

A classifying space for commutativity in Lie groups

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In this article we consider a space $B_{\text{com}}G$ assembled from commuting elements in a Lie group G first defined by Adem, Cohen and Torres-Giese. We describe homotopy-theoretic properties of these spaces using homotopy colimits, and their role as a classifying space for *transitionally commutative* bundles. We prove that $\mathbb{Z} \times B_{\text{com}}U$ is a loop space and define a notion of commutative K-theory for bundles over a finite complex X , which is isomorphic to $[X, \mathbb{Z} \times B_{\text{com}}U]$. We compute the rational cohomology of $B_{\text{com}}G$ for G equal to any of the classical groups $SU(r)$, $U(q)$ and $Sp(k)$, and exhibit the rational cohomologies of $B_{\text{com}}U$, $B_{\text{com}}SU$ and $B_{\text{com}}Sp$ as explicit polynomial rings.

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1 Introduction

Let G denote a topological group and consider the spaces $\{\text{Hom}(\mathbb{Z}^n, G)\}_{n \geq 0}$ of ordered commuting n -tuples in G . Adem, Cohen and Torres-Giese [3] showed that they can be assembled into a simplicial space where the resulting geometric realization, denoted here by $B_{\text{com}}G$, is the first term in an increasing filtration of BG . The universal bundle over BG pulls back to a principal bundle over $B_{\text{com}}G$ with total space $E_{\text{com}}G$ that can also be described simplicially (see Section 2). In this paper we analyze the properties of $E_{\text{com}}G$ and $B_{\text{com}}G$ when G is a Lie group. We also study variants of our constructions, denoted $E_{\text{com}}G_{\mathbf{1}}$ and $B_{\text{com}}G_{\mathbf{1}}$, which arise from the components of the identity in the commuting varieties (see Section 3 for details). These two constructions agree if the spaces $\text{Hom}(\mathbb{Z}^n, G)$ are path-connected for every $n \geq 0$. It can be shown that for a compact Lie group G , this condition is equivalent to the property that the maximal abelian subgroups of G are precisely the maximal tori (see the proof of Adem and Gómez [4, Proposition 2.5]). For example, this condition holds for the classical groups $SU(r)$, $U(q)$ and $Sp(k)$ (see Borel [9, Theorem 5.2]), and therefore for any of their finite cartesian products.

We start by applying the recent work of Pettet and Souto [23] to reduce matters to compact Lie groups:

Theorem 3.1 *If G is a real or complex reductive algebraic group with maximal compact subgroup K , then the inclusion map $K \subset G$ induces homotopy equivalences $B_{\text{com}}K \simeq B_{\text{com}}G$ and $E_{\text{com}}K \simeq E_{\text{com}}G$.*

A similar statement is true for the variants $E_{\text{com}}G_{\mathbf{1}}$ and $B_{\text{com}}G_{\mathbf{1}}$ (see Section 3 for details). Based on this we can focus on the case of a compact connected Lie group G . The connected component of the identity in the commuting variety $\text{Hom}(\mathbb{Z}^n, G)$ has the key feature that any n -tuple in it can be conjugated into a maximal torus in G . Using this we obtain a natural identification $B_{\text{com}}G_{\mathbf{1}} \cong \text{colim}_{S \in \mathcal{T}(G)} BS$, where $\mathcal{T}(G)$ is the topological poset formed by the maximal tori and their intersections under inclusion (see Definition 5.3). We describe the homotopy of these spaces using more tractable homotopy colimits defined over a discrete category. Let $Z = Z(G)$ be the center of G and write $n = \text{rank}(G) - \text{rank}(Z) \geq 0$. Consider the poset $\mathcal{S}(n)$ consisting of all the nonempty subsets of $\{0, 1, \dots, n\}$, with the order given by the reverse inclusion of sets. For each G we have functors $\mathcal{F}_G, \mathcal{H}_G: \mathcal{S}(n) \rightarrow \text{Top}$ such that the following holds (see Section 6 for the definitions of \mathcal{F}_G and \mathcal{H}_G).

Theorem 6.3 *Suppose that G is a compact, connected Lie group. Then there is a natural homotopy equivalence $\text{hocolim}_{\mathbf{i} \in \mathcal{S}(n)} \mathcal{F}_G(\mathbf{i}) \simeq B_{\text{com}}G_{\mathbf{1}}$.*

Theorem 6.5 *Suppose that G is a compact connected Lie group. Then there is a natural G -equivariant homotopy equivalence $\text{hocolim}_{\mathbf{i} \in \mathcal{S}(n)} \mathcal{H}_G(\mathbf{i}) \simeq E_{\text{com}}G_{\mathbf{1}}$.*

In terms of bundle theory, we prove that $B_{\text{com}}G$ is a classifying space for bundles that are *transitionally commutative*.

Theorem 2.2 *Suppose that G is a Lie group and let $f: X \rightarrow BG$ denote the classifying map of a principal G -bundle $q: E \rightarrow X$ over the finite CW-complex X . Then up to homotopy, f factors through $B_{\text{com}}G$ if and only if there is an open cover of X on which the bundle is trivial over each open set and such that on intersections the transition functions commute when they are simultaneously defined.*

From this we can define the notion of equivalence between transitionally commutative vector bundles and thus define *commutative K-theory* $K_{\text{com}}(X)$ in a manner analogous to ordinary complex K-theory. Let $U = \text{colim}_{n \rightarrow \infty} U(n)$; then we can establish that $B_{\text{com}}U$ plays a role similar to BU :

Theorem 4.1 *The space $\mathbb{Z} \times B_{\text{com}}U$ is a loop space and for any finite CW-complex X there is a natural isomorphism of groups $K_{\text{com}}(X) \cong [X, \mathbb{Z} \times B_{\text{com}}U]$.*

Adem, Gómez, Lind and Tillmann [5] proved that $\mathbb{Z} \times B_{\text{com}}U$ is in fact an infinite loop space so that commutative K–theory forms part of a generalized cohomology theory. Having established the role played by $B_{\text{com}}G$ in bundle theory, it seems natural to compute its cohomology. As can be seen from the homotopy colimit model and computations for $SU(2)$ (see Example 6.4), we expect these spaces to have rather intricate torsion. Here we focus on calculations for the rational cohomology (inverting the order of the Weyl group W would suffice).

Theorem 7.2 *Suppose that G is a compact, connected Lie group. Then*

$$H^*(B_{\text{com}}G_{\mathbf{1}}; \mathbb{Q})$$

is a free module over $H^(BG; \mathbb{Q})$ of rank $|W|$, where W is the corresponding Weyl group.*

On the other hand, by [3, Theorem 6.1] we have a natural isomorphism

$$H^*(B_{\text{com}}G_{\mathbf{1}}; \mathbb{Q}) \cong (H^*(G/T; \mathbb{Q}) \otimes H^*(BT; \mathbb{Q}))^W,$$

where the Weyl group W acts diagonally on the tensor product. As a corollary we deduce the following algebra isomorphism:

Corollary 7.4 *Suppose that G is a connected compact Lie group with maximal torus T and associated Weyl group W . Then there is a natural isomorphism of rings*

$$H^*(E_{\text{com}}G_{\mathbf{1}}) \cong (H^*(G/T) \otimes H^*(G/T))^W,$$

and the Poincaré series of $B_{\text{com}}G_{\mathbf{1}}$ and $E_{\text{com}}G_{\mathbf{1}}$ satisfy

$$P_{B_{\text{com}}G_{\mathbf{1}}}(t) = P_{BG}(t)P_{E_{\text{com}}G_{\mathbf{1}}}(t).$$

From this we derive the following.

Corollary 7.5 *These statements are equivalent for a compact connected Lie group G :*

- (1) $E_{\text{com}}G_{\mathbf{1}}$ is contractible.
- (2) $E_{\text{com}}G_{\mathbf{1}}$ is rationally acyclic.
- (3) G is abelian.

Using the theory of multisymmetric polynomials in Section 8 we provide combinatorial descriptions and Poincaré series for these algebras in the case of the classical groups $SU(r)$, $U(q)$ and $Sp(k)$. Taking limits we obtain that the algebras $H^*(B_{\text{com}}U; \mathbb{Q})$, $H^*(B_{\text{com}}SU; \mathbb{Q})$ and $H^*(B_{\text{com}}Sp; \mathbb{Q})$ are polynomial algebras on countably many

generators (Corollaries 8.3, 8.5 and 8.9, respectively). For example we have an isomorphism of \mathbb{Q} -algebras

$$H^*(B_{\text{com}}U; \mathbb{Q}) \cong \mathbb{Q}[z_{a,b} \mid (a, b) \in \mathbb{N}^2 \text{ and } b > 0],$$

where the elements $z_{a,b}$ are polynomial generators of degree $2a + 2b$.

This paper is organized as follows. In Section 2 we describe basic properties of the spaces $E_{\text{com}}G$ and $B_{\text{com}}G$ for G a topological group. In Section 3 we focus on the case when G is a Lie group. In Section 4 we introduce commutative K-theory. Section 5 describes the topological poset generated by the maximal tori in a Lie group $\mathcal{T}(G)$. In section Section 6 we derive the decompositions of $B_{\text{com}}G_{\mathbf{1}}$ and $E_{\text{com}}G_{\mathbf{1}}$ as homotopy colimits. Section 7 deals with cohomology calculations. In Section 8 we consider the particular cases when $G = \text{SU}(n)$, $U(n)$ and $\text{Sp}(n)$ and finally in the appendix it is proved that $[B_{\text{com}}G]_*$ is a proper simplicial space for any Lie group G . We are grateful to the referee for providing helpful comments.

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2 Definitions and basic properties of the spaces $B_{\text{com}}G$ and $E_{\text{com}}G$

In this section we study general properties of the spaces $B_{\text{com}}G$ and $E_{\text{com}}G$, which are constructed by assembling the different spaces of ordered commuting k -tuples in a topological group G . These spaces were first introduced in [3] where their basic properties were derived, mostly for the case of finite groups.

Suppose that G is a topological group. For technical reasons we will assume that G is locally compact, Hausdorff and that $1_G \in G$ is a nondegenerate basepoint. We can associate to G a simplicial space, denoted by $[B_{\text{com}}G]_*$, in the following way. For any integer $n \geq 0$ define

$$[B_{\text{com}}G]_n := \text{Hom}(\mathbb{Z}^n, G) \subset G^n.$$

Note that $[B_{\text{com}}G]_n$ can be identified with the subset of G^n consisting of all ordered commuting n -tuples in the group G , and as such it is given the subspace topology. The face and degeneracy maps are defined by

$$s_j(g_1, \dots, g_n) = (g_1, \dots, g_j, 1_G, g_{j+1}, \dots, g_n),$$

$$\partial_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & \text{if } i = 0, \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & \text{if } 0 < i < n, \\ (g_1, \dots, g_{n-1}) & \text{if } i = n. \end{cases}$$

The different s_i and ∂_j are well-defined and satisfy the simplicial identities, as these maps are precisely the restrictions of the degeneracy and face maps in the bar construction $[BG]_*$. We denote by $B_{\text{com}}G$ the geometric realization of the simplicial space $[B_{\text{com}}G]_*$. As shown in [3], the space $B_{\text{com}}G$ is in fact the first space in an increasing filtration of the classifying space of G defined using the descending central series of the free groups. Similarly we can define $[E_{\text{com}}G]_n := \text{Hom}(\mathbb{Z}^n, G) \times G \subset G^{n+1}$ and use the analogous face and degeneracy maps to define a simplicial space $[E_{\text{com}}G]_*$ and its geometric realization $E_{\text{com}}G$.

The projection on the first n -coordinates $[E_{\text{com}}G]_* \rightarrow [B_{\text{com}}G]_*$ defines a simplicial map and therefore at the level of geometric realizations we obtain a continuous map $p_{\text{com}}: E_{\text{com}}G \rightarrow B_{\text{com}}G$. This defines a principal G -bundle that can be seen as the restriction of the universal principal G -bundle $p: EG \rightarrow BG$, and we have a morphism of principal G -bundles that fits into the following diagram:

$$\begin{array}{ccc} E_{\text{com}}G & \longrightarrow & EG \\ p_{\text{com}} \downarrow & & \downarrow p \\ B_{\text{com}} & \xrightarrow{i} & BG \end{array}$$

Note that up to homotopy this gives rise to a fibration sequence $E_{\text{com}}G \rightarrow B_{\text{com}}G \rightarrow BG$. Recall that the bundle $p: EG \rightarrow BG$ is universal in the sense that if $q: E \rightarrow X$ is a principal G -bundle over a CW-complex X , then we can find a continuous map $f: X \rightarrow BG$ such that $q: E \rightarrow X$ is isomorphic to $f^*p: f^*(EG) \rightarrow X$.

Definition 2.1 Suppose that X is a CW-complex. We say that a principal G -bundle $q: E \rightarrow X$ is *transitionally commutative* if and only if we can find an open cover $\{U_i\}_{i \in I}$ of X such that the bundle $q: E \rightarrow X$ is trivial over each U_i and the transition functions $\rho_{i,j}: U_i \cap U_j \rightarrow G$ commute with each other whenever they are simultaneously defined.

When G is a Lie group the bundle $p_{\text{com}}: E_{\text{com}}G \rightarrow B_{\text{com}}G$ is a universal bundle for transitionally commutative principal G -bundles. To make this precise we need to establish the following notation. Let $k \geq 0$ be an integer and consider the standard k -simplex

$$\Delta_k = \left\{ (t_0, \dots, t_k) \in \mathbb{R}^{k+1} \mid t_i \geq 0, \sum_{j=0}^k t_j = 1 \right\}.$$

Note that the symmetric group Σ_{k+1} acts by permutation on the vertices of Δ_k and this action can be extended to a linear action on Δ_k . Suppose that we have a sequence of integers $\mathbf{i} := \{0 \leq i_1 < \dots < i_q \leq k\}$ and let $e_{\mathbf{i}}$ denote the element in Δ_k given by $e_{\mathbf{i}} = \delta_{i_q} \cdots \delta_{i_1} (\frac{1}{k-q+1}, \dots, \frac{1}{k-q+1})$, where $\delta_{i_1}, \dots, \delta_{i_q}$ denote the different face

maps. That is, the element $e_i \in \Delta_k$ has barycentric coordinates (t_0, \dots, t_k) given by

$$t_j = \begin{cases} 0 & \text{if } j \in \{i_1, \dots, i_q\}, \\ \frac{1}{k-q+1} & \text{if } j \notin \{i_1, \dots, i_q\}. \end{cases}$$

The length of such a sequence \mathbf{i} is defined to be the number $|\mathbf{i}| = q$. Observe that if \mathbf{i} runs through all the different sequences of integers $\mathbf{i} := \{0 \leq i_1 < \dots < i_q \leq k\}$ then the collection $\{e_{\mathbf{i}}\}_{\mathbf{i}}$ is precisely the collection of barycenters associated to all the nonempty faces in the standard simplex in Δ_k . Let $\{W_q^k\}_{q=0}^k$ be a collection of open sets in Δ_k satisfying the following properties:

- (1) For each $0 \leq q \leq k$ and each $\mathbf{i} = \{0 \leq i_1, \dots, i_q \leq k\}$ there is an open neighborhood $V_{\mathbf{i}}^k$ of $e_{\mathbf{i}}$ such that $W_q^k = \bigsqcup_{|\mathbf{i}|=q+1} V_{\mathbf{i}}^k$.
- (2) If $t = (t_0, \dots, t_k) \in V_{\mathbf{i}}^k$ for some $\mathbf{i} = \{0 \leq i_1 < \dots < i_q \leq k\}$, then $t_j > 0$ if $j \notin \{i_1, \dots, i_q\}$.
- (3) Each open set W_q^k is invariant under the action of Σ_{k+1} .
- (4) The sets $\{W_q^k\}_{q=0}^k$ form an open cover of Δ_k .

Clearly such an open cover exists and can be constructed in an inductive way. Using this we have the following geometric description for the bundle $p_{\text{com}}: E_{\text{com}}G \rightarrow B_{\text{com}}G$.

Theorem 2.2 *Suppose that G is a Lie group and let $f: X \rightarrow BG$ denote the classifying of a principal G -bundle $q: E \rightarrow X$ over the finite CW-complex X . Then up to homotopy, f factors through $B_{\text{com}}G$ if and only if q is transitionally commutative.*

Proof Let X be a finite CW-complex. Assume that the classifying map of $q: E \rightarrow X$ factors through $B_{\text{com}}G$; that is, suppose that the classifying map of this bundle is of the form $f: X \rightarrow B_{\text{com}}G \subset BG$. We will show first that q is transitionally commutative. By Proposition A.1 in the appendix we have that $[B_{\text{com}}G]_*$ is a proper simplicial space (see the appendix for the definition of a proper simplicial space). It follows that the geometric realization of $[B_{\text{com}}G]_*$ is equivalent to Segal’s fat geometric realization where the equivalences associated to the degeneracy maps are ignored (see [25, Appendix A]). This realization is denoted here by $\mathbb{B}_{\text{com}}G$, and similarly we have $\mathbb{E}_{\text{com}}G$. For the first part of the proof it will be more convenient for us to work with the (equivalent) principal G -bundle $p_{\text{com}}: \mathbb{E}_{\text{com}}G \rightarrow \mathbb{B}_{\text{com}}G$.

