the proof of Case 1, each class F_i has nonnegative intersection with the classes of the components of $\mathcal{C}_t^{\mathcal{F}}$. Hence Proposition 5.1.2 allows negative $\mathcal{C}_0^{\mathcal{F}}$ (resp. $\mathcal{C}_1^{\mathcal{F}}$)–adapted inflation along any class $Y_\mu := \sum \mu_i F_i$, $\mu_i \in [0,1]$, by $-\kappa$ for all κ less than some κ_0 (recall that κ_0 depends only on $\mu_{\max}, \omega, \mathcal{F}$, but not on the class Y itself). Stated differently, $\mathcal{C}_t^{\mathcal{F}}$–adapted negative inflation along classes $\sum \mu_i F_i$, $\mu_i \in [0, \kappa_0]$ are possible for all $\kappa < 1$.

Now choose small constants $\delta_i \in [0, \kappa^0[$ so that

$$[\omega]_\delta = \sum_{i=1}^{K} (\mu_i - \delta_i) a_i,$$

[12] Note that we cannot invoke part (iii) of Proposition 5.1.2 to claim that the forms are constant on $\mathcal{S}$ throughout the deformation because some of the classes Y_t might have nontrivial intersection with $\mathcal{S}$.

is rational. Define ω_t, $t\in[-1,0]$, (resp. $t\in[1,2]$) to be the family of forms obtained from $\omega=\omega_0$ (resp. $\omega'=\omega_1$) by negative $\mathcal{S}\cup\mathcal{C}_0^{\mathcal{F}}$ (resp. $\mathcal{S}\cup\mathcal{C}_1^{\mathcal{F}}$) adapted inflation in class $Y_F:=\sum\delta_i F_i$.

Then $[\omega_{-1}]=[\omega_2]=[\omega]_\delta$ is rational. Hence we may apply the argument of Case 1 to the extended deformation ω_t, $t\in[-1,2]$, that has rational and cohomologous endpoints. The only new point is that we construct the nodal curves $\Sigma^{\pm}_{t,j}$ in classes $A^{\pm}_j$ to be $\mathcal{S}\cup\mathcal{C}_t^{\mathcal{F}}$–adapted rather than $\mathcal{S}$–adapted. This means that, in the notation of the proof of Lemma 5.1.5, the isotopy ρ_t, $t\in[-1,2]$, from ω_{-1} to ω_2 consists of forms that are $\mathcal{S}\cup\mathcal{C}_t^{\mathcal{F}}$–adapted. Hence this isotopy can be positively inflated by a $\mathcal{C}_t^{\mathcal{F}}$–adapted inflation in class $Y_F:=\sum\delta_i F_i$ to an isotopy that joins the original form ω to ω'. This completes the proof. □

5.2 Proof of technical results

It remains to prove Propositions 5.1.2 and Lemma 5.1.4. These use entirely soft methods.

Before embarking on the details of the proof of Proposition 5.1.2, we recall the basic inflation process; see [3; 14; 19]. Given a symplectically embedded surface C with $C\cdot C=n\in\mathbb{Z}$ we normalize ω in some neighborhood $\mathcal{N}$ of C as follows. If r is a radial coordinate in the bundle $\pi:\mathcal{N}\to C$, where $\mathcal{N}=\{r^2/2<\varepsilon\}$, we write

$$\omega=\pi^*(\omega|_C)+\tfrac12 d(r^2\alpha),\quad r\in[0,\sqrt{2\varepsilon}),$$

where α is a connection 1–form with

$$d\alpha=-\frac{n}{\omega(C)}\pi^*(\omega|_C).$$

We then choose a nonincreasing compactly supported function $f:[0,\varepsilon)\to[0,1]$ that is 1 near $s=0$, and define

$$\rho:=-d\Big(f\Big(\frac{r^2}{2}\Big)\alpha\Big).$$

Consider the family of forms

$$\begin{aligned}(5.2.1)\quad \omega+\kappa\rho:=\pi^*(\omega|_C)&+\tfrac12 d(r^2\alpha)-\kappa\, d\Big(f\Big(\frac{r^2}{2}\Big)\alpha\Big)\\ &=\Big(1+\frac{n}{\omega(C)}\Big(\kappa f\Big(\frac{r^2}{2}\Big)-\frac{r^2}{2}\Big)\Big)\pi^*(\omega|_C)+\big(1+\kappa|f'|\big)r\,dr\wedge\alpha.\end{aligned}$$

