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Cohomological non-rigidity of eight-dimensional complex projective towers

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A complex projective tower, or simply $\mathbb{C}P$ tower, is an iterated complex projective fibration starting from a point. In this paper, we classify a certain class of 8–dimensional $\mathbb{C}P$ towers up to diffeomorphism. As a consequence, we show that cohomological rigidity is not satisfied by the collection of 8–dimensional $\mathbb{C}P$ towers: there are two distinct 8–dimensional $\mathbb{C}P$ towers that have the same cohomology rings.

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1 Introduction

Let $\mathcal{M}$ be a collection of diffeomorphism classes of smooth manifolds, and let $\boldsymbol{H}^*\mathcal{M}$ be the isomorphism classes of cohomology rings of manifolds in $\mathcal{M}$. Let $H^*\colon \mathcal{M}\to \boldsymbol{H}^*\mathcal{M}$ be the map defined by $M\in\mathcal{M}\mapsto H^*(M;\mathbb{Z})$. In general, H^* is not bijective. However, if we restrict the class of manifolds then this map sometimes becomes a bijection. For example, if $\mathcal{M}$ is a collection of orientable 2–dimensional manifolds then it is well known that the map H^* is bijective. We say such a collection $\mathcal{M}$ is *cohomologically rigid*, or that $\mathcal{M}$ satisfies *cohomological rigidity*. The problem of whether the map $H^*\colon \mathcal{M}\to \boldsymbol{H}^*\mathcal{M}$ is bijective or not is called the *cohomological rigidity problem*. In this paper, we study the cohomological rigidity problem for *complex projective towers* (or simply $\mathbb{C}P$ *towers*), which we introduced in [7].

A $\mathbb{C}P$ *tower* of height m is a sequence of complex projective fibrations

$$(1)\qquad C_m \xrightarrow{\pi_m} C_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} C_1 \xrightarrow{\pi_1} C_0=\{\text{point}\},$$

where $C_i=P(\xi_{i-1})$ is the projectivization of a complex vector bundle ξ_{i-1} over C_{i-1}. We call each C_i the i^{th} *stage* of the tower. If we forget the tower structure, then we call C_i an *(i–stage)* $\mathbb{C}P$ *manifold*. In [7], we show that the diffeomorphism types of 6–dimensional $\mathbb{C}P$ manifolds are determined by their cohomology rings; ie the collection of 6–dimensional $\mathbb{C}P$ manifolds $\mathbb{C}P\mathcal{M}^6$ is cohomologically rigid. This is a generalization of the fact, due to Choi, Masuda and Suh [5], that the collection $\mathcal{GBM}^6$

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of 6–dimensional generalized Bott manifolds is cohomologically rigid. It is also known that the collection $\mathcal{GBM}_2^{2n}$ of $2n$–dimensional 2–stage generalized Bott manifolds is cohomologically rigid. The purpose of this paper is to show that the collection $\mathbb{C}\mathrm{P}\mathcal{M}_2^8$ of 8–dimensional 2–stage $\mathbb{C}\mathrm{P}$ manifolds is not cohomologically rigid.

To state our main theorem, let us recall a theorem of Atiyah and Rees [1, Theorem 2.8]. Let $\mathrm{Vect}_2(\mathbb{C}\mathrm{P}^3)$ be the collection of isomorphism classes of 2–dimensional complex vector bundles over $\mathbb{C}\mathrm{P}^3$.

Theorem 1.1 (Atiyah–Rees) *There exists an injective map*

$$\phi\colon \mathrm{Vect}_2(\mathbb{C}\mathrm{P}^3) \to \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}, \quad \xi \mapsto (\alpha(\xi), c_1(\xi), c_2(\xi)),$$

where $c_1(\xi)$ and $c_2(\xi)$ are the first and the second Chern classes of ξ, and $\alpha(\xi) \in \mathbb{Z}_2$ is 0 when $c_1(\xi)$ is odd.