By definition

$$\mathbb{B}_{\text{com}}G := \left(\bigsqcup_{n \geq 0} \text{Hom}(\mathbb{Z}^n, G) \times \Delta_n \right) / \sim,$$

where $((g_1, \dots, g_k), \delta_i u) \sim (\partial_i(g_1, \dots, g_k), u)$ with $u \in \Delta_{k-1}$. For each $k \geq 0$, let

$$F_k \mathbb{B}_{\text{com}} G = \text{Im} \left\{ \bigsqcup_{0 \leq n \leq k} \text{Hom}(\mathbb{Z}^n, G) \times \Delta_n \right\} \subset \mathbb{B}_{\text{com}} G.$$

In this way we obtain an increasing filtration of $\mathbb{B}_{\text{com}} G$

$$F_0 \mathbb{B}_{\text{com}} G \subset F_1 \mathbb{B}_{\text{com}} G \subset \dots \subset F_k \mathbb{B}_{\text{com}} G \subset \dots \subset \mathbb{B}_{\text{com}} G$$

and $\mathbb{B}_{\text{com}} G = \text{colim}_{k \rightarrow \infty} F_k \mathbb{B}_{\text{com}} G$. Since X is a finite CW-complex, we can find some $k \geq 0$ such that the map f factors through $F_k \mathbb{B}_{\text{com}} G$; that is, $f: X \rightarrow F_k \mathbb{B}_{\text{com}} G \subset \mathbb{B}_{\text{com}} G$. It suffices to show the result for the restriction of the universal principal G -bundle $p_{\text{com}}: \mathbb{E}_{\text{com}} G \rightarrow \mathbb{B}_{\text{com}} G$ over $F_k \mathbb{B}_{\text{com}} G$ for each $k \geq 0$ fixed. For this fix $\mathbf{i} := \{0 \leq i_1 < \dots < i_q \leq k\}$ a sequence of integers and assume that $(g_1, \dots, g_k) \in \text{Hom}(\mathbb{Z}^k, G) = [B_{\text{com}} G]_k$. Then we can see $(g_1, \dots, g_k, 1) \in \text{Hom}(\mathbb{Z}^k, G) \times G = [E_{\text{com}} G]_k$ and we define

$$\varphi_{\mathbf{i}}(g_1, \dots, g_k) := \pi_{k-q}(\partial_{i_1} \dots \partial_{i_q}(g_1, \dots, g_k, 1))^{-1}.$$

In this equation, $\partial_{i_1}, \dots, \partial_{i_q}$ denote the face maps in the simplicial space $[E_{\text{com}} G]_*$ and $\pi_{n-k}: \text{Hom}(\mathbb{Z}^{n-k-1}) \times G \rightarrow G$ is the projection onto the last coordinate. For example if $(g_1, g_2, g_3) \in \text{Hom}(\mathbb{Z}^3, G)$ and $\mathbf{i} = \{2, 3\}$, then $\varphi_{\mathbf{i}}(g_1, g_2, g_3) = g_3^{-1} g_2^{-1}$ and if $\mathbf{j} = \{1, 3\}$ then $\varphi_{\mathbf{j}}(g_1, g_2, g_3) = g_3^{-1}$. For any $(g_1, \dots, g_k) \in \text{Hom}(\mathbb{Z}^k, G)$ and any sequence \mathbf{i} we have $\varphi_{\mathbf{i}}(g_1, \dots, g_k) = g_{j_1}^{-1} \dots g_{j_r}^{-1}$ for suitable integers j_1, \dots, j_r . The functions $\varphi_{\mathbf{i}}$ can be used to define local sections of the restriction of the bundle $p_{\text{com}}: \mathbb{E}_{\text{com}} G \rightarrow \mathbb{B}_{\text{com}} G$ over $F_k \mathbb{B}_{\text{com}} G$. Indeed, for each $0 \leq q \leq k$ let U_q^k be the image of $\text{Hom}(\mathbb{Z}^k, G) \times W_q^k$ in $F_k \mathbb{B}_{\text{com}} G$. Thus defined, each U_q^k is an open set in $F_k \mathbb{B}_{\text{com}} G$ and the collection $\{U_q^k\}_{q=0}^k$ forms an open cover of $F_k \mathbb{B}_{\text{com}} G$. Define $\sigma_q: U_q^k \rightarrow p_{\text{com}}^{-1}(U_q^k)$ in the following way. Let $x \in U_q^k$ and write $x = [(g_1, \dots, g_k), t]$ for some $(g_1, \dots, g_k) \in \Delta_k$ and $t \in W_q^k$. We define $\sigma_q(x) := [(g_1, \dots, g_k, \varphi_{\mathbf{i}}(g_1, \dots, g_k)), t]$, provided that $t \in V_{\mathbf{i}}^k$. The functions $\varphi_{\mathbf{i}}$ are defined so that the function σ_q is well defined and continuous over U_q^k . Thus σ_q is a continuous section of the restriction of p_{com} over U_q^k making it a trivial principal G -bundle. With the trivializations provided by these sections, if $x = [(g_1, \dots, g_k), t]$ in $U_r^k \cap U_q^k$, then the transition function $\rho_{r,q}: U_r^k \cap U_q^k \rightarrow G$ is such that

$$x = [(g_1, \dots, g_k), t] \mapsto g_{j_0}^{\pm 1} \dots g_{j_r}^{\pm 1}$$

for a suitable sequence of integers j_0, \dots, j_r . In particular it follows that the different transition functions $\rho_{r,q}$ are pairwise commutative whenever they are simultaneously defined as $(g_1, \dots, g_k) \in \text{Hom}(\mathbb{Z}^k, G)$ for any $x = [(g_1, \dots, g_k), t] \in U_r^k \cap U_s^k$.

Conversely, suppose that $q: E \rightarrow X$ is a transitionally commutative principal G -bundle. Then we can find an open cover $\mathcal{U} := \{U_i\}_{i \in I}$ of X such that the bundle $q: E \rightarrow X$ is trivial over each U_i and the transition functions $\rho_{i,j}: U_i \cap U_j \rightarrow G$ commute with

each other whenever they are simultaneously defined. By passing to a refinement of \mathcal{U} , we can assume that each nonempty intersection of the sets U_i is contractible. Moreover, this cover can be reduced to a countable cover. Let $\mathcal{U} = \{U_i\}_{i \geq 0}$ be the resulting open cover of X . For each $i \geq 0$, fix $\varphi_i: q^{-1}(U_i) \rightarrow U_i \times G$ a trivialization of the restriction of q over U_i . These trivializations define transition functions $\rho_{i,j}: U_i \cap U_j \rightarrow G$ whenever $U_i \cap U_j \neq \emptyset$ and satisfy the cocycle condition $\rho_{ik} = \rho_{ij} \rho_{jk}$ whenever they are defined and are pairwise commutative by assumption. Consider the nerve $N_*(\mathcal{U})$ of the cover \mathcal{U} . This is a simplicial set with $N_k(\mathcal{U}) = \bigsqcup_{0 \leq i_1 \leq \dots \leq i_{k+1}} U_{i_1} \cap \dots \cap U_{i_{k+1}}$. The different transition functions can be used to define a map of simplicial spaces $\rho_*: N_*(\mathcal{U}) \rightarrow [B_{\text{com}}G]_*$ in the following way. Suppose that $x \in U_{i_1} \cap \dots \cap U_{i_{k+1}}$ for some $0 \leq i_1 \leq \dots \leq i_{k+1}$. Define $\rho_k(x) = (\rho_{i_1 i_2}(x), \rho_{i_2 i_3}(x), \dots, \rho_{i_k i_{k+1}}(x)) \in \text{Hom}(\mathbb{Z}^k, G)$. It is easy to see that this defines a map of simplicial spaces and in particular it induces a continuous map $g = |\rho_*|: N(\mathcal{U}) \rightarrow B_{\text{com}}G$. Since the cover \mathcal{U} was chosen so that each nonempty intersection of sets in \mathcal{U} is contractible, then the natural map $\alpha: N(\mathcal{U}) \rightarrow X$ is a homotopy equivalence (see for example [14, Corollary 4G.3]). Let $\beta: X \rightarrow N(\mathcal{U})$ be a homotopy inverse of α . Then $f := g \circ \beta: X \rightarrow B_{\text{com}}G$ is a continuous map that classifies the principal G -bundle $q: E \rightarrow X$. \square

As a consequence of the proof of the previous theorem, we have that the restriction of the bundle $p_{\text{com}}: \mathbb{E}_{\text{com}}G \rightarrow \mathbb{B}_{\text{com}}G$ to each $F_k \mathbb{B}_{\text{com}}G$ defines a transitionally commutative principal G -bundle. From this we infer that the bundle $p_{\text{com}}: \mathbb{E}_{\text{com}}G \rightarrow \mathbb{B}_{\text{com}}G$ is itself transitionally commutative, as is the equivalent bundle $p_{\text{com}}: E_{\text{com}}G \rightarrow B_{\text{com}}G$.

As an application of the previous theorem suppose that G is a Lie group and that X is a finite CW-complex for which we can find an open cover $X = U \cup V$ with both U and V contractible. Let $q: E \rightarrow X$ be any principal G -bundle over X . Then the restriction of q over U and V is trivial since U and V are contractible. Over this trivialization there is only one transition function and thus any such principal G -bundle over X is transitionally commutative. By the previous theorem we conclude that the classifying map of the bundle $q: E \rightarrow X$ factors through $B_{\text{com}}G$ up to homotopy. This situation applies in particular to $X = \mathbb{S}^n$ for any $n \geq 0$. Therefore the inclusion map $i: B_{\text{com}}G \hookrightarrow BG$ induces a surjective map $i_{\#}: [\mathbb{S}^n, B_{\text{com}}G] \rightarrow [\mathbb{S}^n, BG]$. The following corollary is an immediate consequence after modifying for basepoints and using the fibration $E_{\text{com}}G \rightarrow B_{\text{com}}G \rightarrow BG$.

Corollary 2.3 *Let G be a Lie group. Then the map $i: B_{\text{com}}G \rightarrow BG$ induces a surjection $i_*: \pi_n(B_{\text{com}}G) \rightarrow \pi_n(BG)$ for every $n \geq 0$ and in particular, for every $n \geq 0$ we have a short exact sequence*

$$1 \rightarrow \pi_n(E_{\text{com}}G) \rightarrow \pi_n(B_{\text{com}}G) \xrightarrow{i_*} \pi_n(BG) \rightarrow 1.$$

Remark 2.4 Suppose that G is a connected Lie group. Then by [3, Theorem 6.3] the fibration sequence $\Omega E_{\text{com}}G \rightarrow \Omega B_{\text{com}}G \rightarrow \Omega BG$ has a natural continuous section $\sigma(G): \Omega BG \rightarrow \Omega B_{\text{com}}G$. This implies that for such groups and every $n \geq 0$ the short exact sequence of homotopy groups obtained in the previous corollary splits naturally. Moreover, by [3, Theorem 6.3] there is a natural homotopy equivalence $\theta(G): G \times \Omega E_{\text{com}}G \rightarrow \Omega B_{\text{com}}G$.

Suppose that X is a finite CW-complex. Note that two principal G -bundles $q_0: E_0 \rightarrow X$ and $q_1: E_1 \rightarrow X$ are isomorphic if and only if we can find a principal G -bundle $p: E \rightarrow X \times [0, 1]$ such that $q_0 = p|_{p^{-1}(X \times \{0\})}$ and $q_1 = p|_{p^{-1}(X \times \{1\})}$. Suppose now that $q_0: E_0 \rightarrow X$ and $q_1: E_1 \rightarrow X$ are two transitionally commutative principal G -bundles. Then we say that these bundles are *transitionally commutative isomorphic* if we can find a transitionally commutative principal G -bundle $p: E \rightarrow X \times [0, 1]$ such that $q_0 = p|_{p^{-1}(X \times \{0\})}$ and $q_1 = p|_{p^{-1}(X \times \{1\})}$. Thus we can identify the set $[X, B_{\text{com}}G]$ with the set of transitionally commutative isomorphism classes of transitionally commutative principal G -bundles over X . In other words, the space $B_{\text{com}}G$ is a classifying space for transitionally commutative bundles. If two transitionally commutative principal G -bundles are transitionally commutative isomorphic then they are isomorphic as principal G -bundles. However, the converse is not true as is demonstrated in the next example.

Example 2.5 Let $G = \text{SU}(2)$ and $T \subset G$ the maximal torus, which in this case is a circle. The quotient G/T can be identified with the sphere \mathbb{S}^2 , let $f: \mathbb{S}^2 \rightarrow G/T$ be a fixed homeomorphism. Now the action map $G \times T^n \rightarrow \text{Hom}(\mathbb{Z}^n, G)$ defined by $(g, t_1, \dots, t_n) \mapsto (gt_1g^{-1}, \dots, gt_ng^{-1})$ factors through $G/T \times T^n$. Looking at the realizations of the respective simplicial spaces, this defines a map $\theta: G/T \times BT \rightarrow B_{\text{com}}G$. According to [3, Theorem 6.1], this gives rise to a rational cohomology isomorphism $H^*(B_{\text{com}}G, \mathbb{Q}) \rightarrow H^*(G/T \times BT, \mathbb{Q})^W$, where $W = \mathbb{Z}/2\mathbb{Z}$ is the Weyl group. This group acts through the sign representation both on the generator $a \in H^2(BT, \mathbb{Q})$ and on the top class $b \in H^2(G/T, \mathbb{Q})$. The invariant classes a^2 and ab correspond to a basis for $H^4(B_{\text{com}}G, \mathbb{Q})$. Now let $g: \mathbb{S}^2 \rightarrow BT \cong \mathbb{C}\mathbb{P}^\infty$ be a representative of a generator of $\pi_2(\mathbb{C}\mathbb{P}^\infty) \cong \mathbb{Z}$. Consider the map $h: \mathbb{S}^2 \times \mathbb{S}^2 \rightarrow B_{\text{com}}\text{SU}(2)$ given by the composition $\theta \circ (f \times g)$. In the next section we shall see that $B_{\text{com}}G$ is 3-connected for $G = \text{SU}(2)$, which implies that this map is nullhomotopic on $\mathbb{S}^2 \vee \mathbb{S}^2$ and so defines a map $\tilde{h}: \mathbb{S}^4 \rightarrow B_{\text{com}}G$. By construction this map is nontrivial in rational cohomology, corresponding to the element ab . Moreover, if $i: B_{\text{com}}G \rightarrow BG$ denotes the inclusion map, then the composition $i \circ \tilde{h}$ is trivial in cohomology (as the Chern class in dimension four corresponds to b^2) and so is nullhomotopic. It follows that the principal G -bundle over \mathbb{S}^4 induced by \tilde{h} is trivial as a principal G -bundle but not as a transitionally commutative principal G -bundle.

Remark 2.6 If $G = U(k)$ with $k > 1$, then the map $i_*: \pi_n(B_{\text{com}}U(k)) \rightarrow \pi_n(BU(k))$ cannot be an isomorphism for every $n \geq 0$. If this were true i would be a homotopy equivalence; however it follows from [3, Theorem 6.1] that the rational cohomology of $B_{\text{com}}U(k)$ is not isomorphic to that of $BU(k)$. More generally we shall see that if G is a compact connected Lie group which is not abelian, then $E_{\text{com}}G$ is *not contractible*, unlike the classical universal space EG .

3 Properties of $B_{\text{com}}G$ and $E_{\text{com}}G$ when G is a Lie group

In this section we focus our attention on the case when G is a real or complex reductive algebraic group. We can consider G as a real or complex Lie group, respectively. Let $K \subset G$ be a maximal compact subgroup; it is well known that such a group always exists and the inclusion map $i: K \hookrightarrow G$ is a strong deformation retract. However, in general there is no retraction $r: G \rightarrow K$ that preserves commutativity; see for example [26] where the nonexistence of such a retraction was proved for the groups $SL_n(\mathbb{C})$ with $n \geq 8$. On the other hand, by [23, Corollary 1.2] the inclusion $\text{Hom}(\mathbb{Z}^n, K) \hookrightarrow \text{Hom}(\mathbb{Z}^n, G)$ is a strong deformation retract. We show here that this can be used to prove that the inclusion $i: B_{\text{com}}K \hookrightarrow B_{\text{com}}G$ is also a strong deformation retract.

Theorem 3.1 *Suppose that G is a real or complex reductive algebraic group and let K be a maximal compact subgroup. Then the inclusion map $i: K \hookrightarrow G$ induces homotopy equivalences $i: B_{\text{com}}K \rightarrow B_{\text{com}}G$ and $i: E_{\text{com}}K \rightarrow E_{\text{com}}G$.*

Proof We can view G as a (real or complex) Lie group and thus by Proposition A.1 in the appendix $[B_{\text{com}}G]_*$ and $[B_{\text{com}}K]_*$ are proper simplicial spaces. The inclusion map $i: K \hookrightarrow G$ induces a map of simplicial spaces $i_*: [B_{\text{com}}K]_* \rightarrow [B_{\text{com}}G]_*$ that is a level-wise homotopy equivalence. By [19, Theorem A.4] we conclude that the induced map at the level of geometric realizations $i: B_{\text{com}}K \rightarrow B_{\text{com}}G$ is a homotopy equivalence. Next we prove that $i: E_{\text{com}}K \rightarrow E_{\text{com}}G$ is a homotopy equivalence. For this, note that the inclusion map $B_{\text{com}}K \rightarrow B_{\text{com}}G$ induces a morphism of the corresponding fibrations:

$$\begin{array}{ccccc}
 E_{\text{com}} & \longrightarrow & B_{\text{com}}K & \longrightarrow & BK \\
 i \downarrow & & i \downarrow & & i \downarrow \\
 E_{\text{com}} & \longrightarrow & B_{\text{com}} & \longrightarrow & BG
 \end{array}$$

This diagram induces a commutative diagram between the corresponding long exact sequences in homotopy groups. Since the inclusion maps $i: B_{\text{com}}K \rightarrow B_{\text{com}}G$ and $i: BK \rightarrow BG$ are homotopy equivalences, by the five lemma it follows that the inclusion map $i: E_{\text{com}}K \rightarrow E_{\text{com}}G$ is also a homotopy equivalence. □

If G is a compact Lie group then it can be given the structure of a real algebraic variety that is reductive by complete reducibility. Let $G_{\mathbb{C}}$ denote its complexification. Then by the previous theorem it follows that $B_{\text{com}}G$ and $B_{\text{com}}G_{\mathbb{C}}$ are homotopy equivalent and similarly for $E_{\text{com}}G$ and $E_{\text{com}}G_{\mathbb{C}}$. This shows that we can work in the category of compact Lie groups without loss of generality whenever we want to study the spaces $B_{\text{com}}G$ and $E_{\text{com}}G$ for a real or complex reductive algebraic group G .

Suppose that G is a topological group; $\text{Hom}(\mathbb{Z}^n, G)$ may fail to be path-connected even if we assume that G is path-connected or simply connected. For every $n \geq 0$ define $\text{Hom}(\mathbb{Z}^n, G)_{\mathbf{1}}$ to be the path-connected component of $\text{Hom}(\mathbb{Z}^n, G)$ containing the trivial representation $\mathbf{1}: \mathbb{Z}^n \rightarrow G$. It is easy to see that the collection $\{\text{Hom}(\mathbb{Z}^n, G)_{\mathbf{1}}\}_{n \geq 0}$ forms a simplicial subspace of $[B_{\text{com}}G]_*$. We denote by $B_{\text{com}}G_{\mathbf{1}}$ its geometric realization. When G is a compact Lie group the path-connected component $\text{Hom}(\mathbb{Z}^n, G)_{\mathbf{1}}$ has the following important feature as already pointed out in [8, Lemma 4.2]. An n -tuple (g_1, \dots, g_n) of elements in G belongs to $\text{Hom}(\mathbb{Z}^n, G)_{\mathbf{1}}$ if and only if there is a maximal torus $T \subset G$ that contains g_1, \dots, g_n . On the other hand, if G is a complex reductive algebraic variety then a commuting tuple (g_1, \dots, g_n) belongs to $\text{Hom}(\mathbb{Z}^n, G)_{\mathbf{1}}$ if and only if there is a torus $T \subset G$ containing the semisimple part of the Jordan decomposition of g_i for all $1 \leq i \leq n$. The spaces $B_{\text{com}}G$ and $B_{\text{com}}G_{\mathbf{1}}$ agree if $\text{Hom}(\mathbb{Z}^n, G)$ is path-connected for all $n \geq 0$. When G is a compact Lie group this is the case if and only if a subgroup $A \subset G$ is a maximal abelian subgroup in G if and only if A is a maximal torus in G by [4, Proposition 2.5]. This is true for Lie groups that arise as finite cartesian products of the groups $SU(r)$, $U(q)$ and $Sp(k)$ by [9, Theorem 5.2] and thus $\text{Hom}(\mathbb{Z}^n, G)$ is path-connected for every $n \geq 0$. The same is true for their corresponding complexifications $SL_r(\mathbb{C})$, $GL_q(\mathbb{C})$ and $Sp_k(\mathbb{C})$. Thus $B_{\text{com}}G = B_{\text{com}}G_{\mathbf{1}}$ for such groups. Note that the argument provided in Theorem 3.1 works exactly in the same way if we replace $B_{\text{com}}G$ by $B_{\text{com}}G_{\mathbf{1}}$. Thus if G is a real or complex reductive algebraic group and $K \subset G$ is a maximal compact subgroup then $B_{\text{com}}K_{\mathbf{1}}$ is homotopy equivalent to $B_{\text{com}}G_{\mathbf{1}}$. On the other hand, define $E_{\text{com}}G_{\mathbf{1}} := p^{-1}(B_{\text{com}}G_{\mathbf{1}})$. Note that $E_{\text{com}}G_{\mathbf{1}}$ is the geometric realization of the simplicial subspace of $[E_{\text{com}}G]_*$ defined by $[E_{\text{com}}G_{\mathbf{1}}]_n = \text{Hom}(\mathbb{Z}^n, G)_{\mathbf{1}} \times G$. We have a commutative diagram:

$$\begin{array}{ccc} E_{\text{com}}G_{\mathbf{1}} & \longrightarrow & EG \\ p \downarrow & & \downarrow p \\ B_{\text{com}}G_{\mathbf{1}} & \longrightarrow & BG \end{array}$$

Here the lower horizontal map is the inclusion map $i: B_{\text{com}}G_{\mathbf{1}} \rightarrow BG$. After replacing i with a fibration we obtain a fibration sequence $E_{\text{com}}G_{\mathbf{1}} \rightarrow B_{\text{com}}G_{\mathbf{1}} \rightarrow BG$ and in

the same way as was done in [Theorem 3.1](#), we can prove that if G is a real or complex reductive algebraic group and $K \subset G$ is a maximal compact subgroup then $E_{\text{com}}K_{\mathbf{1}}$ is homotopy equivalent to $E_{\text{com}}G_{\mathbf{1}}$.