By construction, this form lies in the class $[\omega]+\kappa\,\mathrm{PD}(C)$. If $n\ge0$ it is nondegenerate for all $\kappa\ge0$, and is also nondegenerate in some interval $-\kappa^0\le\kappa<0$, where the bounds on κ^0 come from both terms: in particular, because we need $1+\kappa|f'|>0$ the bound κ^0 depends on the size of ε and hence of the neighborhood $\mathcal{N}$. If $n<0$ the first term also presents a significant obstruction, and we can only inflate for $\kappa<\kappa^1$ where $|n|\kappa^1<\omega(C)$.

In the situation of Lemma 5.1.4, we assume that $\mathcal{C}_t$, $t\in[0,1]$, is a smooth family of symplectic submanifolds satisfying Condition 5.1.1 with respect to the forms ω_t, with an associated family of local fibered structures $\mathcal{F}_t$ on a neighborhood $\mathcal{N}(\mathcal{C}_t)$ as described just after Definition 1.2.1. In particular, each intersection point q_t of $C_t^{T_i}$ with $C_t^{T_j}$ has a neighborhood $\mathcal{N}_{q_t}$, which is a connected component of $\mathcal{N}(C_t^{T_i})\cap\mathcal{N}(C_t^{T_j})$ with product structure given by the projections to $C_t^{T_i}$ and $C_t^{T_j}$. We fix corresponding polar coordinates $r_{t,i},\theta_{t,i},r_{t,j},\theta_{t,j}$ in the fibers of $\mathcal{L}_{t,i}$ and $\mathcal{L}_{t,j}$ that vary smoothly with t. We assume that these neighborhoods $\mathcal{N}_{q_t}$ have disjoint closures for $q_t\in\bigcup_{i\neq j}(C_t^{T_i}\cap C_t^{T_j})$, and then extend each radial function $r_{t,i}$ smoothly over $\mathcal{N}(C_t^{T_i})$. (This amounts to choosing a restriction of the structural group of $\mathcal{L}_{t,i}$ to S^1.) We assume that for suitable constants $\varepsilon_i>0$

$$\mathcal{N}(C_t^{T_i})=\{x\in\mathcal{L}_i : r_{t,i}(x)\le\sqrt{2\varepsilon_i}\}. \tag{5.2.2}$$

We also define

$$\omega_t^{T_i}:=\omega_t\big|_{C_t^{T_i}}. \tag{5.2.3}$$

Finally, we shrink the neighborhoods as necessary so that for negative $C_t^{T_i}$ we have

$$\sum_{q_t\in C_t^{T_i}\cap C_t^{T_j},\, j\neq i}\int_{C_t^{T_i}\cap\mathcal{N}_{q_t}}\omega_t^{T_i}\le\frac{1}{2}\int_{C_t^{T_i}}\omega_t^{T_i}. \tag{5.2.4}$$

Lemma 5.2.1 *For each $C_t^{T_i}\in\mathcal{C}_t$ there are constants $\kappa_i^0,\kappa_i^1>0$ and a family of forms $\rho_{t,i}$ with the following properties:*

- $[\rho_{t,i}]=\mathrm{PD}(T_i)$.
- *$\rho_{t,i}$ is supported in the fibered neighborhood $\mathcal{N}(C_t^{T_i})$ for each t,i.*
- *$\rho_{t,i}$ is compatible with the product structure of the form $df_{t,i}(\frac{1}{2}r_{t,i}^2)\wedge d\theta_{t,i}$ on the product neighborhoods $\mathcal{N}_p$ of each $p\in C_t^{T_i}\cap(\mathcal{C}\smallsetminus C_t^{T_i})$.*
- *$\omega_t+\kappa\rho_{t,i}$ is symplectic and $\mathcal{S}\cup\mathcal{C}_t$–adapted for $-\kappa_i^0\le\kappa<\kappa_i^1$ and all t.*

Moreover, κ_i^1 can be arbitrarily large if $n_i:=T_i\cdot T_i\ge 0$ and otherwise depends only on cohomological data, namely n_i and $\omega_t(T_i):=\int_{C_t^{T_i}}\omega_{t,i}^{T}$. Moreover, it is an increasing function of $\omega_t(T_i)$.