By Theorem 1.1, any element in $\mathrm{Vect}_2(\mathbb{C}\mathrm{P}^3)$ can be denoted by $\eta_{(\alpha,c_1,c_2)}$, where $(\alpha, c_1, c_2) \in \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$ is such that $\alpha \equiv 0 \pmod 2$ when $c_1 \equiv 1 \pmod 2$. On the other hand, it's easy to see that $P(\eta_{(\alpha,c_1,c_2)})$ is diffeomorphic to $P(\eta_{(0,1,c_2-(c_1^2-1)/4)})$ if $c_1 \equiv 1 \mod 2$, and diffeomorphic to $P(\eta_{(\alpha,0,c_2-c_1^2/4)})$ if $c_1 \equiv 0 \mod 2$; see Lemma 3.2.

We now state the main result of the paper; see Theorem 4.2 for (1) and Theorem 5.2 for a more precise statement of (2).

Theorem 1.2 *Let $N(u) := P(\eta_{(0,1,u)})$ and $\mathcal{N} := \{N(u) \mid u \in \mathbb{Z}\}$. Similarly, let $M_\alpha(u) := P(\eta_{(\alpha,0,u)})$ and $\mathcal{M} := \{M_\alpha(u) \mid \alpha \in \{0,1\}, u \in \mathbb{Z}\}$.*

(1) *$\mathcal{N}$ is cohomologically rigid. In fact, the following are equivalent:*
  - (a) *$N(u)$ is diffeomorphic to $N(u')$.*
  - (b) *$u = u'$.*
  - (c) *$H^*(N(u);\mathbb{Z})$ and $H^*(N(u');\mathbb{Z})$ are isomorphic as graded rings.*

(2) *$\mathcal{M}$ is not cohomologically rigid. In fact, $H^*(M_0(u);\mathbb{Z})$ and $H^*(M_1(u);\mathbb{Z})$ are isomorphic as graded rings for all u, but if $u(u+1)/12 \in \mathbb{Z}$ then $M_0(u)$ is not diffeomorphic, or even homotopic, to $M_1(u)$.*

We prove (2) in Proposition 5.4 by showing that $\pi_6(M_0(u)) \not\cong \pi_6(M_1(u))$ when $u(u+1)/12 \in \mathbb{Z}$.

The organization of this paper is as follows. In Section 2, as examples of $\mathbb{C}\mathrm{P}$ towers, we explain when a flag manifold admits the structure of a $\mathbb{C}\mathrm{P}$ tower. In Section 3, we recall some basic facts from [7]. In Section 4, we show that $\mathcal{N}$ satisfies cohomological rigidity. In Section 5, we compute the 6–dimensional homotopy group of the elements in some class of $\mathcal{M}$ and show that $\mathcal{M}$ does not satisfy cohomological rigidity.

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2 Flag manifolds of type A and C

$\mathbb{C}P$ towers include many interesting classes of manifolds. In a previous paper [7], we showed that generalized Bott manifolds and the Milnor hypersurface admit a $\mathbb{C}P$ tower structure. We first introduce two other examples of $\mathbb{C}P$ towers. Let $\mathbb{C}P\mathcal{M}_m^{2n}$ be the collection of $2n$–dimensional m–stage $\mathbb{C}P$ manifolds up to diffeomorphism.

Example 2.1 A *partial flag manifold* $\mathcal{F}(d_1, d_2, \dots, d_k)$, where $0 = d_0 < d_1 < d_2 < \cdots < d_{k-1} < d_k = n+1$, is defined by the set of partial flags

$$\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_{k-1} \subset V_k = \mathbb{C}^{n+1},$$

where V_i is a complex subspace of $\mathbb{C}^{n+1}$ of complex dimension d_i. This is well known to be diffeomorphic to the homogeneous space $U(n+1)/(U(n_1)\times\cdots\times U(n_k))$, where $n_i = d_i - d_{i-1}$ for $i = 1, \dots, k$. Denote the partial flag manifold $\mathcal{F}(i, i+1, \dots, n+1)$ by $\mathcal{F}_i$. In particular, we call $\mathcal{F}_1 = \mathcal{F}(1, 2, \dots, n+1)$ a *flag manifold of type A* (or a *complete flag manifold*), and denote it by $\mathcal{F}l(\mathbb{C}^{n+1})$. We will show that the flag manifold of type A has the structure of a $\mathbb{C}P$ tower with height n. We first define a map $p_i\colon \mathcal{F}_i \to \mathcal{F}_{i+1}$ by

$$p_i\colon \{0\} \subset V_i \subset V_{i+1} \subset \cdots \subset V_n \subset \mathbb{C}^{n+1} \mapsto \{0\} \subset V_{i+1} \subset \cdots \subset V_n \subset \mathbb{C}^{n+1}.$$