If G is a connected topological group, the long exact sequence in homotopy groups associated to the fibration sequence $G \rightarrow EG \rightarrow BG$ can be used to show that BG is simply connected. Moreover, if G is a simply connected Lie group, then G is 2-connected since $\pi_2(G) = 0$ for any Lie group G (see [\[10, page 225\]](#) for the compact case). Thus BG is 3-connected for any such group. As a consequence of [\[13, Theorem 1.1\]](#) similar statements are also true for $B_{\text{com}}G$ and $B_{\text{com}}G_{\mathbf{1}}$ as is proved next.

Proposition 3.2 *Suppose that G is a real or complex reductive algebraic group that is connected as a topological space. Then $B_{\text{com}}G$ and $B_{\text{com}}G_{\mathbf{1}}$ are simply connected. Moreover, if G is simply connected then $B_{\text{com}}G_{\mathbf{1}}$ is 3-connected.*

Proof By [Theorem 3.1](#) we only need to prove the theorem for a compact connected Lie group. Also, by [Proposition A.1](#) in the [appendix](#) for any Lie group G the simplicial space $[B_{\text{com}}G]_*$ is a proper simplicial space; in fact it is a strictly proper simplicial space (see [Remark A.2](#)). The same is true for $[B_{\text{com}}G_{\mathbf{1}}]_*$. If G is a connected Lie group then $[B_{\text{com}}G]_0 = [B_{\text{com}}G_{\mathbf{1}}]_0 = *$ is in particular 1-connected and $[B_{\text{com}}G]_1 = [B_{\text{com}}G_{\mathbf{1}}]_1 = G$ is 0-connected. By [\[18, Theorem 11.12\]](#) it follows that $B_{\text{com}}G$ and $B_{\text{com}}G_{\mathbf{1}}$ are simply connected. Suppose now G is simply connected. Then $[B_{\text{com}}G]_0 = *$ is in particular 3-connected, $[B_{\text{com}}G_{\mathbf{1}}]_1 = G$ and thus this space is 2-connected since G is simply connected and thus 2-connected as pointed out before. Also, $[B_{\text{com}}G_{\mathbf{1}}]_2 = \text{Hom}(\mathbb{Z}^2, G)_{\mathbf{1}}$ is 1-connected as it is path-connected and simply connected by [\[13, Theorem 1.1\]](#). Finally, $[B_{\text{com}}G]_3 = \text{Hom}(\mathbb{Z}^3, G)_{\mathbf{1}}$ is path-connected by definition. Using [\[18, Theorem 11.12\]](#) it follows that $B_{\text{com}}G_{\mathbf{1}}$ is 3-connected in this case. \square

Suppose now that G is a Lie group that arises as a finite product of the classical groups $SU(r)$, $U(q)$ and $Sp(k)$. For such groups $E_{\text{com}}G_{\mathbf{1}} = E_{\text{com}}G$ since $\text{Hom}(\mathbb{Z}^n, G)$ is path-connected for all $n \geq 0$ for such groups. Moreover, we have the following.

Proposition 3.3 *Assume that G is a Lie group isomorphic to a finite product of the classical groups $SU(r)$, $U(q)$ and $Sp(k)$ for $r, q, k \geq 1$. Then $E_{\text{com}}G$ is 3-connected.*

Proof Observe that if G and H are topological groups we have a natural homeomorphism $\text{Hom}(\mathbb{Z}^n, G \times H) \cong \text{Hom}(\mathbb{Z}^n, G) \times \text{Hom}(\mathbb{Z}^n, H)$ for every $n \geq 0$. This implies that there is a homeomorphism $E_{\text{com}}(G \times H) \cong E_{\text{com}}G \times E_{\text{com}}H$. Because

of this we only need to prove the proposition when G is one of the groups $SU(r)$, $U(q)$ and $Sp(k)$. For the groups $SU(r)$ and $Sp(k)$ the proposition follows from the previous proposition and [Corollary 2.3](#). Thus we only need to prove the proposition for the case $G = U(q)$. By [Proposition 3.2](#) and [Corollary 2.3](#) it follows that $E_{\text{com}}G$ is simply connected. Hence to show that $E_{\text{com}}G$ is 3-connected it suffices to prove that $\tilde{H}_n(E_{\text{com}}G) = 0$ for $0 \leq n \leq 3$. To see this, recall that the natural filtration of $E_{\text{com}}G$ as the geometric realization of the simplicial space $[E_{\text{com}}G]_*$ induces a spectral sequence

$$E_{p,q}^2 = H_p H_q([E_{\text{com}}G]_*) \Rightarrow H_{p+q}(E_{\text{com}}G; \mathbb{Z}).$$

The term $E_{p,q}^2$ in this spectral is obtained by taking the p^{th} homology group of the simplicial group $H_q([E_{\text{com}}G]_*)$. Trivially we have $E_{0,0}^2 = \mathbb{Z}$. We show next that $E_{p,q}^2 = 0$ for all $p, q \geq 0$ with $0 < p + q \leq 3$. To prove this, we claim that the map of simplicial spaces $i_*: [E_{\text{com}}G]_* \rightarrow [EG]_*$ induced by the inclusion map induces an isomorphism $i_*: H_p H_q([E_{\text{com}}G]_*) \rightarrow H_p H_q([EG]_*)$ for $0 \leq p + q \leq 3$. Since $H_p H_q([EG]_*) = 0$ for all $p + q > 0$ then the proposition follows. The claim is trivial for $q = 0$ because $[E_{\text{com}}G]_k$ is connected for all $k \geq 0$. When $q = 1$ the simplicial groups $H_1([E_{\text{com}}G]_*)$ and $H_1([EG]_*)$ are isomorphic by [\[13, Theorem 1.1\]](#). Suppose now that $q = 2$. Let $C_n = H_2(\text{Hom}(\mathbb{Z}^n, G) \times G; \mathbb{Z})$ so that $\{C_n\}_{n \geq 0}$ is the chain complex whose p^{th} homology is $H_p H_2([E_{\text{com}}G]_*)$. Trivially we have that $C_0 = 0$ since $H_2(G; \mathbb{Z}) = 0$. Also, $C_1 \cong \mathbb{Z}$ and the differential $\partial: C_2 \rightarrow C_1 \cong \mathbb{Z}$ is surjective since the inclusion $G \vee G \hookrightarrow \text{Hom}(\mathbb{Z}^2, G)$ induces a split injection at the level of homology by [\[1, Theorem 1.6\]](#). This shows that $H_p H_2([EG]_*) = 0$ for $p = 0, 1$. Finally, $H_0 H_3([E_{\text{com}}G]_*)$ vanishes trivially. \square

4 Commutative K-theory

Suppose that X is a finite CW-complex and let $p: E \rightarrow X$ be an n -plane complex vector bundle. As in the case of a principal bundles, we say that E is transitionally commutative if we can find an open cover $\{U_i\}_{i \in I}$ of X such that E is trivial over each U_i and the corresponding transition functions commute with each other whenever they are simultaneously defined. This is equivalent to saying that, with a Hermitian metric in sight, the corresponding frame bundle is a transitionally commutative principal $U(n)$ -bundle. Similarly, two such complex vector bundles $q_0: E_0 \rightarrow X$ and $q_1: E_1 \rightarrow X$ are said to be transitionally commutative isomorphic if their frame bundles are transitionally commutative isomorphic. This means that we can find a transitionally commutative vector bundle $p: E \rightarrow X \times [0, 1]$ such that $q_0 = p|_{p^{-1}(X \times \{0\})}$ and $q_1 = p|_{p^{-1}(X \times \{1\})}$. Let $\text{Vect}_{\text{com}}(X)$ be the set of transitionally commutative isomorphism classes of transitionally commutative vector bundles over X . The Whitney sum of two transitionally

commutative vector bundles is also transitionally commutative and thus $\text{Vect}_{\text{com}}(X)$ has the structure of a monoid. We define the commutative K–theory of X to be $K_{\text{com}}(X) := \text{Gr}(\text{Vect}_{\text{com}}(X))$, where Gr denotes the Grothendieck construction. It is easy to see that if E and F are two transitionally commutative vector bundles over X , then $E \oplus F$ is transitionally commutative isomorphic to $F \oplus E$. This shows that, as in the case of classical K–theory, this construction defines a functor from the category of topological spaces to the category of abelian groups.

By [Theorem 2.2](#) any transitionally commutative n –plane complex vector bundle is classified by a map $f: X \rightarrow B_{\text{com}}U(n)$. Moreover, two such vector bundles classified by maps $f, g: X \rightarrow B_{\text{com}}U(n)$ are transitionally commutative isomorphic if and only if f is homotopic to g . Let $U = \text{colim}_{n \rightarrow \infty} U(n)$, where the colimit is taken over the natural inclusions $i_n: U(n) \rightarrow U(n+1)$. We conclude, in an analogous way to the case of K–theory, that there is a natural isomorphism of groups $K_{\text{com}}(X) \cong [X, \mathbb{Z} \times B_{\text{com}}U]$.

Theorem 4.1 *The space $\mathbb{Z} \times B_{\text{com}}U$ is a loop space and for any finite CW–complex X there is a natural isomorphism of groups $K_{\text{com}}(X) \cong [X, \mathbb{Z} \times B_{\text{com}}U]$.*

Proof As pointed out above we have a natural isomorphism $K_{\text{com}}(X) \cong [X, \mathbb{Z} \times B_{\text{com}}U]$ for any finite CW–complex X . Consider $M := \bigsqcup_{n \geq 0} B_{\text{com}}U(n)$; this space has the structure of a topological monoid defined as follows. For each $n, m \geq 0$ consider the homomorphism of topological groups

$$\begin{aligned} \iota_{n,m}: U(n) \times U(m) &\rightarrow U(n+m), \\ (A, B) &\mapsto \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}. \end{aligned}$$

This homomorphism induces a continuous map

$$\Gamma_{n,m}: B_{\text{com}}U(n) \times B_{\text{com}}U(m) = B_{\text{com}}(U(n) \times U(m)) \rightarrow B_{\text{com}}U(n+m).$$

The different maps $\{\Gamma_{n,m}\}_{n,m \geq 0}$ can be assembled to obtain a map $\Gamma: M \times M \rightarrow M$ giving M the structure of a strictly associative topological monoid. Moreover, this monoid is commutative up to homotopy. Indeed, for each $n, m \geq 0$ fix a continuous path $\beta_{n,m}: [0, 1] \rightarrow U(n+m)$ from the identity matrix I_{n+m} to the matrix $\begin{bmatrix} 0 & I_m \\ I_n & 0 \end{bmatrix}$. Such a path exists because $U(n+m)$ is path-connected. These paths induce a continuous family of homomorphisms

$$\begin{aligned} h_{n,m}(t): U(n) \times U(m) &\rightarrow U(n+m), \\ (A, B) &\mapsto \beta_{n,m}(t) \iota_{n,m}(A, B) \beta_{n,m}(t)^{-1} \end{aligned}$$

defined for $0 \leq t \leq 1$. After applying the functor B_{com} , these maps induce a homotopy $h: M \times M \times I \rightarrow M$ such that $h(A, B, 0) = \Gamma(A, B)$ and $h(A, B, 1) = \Gamma(B, A)$. The

above proves that M is a strict topological monoid that is commutative up to homotopy. On the other hand, observe that $\pi_0(M) = \mathbb{N}$. Fix an element $m \in B_{\text{com}}U(1)$ and consider the mapping telescope

$$M_\infty = \text{Tel}(M \xrightarrow{*m} M \xrightarrow{*m} M \xrightarrow{*m} \dots) \cong \mathbb{Z} \times B_{\text{com}}U.$$

By the group completion theorem (see for example [21, Proposition 1]), it follows that the natural map $M \rightarrow \Omega BM$ induces a map $\eta: M_\infty \cong \mathbb{Z} \times B_{\text{com}}U \rightarrow \Omega BM$ that is an isomorphism in homology. Let $(\Omega BM)_0$ be the path-connected component of ΩBM containing the trivial loop. Then the restriction of η , $\eta_0: B_{\text{com}}U \rightarrow (\Omega BM)_0$, induces an isomorphism in homology with integer coefficients. By Proposition 3.2 the space $B_{\text{com}}U(n)$ is simply connected for every $n \geq 0$. Since $B_{\text{com}}U = \text{colim}_{n \geq 0} B_{\text{com}}U(n)$ the same is true for $B_{\text{com}}U$. Similarly $(\Omega BM)_0$ is simply connected. Therefore $\eta_0: B_{\text{com}}U \rightarrow (\Omega BM)_0$ is a homology isomorphism between simply connected spaces. By the Hurewicz theorem we conclude that η_0 is a homotopy equivalence. On the other hand, since ΩBM is a loop space, all of its connected component are homotopy equivalent. We conclude that there is a homotopy equivalence $\mathbb{Z} \times B_{\text{com}}U \simeq \Omega BM$ and thus $\mathbb{Z} \times B_{\text{com}}U$ is a loop space. \square

Proposition 4.2 *If X is a connected finite CW-complex, there is a natural isomorphism of groups $K_{\text{com}}(\Sigma X) \cong K^0(\Sigma X) \times [\Sigma X, E_{\text{com}}U]$.*

Proof By [3, Theorem 6.3], for any $n \geq 0$ there is a natural homotopy equivalence given by $\theta(U(n)): U(n) \times \Omega E_{\text{com}}U(n) \xrightarrow{\cong} \Omega B_{\text{com}}U(n)$. Since $B_{\text{com}}U = \text{colim}_{n \rightarrow \infty} B_{\text{com}}U(n)$ and $E_{\text{com}}U = \text{colim}_{n \rightarrow \infty} E_{\text{com}}U(n)$, by passing to the colimit as $n \rightarrow \infty$ we obtain a homotopy equivalence $\Omega B_{\text{com}}U \simeq U \times \Omega E_{\text{com}}U$. Using this homotopy equivalence and adjunction between the functors Σ and Ω , we obtain natural isomorphisms

$$\begin{aligned} K_{\text{com}}(\Sigma X) &\cong [\Sigma X, \mathbb{Z} \times B_{\text{com}}U] \cong \mathbb{Z} \times [\Sigma X, B_{\text{com}}U] \\ &\cong \mathbb{Z} \times [X, U \times \Omega E_{\text{com}}U] \cong \mathbb{Z} \times [X, U] \times [X, \Omega E_{\text{com}}U] \\ &\cong [\Sigma X, \mathbb{Z} \times BU] \times [X, \Omega E_{\text{com}}U] \cong K^0(\Sigma X) \times [\Sigma X, E_{\text{com}}U]. \quad \square \end{aligned}$$

Example 4.3 Using this proposition we see that $K_{\text{com}}(\mathbb{S}^m) \cong K^0(\mathbb{S}^m)$ for $0 \leq m \leq 3$. For $m = 0$ this is trivial; for $1 \leq m \leq 3$, by the above computation there is an isomorphism $K_{\text{com}}(\mathbb{S}^m) \cong K^0(\mathbb{S}^m) \times [\mathbb{S}^m, E_{\text{com}}U]$. The space $E_{\text{com}}U(n)$ is 3-connected by Proposition 3.3 for all $n \geq 0$. By passing to the colimit as $n \rightarrow \infty$ it follows that the same is true for $E_{\text{com}}U$. Therefore for $1 \leq m \leq 3$ we have that $\pi_m(E_{\text{com}}U) \cong [\mathbb{S}^m, E_{\text{com}}U]$ is trivial and we conclude that $K_{\text{com}}(\mathbb{S}^m) \cong K^0(\mathbb{S}^m)$ for $0 \leq m \leq 3$. However, $K_{\text{com}}(\mathbb{S}^4) \not\cong K^0(\mathbb{S}^4)$. To see this note that the cohomological computations derived in

Section 8 imply that $H^4(E_{\text{com}}U; \mathbb{Q}) \neq 0$. This together with the universal coefficient theorem and the Hurewicz theorem imply that $\pi_4(E_{\text{com}}U) \cong [S^4, E_{\text{com}}U] \neq 0$ and thus $K_{\text{com}}(\mathbb{S}^4) \not\cong K^0(\mathbb{S}^4)$. This in particular shows that the functor K_{com} does not satisfy Bott periodicity for its values on spheres. We should also mention that the nontrivial element in $\pi_4(B_{\text{com}}SU(2))$, which we described in Example 2.5, is mapped nontrivially to $\pi_4(B_{\text{com}}U(2))$ and this defines a nontrivial commutative vector bundle over \mathbb{S}^4 , which is trivial as an ordinary bundle.

Remark 4.4 In [5] it is proved that $\mathbb{Z} \times B_{\text{com}}U$ is in fact an infinite loop space. In particular it follows that the definition of commutative K–theory can be extended to obtain a generalized cohomology theory. Moreover, it is shown there that the fibration sequence $E_{\text{com}}U \rightarrow B_{\text{com}}U \rightarrow BU$ splits and that $\mathbb{Z} \times B_{\text{com}}U \simeq (\mathbb{Z} \times BU) \times E_{\text{com}}U$ as infinite loop spaces. This implies in particular that commutative K–theory contains topological K–theory as a summand. Note however that, as proved in Example 4.3, commutative K–theory is not 2–periodic, unlike classical K–theory. The homotopy type of $\mathbb{Z} \times B_{\text{com}}U$ remains to be determined.