Proof Step 1 We may assume that there are connection 1–forms $\alpha_{t,i}$ on the bundles $\pi_{t,i}:\mathcal{L}_{t,i}\to C_t^{T_i}$ such that

$$\omega_t\big|_{\mathcal{N}(C_t^{T_i})}=\pi_{t,i}^*(\omega_t^{T_i})+\tfrac12 d(r_{t,i}^2\alpha_{t,i}),\quad 1\le i\le L, \tag{5.2.5}$$

where $r_{t,i}$ is the radial coordinate in the fiber of $\mathcal{L}_{t,i}$ described above.

After possibly shrinking the neighborhoods $\mathcal{N}(C_t^{T_i})$, this can be achieved by a standard Moser type argument. □

Next, denote by $g_{t,i}:C_t^{T_i}\to\mathbb{R}$ the curvature function of $\alpha_{t,i}$; thus

$$d\alpha_{t,i}=-n_i\pi_i^*(g_{t,i}\omega_{t,i}^T),\quad 1\le i\le N. \tag{5.2.6}$$

Note that $g_{t,i}=0$ in each product neighborhood because ω_t is a product there.

Step 2 We may assume that $g_{t,i}(x)\ge 0$ for all $x\in C^{T_i}$, and satisfy the following pointwise upper bound on the negative curves (those with $n_i<0$):

$$g_{t,i}(x)\le\frac{2}{\omega_t(T_i)}. \tag{5.2.7}$$

Again this follows by a standard Moser argument. Note that to achieve this bound we must use condition (5.2.4) because $\int_{C_t^{T_i}} d\alpha_{t,i}=-n_i$ is fixed, while $g_{t,i}=0$ in each product neighborhood. □

Step 3: Completion of the proof Choose a family of smooth compactly supported functions $f_{t,i}:[0,\sqrt{2\varepsilon_i})\to\mathbb{R}$ (where ε_i is as in (5.2.2)) that equal 1 near $r=0$. With $\rho_{i,t}:=d(f_{t,i}(\frac12 r^2)\,\alpha_{t,i})$, we have

$$\begin{aligned}\omega_t+\kappa\rho_{i,t}&=\pi_i^*(\omega_{t,i}^T)+\tfrac12 d(r^2\alpha_{t,i})-\kappa d(f_{t,i}(\tfrac12 r^2)\alpha_{t,i})\\&=\big(1+n_i\pi_i^*(g_{t,i})\big(\kappa f_{t,i}(\tfrac12 r^2)-\tfrac12 r^2\big)\big)\,\pi_i^*(\omega_{t,i}^T)+\big(1+\kappa|f_{t,i}'|\big)r\,dr\wedge\alpha_{t,i}.\end{aligned}$$

As before, when $n_i\ge 0$ these forms are nondegenerate for all $\kappa\ge 0$ and for $\kappa>-\kappa_i^0$, where κ_i^0 depends only on the size ε_i of $\mathcal{N}_{t,i}$. When $n_i<0$ we have similar limits for κ^0, but now must only consider $\kappa<\kappa_i^1$, where the size of κ_i^1 is determined by the requirement that the form $\omega_t+\kappa\rho_{i,t}$ restrict positively to $C_t^{T_i}=\{r=0\}$. Since $f_{t,i}(0)=1$ and $g_{t,i}$ satisfies (5.2.7) this depends only on n_i and $\omega_t(T_i)$. The other properties of these forms are clear. □

Proof of Proposition 5.1.2 This proposition states the following.

Let $Y:=\sum_{i=1}^L\lambda_iT_i$, where $\lambda_i\ge 0$. Then there are constants $\kappa^0,\kappa^1>0$, depending on Y and $\mathcal{C}$, and a smooth family of symplectic forms $\omega_{\kappa,Y}$ on M such that the following holds for all $\kappa\in[-\kappa^0,\kappa^1]$:

(i) $[\omega_{\kappa,Y}] = [\omega_0] + \kappa\,\mathrm{PD}(Y)$, *where* $\mathrm{PD}(Y)$ *denotes the Poincaré dual of* Y.