As the pull-back of a point in $\mathcal{F}_{i+1}$ by p_i can be regarded as the set of codimension-one subspaces $V_i \subset V_{i+1}$, $\mathcal{F}_i$ is a $\mathrm{Gr}_i(V_{i+1})$–bundle over $\mathcal{F}_{i+1}$. Here, $\mathrm{Gr}_i(V_{i+1})$

is the complex Grassmaniann of i–dimensional subspaces in V_{i+1}; ie $\mathcal{F}(i, i+1)$. Because the normal subspace of a codimension-one subspace $V_i \subset V_{i+1}$ is just a line through the origin, the complex Grassmaniann $\mathrm{Gr}_i(V_{i+1})$ may be regarded as the i–dimensional complex projective space $\mathbb{C}\mathrm{P}(V_{i+1}) = (V_{i+1} \setminus \{0\})/\mathbb{C}^*$. Using this fact, it is easy to check that $\mathcal{F}_i$ is the projectivization of the tautological bundle over $\mathcal{F}_{i+1}$; ie $\mathcal{F}_i = \mathbb{C}\mathrm{P}(\eta_{i+1})$, where the tautological bundle η_{i+1} is the complex $(i+1)$–dimensional vector bundle defined by the subset

$$\big\{(\{0\} \subset V_{i+1} \subset \cdots \subset V_n \subset \mathbb{C}^{n+1}, x) \mid x \in V_{i+1}\big\}$$

of $\mathcal{F}_{i+1} \times \mathbb{C}^{n+1}$. Therefore, $\mathcal{F}l(\mathbb{C}^{n+1})$ has the structure of a $\mathbb{C}\mathrm{P}$ tower:

$$\mathcal{F}l(\mathbb{C}^{n+1}) = P(\eta_2) \xrightarrow{\mathbb{C}\mathrm{P}^1} \mathcal{F}_2 = P(\eta_3) \xrightarrow{\mathbb{C}\mathrm{P}^2} \cdots \xrightarrow{\mathbb{C}\mathrm{P}^{n-1}} \mathcal{F}_n \simeq \mathbb{C}\mathrm{P}^n \longrightarrow \{*\}.$$

Hence the flag manifold of type A is an element of $\mathbb{C}\mathrm{P}\mathcal{M}_n^{n^2+n}$.

Example 2.2 Let $(\mathbb{C}^{2n}, \omega)$ be a complex vector space with a symplectic structure ω given by the skew-symmetric bilinear form

$$\Omega = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix},$$

where O is the $n \times n$ zero matrix and I_n is the $n \times n$ identity matrix. Let V be a complex linear subspace in $\mathbb{C}^{2n}$. Define the *ω–perpendicular space* of V to be the subspace

$$V^{\omega} = \{w \in \mathbb{C}^{2n} \mid \omega(v, w) = v^T \Omega w = 0 \text{ for all } v \in V\}.$$

Note that $(V^{\omega})^{\omega} = V$ and $\dim_{\mathbb{C}} V + \dim_{\mathbb{C}} V^{\omega} = 2n$. We call V *isotropic* or *coisotropic* if $V \subset V^{\omega}$ or $V^{\omega} \subset V$, respectively. A *symplectic partial flag manifold* $\mathrm{Sp}^n\,\mathcal{F}(d_1, d_2, \ldots, d_k)$, where $0 = d_0 < d_1 < d_2 < \cdots < d_{k-1} < d_k \le n$, is defined by the set of (isotropic) partial flags

$$\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_{k-1} \subset V_k \subset \mathbb{C}^{2n},$$

where V_i is a complex isotropic subspace of $(\mathbb{C}^{2n}, \omega)$ of complex dimension d_i. It is easy to check that this is equivalent to the set of partial flags

$$\{0\} \subset V_1 \subset \cdots \subset V_{k-1} \subset V_k \subset V_k^{\omega} \subset V_{k-1}^{\omega} \subset \cdots \subset V_1^{\omega} \subset \mathbb{C}^{2n}.$$