5 The topological poset associated to maximal tori in a Lie group

Our next goal is to provide a description of the spaces $B_{\text{com}}G_{\mathbf{1}}$ as suitable homotopy colimits for any Lie groups G that are compact and connected. To achieve this decomposition we first need to study the poset generated by all maximal tori in a compact Lie group.

We begin by recalling the definition of a topological poset.

Definition 5.1 A topological poset is a partially ordered set (\mathcal{R}, \leq) together with a topology on the set of objects \mathcal{R} in such a way that the order space $\mathcal{O}_{\mathcal{R}} := \{(x, y) \in \mathcal{R} \times \mathcal{R} \mid x \leq y\}$ is a closed subset of $\mathcal{R} \times \mathcal{R}$.

A topological poset can be seen as a topological category where the space of objects is \mathcal{R} and the space of morphisms is the order space $\mathcal{O}_{\mathcal{R}}$. In this article, by a topological category we mean a small category \mathcal{D} for which the sets $\text{Ob}(\mathcal{D})$ and $\text{Mor}(\mathcal{D})$ come equipped with topologies in such a way that the structural maps source, target, composition and identity are continuous maps.

Example 5.2 Let $n \geq 0$ be a fixed integer. Given $0 \leq k \leq n$, denote by $G_k(\mathbb{C}^n)$ the Grassmannian manifold consisting of all those k –dimensional \mathbb{C} –vector spaces in \mathbb{C}^n .

Let $\mathcal{G}r(n)$ be the poset of all \mathbb{C} -vector subspaces in \mathbb{C}^n . This set naturally has the structure of a poset by inclusion. Note that $\mathcal{G}r(n) = \bigsqcup_{0 \leq k \leq n} G_k(\mathbb{C}^n)$. We can use this identification to give $\mathcal{G}r(n)$ a topology making it into a topological poset.

The maximal tori in a compact Lie group G define a topological poset in the following natural way.

Definition 5.3 Suppose that G is a Lie group. Define $\mathcal{T}(G)$ to be the poset whose objects are closed subspaces $S \subset G$ arising as the intersection of a collection of maximal tori in G , with the order in $\mathcal{T}(G)$ given by inclusion.

The set $\mathcal{T}(G)$ can be given a topology making it into a topological poset as follows. Let $\mathcal{C}(G)$ denote the set of all closed subspaces in G . Suppose that $\mathcal{U} := \{U_1, \dots, U_n\}$ is a finite collection of open sets in G . Define $\mathcal{C}(G, \mathcal{U})$ to be the set of elements $A \in \mathcal{C}(G)$ such that $A \subset \bigcup_{i=1}^n U_i$ and $A \cap U_i \neq \emptyset$ for all $1 \leq i \leq n$. The sets of the form $\mathcal{C}(G, \mathcal{U})$ form a basis for a topology in $\mathcal{C}(G)$ called the finite topology (see [22] for details). Note that $\mathcal{T}(G) \subset \mathcal{C}(G)$ and in this way we can give $\mathcal{T}(G)$ the subspace topology making it into a topological poset. Our next goal is to describe the structure of $\mathcal{T}(G)$ as a topological space for any compact connected Lie group G .

Let \mathfrak{g} denote the Lie algebra of G and fix $T \subset G$ a maximal torus in G with Lie algebra \mathfrak{t} . Let Φ be the root system associated to $(\mathfrak{g}, \mathfrak{t})$ and fix a subset Φ^+ of positive roots of Φ . For each $\alpha \in \Phi^+$ and any integer n , define

$$\mathfrak{t}_{\alpha} := \{X \in \mathfrak{t} \mid \alpha(X) \in 2\pi i \mathbb{Z}\}, \quad \mathfrak{t}_{\alpha,n} := \{X \in \mathfrak{t} \mid \alpha(X) = 2\pi i n\}.$$

Each $\mathfrak{t}_{\alpha,n}$ is a hyperplane of codimension one and the set $D(G) := \bigcup_{\alpha \in \Phi^+} \mathfrak{t}_{\alpha}$ is called the Stiefel diagram of G . Recall that an element $g \in G$ is called singular if it belongs to more than one maximal torus in G . Equivalently, $g \in G$ is singular if and only if $\dim(Z_G(g)) > \dim(T)$. Let $G_s \subset G$ be the set of singular elements in G and let $T_s = T \cap G_s$ be the set of singular elements in T . Consider the restriction of the exponential map $\exp: \mathfrak{t} \rightarrow T$. We have $\exp^{-1}(T_s) = D(G)$; that is, if $X \in \mathfrak{t}$ then $\exp(X)$ is singular if and only if $X \in D(G)$. Given a set of positive roots $I = \{\alpha_1, \dots, \alpha_k\}$ define

$$t_I = \bigcap_{i=1}^k \mathfrak{t}_{\alpha_i} \quad \text{and} \quad T_I := \exp(t_I) \subset T.$$

In the previous definition we allow the case $k = 0$. In this case we take the convention that $t_{\emptyset} = \mathfrak{t}$ and thus $T_{\emptyset} = T$. Let $\Delta = \{\alpha_1, \dots, \alpha_r\}$ be a set of simple roots for the root system Φ . Recall that the Weyl group W is a reflection subgroup generated by

the reflections s_α corresponding to elements $\alpha \in \Delta$. Given $I \subset \Delta$, the subgroup of W_I of W generated by the reflections s_α corresponding to elements $\alpha \in I$ is called a parabolic subgroup of W . Note that each parabolic subgroup is itself a reflection subgroup. We are interested in the different parabolic subgroups of W up to conjugacy. If $I, J \subset \Delta$, then W_I is conjugated to W_J if and only if $I = wJ$ for some $w \in W$. In that case we say that I and J are in the same Coxeter class and write $I \sim_W J$. The relation \sim_W defines an equivalence relation on the set of subsets of Δ . We denote by \mathcal{E}_W the set of equivalence classes of subsets of Δ under this equivalence relation and by $[I]$ the equivalence class in \mathcal{E}_W that contains $I \subset \Delta$.

Theorem 5.4 *Suppose that G is a compact connected Lie group. Fix $\Delta = \{\alpha_1, \dots, \alpha_r\}$ a set of simple roots. Then any element $S \in \mathcal{T}(G)$ is conjugated to T_I for some $I \subset \Delta$. Moreover, there is a G -equivariant homeomorphism $\mathcal{T}(G) \cong \bigsqcup_{[I] \in \mathcal{E}_W} G/N_G(T_I)$.*

Proof Fix a maximal torus $T \subset G$ and suppose that $S \in \mathcal{T}(G)$. Since any two maximal tori in G are conjugated, after replacing S with a suitable conjugate we may assume that $S \subset T$. If $S = T$, then $S = T_\emptyset$ and there is nothing to prove. Assume then that $S \subsetneq T$. We will show that under this assumption S is conjugated to T_I for some set of simple roots $I \subset \Delta$. Let S_0 be the identity component of S . Then S_0 is a compact, connected and abelian subgroup of G . Thus S_0 is a torus and in particular we can find an element $x_0 \in S_0$ such that $S_0 = \overline{\langle x_0 \rangle}$. Let β_1, \dots, β_l be the set of positive roots α with $x_0 \in T_\alpha$. It follows that $S_0 \subset T_{\beta_1} \cap \dots \cap T_{\beta_l} = T_J$, where $J = \{\beta_1, \dots, \beta_l\}$. In fact $S_0 \subset T_{J,0}$, where $T_{J,0}$ denotes the identity component of T_J . As a first step we show that $S_0 = T_{J,0}$. To see this recall that the adjoint representation provides a decomposition of the complexification of \mathfrak{g} into a direct sum of weight spaces $\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus (\bigoplus_{\alpha \in \Phi} L_\alpha)$. This in turn provides a decomposition of \mathfrak{g} in the form $\mathfrak{g} = \mathfrak{t} \oplus (\bigoplus_{\alpha \in \Phi^+} M_\alpha)$, where $M_\alpha = (L_\alpha \oplus L_{-\alpha}) \cap \mathfrak{g}$. Note that $Z_G(x_0) = Z_G(S_0)$ and in particular this group is connected since the centralizer of any torus in G is connected. By [10, Proposition V 2.3] the Lie algebra of $Z_G(x_0)$ is $\mathfrak{z}_\mathfrak{g}(x_0) = \mathfrak{t} \oplus (\bigoplus_{i=1}^l M_{\beta_i})$. On the other hand, the Lie algebra of $T_{J,0}$ is $\mathfrak{t}_{J,0} := \mathfrak{t}_{\beta_1,0} \cap \dots \cap \mathfrak{t}_{\beta_l,0}$. It follows that the Lie algebra of $Z_G(T_{J,0})$ is $\mathfrak{z}_\mathfrak{g}(\mathfrak{t}_{J,0}) = \mathfrak{t} \oplus (\bigoplus_{i=1}^l M_{\beta_i})$. This proves that $Z_G(x_0)$ and $Z_G(T_{J,0})$ are connected subgroups of G that have the same Lie algebra, which in turn implies that $Z_G(x_0) = Z_G(T_{J,0})$. Suppose now that T' is a maximal torus that contains x_0 . Then $T' \subset Z_G(x_0) = Z_G(T_{J,0})$, which implies that $T_{J,0} \subset T'$ since the centralizer of a connected abelian subgroup of G is the union of all maximal tori containing it. This shows that $T_{J,0}$ is contained in the intersection of all maximal tori that contain x_0 . Since S is the intersection of a family of maximal tori, it follows that $T_{J,0} \subset S$, and by connectedness we have $T_{J,0} \subset S_0$. We conclude that $S_0 = T_{J,0}$. To show that $T_J = S$ recall that the center of G is the intersection of all maximal tori in G .

Therefore S contains the center of G . This is also true for T_J . Moreover, it is easy to see that the center of G intersects all the connected components of S and T_J . From here it follows that $S = T_J$ and that the Lie algebra of S is $\mathfrak{t}_{J,0}$. We will show now that after replacing S with some further conjugate, we can choose the β_j to be simple roots. Choose a minimal set of positive roots $\gamma_1, \dots, \gamma_k$ with $\mathfrak{t}_{J,0} = \mathfrak{t}_{\gamma_1,0} \cap \dots \cap \mathfrak{t}_{\gamma_k,0}$. Then we have proper inclusions $\mathfrak{t}_{\gamma_1,0} \cap \dots \cap \mathfrak{t}_{\gamma_k,0} \subset \mathfrak{t}_{\gamma_1,0} \cap \dots \cap \mathfrak{t}_{\gamma_{k-1},0} \subset \dots \subset \mathfrak{t}_{\gamma_1,0}$. As $\mathfrak{t} \setminus \bigcup_{\alpha \in \Phi^+} \mathfrak{t}_{\alpha,0}$ is a union of Weyl chambers, then we can find some (closed) Weyl chamber \mathfrak{C} in such a way that each $\mathfrak{t}_{\gamma_i,0}$ is a face of \mathfrak{C} for every $1 \leq i \leq k$. Associated to the Weyl chamber \mathfrak{C} there is a set of simple roots of Φ . Since each $\mathfrak{t}_{\gamma_i,0}$ is a face of the chamber \mathfrak{C} then, after replacing the sign of the γ_i if necessary, the roots $\gamma_1, \dots, \gamma_k$ are roots in some base of Φ . Since the Weyl group acts transitively on the set of all bases on Φ , it follows that we can find some $w \in W$ such that $w\gamma_1, \dots, w\gamma_k$ are in Δ . This shows that S is conjugated to T_I , where $I = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$ is some set of simple roots.

On the other hand, it is easy to see that for any $I \subset \Delta$, the closed subspace T_I is the intersection of all the maximal tori containing it and thus $T_I \in \mathcal{T}(G)$. Also, if I and J are subsets of Δ , then T_I is conjugated to T_J if and only if $J = wI$ for some $w \in W$; that is, T_I is conjugated to T_J if and only if I and J are in the same Coxeter class. To finish note that the space of subgroups in G that are conjugated to T_I can be identified with $G/N_G(T_I)$ and thus the theorem follows. \square

Example 5.5 Suppose that $G = U(n)$ for $n \geq 1$. For this group a maximal torus T can be chosen to be the set of all diagonal matrices with diagonal entries $x_1, \dots, x_n \in \mathbb{S}^1$. The Weyl group $W = \Sigma_n$ acts by permutation on the diagonal entries. The Lie algebra \mathfrak{t} can be identified with $\mathfrak{t} = \{(X_1, \dots, X_n) \mid X_i \in i\mathbb{R} \text{ for all } 1 \leq i \leq n\}$. The root system Φ consists of all functions $\alpha_{i,j}(X_1, \dots, X_n) = X_i - X_j$ for $1 \leq i, j \leq n$ with $i \neq j$. A choice of positive roots is the set of roots $\alpha_{i,j}$ with $i < j$ and the roots

$$\Delta := \{\alpha_1 := \alpha_{1,2}, \alpha_2 := \alpha_{2,3}, \dots, \alpha_{n-1} := \alpha_{n-1,n}\}$$

form a set of simple roots. By the previous theorem any $S \in \mathcal{T}(U(n))$ is conjugated to some T_I , where $I := \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$ and $1 \leq i_1 < \dots < i_k \leq n - 1$. Unraveling the definition, we see that T_I consists of all diagonal matrices with entries x_1, \dots, x_n and with $x_{i_r} = x_{i_r+1}$ for all $1 \leq r \leq k$. In other words, the roots $\alpha_{i_1}, \dots, \alpha_{i_k}$ determine the number of repeated diagonal entries in the elements of T_I . The conjugacy classes of such tori can be parametrized using partitions of the number n . Recall that a nondecreasing sequence of integers $\lambda = (\lambda_1, \dots, \lambda_k)$ is a partition of n if $n = \lambda_1 + \lambda_2 + \dots + \lambda_k$. We write $\lambda \vdash n$ to mean that λ is a partition of n . The set of Coxeter classes in Δ are in bijective correspondence with the set of partitions λ of

the number n . This can be seen by corresponding for a partition λ of n the Coxeter class represented by the set of simple roots $I(\lambda) := \Delta \setminus \{\alpha_{\lambda_1}, \dots, \alpha_{\lambda_1 + \dots + \lambda_{k-1}}\}$. Note that if $\lambda \vdash n$ then the associated torus $T_{I(\lambda)}$ is the subspace of T consisting of those diagonal matrices with entries of the form

$$\underbrace{(x_1, \dots, x_1)}_{\lambda_1}, \dots, \underbrace{(x_k, \dots, x_k)}_{\lambda_k}.$$

Therefore each $S \in \mathcal{T}(U(n))$ is conjugated to $T_{I(\lambda)}$ for some unique $\lambda \vdash n$. On the other hand, given a partition λ of n , let $Fl(\lambda) := U(n)/(U(\lambda_1) \times \dots \times U(\lambda_k))$. The space $Fl(\lambda)$ is the flag manifold consisting of all flags of the form $V_1 \subset V_2 \subset \dots \subset V_k = \mathbb{C}^n$, where V_i is a \mathbb{C} -vector subspace of \mathbb{C}^n of dimension $\dim_{\mathbb{C}}(V_i) = \lambda_1 + \dots + \lambda_i$. We can see the partition λ as an ordered k -tuple $(\lambda_1, \dots, \lambda_k)$. The symmetric group Σ_k acts on the set of such k -tuples by permutation. We denote by Σ_λ the isotropy of Σ_k at λ under this action. With this notation, the group $N_{U(n)}(T_{I(\lambda)})$ fits in a short exact sequence

$$1 \rightarrow U(\lambda_1) \times \dots \times U(\lambda_k) \rightarrow N_{U(n)}(T_{I(\lambda)}) \rightarrow \Sigma_\lambda \rightarrow 1.$$

It follows that if $\lambda \vdash n$, then

$$U(n)/N_{U(n)}(T_{I(\lambda)}) \cong (U(n)/(U(\lambda_1) \times \dots \times U(\lambda_r)))/\Sigma_\lambda = Fl(\lambda)/\Sigma_\lambda.$$

We conclude that $\mathcal{T}(U(n)) = \bigsqcup_{\lambda \vdash n} Fl(\lambda)/\Sigma_\lambda$, where λ runs through all partitions of n .

6 The space $B_{\text{com}}G_1$ as a homotopy colimit

In this section we derive a description of $B_{\text{com}}G_1$ as a suitable homotopy colimit for a real or complex reductive algebraic group G . Note that by [Theorem 3.1](#) we can work without loss of generality in the category of compact Lie groups.

To start we show that the space $B_{\text{com}}G_1$ is a colimit over the poset $\mathcal{T}(G)$.

Proposition 6.1 *For any compact connected Lie group G we have*

$$B_{\text{com}}G_1 \cong \text{colim}_{S \in \mathcal{T}(G)} BS.$$

Proof Suppose that $T \subset G$ is a maximal torus. Then $T^n \subset \text{Hom}(\mathbb{Z}^n, G)_1$ for all $n \geq 0$ and thus $BT \subset B_{\text{com}}G_1$. This proves that $\bigcup_{T \in \mathcal{T}(G)} BT \subset B_{\text{com}}G_1$. Suppose now that $x \in B_{\text{com}}G_1$. Then x can be represented in the form $x = [(g_1, \dots, g_n, t)]$ for some $(g_1, \dots, g_n) \in \text{Hom}(\mathbb{Z}^n, G)_1$ and $t \in \Delta_n$. By [\[8, Lemma 4.2\]](#) there is a maximal torus $T \subset G$ such that $g_i \in T$ for all $1 \leq i \leq n$. Therefore $x \in BT$. This proves that $B_{\text{com}}G_1 = \bigcup_{T \in \mathcal{T}(G)} BT$ and thus $B_{\text{com}}G_1 = \bigcup_{T \in \mathcal{T}(G)} BT \cong \text{colim}_{S \in \mathcal{T}(G)} BS$. \square

As is well known, homotopy colimits are better suited for homotopical computations than colimits. Therefore we would like to obtain a decomposition $B_{\text{com}}G_1$ as a homotopy colimit over a suitable category. It can be seen that the space $B_{\text{com}}G_1$ can be described as the homotopy colimit of the functor $B: \mathcal{T}(G) \rightarrow \text{Top}$. However, this decomposition is not very helpful as the category $\mathcal{T}(G)$ is a topological category (see [16; 17] for a discussion of homotopy colimits over topological categories). In particular the usual Bousfield–Kan spectral sequence does not apply to such homotopy colimits. We will get around this issue by obtaining a decomposition of $B_{\text{com}}G_1$ as a homotopy colimit over a *discrete* category. A key element in this decomposition is the rank function defined over the poset $\mathcal{T}(G)$. To be more precise, suppose that G is a compact connected Lie group. Let $Z = Z(G)$ be the center of G and write $n = \text{rank}(G) - \text{rank}(Z) \geq 0$. If $S \in \mathcal{T}(G)$ then S is a closed subgroup of G and in particular it is a compact Lie group. Define

$$\rho(S) := \text{rank}(S) - \text{rank}(Z).$$

This way for every $S \in \mathcal{T}(G)$ we obtain $0 \leq \rho(S) \leq n$.