(ii) $\omega_{\kappa,Y}$ *is* $\mathcal{S}\cup\mathcal{C}$*–adapted.*

(iii) *If* $Y\cdot T_j = 0$ *for some* j *the restrictions of* $\omega_{\kappa,Y}$ *and* ω *to a neighborhood of* C^{T_j} *are equal.*

(iv) *The constant* κ^0 *depends on geometric information, namely* $\omega, \mathcal{C}$ *and* $\lambda_{\max}$, *while* κ^1 *depends only on* $[\omega]$ *and the homology classes* T_i, Y. *Moreover, if* $Y\cdot T_i \ge 0$ *for all* i *then* κ^1 *can be arbitrarily large.*

We use the notation of Lemma 5.2.1 omitting t since for the moment we are considering single forms. First consider the forms

$$\omega'_{\kappa,Y} := \omega + \sum_{i=1}^{L} \lambda_i \kappa \rho_i .$$

Because the supports of two forms ρ_i, ρ_j, $i \ne j$, intersect only in the neighborhoods $\mathcal{N}_p$ in which the ρ_i are products, the form $\omega'_{\kappa,Y}$ is nondegenerate provided that each form $\omega + \lambda_i\kappa\rho_i$ is nondegenerate. Therefore, if these forms are nondegenerate for $-\kappa_i^0 \le \kappa \le \kappa_i^1$, we may take the lower bound $-\kappa^0$ to be $\max_i -\kappa_i^0/\lambda_i$ and the upper bound to be $\varepsilon^1 := \min_i \kappa_i^1/\lambda_i$. This form satisfies (i) and (ii). Also, as explained in Step 3 of the proof of Lemma 5.2.1 the bound on κ^0 depends on the size of the neighborhoods $\mathcal{N}_i$ of the curves C^{T_i}, and hence on geometric information about $\mathcal{C}$ and ω.

If $Y\cdot T_i \ge 0$, for each i the quantity $\omega'_{\kappa,Y}(T_i)$ is a nondecreasing function of κ. But notice that as κ increases the area of C^{T_i} is redistributed so that (5.2.4) eventually ceases to hold. Thus when $\kappa = \varepsilon^1$ we isotop the form $\omega^1 := \omega'_{\varepsilon^1,Y}$ near $\mathcal{C}$ pushing area out of the product neighborhoods to make (5.2.4) valid again. We saw in Lemma 5.2.1 that the upper limit κ_i^1 for positive inflation by ρ_i increases as $\omega'_{\kappa,Y}(T_i)$ increases. Therefore, we may now repeat this process, starting with ω^1 and inflating by adding a suitable form in class $\kappa PD(Y)$ for $\kappa \in [0,\varepsilon^2]$, where $\varepsilon^2 \ge \varepsilon^1$. After a finite number of such steps, we arrive at a form in class $[\omega_0] + \kappa\,\mathrm{PD}(Y)$ for arbitrarily large κ. If $Y\cdot T_i < 0$, for some i, then $\omega'_{\kappa,Y}(T_i)$ decreases and it follows from Lemma 5.2.1 that the bound on κ^1 depends on cohomological data, namely $Y\cdot T_i$ and $\omega(T_i)$.

This gives a family of forms $\omega_{\kappa,Y}$ that satisfies (ii) and (iv), and nearly satisfies (i): The problem here is that we paused the inflation at $\kappa = \varepsilon^1, \varepsilon^1+\varepsilon^2$ and so on, while we readjusted the area distribution. However, one can easily combine these two deformations and then reparametrize with respect to κ so as to satisfy (i). Finally, note that when $Y\cdot T_j = 0$ the total area of the curve C^{T_j} is constant throughout the isotopy, although the distribution of area changes with κ. Hence to achieve (iii) we alter the isotopy near each such component C^{T_j} so that it is constant near that component. Again this is a standard Moser type argument: one should begin by adjusting the forms near each intersection point $C^{T_i} \cap C^{T_j}$, keeping the product structure, and then adjust near the rest of $\mathcal{C}$. □

Proof of Lemma 5.1.4 The proof of part (i) is similar, and will be left to the reader. It uses the full force of Lemma 5.2.1. Moreover part (ii) holds because at each step of the construction in Lemma 5.2.1 the set of possible choices (for example, of the size of the neighborhoods $\mathcal{N}(C^{T_i})$ or of the precise normal form chosen for ω_t as in (5.2.5)) is contractible. Further, if one constructs two paths ω_{κ^s}, $s = 0, 1$, using the same fibered structure (choice of projections π_i, radial coordinates r, and neighborhoods $\mathcal{N}(C_t^{T_i})$) then the linear isotopy

$$(1-s)\omega_{\kappa,0} + s\omega_{\kappa,1}, \quad 0 \le s \le 1,$$

between them consists of nondegenerate forms. □

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