This is well known to be diffeomorphic to the homogeneous space $\mathrm{Sp}(n)/(U(n_1) \times \cdots \times U(n_k) \times \mathrm{Sp}(n_{k+1}))$, where $n_i = d_i - d_{i-1}$ for $i = 1, \ldots, k$ and $n_{k+1} = \frac{1}{2}(\dim V_k^{\omega} - \dim V_k) = n - d_k$. If $d_k = \frac{1}{2}\dim V_k = n$, ie $V_k = V_k^{\omega}$ is a Lagrangian subspace, then $\mathrm{Sp}^n\,\mathcal{F}(d_1, d_2, \ldots, d_{k-1}, n)$ is diffeomorphic to $\mathrm{Sp}(n)/(U(n_1) \times \cdots \times U(n_k))$. Denote the symplectic partial flag manifold $\mathrm{Sp}^n\,\mathcal{F}(1, 2, \ldots, i)$ by $\mathrm{Sp}^n\,\mathcal{F}_i$ for $i \ge 1$.

In particular, we call $\mathrm{Sp}^n\,\mathcal{F}_n = \mathrm{Sp}^n\,\mathcal{F}(1,2,\dots,n)$ a *flag manifold of type C* (or a *symplectic flag manifold*), and denote it by $\mathrm{Sp}\,\mathcal{F}l(\mathbb{C}^{2n})$. We will show that the flag manifold of type C has the structure of a $\mathbb{C}$P tower with height n. We first define a map $q_i\colon \mathrm{Sp}^n\,\mathcal{F}_{i+1}\to \mathrm{Sp}^n\,\mathcal{F}_i$ by

$$q_i\colon \{0\}\subset V_1\subset\cdots\subset V_i\subset V_{i+1}\subset V_{i+1}^{\omega}\subset V_i^{\omega}\subset\cdots\subset V_1^{\omega}\subset\mathbb{C}^{2n}$$
$$\mapsto \{0\}\subset V_1\subset\cdots\subset V_i\subset V_i^{\omega}\subset\cdots\subset V_1^{\omega}\subset\mathbb{C}^{2n}.$$

The pull-back of a point in $\mathrm{Sp}^n\,\mathcal{F}_i$ by q_i can be regarded as the set of isotropic subspaces V_{i+1} in $\mathbb{C}^{2n}$ which contain the isotropic subspace V_i as a codimension-one subspace. Note that for any vectors $v\in V_i^{\omega}\setminus V_i$, the subspace $V_i\oplus \mathrm{span}_{\mathbb{C}}(v)$ is an isotropic subspace which contains V_i as a codimension-one subspace. Therefore, there exists a one-to-one correspondence between the pull-back of a point in $\mathrm{Sp}^n\,\mathcal{F}_i$ by q_i and all complex lines in the quotient vector space $V_i^{\omega}/V_i\simeq\mathbb{C}^{2n-2i}$; ie $\mathrm{Sp}^n\,\mathcal{F}_{i+1}$ is a $\mathbb{C}\mathrm{P}^{2n-2i-1}$–bundle over $\mathrm{Sp}^n\,\mathcal{F}_i$. Using this fact, it is easy to check that $\mathrm{Sp}^n\,\mathcal{F}_{i+1}$ is the projectivization of the quotient bundle over $\mathrm{Sp}^n\,\mathcal{F}_i$; ie $\mathrm{Sp}^n\,\mathcal{F}_{i+1}=P(\zeta_i^{\omega}/\zeta_i)$, where the two tautological bundles ζ_i^{ω} and ζ_i are defined by the following subsets in $\mathrm{Sp}^n\,\mathcal{F}_i\times\mathbb{C}^{2n}$, respectively:

$$\big\{(\{0\}\subset V_1\subset\cdots\subset V_i\subset V_i^{\omega}\subset\cdots\subset V_1^{\omega}\subset\mathbb{C}^{2n},x)\mid x\in V_i^{\omega}\big\},$$
$$\big\{(\{0\}\subset V_1\subset\cdots\subset V_i\subset V_i^{\omega}\subset\cdots\subset V_1^{\omega}\subset\mathbb{C}^{2n},x)\mid x\in V_i\big\}.$$