Proposition 6.2 *If G is a compact connected Lie group the function $\rho: \mathcal{T}(G) \rightarrow \mathbb{N}$ is strictly increasing and continuous. Moreover, ρ attains any value $0 \leq m \leq n$.*

Proof Suppose that $S_2 \subsetneq S_1$ are elements in $\mathcal{T}(G)$. Fix a maximal torus $T \subset G$ and a set Φ^+ of positive roots. By [Theorem 5.4](#) we can find simple roots $I := \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$ and $J := \{\beta_{j_1}, \dots, \beta_{j_l}\}$ and $g_1, g_2 \in G$ such that $S_2 = g_2 T_J g_2^{-1} \subsetneq S_1 = g_1 T_I g_1^{-1}$. Since the exponential map is surjective this implies $g_2 t_J g_2^{-1} \subsetneq g_1 t_I g_1^{-1}$. In fact we must have $g_2 t_{J,0} g_2^{-1} \subsetneq g_1 t_{I,0} g_1^{-1}$. Therefore $\text{rank}(S_2) = \dim_{\mathbb{R}}(g_2 t_{J,0} g_2^{-1}) < \dim_{\mathbb{R}}(g_1 t_{I,0} g_1^{-1}) = \text{rank}(S_1)$ and thus $\rho(S_2) < \rho(S_1)$. Also recall that by [Theorem 5.4](#) we have $\mathcal{T}(G) \cong \bigsqcup_{[I] \in \mathcal{E}_W} G/N_G(T_I)$. The map ρ is constant on each connected component of the form $G/N_G(T_I)$. Thus $\rho: \mathcal{T}(G) \rightarrow \mathbb{N}$ is a continuous function. Finally, fix $0 \leq m \leq n$ and let r be the rank of the center of G . Choose any set I consisting of $m + r$ simple roots. Then the corresponding element $T_I \in \mathcal{T}(G)$ is such that $\rho(T_I) = m + r - r = m$. □

Choose $n \geq 0$ as in the previous proposition and let $\mathcal{S}(n)$ be the poset consisting of all the nonempty subsets of $\{0, 1, \dots, n\}$, with the order given by the *reverse* inclusion of sets. We see an element in $\mathcal{S}(n)$ of the form $\mathbf{i} := \{i_0, \dots, i_k\}$, with $0 \leq i_0 < i_1 < \dots < i_k \leq n$. Associated to the group G we have a functor $\mathcal{F}_G: \mathcal{S}(n) \rightarrow \text{Top}$ defined in the following way. Suppose that $\mathbf{i} := \{i_0, \dots, i_k\}$ is an object in $\mathcal{S}(n)$. Then we define

$$\mathcal{F}_G(\mathbf{i}) := \{(S_0, \dots, S_k, a) \mid S_0 \subset \dots \subset S_k \in \mathcal{T}(G), \rho(S_r) = i_r \text{ for } 0 \leq r \leq k, a \in B S_0\}.$$

Note that we have a natural inclusion $\mathcal{F}_G(\mathbf{i}) \subset \mathcal{T}(G)^{k+1} \times BG$ and we give $\mathcal{F}_G(\mathbf{i})$ the subspace topology. If \mathbf{j} is a subset of \mathbf{i} then the natural projection maps induce continuous functions $p_{\mathbf{i}, \mathbf{j}}: \mathcal{F}_G(\mathbf{i}) \rightarrow \mathcal{F}_G(\mathbf{j})$. This defines a functor $\mathcal{F}_G: \mathcal{S}(n) \rightarrow \text{Top}$. To simplify matters, we use the following notation for elements in $\mathcal{F}_G(\mathbf{i})$. Given a chain $S_0 \subset \dots \subset S_k$ in $\mathcal{T}(G)$ with $\rho(S_k) = i_r$ for $0 \leq r \leq k$, we denote $\mathbf{S}_i = (S_0, \dots, S_k)$. With this notation the objects in $\mathcal{F}_G(\mathbf{i})$ are pairs of the form (\mathbf{S}_i, a) with $a \in BS_0$.

Theorem 6.3 *Suppose that G is a compact connected Lie group. Then there is a natural homotopy equivalence*

$$\text{hocolim}_{\mathbf{i} \in \mathcal{S}(n)} \mathcal{F}_G(\mathbf{i}) \simeq B_{\text{com}}G_1.$$

Proof The proof of this theorem will be divided into two steps. As a first step we construct a topological category \mathcal{D} in such a way that there is a homotopy equivalence $B\mathcal{D} \simeq B_{\text{com}}G_1$.

Let \mathcal{C} be the topological category whose objects are the elements in $B_{\text{com}}G_1$ and the only morphisms are the identity maps. Since there are no nontrivial morphisms in \mathcal{C} we have $B\mathcal{C} = B_{\text{com}}G_1$ as topological spaces. Also, consider the topological category \mathcal{D} defined as follows. The objects in \mathcal{D} are pairs of the form (S, a) , where $S \in \mathcal{T}(G)$ and $a \in BS$. If (S_1, a) and (S_2, b) are two objects in \mathcal{D} , then there is a unique morphism $i: (S_1, a) \rightarrow (S_2, b)$ if and only if $a = b$ and $S_1 \subset S_2$. We have a functor $F: \mathcal{D} \rightarrow \mathcal{C}$ that sends an object (S, a) in \mathcal{D} to $F(S, a) = a \in BS \subset B_{\text{com}}G_1$. The functor F sends every morphism in \mathcal{D} to the corresponding identity morphism in \mathcal{C} . Fix an element $a \in B_{\text{com}}G_1$ and consider the under category $a \setminus F$. The objects in this category are tuples of the form $((S, a), \text{id}_a)$, where $S \in \mathcal{T}(G)$ is such that $a \in BS$ and $\text{id}_a: a \rightarrow a$ is the identity morphism in \mathcal{C} . There is a unique morphism $((S_1, a), \text{id}_a) \rightarrow ((S_2, a), \text{id}_a)$ in $a \setminus F$ whenever $S_1 \subset S_2$. We observe that the category $a \setminus F$ has an initial object. Indeed, let $S_a = \bigcap_{S \in \mathcal{T}(G), a \in BS} S$. Then S_a is the smallest element in $\mathcal{T}(G)$ such that $a \in BS_a$ and this implies that $((S_a, a), \text{id}_a)$ is an initial object in $a \setminus F$. We conclude that the category $a \setminus F$ is a contractible category since it has an initial object. This means that its classifying space is contractible. Therefore as a particular case of [17, Lemma A.5] we obtain that the map $BF: B\mathcal{D} \rightarrow B\mathcal{C} \cong B_{\text{com}}G_1$ is a homotopy equivalence.

As a second step we show that there is a natural homeomorphism

$$\phi: \text{hocolim}_{\mathbf{i} \in \mathcal{S}(n)} \mathcal{F}_G(\mathbf{i}) \rightarrow B\mathcal{D}.$$

This will finish the proof. For this recall that by definition $\text{hocolim}_{\mathbf{i} \in \mathcal{S}(n)} \mathcal{F}_G(\mathbf{i}) = B\mathcal{G}$, where \mathcal{G} is the topological category whose objects are sequences of the form $(\mathbf{i}, \mathbf{S}_i, a)$.

Here $\mathbf{i} = \{i_0, \dots, i_k\}$ is an element in $\mathcal{S}(n)$, $\mathbf{S}_i = (S_0, \dots, S_k)$ is a chain in $\mathcal{T}(G)$ with $\rho(S_r) = i_r$ for $0 \leq r \leq k$ and $a \in BS_0$. Whenever $\mathbf{j} \subset \mathbf{i}$ there is a unique morphism in \mathcal{G} , $(\mathbf{i}, \mathbf{S}_i, a) \rightarrow (\mathbf{j}, \mathbf{S}_j, a)$, which is induced by the corresponding projections. An element in $B\mathcal{G}$ is of the form $w = [(g_1, \dots, g_l), t]$, where $t \in \Delta_l$ and (g_1, \dots, g_l) is a sequence of composable morphisms in \mathcal{G} of the form

$$(\mathbf{i}_l, \mathbf{S}_{i_l}, a) \xrightarrow{g_l} (\mathbf{i}_{l-1}, \mathbf{S}_{i_{l-1}}, a) \xrightarrow{g_{l-1}} \dots \xrightarrow{g_1} (\mathbf{i}_0, \mathbf{S}_{i_0}, a).$$

This implies in particular that $\mathbf{i}_0 \subset \dots \subset \mathbf{i}_l$ is a nested sequence of nonempty sets. Write $\mathbf{i} := \mathbf{i}_l = \{i_0, \dots, i_k\}$ and $\mathbf{S}_i = (S_0, \dots, S_k)$. Therefore the morphisms g_1, \dots, g_l induce composable morphisms in \mathcal{D}

$$(S_0, a) \xrightarrow{f_1} (S_1, a) \xrightarrow{f_2} \dots \xrightarrow{f_k} (S_k, a),$$

where f_r is the unique morphism in \mathcal{D} from (S_{r-1}, a) to (S_r, a) . Consider the standard k -simplex Δ_k that corresponds to the composable sequence (f_k, \dots, f_1) in \mathcal{D} . We identify the vertices of Δ_k with the numbers i_0, \dots, i_k by corresponding to each i_r the vertex $v_{i_r} = (0, \dots, 1, \dots, 0) \in \Delta_k$, with the entry 1 in the r^{th} position. If $\mathbf{j} = \{i_{r_0}, \dots, i_{r_m}\} \subset \mathbf{i}$, we denote by $v_{\mathbf{j}}$ the barycenter of the simplex generated by the vertices $v_{i_{r_0}}, \dots, v_{i_{r_m}}$. Thus with this notation, all the vertices in the barycentric subdivision of Δ_k , $B\Delta_k$, are of the form $v_{\mathbf{j}}$, where \mathbf{j} is a nonempty subset of \mathbf{i} . We can associate to the nested sequence $\mathbf{i}_0 \subset \dots \subset \mathbf{i}_l$ the face of $B\Delta_k$ generated by the vertices v_{i_0}, \dots, v_{i_l} . Given $t = (t_0, \dots, t_l) \in \Delta_l$, define $\gamma(t) \in \Delta_k$ by

$$\gamma(t) = \gamma(t_0, \dots, t_l) = t_0 v_{i_l} + i_1 v_{i_{l-1}} + \dots + t_l v_{i_0}.$$

Using this convention we define

$$\phi([(g_1, \dots, g_l), t]) := [(f_k, \dots, f_1), \gamma(t)] \in B\mathcal{D}.$$

In other words, the map ϕ is a linear isomorphism from the standard simplex Δ_l , that corresponds to the composable sequence (g_1, \dots, g_l) in \mathcal{G} , onto the face of $B\Delta_k$ that corresponds to the chain $\mathbf{i}_0 \subset \mathbf{i}_2 \subset \dots \subset \mathbf{i}_l$, where Δ_k is the simplex associated to the composable sequence (i_k, \dots, i_1) in \mathcal{D} . It can be seen that the map ϕ is well defined and is in fact a homeomorphism. \square

The values of the functor \mathcal{F}_G can be described explicitly as follows. Fix $\mathbf{i} = \{i_0, \dots, i_k\}$ an element in $\mathcal{S}(n)$. The conjugation action of G defines an equivalence relation on the set of chains in $\mathcal{T}(G)$ in the following way. Suppose that $\mathbf{S}_i = \{S_0 \subset \dots \subset S_k\}$ and $\mathbf{S}'_i = \{S'_0 \subset \dots \subset S'_k\}$ are two chains with $\rho(S_r) = \rho(S'_r) = i_r$ for $0 \leq r \leq k$. Then we say that $\mathbf{S}_i \sim \mathbf{S}'_i$ if and only if we can find some $g \in G$ such that $\mathbf{S}_i = g\mathbf{S}'_i g^{-1}$; that is, $S_r = gS'_r g^{-1}$ for all $0 \leq r \leq k$. Denote by $\mathcal{E}(\mathbf{i})$ the set of all equivalence

classes of such chains and by $[\mathbf{S}_i]$ the equivalence class representing \mathbf{S}_i in $\mathcal{E}(i)$. Fix a chain $\mathbf{S}_i := \{S_0 \subset \dots \subset S_k\}$ in $\mathcal{T}(G)$ with $\rho(S_r) = i_r$ for $0 \leq r \leq k$. The conjugation action of G induces a continuous map

$$\bar{\mu}_{\mathbf{S}_i}: G \times BS_0 \rightarrow \mathcal{F}_G(i), \quad (g, a) \mapsto (g\mathbf{S}_i g^{-1}, gag^{-1}).$$

Let $N_G(\mathbf{S}_i)$ be the normalizer of \mathbf{S}_i in G ; that is, the subgroup of G consisting of elements $g \in G$ such that $g\mathbf{S}_i g^{-1} = \mathbf{S}_i$. The group $N_G(\mathbf{S}_i)$ acts by conjugation on BS_0 and on the left on G by the assignment $n \cdot g = gn^{-1}$. This induces a diagonal action of $N_G(\mathbf{S}_i)$ on $G \times BS_0$ and the map $\bar{\mu}_{\mathbf{S}_i}$ is invariant under this action. Therefore $\bar{\mu}_{\mathbf{S}_i}$ induces a continuous function

$$\mu_{\mathbf{S}_i}: G \times_{N_G(\mathbf{S}_i)} BS_0 \rightarrow \mathcal{B}_{\mathcal{T}(G)}(i).$$

If we vary \mathbf{S}_i through the different equivalence classes in $\mathcal{E}(i)$, then we obtain a continuous map

$$\mu_i = \bigsqcup_{[\mathbf{S}_i] \in \mathcal{E}(i)} \mu_{\mathbf{S}_i}: \bigsqcup_{[\mathbf{S}_i] \in \mathcal{E}(i)} G \times_{N_G(\mathbf{S}_i)} BS_0 \rightarrow \mathcal{F}_G(i).$$

This map is clearly surjective. In fact this map is also injective. Indeed, suppose that

$$\mu_i(g, a) = (g\mathbf{S}_i g^{-1}, gag^{-1}) = (g_1\mathbf{S}'_i g_1^{-1}, g_1 a_1 g_1^{-1}) = \mu_i(g_1, a_1).$$

Then $g\mathbf{S}_i g^{-1} = g_1\mathbf{S}'_i g_1^{-1}$, which means that $[\mathbf{S}_i] = [\mathbf{S}'_i]$. Thus we can assume without loss of generality that $\mathbf{S}_i = \mathbf{S}'_i$. Also we have $g_1^{-1}g\mathbf{S}_i(g_1^{-1}g)^{-1} = \mathbf{S}_i$ and $gag^{-1} = g_1 a_1 g_1^{-1}$. Therefore $n := g_1^{-1}g \in N_G(\mathbf{S}_i)$ is such that $nan^{-1} = a_1$. We conclude that in $G \times_{N_G(\mathbf{S}_i)} BS_0$ we have $[(g, a)] = [gn^{-1}, nan^{-1}] = [(g_1, a_1)]$, proving that μ_i is injective. Moreover, it can easily be seen that μ_i^{-1} is also continuous and thus μ_i is a homeomorphism. We conclude that for every element i in $\mathcal{S}(n)$ there is a natural homeomorphism

$$\mathcal{F}_G(i) \cong \bigsqcup_{[\mathbf{S}_i] \in \mathcal{E}(i)} G \times_{N_G(\mathbf{S}_i)} BS_0.$$

The sets $\mathcal{E}(i)$ that appear in the previous description can be expressed in terms of the root system Φ associated to a maximal torus $T \subset G$ in the following way. Let $i = \{i_0, \dots, i_k\}$ be an object in $\mathcal{S}(n)$ and $\mathbf{S}_i = \{S_0 \subset \dots \subset S_k\}$ a chain in $\mathcal{T}(G)$ with $\rho(S_r) = i_r$ for $0 \leq r \leq k$. After replacing \mathbf{S}_i with a suitable conjugate we may assume that \mathbf{S}_i is such that $S_0 \subset \dots \subset S_k \subset T$. By [Theorem 5.4](#), for every $0 \leq r \leq k$ we can find a set of simple roots I_r in such a way that $S_r = g_r T_{I_r} g_r^{-1}$ for some $g_r \in G$. Let $T_{I_r, 0}$ denote the connected component of T_{I_r} that contains the identity. Then $T_{I_r, 0}$ is a torus since it is a compact, connected abelian Lie group. Therefore for each $0 \leq r \leq k$ we can find some element x_r such that $T_{I_r, 0} = \overline{\langle x_r \rangle}$. Each x_r

is such that $x_r \in T_{I_r} \subset T$ and also $g_r x_r g_r^{-1} \in T$. By [10, Lemma IV 2.5], for each $1 \leq r \leq k$ we can find some $w_r \in W$ such that $w_r x_r = g_r x_r g_r^{-1}$. We conclude then that $S_{r,0} = w_r T_{I_r,0}$ and this in turn implies that $S_r = w_r T_{I_r} = T_{w_r^{-1} I_r}$ for all $1 \leq r \leq k$. This proves that any chain $\mathcal{S}_i = \{S_0 \subset \dots \subset S_k\}$ in $\mathcal{T}(G)$ is conjugated to a chain of the form $T_{J_0} \subset T_{J_1} \subset \dots \subset T_{J_k}$ for a collection of sets of roots J_0, \dots, J_k . Moreover, two chains such chains $T_{I_0} \subset T_{I_1} \subset \dots \subset T_{I_k}$ and $T_{J_0} \subset T_{J_1} \subset \dots \subset T_{J_k}$ are conjugated if and only if we can find some $w \in W$ such that $T_{I_r} = T_w J_r$ for $0 \leq r \leq k$. This proves that the set $\mathcal{E}(i)$ can be identified with the set of equivalence classes of sequences of sets of roots of the form (J_0, \dots, J_k) with $\rho(T_{J_r}) = i_r$ for $0 \leq r \leq k$, where $(J_0, \dots, J_k) \sim (I_0, \dots, I_k)$ if and only if we can find some $w \in W$ such that $T_{I_r} = T_w J_r$ for $0 \leq r \leq k$.