Note that ζ_i^{ω} is a $\mathbb{C}^{2n-i}$–vector bundle and ζ_i is a $\mathbb{C}^i$–vector bundle; therefore, the quotient bundle ζ_i^{ω}/ζ_i is a $\mathbb{C}^{2n-2i}$–vector bundle. Therefore, $\mathrm{Sp}\,\mathcal{F}l(\mathbb{C}^{2n})$ has the structure of a $\mathbb{C}$P tower:

$$\mathrm{Sp}\,\mathcal{F}l(\mathbb{C}^{2n})=P(\zeta_{n-1}^{\omega}/\zeta_{n-1})\xrightarrow{\mathbb{C}\mathrm{P}^1}\mathrm{Sp}^n\,\mathcal{F}_{n-1}=P(\zeta_{n-2}^{\omega}/\zeta_{n-2})\xrightarrow{\mathbb{C}\mathrm{P}^3}\cdots\xrightarrow{\mathbb{C}\mathrm{P}^{2n-3}}\mathrm{Sp}^n\,\mathcal{F}_1$$
$$\simeq\mathbb{C}\mathrm{P}^{2n-1}\longrightarrow\{*\}.$$

Hence the flag manifold of type C is an element of $\mathbb{C}\mathrm{P}\mathcal{M}_n^{2n^2}$.

Remark 1 As is well known, both of the flag manifolds $\mathcal{F}l(\mathbb{C}^{n+1})\simeq U(n+1)/T^{n+1}$ and $\mathrm{Sp}\,\mathcal{F}l(\mathbb{C}^{2n})\simeq\mathrm{Sp}(n)/T^n$ with $n\geq 2$ do not admit the structure of a *toric manifold*; see [3], for example. On the other hand, $U(2)/T^2\cong\mathrm{Sp}(1)/T^1\cong\mathbb{C}\mathrm{P}^1$ is a toric manifold.

Moreover, by computing the generators of flag manifolds of other types— B_n $(n\geq 3)$, D_n $(n\geq 4)$, G_2, F_4, E_6, E_7, E_8 —we see that not all flag manifolds admit the structure of a $\mathbb{C}$P tower; see [2], or [6] for classical types. This leads us to the following proposition.

Proposition 2.3 *Let $M = G/T$ be a flag manifold, where G is a compact simple Lie group and T is its maximal torus. If M admits the structure of a $\mathbb{C}\mathrm{P}$ tower, then G must be a compact Lie group of type A or C.*

The following open problem naturally arises (also see Remark 2).

Problem 1 Let $H^*\colon \mathbb{C}\mathrm{P}\mathcal{M} \to \boldsymbol{H}^*\mathbb{C}\mathrm{P}\mathcal{M}$ be the map defined by taking cohomology rings. Classify the diffeomorphism types of all manifolds in the classes

$$(H^*)^{-1}\big(H^*(U(n+1)/T^{n+1})\big) \quad \text{and} \quad (H^*)^{-1}\big(H^*(\mathrm{Sp}(n)/T^n)\big).$$

3 Some preliminaries

3A Preliminaries from [7]

We first recall some basic facts from [7, Section 2].

Let ξ be an n–dimensional complex vector bundle over a topological space X, and let $P(\xi)$ denote its projectivization. Then

$$(2) \qquad H^*(P(\xi);\mathbb{Z}) \cong H^*(X;\mathbb{Z})[x] \Big/ \Big\langle x^{n+1} + \sum_{i=1}^{n} (-1)^i c_i(\pi^*\xi) x^{n+1-i} \Big\rangle,$$

where $\pi^*\xi$ is the pull-back of ξ along $\pi\colon P(\xi) \to X$ and $c_i(\pi^*\xi)$ is the i^{th} Chern class of $\pi^*\xi$ [7]. Here x can be viewed as the first Chern class of the canonical line bundle over $P(\xi)$; ie the complex 1–dimensional sub-bundle γ_ξ in $\pi^*\xi \to P(\xi)$ such that the restriction $\gamma_\xi|_{\pi^{-1}(a)}$ is the canonical line bundle over $\pi^{-1}(a) \cong \mathbb{C}\mathrm{P}^{n-1}$ for all $a \in X$. Therefore $\deg x = 2$. Since it is well known that the induced homomorphism $\pi^*\colon H^*(X;\mathbb{Z}) \to H^*(P(\xi);\mathbb{Z})$ is injective, we often abuse the notation $c_i(\pi^*\xi)$ by writing $c_i(\xi)$. The formula (2) is called the *Borel–Hirzebruch formula*.

To prove the main theorem, we often use the following two lemmas.

Lemma 3.1 *Let γ be any complex line bundle over M and let $P(\xi)$ be the projectivization of a complex vector bundle ξ over M. Then $P(\xi)$ is diffeomorphic to $P(\xi \otimes \gamma)$.*

Lemma 3.2 *Let γ be a complex line bundle and let ξ be a 2–dimensional complex vector bundle over a manifold M. Then the Chern classes of the tensor product $\xi \otimes \gamma$ are*

$$c_1(\xi \otimes \gamma) = c_1(\xi) + 2c_1(\gamma),$$
$$c_2(\xi \otimes \gamma) = c_1(\gamma)^2 + c_1(\gamma)c_1(\xi) + c_2(\xi).$$

3B The Atiyah–Rees theorem

By Theorem 1.1, all of the complex 2–plane bundles over $\mathbb{C}P^3$ can be written $\eta_{(\alpha,c_1,c_2)}$ for some $(\alpha, c_1, c_2) \in \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}$. Using Lemma 3.1, its projectivization $P(\eta_{(\alpha,c_1,c_2)})$ is diffeomorphic to $P(\eta_{(\alpha,c_1,c_2)} \otimes \gamma)$ for any complex line bundle γ over $\mathbb{C}P^3$. By Lemma 3.2 and the proof of [1, Theorem 2.8] (Theorem 1.1 here), we also have

$$\eta_{(\alpha,c_1,c_2)} \otimes \gamma \equiv \eta_{(\alpha,\, c_1+2c_1(\gamma),\, c_1(\gamma)^2+c_1(\gamma)c_1+c_2)}.$$

Thus we may assume $c_1 \in \{0,1\}$. Consequently, to classify all $P(\eta_{(\alpha,c_1,c_2)})$ up to diffeomorphisms, it is enough to classify

$$M_0(u) = P(\eta_{(0,0,u)}),$$
$$M_1(u) = P(\eta_{(1,0,u)}),$$
$$N(u) = P(\eta_{(0,1,u)}),$$

with $u \in \mathbb{Z}$. We denote the class of $M_0(u)$, $M_1(u)$ up to diffeomorphism by $\mathcal{M}$ and that of $N(u)$ by $\mathcal{N}$. Then both classes $\mathcal{M}$ and $\mathcal{N}$ are subclasses of $\mathbb{C}P\mathcal{M}^8_2$ consisting of 8–dimensional 2–stage $\mathbb{C}$P manifolds.

3C The intersection of $\mathcal{M}$ and $\mathcal{N}$ is empty

We prove that $\mathcal{M} \cap \mathcal{N} = \varnothing$ by comparing cohomology rings. Namely, we prove the following lemma.

Lemma 3.3 *Two cohomology rings $H^*(M_\alpha(u))$ and $H^*(N(u'))$ are not isomorphic for any $u, u' \in \mathbb{Z}$.*

Proof Using the Borel–Hirzebruch formula (2), we have the cohomology rings with $\mathbb{Z}_2$–coefficients

$$H^*(M_\alpha(u);\mathbb{Z}_2) \cong \mathbb{Z}_2[X,Y]/\langle X^4, uX^2+Y^2\rangle,$$
$$H^*(N(u');\mathbb{Z}_2) \cong \mathbb{Z}_2[x,y]/\langle x^4, u'x^2+xy+y^2\rangle.$$

Now, the element $uX+Y$ in $H^2(M_\alpha(u);\mathbb{Z}_2)$ satisfies

$$(uX+Y)^2 = u^2X^2 + 2uXY + Y^2 \equiv uX^2 + Y^2 (=0) \mod 2.$$

However, the squares of all non-zero elements x, y, $x+y$ in $H^2(N(u');\mathbb{Z}_2)$ are not zero because of its ring structure. Hence

$$H^*(M_\alpha(u)) \ncong H^*(N(u')) \quad \text{for all } u, u' \in \mathbb{Z}. \qquad \square$$

Corollary 3.4 *The classes $\mathcal{M}$ and $\mathcal{N}$ are disjoint.*

4 Cohomological rigidity of $\mathcal{N}$

In this section, we prove the cohomological rigidity of the class $\mathcal{N}$. It is enough to prove the following lemma.

Lemma 4.1 *The following statements are equivalent for integers u, u':*

(1) $H^*(N(u)) \cong H^*(N(u'))$.