Example 6.4 Take $G = \text{SU}(2)$, which is a Lie group of rank 1. In this case the poset $\mathcal{T}(G)$ has one element corresponding to the center of G (isomorphic to $\mathbb{Z}/2$) which has rank zero and an element for every maximal torus in G . Fix $T \subset G$ the maximal torus consisting of those 2×2 diagonal matrices in G . The Weyl group $W = \mathbb{Z}/2$ acts by permutation on the diagonal entries for such matrices. The space of all maximal tori in G is homeomorphic to $G/N_G(T)$. Therefore $\mathcal{T}(G) = * \sqcup G/N_G(T) \cong * \sqcup \mathbb{R}P^2$. By Theorem 6.3 it follows that $B_{\text{com}}G \simeq \text{hocolim}_{i \in \mathcal{S}(1)} \mathcal{F}_G(i)$. In this case we have

$$\begin{aligned} \mathcal{F}_G(0) &= BZ(G) = B\mathbb{Z}/2 = \mathbb{R}P^\infty, \\ \mathcal{F}_G(1) &= G/T \times_W BT = \mathbb{S}^2 \times_{\mathbb{Z}/2} \mathbb{C}P^\infty, \\ \mathcal{F}_G(0, 1) &= G/N_G(T) \times B\mathbb{Z}/2 = \mathbb{R}P^2 \times \mathbb{R}P^\infty. \end{aligned}$$

Therefore $B_{\text{com}}G$ is homotopy equivalent to the homotopy pushout of the diagram

$$\mathcal{F}_G(0) \xleftarrow{p_0} \mathcal{F}_G(0, 1) \xrightarrow{p_1} \mathcal{F}_G(1),$$

where

$$p_0: \mathcal{F}_G(0, 1) \cong \mathbb{R}P^2 \times \mathbb{R}P^\infty \rightarrow \mathcal{F}_G(0) \cong \mathbb{R}P^\infty$$

corresponds to second projection and

$$p_1: \mathcal{F}_G(0, 1) \cong \mathbb{R}P^2 \times \mathbb{R}P^\infty \rightarrow \mathcal{F}_G(1) \cong \mathbb{S}^2 \times_{\mathbb{Z}/2} \mathbb{C}P^\infty$$

is the map induced by the inclusion $\mathbb{Z}/2 \hookrightarrow T$. Using the associated Mayer–Vietoris sequence we obtain

$$H^k(B_{\text{com}}\text{SU}(2); \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ 0 & \text{if } k = 2 \text{ or } k \text{ odd,} \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } k \equiv 0 \pmod{4}, k > 0, \\ \mathbb{Z}/2 & \text{if } k \equiv 2 \pmod{4}, k > 2. \end{cases}$$

Note in particular the presence of a $\mathbb{Z}/2$ -factor in cohomological degrees $k \equiv 2 \pmod{4}$ and $k > 2$. The existence of this 2-torsion is particularly intriguing and we are interested in finding a suitable geometric interpretation.

Suppose now that G is a Lie group that is compact and connected Lie group with center Z . Let $n = \text{rank}(G) - \text{rank}(Z)$. Define a functor $\mathcal{H}_G: \mathcal{S}(n) \rightarrow \text{Top}$ as follows. If $\mathbf{i} = \{i_0, \dots, i_k\}$ is an object in $\mathcal{S}(n)$ then we define

$$\mathcal{H}_G(\mathbf{i}) := \{(S_0, \dots, S_k, x) \mid S_0 \subset \dots \subset S_k \in \mathcal{T}(G), \rho(S_r) = i_r \text{ for } 0 \leq r \leq k, x \in G/S_0\}.$$

If \mathbf{j} is a subset of \mathbf{i} then the corresponding function is the map $p_{\mathbf{i}, \mathbf{j}}: \mathcal{H}_G(\mathbf{i}) \rightarrow \mathcal{H}_G(\mathbf{j})$ induced by the projection maps and the quotient map.

Theorem 6.5 *Suppose that G is a compact connected Lie group. Then there is a natural G -equivariant homotopy equivalence*

$$\text{hocolim}_{\mathbf{i} \in \mathcal{S}(n)} \mathcal{H}_G(\mathbf{i}) \simeq E_{\text{com}}G_{\mathbf{1}}.$$

Proof Define a functor $\tilde{\mathcal{H}}_G: \mathcal{S}(n) \rightarrow \text{Top}$ that associates an object \mathbf{i} in $\mathcal{S}(n)$ with

$$\tilde{\mathcal{H}}_G(\mathbf{i}) := \{(S_0, \dots, S_k, x) \mid S_0 \subset \dots \subset S_k, \rho(S_r) = i_r \text{ for } 0 \leq r \leq k, x \in p_{\text{com}}^{-1}(BS_0)\},$$

where $p_{\text{com}}: E_{\text{com}}G_{\mathbf{1}} \rightarrow B_{\text{com}}G$ is the projection map. Note that $p_{\text{com}}^{-1}(BS_0)$ is the geometric realization of the subsimplicial space of $[E_{\text{com}}G]_*$ whose n^{th} space is $\text{Hom}(\mathbb{Z}^n, S_0) \times G = S_0^n \times G$. In particular we have a homotopy equivalence $p_{\text{com}}^{-1}(BS_0) \simeq ES_0 \times_{S_0} G \simeq G/S_0$. Using this equivalence we obtain a natural transformation $\mu: \mathcal{H}_G \rightarrow \tilde{\mathcal{H}}_G$ such that $\mu_{\mathbf{i}}: \mathcal{H}_G(\mathbf{i}) \rightarrow \tilde{\mathcal{H}}_G(\mathbf{i})$ is a G -equivariant homotopy equivalence for every \mathbf{i} . We conclude then that there is a G -equivariant homotopy equivalence

$$\text{hocolim}_{\mathbf{i} \in \mathcal{S}(n)} \mathcal{H}_G(\mathbf{i}) \simeq \text{hocolim}_{\mathbf{i} \in \mathcal{S}(n)} \tilde{\mathcal{H}}_G(\mathbf{i}).$$

On the other hand, let \mathcal{D} be the topological category whose objects are pairs of the form (S, x) , where $S \in \mathcal{T}(G)$ and $x \in p_{\text{com}}^{-1}(BS)$. There is a unique morphism $(S_1, x) \rightarrow (S_2, y)$ in \mathcal{D} if and only if $x = y$ and $S_1 \subset S_2$. An argument similar to the one provided in [Theorem 6.3](#) shows that there is a G -equivariant homeomorphism $\text{hocolim}_{\mathbf{i} \in \mathcal{S}(n)} \mathcal{H}_G(\mathbf{i}) \cong B\mathcal{D}$. Finally, let \mathcal{C} be the topological category whose objects are the elements in $E_{\text{com}}G_{\mathbf{1}}$ and the only morphisms are the identity morphisms. Thus we have $B\mathcal{C} = E_{\text{com}}G_{\mathbf{1}}$. Let $F: \mathcal{D} \rightarrow \mathcal{C}$ be the functor that for an object (S, x) in \mathcal{D} corresponds $F(S, x) = x \in p_{\text{com}}^{-1}(BS) \subset E_{\text{com}}G_{\mathbf{1}}$. The functor F sends any morphism in \mathcal{D} to the corresponding identity morphism in \mathcal{C} . Using the same argument as in

Theorem 6.3 we conclude that the map $BF: BD \rightarrow BC$ is a G -equivariant homotopy equivalence. \square

Remark 6.6 Let $Y(G) := \text{hocolim}_{i \in \mathcal{S}(n)} \mathcal{H}_G(i)$. Note that $Y(G)$ is a finite G -CW-complex and by **Theorem 6.5** we have $B_{\text{com}}G_{\mathbf{1}} \simeq EG \times_G Y(G)$. When G is abelian $Y(G)$ is a contractible space, and so it can be seen as measuring how far G is from being a commutative group. Using the Atiyah–Segal completion theorem we conclude that $K^*(B_{\text{com}}G_{\mathbf{1}})$ is the completion of $K^*_G(Y(G))$ with respect to the augmentation ideal I_G in the complex representation ring $R(G)$ of G . This can be seen as a generalization of the classical computation $K^*(BG) \cong R(G)_{I_G}^\wedge$ for a compact Lie group G .

7 Rational cohomology of $B_{\text{com}}G_{\mathbf{1}}$

In this section we provide computations for the cohomology of the spaces $B_{\text{com}}G_{\mathbf{1}}$ with rational coefficients for a real or complex reductive algebraic group G that is connected as a topological space. By **Theorem 3.1** we can work with compact connected Lie groups without loss of generality. Throughout this section we take the rational numbers as the ground field for all computations unless otherwise specified.

Fix a compact connected Lie group G . Let $T \subset G$ be a maximal torus and let W be the corresponding Weyl group. For every $n \geq 0$ consider the map

$$\begin{aligned} \bar{\varphi}_n: G \times T^n &\rightarrow \text{Hom}(\mathbb{Z}^n, G)_{\mathbf{1}}, \\ (g, t_1, \dots, t_n) &\mapsto (gt_1g^{-1}, \dots, gt_ng^{-1}). \end{aligned}$$

The group $N_G(T)$ acts naturally on $G \times T^n$ by

$$n \cdot (g, t_1, \dots, t_n) := (gn^{-1}, nt_1n^{-1}, \dots, nt_nn^{-1}).$$

The map $\bar{\varphi}_n$ is invariant under this action; as a result we obtain a continuous map

$$\begin{aligned} \varphi_n: G/T \times_W T^n &= G \times_{N_G(T)} T^n \rightarrow \text{Hom}(\mathbb{Z}^n, G)_{\mathbf{1}}, \\ [(g, t_1, \dots, t_n)] &\mapsto (gt_1g^{-1}, \dots, gt_ng^{-1}). \end{aligned}$$

Here W acts diagonally on T^n . This map is surjective as any n -tuple (g_1, \dots, g_n) of elements in G belongs to $\text{Hom}(\mathbb{Z}^n, G)_{\mathbf{1}}$ if and only if there is a maximal torus in G that contains g_1, \dots, g_n and all maximal tori in G are conjugated. By [8, Lemma 3.2] it follows that the fibers of φ_n are rationally acyclic and thus φ_n induces an isomorphism in cohomology with rational coefficients. It is easy to see that the collection $\{\varphi_n\}_{n \geq 0}$ defines a map of simplicial spaces and by passing to the geometric realization we obtain a continuous surjective map

$$\varphi: G/T \times_W BT \rightarrow B_{\text{com}}G_{\mathbf{1}}.$$

In the same way as in [3, Theorem 6.1], we conclude that the map φ induces an isomorphism in cohomology with rational coefficients and thus we obtain an isomorphism

$$(7-1) \quad \varphi^*: H^*(B_{\text{com}}G_1) \xrightarrow{\cong} (H^*(G/T) \otimes H^*(BT))^W,$$

with W acting diagonally on $H^*(G/T) \otimes H^*(BT)$. This can be used to provide the following useful identification of the rational cohomology of $B_{\text{com}}G_1$.

Proposition 7.1 *Suppose that G is a compact connected Lie group and let $T \subset G$ be a maximal torus. Then there is a natural isomorphism of rings*

$$\alpha_G: H^*(B_{\text{com}}G_1) \xrightarrow{\cong} (H^*(BT) \otimes H^*(BT))^W / J_G.$$

In the above equation W acts diagonally on $H^*(BT) \otimes H^*(BT)$ and J_G is the ideal generated by the elements of positive degrees in the image of

$$i_1: H^*(BG) \rightarrow (H^*(BT) \otimes H^*(BT))^W, \\ x \mapsto x \otimes 1.$$

Proof The Eilenberg–Moore spectral sequence with \mathbb{Q} –coefficients associated to the fibration

$$G/T \rightarrow BT \xrightarrow{i} BG$$

collapses at the E_2 –term (see [20, page 278]). Also, there is a W –equivariant isomorphism of graded rings $H^*(G/T) \cong H^*(BT)/I_G$, where I_G is the ideal in $H^*(BT)$ generated by the elements of positive degree in the image of $i^*: H^*(BG) \rightarrow H^*(BT)$. Consider now the natural map

$$\pi: H^*(BT) \otimes H^*(BT) \rightarrow (H^*(BT)/I_G) \otimes H^*(BT).$$

This is a surjective map whose kernel is the ideal \tilde{I}_G generated by the elements of positive degree in the image of the map $i_1: H^*(BG) \rightarrow H^*(BT) \otimes H^*(BT)$ given by $x \mapsto x \otimes 1$. Thus we have a short exact sequence

$$0 \rightarrow \tilde{I}_G \rightarrow H^*(BT) \otimes H^*(BT) \rightarrow (H^*(BT)/I_G) \otimes H^*(BT) \rightarrow 0.$$

Since we are working in characteristic zero and W is a finite group, the exactness of this sequence is preserved at the level of W –invariants; that is, there is a short exact sequence

$$(7-2) \quad 0 \rightarrow \tilde{I}_G^W \rightarrow (H^*(BT) \otimes H^*(BT))^W \rightarrow ((H^*(BT)/I_G) \otimes H^*(BT))^W \rightarrow 0.$$

Note that $J_G = \tilde{I}_G^W$; thus we obtain a natural isomorphism

$$\psi: (H^*(BT) \otimes H^*(BT))^W / J_G \rightarrow (H^*(G/T) \otimes H^*(BT))^W.$$

The required isomorphism is then $\alpha_G := \psi^{-1} \circ \varphi$, where φ is as in (7-1). □

The previous proposition has a number of interesting applications. To start note that we have a natural inclusion $BT \subset B_{\text{com}}G_1 \subset BG$. At the level of cohomology groups this induces a natural monomorphism $H^*(BG) \hookrightarrow H^*(B_{\text{com}}G_1)$ and thus we can consider $H^*(B_{\text{com}}G_1)$ as a module over $H^*(BG)$. Under the identification

$$\alpha: H^*(B_{\text{com}}G_1) \xrightarrow{\cong} (H^*(BT) \otimes H^*(BT))^W / J_G$$

provided in the previous proposition, this structure as $H^*(BG)$ -module corresponds to the structure on $(H^*(BT) \otimes H^*(BT))^W / J_G$ given by $g \cdot [x \otimes y] := [x \otimes gy]$. As a consequence of this we derive the following theorem.

Theorem 7.2 *Suppose that G is a compact, connected Lie group. Then*

$$H^*(B_{\text{com}}G_1)$$

is a free module over $H^(BG)$ of rank $|W|$, where W is the corresponding Weyl group.*

Proof Fix a maximal torus $T \subset G$ and let W be the corresponding Weyl group. Let $S = H^*(BG)$ and $A = H^*(BT)$. These are graded rings, W acts on A with degree-preserving ring automorphisms and we have a natural isomorphism $S \cong A^W$. The ring S is a polynomial ring in finitely many commuting variables. Also, the ring A can be seen as a graded module over S and this is in fact a free module of rank $|W|$. Consider $M^W := (A \otimes A)^W$. This is a graded ring that contains $R := S \otimes S$ as a subring. Thus M^W can be seen as a graded module over R . As a first step we will show that M^W is a finitely generated free R -module. To this end, note that R is a Cohen–Macaulay ring as it is a polynomial ring over \mathbb{Q} . The same is true for $A \otimes A$. Since W acts by degree-preserving ring automorphisms on $A \otimes A$, it follows that $M^W = (A \otimes A)^W$ is also a Cohen–Macaulay ring by [15, Proposition 13]. We observe that M^W is finitely generated as an R -module. Indeed, suppose $\{e_w\}_{w \in W}$ is a free basis of A as a module over S . Then $\{e_v \otimes e_w\}_{v, w \in W}$ is a free basis of $A \otimes A$ as a module over $R = S \otimes S$. Define the averaging operator

$$\begin{aligned} \rho: A \otimes A &\rightarrow (A \otimes A)^W = M^W, \\ f &\mapsto \frac{1}{|W|} \sum_{w \in W} w \cdot f. \end{aligned}$$

The map ρ is surjective and R -linear; this implies that the collection $\{\rho(e_v \otimes e_w)\}_{v, w \in W}$ generates M^W as a module over R . Thus M^W is a finitely generated R -module. Since R is a polynomial algebra over \mathbb{Q} , then any finitely generated R -module has finite projective dimension. Using the Auslander–Buchsbaum formula for graded rings we obtain

$$\text{pd}_R(M^W) = \text{depth}(R) - \text{depth}(M^W),$$

where $\text{pd}_R(M^W)$ is the projective dimension of M^W as an R -module. Since both M^W and R are Cohen–Macaulay rings and M^W is finitely generated as an R -module, this implies $\text{pd}_R(M^W) = \dim(R) - \dim(M^W) = 0$. This means that M^W is projective as an R -module and by the Quillen–Suslin theorem (see [24, Theorem 4]), it follows that M^W is a free R -module. Recall that J_G is the ideal in M^W generated by the elements of positive degree of the form $x \otimes 1$ for $x \in S$. Suppose that $\{a_w\}_w$ is a free basis of M^W as a module over R . If $f_w = \bar{a}_w$ is the image of a_w in M^W/J_G under the natural map, then it follows that $\{f_w\}_w$ is a free basis of M^W/J_G as a module over S . By Proposition 7.1 this means that $H^*(B_{\text{com}}G_1)$ is free as a module over $H^*(BG)$. To finish we only need to compute the rank of $H^*(B_{\text{com}}G_1)$. For this recall that we have a natural isomorphism $\varphi^*: H^*(B_{\text{com}}G_1) \xrightarrow{\cong} (H^*(G/T) \otimes H^*(BT))^W$. As an ungraded W -module $H^*(G/T)$ is isomorphic to the regular W -representation. It follows that as an ungraded module, $H^*(B_{\text{com}}G_1)$ is isomorphic to $H^*(BT)$ and the latter has rank $|W|$ as a module over $H^*(BG)$. This implies that as a graded $H^*(BG)$ -module $H^*(B_{\text{com}}G_1)$ is free and of rank $|W|$. \square

Remark 7.3 The previous theorem is not true in general if we use integer coefficients. For example if $G = \text{SU}(2)$ then $H^*(B_{\text{com}}\text{SU}(2); \mathbb{Z})$ is not free as a module over $H^*(B\text{SU}(2); \mathbb{Z})$ because the former contains 2-torsion as we proved in Example 6.4 and the latter does not contain torsion.

Consider now the inclusion map $i: B_{\text{com}}G_1 \rightarrow BG$. Up to homotopy we have a fibration sequence

$$(7-3) \quad E_{\text{com}}G_1 \xrightarrow{p_{\text{com}}} B_{\text{com}}G_1 \xrightarrow{i} BG.$$

Since G is assumed to be connected then the base space, BG , is simply connected. The E_2 -term of the Eilenberg–Moore spectral sequence with \mathbb{Q} -coefficients associated to the fibration (7-3) is

$$E_2^{*,*} = \text{Tor}_{H^*(BG)}(\mathbb{Q}, H^*(B_{\text{com}}G_1))$$

and this spectral sequence converges strongly to $H^*(E_{\text{com}}G_1)$. By the previous theorem, if G is a compact connected Lie group then $H^*(B_{\text{com}}G_1)$ is a free module over $H^*(BG)$. It follows that

$$\text{Tor}_{H^*(BG)}(\mathbb{Q}, H^*(B_{\text{com}}G_1)) \cong \mathbb{Q} \otimes_{H^*(BG)} H^*(B_{\text{com}}G_1)$$

and the Eilenberg–Moore spectral sequence collapses to the $E_2^{0,*}$ -column. The map $p_{\text{com}}^*: H^*(B_{\text{com}}G_1) \rightarrow H^*(E_{\text{com}}G_1)$ is surjective since $\text{Im}(p_{\text{com}}^*) = E_{\infty}^{0,*}$ and the sequence collapses at the $E_2^{0,*}$ -column. Let K_G denote the ideal in $H^*(B_{\text{com}}G_1)$ generated by the elements in $H^*(BG)$ of positive degree. Then $\text{Ker}(p_{\text{com}}^*) = K_G$ and

we conclude that there is an isomorphism of rings $H^*(E_{\text{com}}G_1) \cong H^*(B_{\text{com}}G_1)/K_G$. Using the isomorphism provided in Proposition 7.1 we conclude that if L_G is the ideal in $(H^*(BT) \otimes H^*(BT))^W$ generated by the elements of positive degree in the image of $H^*(BG) \otimes H^*(BG)$, then there is a natural isomorphism $H^*(E_{\text{com}}G_1) \cong (H^*(BT) \otimes H^*(BT))^W/L_G \cong (H^*(G/T) \otimes H^*(G/T))^W$. This proves the following corollary.