(2) $u = u'$.

Proof Because (2) $\Rightarrow$ (1) is trivial, it is enough to show (1) $\Rightarrow$ (2). Assume there is an isomorphism $f\colon H^*(N(u)) \cong H^*(N(u'))$, where

$$H^*(N(u)) \cong \mathbb{Z}[X, Y]/\langle X^4, uX^2 + xy + Y^2\rangle,$$
$$H^*(N(u')) \cong \mathbb{Z}[x, y]/\langle x^4, u'x^2 + xy + y^2\rangle.$$

Here we may set

$$f(X) = ax + by \quad \text{and} \quad f(Y) = cx + dy$$

for some $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = \epsilon = \pm 1$. By taking its inverse, we also have

$$f^{-1}(x) = d\epsilon X - b\epsilon Y \quad \text{and} \quad f^{-1}(y) = -c\epsilon X + a\epsilon Y.$$

Because $f(Y^2 + XY + uX^2) = 0$ and $f^{-1}(y^2 + xy + u'x^2) = 0$, we get

$$c^2 - d^2u' = -ua^2 + b^2uu' - ac + bdu', \tag{3}$$

$$2cd - d^2 = -2abu + b^2u - ad - bc + bd. \tag{4}$$

Because $f(X^4) = 0$ and $f^{-1}(x^4) = 0$, one of the following holds:

(1) $b = 0$.

(2) $b \neq 0$ and $4a^3 - 6a^2b + 4ab^2(1-u') + b^3(2u'-1) = -4d^3 - 6d^2b - 4db^2(1-u) + b^3(2u-1) = 0$.

If $b = 0$, then $|a| = |d| = 1$. Therefore, by (4), $2c = d - a$; ie $c = 0$ if $d = a$ or $c = -a$ if $d = -a$. Because $c^2 - u' = -u - ac$ by (3), we have that $u = u'$.

Assume $b \neq 0$. Because $4a^3 - 6a^2b + 4ab^2(1-u') + b^3(2u'-1) = 0$, b is even. Therefore, since $ad - bc = \pm 1$, a is odd. We note that the equation $4a^3 - 6a^2b + 4ab^2(1-u') + b^3(2u'-1) = 0$ can be written

$$(2a - b)(2a^2 - 2ab + b^2 - 2b^2u') = 0.$$

Because a is odd and b is even, the second factor is not zero; therefore

$$b = 2a.$$

Since $ad-bc=\pm 1$, we conclude $(a,b)=\pm(1,2)$. The same argument applied to the equation $-4d^3-6d^2b-4db^2(1-u)+b^3(2u-1)=0$ shows that $-b=2d$ and $(d,b)=\pm(-1,2)$. Therefore, (a,b,d) must be either $(1,2,-1)$ or $(-1,-2,1)$. Then $c=0$ or -1 in the former case while $c=0$ or 1 in the latter because $ad-bc=\pm 1$. In any case, it follows from (3) that $u'+u=4uu'$, an identity which holds only when $u=u'=0$ since $u,u'\in\mathbb{Z}$. This completes the case where $b\neq 0$. □

Theorem 1.1 and Lemma 4.1 give the next theorem, which establishes Theorem 1.2(1).

Theorem 4.2 *The following three statements are equivalent:*

(1) *$N(u)$ and $N(u')$ are diffeomorphic.*

(2) *The cohomology rings $H^*(N(u))$ and $H^*(N(u'))$ are isomorphic.*

(3) *$u=u'\in\mathbb{Z}$.*

In particular, the class $\mathcal{N}$ is cohomologically rigid.

5 Cohomological non-rigidity of $\mathbb{C}\mathrm{P}\mathcal{M}_2^8$

Lemma 5.1 *The following two statements are equivalent for integers u,u':*

(1) *$H^*(M_\alpha(u))\cong H^*(M_{\alpha'}(u'))$, where $\alpha,\alpha'\in\{0,1\}$.*

(2) *$u=u'$.*

Proof Because $(2)\Rightarrow(1)$ is trivial, it is enough to show $(1)\Rightarrow(2)$. Assume there is an isomorphism $f\colon H^*(M_\alpha(u))\to H^*(M_{\alpha'}(u'))$, where

$$H^*(M_\alpha(u))\cong\mathbb{Z}[X,Y]/\langle X^4,uX^2+Y^2\rangle,$$
$$H^*(M_{\alpha'}(u'))\cong\mathbb{Z}[x,y]/\langle x^4,u'x^2+y^2\rangle.$$