Corollary 7.4 *Suppose that G is a connected compact Lie group with maximal torus T and associated Weyl group W . Then there is a natural isomorphism of rings*

$$H^*(E_{\text{com}}G_1) \cong (H^*(G/T) \otimes H^*(G/T))^W,$$

and the Poincaré series of $B_{\text{com}}G_1$ and $E_{\text{com}}G_1$ satisfy

$$P_{B_{\text{com}}G_1}(t) = P_{BG}(t)P_{E_{\text{com}}G_1}(t).$$

Note that $G/T \times G/T$ is a compact, orientable manifold and that W preserves orientation. Hence we infer that the fundamental class is W -invariant. This yields the following:

Corollary 7.5 *These statements are equivalent for a compact connected Lie group G :*

- (1) $E_{\text{com}}G_1$ is contractible.
- (2) $E_{\text{com}}G_1$ is rationally acyclic.
- (3) G is abelian.

Proof If $E_{\text{com}}G_1$ is contractible then it is acyclic. If it is acyclic, then G/T must be zero-dimensional, hence $G = T$ and so G is abelian. If G is abelian $B_{\text{com}}G_1 = BG$ and so $E_{\text{com}}G_1$ is contractible. □

From the description given in Corollary 7.4, it follows that the Poincaré series for $E_{\text{com}}G_1$ encodes information about all the complex irreducible representations of the Weyl group W associated to the pair (G, T) . To see this recall that as an ungraded W -representation $H^*(G/T; \mathbb{C})$ is isomorphic to the regular representation and thus it contains all the irreducible representations of W . For every irreducible representation λ of W , consider its character χ^λ . The multiplicity of λ in the regular representation equals its degree, which we denote by $f^\lambda = \chi^\lambda(e)$. The multiplicity of the representation λ in the representations $H^i(G/T; \mathbb{C})$ for $i \geq 0$ can be described by the fake degree polynomial $f^\lambda(t)$ defined by

$$f^\lambda(t) := \sum_{i \geq 0} t^i \langle \chi^\lambda, H^i(G/T; \mathbb{C}) \rangle.$$

In other words, the coefficient of t^i in $f^\lambda(t)$ is exactly the multiplicity of λ in $H^i(G/T; \mathbb{C})$. The Poincaré polynomial of the flag manifold G/T is then given by

$$P_{G/T}(t) = \sum_{\lambda} f^\lambda f^\lambda(t) = \sum_{\lambda} f^\lambda(1) f^\lambda(t),$$

where λ runs through all complex irreducible representations of W . On the other hand, the Poincaré polynomial of $E_{\text{com}}G_1$ is given by

$$P_{E_{\text{com}}G_1}(t) = \sum_{\lambda} f^{\bar{\lambda}}(t) f^\lambda(t),$$

where λ runs through all complex irreducible representations of W , and $\bar{\lambda}$ is the complex conjugate of λ . To see this, note that by [Corollary 7.4](#) and the universal coefficient theorem we have $H^*(E_{\text{com}}G_1; \mathbb{C}) \cong (H^*(G/T; \mathbb{C}) \otimes H^*(G/T; \mathbb{C}))^W$. On the other hand, for each $0 \leq k \leq n$ we have an isomorphism

$$\begin{aligned} (H^k(G/T; \mathbb{C}) \otimes H^{n-k}(G/T; \mathbb{C}))^W \\ \cong \text{Hom}_W(\text{Hom}(H^k(G/T; \mathbb{C}), \mathbb{C}), H^{n-k}(G/T; \mathbb{C})). \end{aligned}$$

This together with Schur's lemma shows that $H^n(E_{\text{com}}G_1; \mathbb{C})$ is a vector space over \mathbb{C} of dimension

$$\sum_{0 \leq k \leq n} \sum_{\lambda} \langle \chi^{\bar{\lambda}}, H^k(G/T; \mathbb{C}) \rangle \langle \chi^{\lambda}, H^{n-k}(G/T; \mathbb{C}) \rangle$$

and thus $P_{E_{\text{com}}G_1}(t) = \sum_{\lambda} f^{\bar{\lambda}}(t) f^\lambda(t)$.

8 The cases $SU(n)$, $U(n)$ and $Sp(n)$

In this section we study in detail the cohomology with rational coefficients of the space $B_{\text{com}}G$ when G is one of the classical groups $SU(n)$, $U(n)$ and $Sp(n)$ and also for their corresponding complexifications $SL_n(\mathbb{C})$, $GL_n(\mathbb{C})$ and $Sp_n(\mathbb{C})$. In particular we provide explicit free bases of $H^*(B_{\text{com}}G; \mathbb{Q})$ as a module over $H^*(BG; \mathbb{Q})$. As in the previous section, we take the rational numbers as the ground field unless otherwise specified.

To start, recall that by [\[4, Proposition 2.5\]](#) we have that $\text{Hom}(\mathbb{Z}^n, G)$ is path-connected for all $n \geq 0$ when G is one of the groups $SU(n)$, $U(n)$ and $Sp(n)$. Thus for such groups G and their corresponding complexifications we have $B_{\text{com}}G = B_{\text{com}}G_1$ and $E_{\text{com}}G = E_{\text{com}}G_1$.

8.1 Case $G = U(n)$

Suppose $G = U(n)$. In this case we can choose a maximal torus $T \subset U(n)$ to be the set of diagonal matrices with entries in \mathbb{S}^1 . We have $H^*(BT) \cong \mathbb{Q}[x]$, where $x := \{x_1, \dots, x_n\}$ and $\deg(x_i) = 2$ for $1 \leq i \leq n$. The Weyl group is the symmetric group $W = \Sigma_n$ acting by permutation on the variables x_1, \dots, x_n . Therefore by Proposition 7.1 we have an isomorphism

$$\alpha_n := \alpha_{U(n)}: H^*(B_{\text{com}}U(n)) \xrightarrow{\cong} (\mathbb{Q}[x] \otimes \mathbb{Q}[y])^{\Sigma_n} / J_{U(n)},$$

where Σ_n acts diagonally by permuting the variables $x = \{x_1, \dots, x_n\}$ and $y = \{y_1, \dots, y_n\}$. The algebra $M^{\Sigma_n} := (\mathbb{Q}[x] \otimes \mathbb{Q}[y])^{\Sigma_n}$ is known as the algebra of multisymmetric polynomials. In this case $J_n := J_{U(n)}$ is the ideal in M^{Σ_n} generated by the elementary symmetric polynomials

$$e_k(x_1, \dots, x_n) = \sum_{\substack{1 \leq i_1 < i_2 \\ < \dots < i_k \leq n}} x_{i_1} x_{i_2} \cdots x_{i_k}$$

for $1 \leq k \leq n$. Since we are working with rational coefficients, the ideal J_n is also the ideal generated by the power sums $p_n(a, 0) := x_1^a + \dots + x_n^a$ for $1 \leq a \leq n$. These classical power sums have analogues in the ring of multisymmetric polynomials. For every pair of integers $a, b \geq 0$ define the power sum $p_n(a, b) := x_1^a y_1^b + \dots + x_n^a y_n^b$. Clearly $p_n(a, b) \in M^{\Sigma_n}$ for all $a, b \geq 0$. Moreover, it is well known that the polynomials $p_n(a, b)$ for $0 < a + b \leq n$ generate the \mathbb{Q} -algebra M^{Σ_n} although these polynomials are not algebraically independent. (See for example [27] for a modern account on multisymmetric polynomials). We know by Theorem 7.2 that M^{Σ_n} / J_n is a free module over $H^*(BU(n)) \cong \mathbb{Q}[y]^{\Sigma_n}$. An explicit free basis for M^{Σ_n} / J_n as a module over $\mathbb{Q}[y]^{\Sigma_n}$ can be constructed using the work in [7]. For this consider the averaging operator

$$\begin{aligned} \rho: \mathbb{Q}[x] \otimes \mathbb{Q}[y] &\rightarrow (\mathbb{Q}[x] \otimes \mathbb{Q}[y])^{\Sigma_n} = M^{\Sigma_n}, \\ f(x, y) &\mapsto \frac{1}{n!} \sum_{w \in \Sigma_n} f(wx, wy). \end{aligned}$$

For every $w \in \Sigma_n$ the diagonal descent monomial is defined to be

$$e_w := \prod_{\substack{w^{-1}(i) \\ > w^{-1}(i+1)}} (x_1 \cdots x_i) \otimes \prod_{\substack{w(j) \\ > w(j+1)}} (y_{w(1)} \cdots y_{w(j)}).$$

By [7, Theorem 1.3] the collection $\{\rho(e_w)\}_{w \in \Sigma_n}$ forms a free basis of M^{Σ_n} as a module over $\mathbb{Q}[x]^{\Sigma_n} \otimes \mathbb{Q}[y]^{\Sigma_n}$.

Example 8.1 Suppose that $n = 3$. In this case we obtain the following basis of M^{Σ_3} as a module over $\mathbb{Q}[\mathbf{x}]^{\Sigma_3} \otimes \mathbb{Q}[\mathbf{y}]^{\Sigma_3}$:

$$\begin{aligned} e_1 &= 1, & e_2 &= \rho(x_1 y_2), & e_3 &= \rho(x_1 y_2 y_3), \\ e_4 &= \rho(x_1 x_2 y_3), & e_5 &= \rho(x_1 x_2 y_1 y_3), & e_6 &= \rho(x_1^2 x_2 y_3^2 y_2). \end{aligned}$$

Let f_w be the image of $\rho(e_w)$ in M^{Σ_n}/J_n . Then it follows that $\{f_w\}_{w \in \Sigma_n}$ forms a free basis of $H^*(B_{\text{com}}U(n)) \cong M^{\Sigma_n}/J_n$ as a module over $H^*(BU(n)) \cong \mathbb{Q}[\mathbf{y}]^{\Sigma_n}$. For each $w \in \Sigma_n$ define the descent of w to be the set

$$\text{Des}(w) := \{1 \leq i \leq n - 1 \mid w(i) > w(i + 1)\}.$$

The major index of w , denoted by $\text{maj}(w)$, is defined to be

$$\text{maj}(w) := \sum_{i \in \text{Des}(w)} i = \sum_{w(i) > w(i+1)} i.$$

For every $w \in \Sigma_n$ we have $\text{deg } f_w = 2(\text{maj}(w) + \text{maj}(w^{-1}))$. As a corollary we obtain the following.

Corollary 8.2 Suppose that $n \geq 1$. Then

$$\begin{aligned} P_{E_{\text{com}} \text{GL}_n(\mathbb{C})} &= P_{E_{\text{com}} U(n)}(t) = \sum_{w \in \Sigma_n} t^{2(\text{maj}(w) + \text{maj}(w^{-1}))}, \\ P_{B_{\text{com}} \text{GL}_n(\mathbb{C})} &= P_{B_{\text{com}} U(n)}(t) = \frac{\sum_{w \in \Sigma_n} t^{2(\text{maj}(w) + \text{maj}(w^{-1}))}}{\prod_{1 \leq i \leq n} (1 - t^{2i})}. \end{aligned}$$

Consider now the standard inclusion $i_n: U(n) \rightarrow U(n + 1)$. Let $U := \text{colim}_{n \rightarrow \infty} U(n)$. Then $B_{\text{com}}U = \text{colim}_{n \rightarrow \infty} B_{\text{com}}U(n)$ and $H^*(B_{\text{com}}U(n)) \cong \varprojlim H^*(B_{\text{com}}U(n))$. The isomorphisms α_n and the maps i_n are compatible in the sense that

$$\begin{array}{ccc} H^*(B_{\text{com}}U(n + 1)) & \xrightarrow{\alpha_{n+1}} & M^{\Sigma_{n+1}}/J_{n+1} \\ i_n^* \downarrow & & \downarrow j_n^* \\ H^*(B_{\text{com}}U(n)) & \xrightarrow{\alpha_n} & M^{\Sigma_n}/J_n \end{array}$$

is a commutative diagram, where $j_n^*: M^{\Sigma_{n+1}} \rightarrow M^{\Sigma_n}$ is the map obtained by sending $x_i \mapsto x_i, y_i \mapsto y_i$ for $1 \leq i \leq n$ and $x_{n+1}, y_{n+1} \mapsto 0$. Define

$$M^{\Sigma_\infty} := \varprojlim M^{\Sigma_n} \quad \text{and} \quad J_\infty := \varprojlim J_n.$$

Then we obtain an isomorphism of graded \mathbb{Q} -algebras

$$H^*(B_{\text{com}}U) \cong \varprojlim M^{\Sigma_n} J_n \cong M^{\Sigma_\infty} / J_\infty.$$

The last isomorphism follows from the fact that $j_n^*: J_{n+1} \rightarrow J_n$ is surjective for every $n \geq 1$ and thus $\varprojlim^1 J_n$ vanishes. We show next that this algebra is a polynomial algebra. For every pair of integers $a, b \geq 0$ we have $j_n^*(p_{n+1}(a, b)) = p_n(a, b)$ and thus these polynomials define an element in M^{Σ_∞} , which we denote by $p(a, b)$. By [27, Theorem 2] the algebra M^{Σ_∞} is a polynomial algebra over \mathbb{Q} generated by the elements $p(a, b)$ for $(a, b) \neq 0$. On the other hand, since J_n is the ideal in M^{Σ_n} generated by the power sums $p_n(1, 0), \dots, p_n(n, 0)$, it follows that J_∞ is the ideal generated by $p(a, 0)$ for $a \geq 1$. For each pair of integers $a, b \geq 0$ not both zero let $z_{a,b}$ be a $2(a + b)$ -dimensional variable such that the collection $\{z_{a,b}\}_{a,b}$ is a collection of commuting independent variables. Define $\mathfrak{M}_U = \{(a, b) \in \mathbb{N}^2 \mid b > 0\}$; we conclude that the assignment

$$\begin{aligned} \mathbb{Q}[z_{a,b} \mid (a, b) \in \mathfrak{M}_U] &\rightarrow M^{\Sigma_\infty} / J_n \cong H^*(B_{\text{com}}U), \\ z_{a,b} &\mapsto p(a, b) \end{aligned}$$

is an isomorphism of algebras over \mathbb{Q} . Also define $\text{GL}_\infty(\mathbb{C}) := \text{colim}_{n \rightarrow \infty} \text{GL}_n(\mathbb{C})$. Since $B_{\text{com}}U(n) \simeq B_{\text{com}}\text{GL}_n(\mathbb{C})$ for every $n \geq 0$, it follows that

$$B_{\text{com}}U \simeq B_{\text{com}}\text{GL}_\infty(\mathbb{C}).$$

This proves the following corollary.

Corollary 8.3 *Let $\mathfrak{M}_U = \{(a, b) \in \mathbb{N}^2 \mid b > 0\}$. Then we have isomorphisms of \mathbb{Q} -algebras*

$$H^*(B_{\text{com}}\text{GL}_\infty(\mathbb{C})) \cong H^*(B_{\text{com}}U) \cong \mathbb{Q}[z_{a,b} \mid (a, b) \in \mathfrak{M}_U].$$

8.2 Case $G = \text{SU}(n)$

The case of the special unitary groups $G = \text{SU}(n)$ can be handled in a similar way. In this case we can choose $T \subset \text{SU}(n)$ to be the set of diagonal matrices with entries in \mathbb{S}^1 and determinant one. The Weyl group is the symmetric group $W = \Sigma_n$ acting by permutation on the diagonal entries and $H^*(BT) \cong \mathbb{Q}[\mathbf{x}] / (p_n(1, 0))$, where as before we use the notation $\mathbf{x} = \{x_1, \dots, x_n\}$. Using an argument similar to that in Proposition 7.1, we conclude that

$$H^*(B_{\text{com}}\text{SU}(n)) \cong (\mathbb{Q}[\mathbf{x}] \otimes \mathbb{Q}[\mathbf{y}])^{\Sigma_n} / K_n = M^{\Sigma_n} / K_n,$$

where K_n is the ideal in M^{Σ_n} generated by the multisymmetric polynomials $p_n(a, 0)$ for $1 \leq a \leq n$ and $p_n(0, 1)$. As it was pointed out before the collection $\{\rho(e_w)\}_{w \in \Sigma_n}$

forms a free basis for M^{Σ_n} as a module over $\mathbb{Q}[\mathbf{x}]^{\Sigma_n} \otimes \mathbb{Q}[\mathbf{y}]^{\Sigma_n}$. Let g_w denote the image of $\rho(e_w)$ in M^{Σ_n}/K_n . Then it follows that $\{g_w\}_{w \in \Sigma_n}$ forms a free basis for $H^*(B_{\text{com}} \text{SU}(n)) \cong M^{\Sigma_n}/K_n$ as a module over

$$H^*(BSU(n)) = \mathbb{Q}[p_n(0, 2), \dots, p_n(0, n)].$$

As a corollary we get the following.

Corollary 8.4 *Suppose that $n \geq 1$. Then*

$$P_{E_{\text{com}} \text{SL}_n(\mathbb{C})}(t) = P_{E_{\text{com}} \text{SU}(n)}(t) = \sum_{w \in \Sigma_n} t^{2(\text{maj}(w) + \text{maj}(w^{-1}))},$$

$$P_{B_{\text{com}} \text{SL}_n(\mathbb{C})}(t) = P_{B_{\text{com}} \text{SU}(n)}(t) = \frac{\sum_{w \in \Sigma_n} t^{2(\text{maj}(w) + \text{maj}(w^{-1}))}}{\prod_{2 \leq i \leq n} (1 - t^{2i})},$$

where $\text{maj}(w)$ is the major index of w as defined above.

As in the case of the unitary groups we have a stabilization process given by the standard inclusions $i_n: \text{SU}(n) \rightarrow \text{SU}(n + 1)$. Let $\text{SU} := \text{colim}_{n \rightarrow \infty} \text{SU}(n)$. It follows that

$$H^*(B_{\text{com}} \text{SU}) = \varprojlim H^*(B_{\text{com}} \text{SU}(n)) = \varprojlim M^{\Sigma_n}/K_n.$$

Let $K_\infty \subset M^{\Sigma_\infty}$ denote the ideal corresponding to the ideals $K_n \subset M^{\Sigma_n}$ for $n \geq 0$. For SU we have $H^*(B_{\text{com}} \text{SU}) \cong M^{\Sigma_\infty}/K_\infty$. Note that K_∞ is precisely the ideal in M^{Σ_∞} generated by $p(a, 0)$ for $a \geq 1$ and $p(0, 1)$. Similarly define $\text{SL}_\infty(\mathbb{C}) := \text{colim}_{n \rightarrow \infty} \text{SL}_n(\mathbb{C})$. As a corollary we get the following.

Corollary 8.5 *Let $\mathfrak{M}_{\text{SU}} = \{(a, b) \in \mathbb{N}^2 \mid (a, b) \neq (0, 1), b > 0\}$. Then we have isomorphisms of \mathbb{Q} -algebras*

$$H^*(B_{\text{com}} \text{SL}_\infty(\mathbb{C})) \cong H^*(B_{\text{com}} \text{SU}) \cong \mathbb{Q}[z_{a,b} \mid (a, b) \in \mathfrak{M}_{\text{SU}}].$$

8.3 Case $G = \text{Sp}(n)$

Finally, suppose that $G = \text{Sp}(n)$. In this case $H^*(BT) \cong \mathbb{Q}[\mathbf{x}]$, where $\mathbf{x} = \{x_1, \dots, x_n\}$ and $\text{deg}(x_i) = 2$. The Weyl group is a semi-direct product $W = \Sigma_n \ltimes (\mathbb{Z}/2)^n$. This group can be identified with the group of signed permutations. More precisely, let

$$\mathbb{I}_n := \{-n, -n + 1, \dots, -1, 1, \dots, n - 1, n\}.$$

Let B_n denote the group of bijections $\sigma: \mathbb{I}_n \rightarrow \mathbb{I}_n$ such that $\sigma(-k) = -\sigma(k)$ for all $k \in \mathbb{I}_n$, with the composition of functions as the group operation. Under this identification the group $W \cong B_n$ acts on $\mathbb{Q}[\mathbf{x}]$ by signed permutations. In this case $H^*(BG)$ is a

polynomial algebra generated by $e_1(x_1^2, \dots, x_n^2), \dots, e_n(x_1^2, \dots, x_n^2)$, or equivalently, by the power sums $p_n(2, 0), \dots, p_n(2n, 0)$. Also $M^{B_n} := (\mathbb{Q}[x] \otimes \mathbb{Q}[y])^{B_n}$ is the ring of diagonally signed-symmetric or signed-invariant multisymmetric polynomials. Note in particular that M^{B_n} is a subalgebra of the algebra of multisymmetric polynomials M^{Σ_n} . By Proposition 7.1 we have an isomorphism

$$\alpha_{\text{Sp}(n)}: H^*(B_{\text{com}} \text{Sp}(n)) \xrightarrow{\cong} M^{B_n}/L_n,$$

where $L_n = J_{\text{Sp}(n)}$ is the ideal in M^{B_n} generated by the power sums $p_n(2, 0), \dots, p_n(2n, 0)$. An explicit basis for M^{B_n}/L_n as a module over $\mathbb{Q}[y]^{B_n}$ can be found using the work in [12]. As before, given $w \in B_n$, define its descent to be the set

$$\text{Des}(w) := \{1 \leq i \leq n-1 \mid w(i) > w(i+1)\}.$$

For $1 \leq i \leq n$ let

$$d_i(w) := |\{j \in \text{Des}(w) \mid j \geq i\}|, \quad \varepsilon_i(w) := \begin{cases} 0 & \text{if } w(i) > 0, \\ 1 & \text{if } w(i) < 0. \end{cases}$$

$$f_i(w) := 2d_i(w) + \varepsilon_i(w),$$

The diagonal signed descent monomial associated to w is defined to be

$$c_w := \left(\prod_{i=1}^n x_i^{f_i(w^{-1})} \right) \left(\prod_{i=1}^n y_{|w(i)|}^{f_i(w)} \right) = \prod_{i=1}^n x_i^{f_i(w^{-1})} y_i^{f_{|w^{-1}(i)}(w)}.$$

By [12, Theorem 1.1] the collection $\{\rho(c_w)\}_{w \in \Sigma_n}$ forms a free basis of M^{B_n} as a module over $\mathbb{Q}[x]^{B_n} \otimes \mathbb{Q}[y]^{B_n}$.

Example 8.6 Suppose that $n = 2$. In this case we obtain the following basis of M^{B_2} as a module over $\mathbb{Q}[x]^{B_2} \otimes \mathbb{Q}[y]^{B_2}$:

$$c_1 = 1, \quad c_2 = \rho(x_1 y_1), \quad c_3 = \rho(x_1^2 y_2^2), \quad c_4 = \rho(x_1 y_1 y_2^2),$$

$$c_5 = \rho(x_1^2 x_2 y_2), \quad c_6 = \rho(x_1 x_2 y_1 y_2), \quad c_7 = \rho(x_1^2 x_2 y_1^2 y_2), \quad c_8 = \rho(x_1^3 x_2 y_1^3 y_2).$$

Let h_w be the image of $\rho(c_w)$ in M^{B_n}/L_n . It follows that $\{h_w\}_{w \in B_n}$ forms a free basis of $H^*(B_{\text{com}} \text{Sp}(n)) \cong M^{B_n}/L_n$ as a module over $H^*(B\text{Sp}(n)) \cong \mathbb{Q}[y]^{B_n}$. The flag major index of a signed permutation w was defined in [6] to be

$$\text{fmaj}(w) = \sum_{i=1}^n f_i(w) = 2 \text{maj}(w) + \text{neg}(w),$$

where $\text{maj}(w) := \sum_{i \in \text{Des}(w)} i = \sum_{w(i) > w(i+1)} i$ and $\text{neg}(w) := |\{1 \leq i \leq n \mid w(i) < 0\}|$. Note that for every $w \in B_n$ we have $\text{deg } h_w = 2(\text{fmaj}(w^{-1}) + \text{fmaj}(w))$. As a corollary we get the following.

Corollary 8.7 *Suppose that $n \geq 1$. Then*

$$P_{E_{\text{com}} \text{Sp}_n(\mathbb{C})}(t) = P_{E_{\text{com}} \text{Sp}(n)}(t) = \sum_{w \in B_n} t^{2(\text{fmaj}(w^{-1}) + \text{fmaj}(w))},$$

$$P_{B_{\text{com}} \text{Sp}_n(\mathbb{C})}(t) = P_{B_{\text{com}} \text{Sp}(n)}(t) = \frac{\sum_{w \in B_n} t^{2(\text{fmaj}(w^{-1}) + \text{fmaj}(w))}}{\prod_{1 \leq i \leq n} (1 - t^{4i})}.$$

As in the case of the unitary groups we have a stabilization process given by the standard inclusions $i_n: \text{Sp}(n) \rightarrow \text{Sp}(n + 1)$. Let $\text{Sp} := \text{colim}_{n \rightarrow \infty} \text{Sp}(n)$. Recall that we have an isomorphism $H^*(B_{\text{com}} \text{Sp}(n)) \cong M^{B_n}/L_n$, where $L_n = J_{\text{Sp}(n)}$ is the ideal in M^{B_n} generated by the power sums $p_n(2, 0), \dots, p_n(2n, 0)$. Thus for Sp we have

$$H^*(B_{\text{com}} \text{Sp}) \cong \varprojlim H^*(B_{\text{com}} \text{Sp}(n)) \cong \varprojlim M^{B_n}/L_n.$$

Define

$$M^{B_\infty} = \varprojlim M^{B_n} \quad \text{and} \quad L_\infty := \varprojlim L_n.$$

Thus for the group Sp we have an isomorphism of algebras over \mathbb{Q}

$$H^*(B_{\text{com}} \text{Sp}) \cong M^{B_\infty}/L_\infty.$$

Next we show that M^{B_∞} is a polynomial algebra. To see this we first show that the power sums $p_n(a, b) = x_1^a y_1^b + \dots + x_n^a y_n^b$, where a, b runs through all nonnegative integers such that $0 < a + b$ and $a + b$ is even, generate M^{B_n} as a \mathbb{Q} -algebra. For this, consider the averaging operator $\rho: \mathbb{Q}[x, y] \rightarrow \mathbb{Q}[x, y]^{B_n}$ corresponding to the group B_n . Note that as a \mathbb{Q} -vector space M^{B_n} is generated by the elements of the form $\rho(m(x, y))$, where $m(x, y) = x_1^{i_1} y_1^{j_1} \dots x_n^{i_n} y_n^{j_n}$ is a monomial. By [12, Lemma 3.2] if $i_k + j_k$ is odd for some $1 \leq k \leq n$ then $\rho(m(x, y)) = 0$. It follows that as a \mathbb{Q} -vector space M^{B_n} is generated by the elements of the form $\rho(m(x, y))$, where $m(x, y) = x_1^{i_1} y_1^{j_1} \dots x_n^{i_n} y_n^{j_n}$ and $i_k + j_k$ is even for $1 \leq k \leq n$. Suppose that $m(x, y)$ is such a monomial. Define the length of $m(x, y)$, $\ell(m(x, y))$, to be the number of tuples (i_k, j_k) that are nonzero for $1 \leq k \leq n$. We can show that $\rho(m(x, y))$ is a polynomial on the different $p_n(a, b)$ in an inductive way on the length of the monomial $m(x, y)$. If $\ell(m(x, y)) = 1$ we have

$$\rho(m(x, y)) = \frac{1}{n}(x_1^i y_1^j + \dots + x_n^i y_n^j) = \frac{p_n(i, j)}{n}$$

and there is nothing to prove. Given any monomial $m(x, y) = x_1^{i_1} y_1^{j_1} \dots x_n^{i_n} y_n^{j_n}$ with length $\ell(m(x, y)) = r$, let $(i_{k_1}, j_{k_1}), \dots, (i_{k_r}, j_{k_r})$ be the corresponding different

tuples that are nonzero. Note that

$$p_n(i_{k_1}, j_{k_1}) \cdots p_n(i_{k_r}, j_{k_r}) = \left(\sum_{i=1}^n x_i^{i_{k_1}} y_i^{j_{k_1}} \right) \cdots \left(\sum_{i=1}^n x_i^{i_{k_r}} y_i^{j_{k_r}} \right) \\ = c\rho(m(\mathbf{x}, \mathbf{y})) + q(\mathbf{x}, \mathbf{y}),$$

where c is a nonzero constant and $q(\mathbf{x}, \mathbf{y})$ is a sum of certain monomials $n(\mathbf{x}, \mathbf{y})$ with $\ell(n(\mathbf{x}, \mathbf{y})) < r$. This proves that the elements $p_n(a, b)$, where $a + b > 0$ is even, generate M^{B_n} . In fact it can be seen that the collection $\{p_n(a, b)\}$, where a, b run through all the nonnegative integers such that $0 < a + b \leq 2n$ and $a + b$ is even, generate M^{B_n} , but we do not need that fact. Recall that for every $n \geq 1$ we have a map $j_n^*: M^{B_{n+1}} \rightarrow M^{B_n}$ obtained by sending $x_i \mapsto x_i, y_i \mapsto y_i$ for $1 \leq i \leq n$ and $x_{n+1}, y_{n+1} \mapsto 0$ and $M^{B_\infty} := \varprojlim M^{B_n}$. Suppose that a, b are nonnegative integers. Note that $j_n^*(p_{n+1}(a, b)) = p_n(a, b)$ and thus the different polynomials $p_n(a, b)$ induce an element $p(a, b)$ in M^{B_∞} . Note that each signed-multisymmetric polynomial is in particular a multisymmetric polynomial; that is, we can see M^{B_∞} as a subset of M^{Σ_∞} . Also, we know that M^{Σ_∞} is a polynomial algebra on the different elements $p(a, b)$ where $a + b > 0$ by [27, Theorem 2]. This implies in particular that the collection $\{p(a, b)\}_{a+b>0, \text{even}}$ is algebraically independent in M^{Σ_∞} and in particular, it is also algebraically independent in M^{B_∞} . As a corollary we obtain:

Corollary 8.8 *The \mathbb{Q} -algebra M^{B_∞} is a polynomial algebra on the generators $p(a, b)$, where a, b run through all nonnegative integers such that $0 < a + b$ is even.*

Using the previous fact we can obtain a description of $H^*(B_{\text{com}} \text{Sp})$ as an algebra. Indeed, recall that

$$H^*(B_{\text{com}} \text{Sp}) \cong M^{B_\infty} / L_\infty,$$

where L_∞ is the ideal generated by the power sums $p(2n, 0)$ for all $n \geq 1$. Similarly define $\text{Sp}_\infty(\mathbb{C}) = \text{colim}_{n \rightarrow \infty} \text{Sp}_n(\mathbb{C})$. As a corollary we obtain the following.

Corollary 8.9 *Define $\mathfrak{M}_{\text{Sp}} = \{(a, b) \in \mathbb{N}^2 \mid b > 0, a + b \text{ even}\}$. Then we have isomorphisms of \mathbb{Q} -algebras*

$$H^*(B_{\text{com}} \text{Sp}_\infty(\mathbb{C})) \cong H^*(B_{\text{com}} \text{Sp}) \cong \mathbb{Q}[z_{a,b} \mid (a, b) \in \mathfrak{M}_{\text{Sp}}].$$

Appendix

The goal of this appendix is to show that for any Lie group G the simplicial space $[B_{\text{com}} G]_*$ is a proper simplicial space. This fact was proved in [1, Theorem 8.3] for

Lie groups G that are closed subgroups of $GL_n(\mathbb{C})$ for some $n \geq 0$ and extended in the equivariant setting in [2] for compact Lie groups. Here we show that the arguments in [1] can be used to prove this result for any Lie group G .

We start by recalling some basic definitions. Recall that a pair of topological spaces (X, A) is said to be an NDR pair if there exist continuous functions $h: X \times [0, 1] \rightarrow X$ and $u: X \rightarrow [0, 1]$ such that the following conditions are satisfied:

- (1) $A = u^{-1}(0)$.
- (2) $h(x, 0) = x$ for all $x \in X$.
- (3) $h(a, t) = a$ for all $a \in A$ and all $t \in [0, 1]$.
- (4) $h(x, 1) \in A$ for all $x \in u^{-1}([0, 1])$.

In this case we say that (h, u) is a representation of (X, A) as an NDR pair. If in addition we have $u(h(x, t)) < 1$ for all $t \in [0, 1]$ whenever $u(x) < 1$, then (X, A) is called a strong NDR pair. A simplicial space X_* is said to be proper if each pair (X_{n+1}, sX_n) is a strong NDR pair, where sX_n is the image of the different degeneracy maps in X_{n+1} .

Proposition A.1 *If G is a Lie group then $[B_{\text{com}}G]_*$ is a proper simplicial space.*

Proof Suppose that G is a Lie group and let \mathfrak{g} denote its Lie algebra endowed with a norm $\|\cdot\|_{\mathfrak{g}}$. Recall that the exponential map $\exp: \mathfrak{g} \rightarrow G$ is a local homeomorphism. Let $U \subset \mathfrak{g}$ be any Ad -invariant open neighborhood of $0 \in \mathfrak{g}$ on which the exponential map is injective. Fix some $\epsilon > 0$ such $\bar{B}_{\epsilon}(0) \subset U$. Then in particular $\exp: \bar{B}_{\epsilon}(0) \rightarrow G$ is a homeomorphism onto its image. Define a function $u: G \rightarrow [0, 1]$ by

$$u(g) = \begin{cases} 2|y|_{\mathfrak{g}}/\epsilon & \text{if } g = \exp(y) \text{ for } y \in \exp(\bar{B}_{\epsilon/2}(0)), \\ 1 & \text{if } g \in G - \exp(B_{\epsilon/2}(0)). \end{cases}$$

Also, let $s: G \rightarrow [0, 1]$ be any bump function satisfying the following conditions

$$s(g) = \begin{cases} 1 & \text{if } g = \exp(y) \text{ for } y \in \exp(\bar{B}_{\epsilon/2}(0)), \\ 0 & \text{if } g \in G - \exp(B_{\epsilon}(0)). \end{cases}$$

Finally, define a homotopy $h: G \times [0, 1] \rightarrow G$ by

$$h(g, t) = \begin{cases} \exp((1-t)y) & \text{if } g = \exp(y) \text{ for } y \in \bar{B}_{\epsilon/2}(0), \\ \exp((1-s(g)t)y) & \text{if } g = \exp(y) \text{ for } y \in \bar{B}_{\epsilon}(0) - B_{\epsilon/2}(0), \\ g & \text{if } g \in G - \exp(B_{\epsilon}(0)). \end{cases}$$

The functions h and u are defined so that (h, u) is representation of $(G, \{1_G\})$ as an NDR pair. This can be seen in the same way as in [1, Proposition 8.2]. Moreover,

we claim that the function h satisfies the following property: for each $g \neq 1_G$ in G and each $0 \leq t < 1$ we have $Z_G(h(g, t)) = Z_G(g)$. Indeed, assume that $g \in G$ with $g \neq 1_G$. Note that if $0 \leq t < 1$ then $h(g, t) = g$ if $g \in G - \exp(B_\epsilon(0))$ and $h(g, t) = \exp(ky)$ for some $0 < k \leq 1$ if $g = \exp(y)$ with $y \in \exp(B_\epsilon(0))$. In the first case we have nothing to prove. Suppose then that $g = \exp(y)$ with $y \in \exp(B_\epsilon(0))$ and thus $h(g, t) = \exp(ky)$ for some $0 < k \leq 1$. Since $y \in \exp(B_\epsilon(0)) \subset U$ and U is an Ad -invariant open set on which the exponential map is injective, then by [11, Lemma 3.2.1] we have $Z_G(g) = Z_G(\exp(y)) = Z_G(\exp(ky)) = Z_G(h(g, t))$ proving that $Z_G(g) = Z_G(h(g, t))$ as claimed. By [1, Theorem 7.3] we conclude that the inclusion map $I_j: S_n(j, G) \hookrightarrow S_n(j-1, G)$ is a cofibration for every $1 \leq j \leq n$ (in the terminology of [1, Definition 6.1], G has cofibrantly commuting elements). Here $S_n(j, G)$ denotes the subspace of $\text{Hom}(\mathbb{Z}^n, G)$ consisting of the commuting n -tuples with at least j coordinates equal to 1_G . This implies in particular that the inclusion map $s([B_{\text{com}}G]_{n-1}) = S_n(1, G) \hookrightarrow \text{Hom}(\mathbb{Z}^n, G) = [B_{\text{com}}G]_n$ is a cofibration. Using the explicit NDR-pair representation of $(\text{Hom}(\mathbb{Z}^n, G), S_n(1, G))$ provided by [1, Theorem 7.3] it can easily be seen that $(\text{Hom}(\mathbb{Z}^n, G), S_n(1, G))$ is actually a strong NDR-pair, proving that $[B_{\text{com}}G]_*$ is a proper simplicial space. \square

Remark A.2 Using the explicit NDR representation provided by [1, Theorem 7.3] for the pair $(\text{Hom}(\mathbb{Z}^n, G), S_n(1, G))$, it can be seen that in fact $[B_{\text{com}}G]_*$ is a strictly proper simplicial space (see [18, Definition 11.2] for the definition of a strictly proper simplicial space).

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