We may use the same representation for f as in the proof of Lemma 4.1. Note that $f(uX^2+Y^2)=0$ and $f^{-1}(u'x^2+y^2)=0$. Using the representation of f, we have

$$ua^2-uu'b^2+c^2-u'd^2=0, \tag{5}$$
$$u'd^2-uu'b^2+c^2-a^2u=0, \tag{6}$$

which lead to

$$c^2=b^2uu', \tag{7}$$
$$ua^2=u'd^2. \tag{8}$$

Because $X^4 = 0$,

$$ab(a^2 - b^2u') = 0.$$

We first assume $ab \neq 0$. Then

$$a^2 = b^2u'.$$

Together with (7) and (8), we have

$$c^2b^2 = b^4uu' = b^2a^2u = b^2d^2u' = a^2d^2.$$

This implies

$$(ad - bc)(ad + bc) = \epsilon(ad + bc) = 0.$$

Hence $ad = -bc$. However this gives a contradiction because $ad - bc = 2ad = \epsilon = \pm 1$. Consequently, $ab = 0$. Since $ad - bc = \epsilon$, if $a = 0$ then $|b| = |c| = 1$, so $u = u' = \pm 1$ by (7), and if $b = 0$ then $|a| = |d| = 1$, so $u = u'$ by (8). This establishes the lemma. □

Lemma 5.1 says that cohomology rings of $\mathcal{M}$ are not affected by $\alpha \in \mathbb{Z}_2$. On the other hand, the goal of this section is to prove the following theorem, that some topological types of $\mathcal{M}$ are affected by $\alpha \in \mathbb{Z}_2$.

Theorem 5.2 *Assume $u(u+1)/12 \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{Z}_2$. The following are equivalent:*

1. *$M_\alpha(u)$ and $M_\beta(u')$ are diffeomorphic.*
2. *$(\alpha, u) = (\beta, u') \in \mathbb{Z}_2 \times \mathbb{Z}$.*
3. *$M_\alpha(u)$ and $M_\beta(u')$ are homotopy equivalent.*

In order to prove Theorem 5.2, we first compute the 6–dimensional homotopy group of $M_\alpha(u)$ in Proposition 5.4. Now $M_\alpha(u)$ can be defined by the following pull-back diagram.

$$\begin{array}{ccc} M_\alpha(u) & \longrightarrow & EU(2) \times_{U(2)} \mathbb{C}P^1 \\ \downarrow & & \downarrow \\ \mathbb{C}P^3 & \xrightarrow{\mu_{\alpha,u}} & BU(2) \end{array}$$

Let $p\colon S^7 \to \mathbb{C}P^3$ be the canonical S^1–fibration and $P(\xi_{\alpha,u})$ be the pull-back of $M_\alpha(u)$ along p. Namely, the following diagram commutes.

(9)
$$\begin{array}{ccccc} P(\xi_{\alpha,u}) & \longrightarrow & M_\alpha(u) & \longrightarrow & EU(2) \times_{U(2)} \mathbb{C}P^1 \\ \downarrow & & \downarrow & & \downarrow \\ S^7 & \xrightarrow{p} & \mathbb{C}P^3 & \xrightarrow{\mu_{\alpha,u}} & BU(2) \end{array}$$

Lemma 5.3 *For* $* \geq 3$, $\pi_*(P(\xi_{\alpha,u})) \cong \pi_*(M_\alpha(u))$.

Proof Because $P(\xi_{\alpha,u})$ is the pull-back of $M_\alpha(u)$, the homotopy exact sequences of $P(\xi_{\alpha,u})$ and $M_\alpha(u)$ satisfy the following commutative diagram.

$$\begin{array}{ccccccccc}
\pi_{*+1}(S^7) & \longrightarrow & \pi_*(\mathbb{C}\mathrm{P}^1) & \longrightarrow & \pi_*(P(\xi_{\alpha,u})) & \longrightarrow & \pi_*(S^7) & \longrightarrow & \pi_{*-1}(\mathbb{C}\mathrm{P}^1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_{*+1}(\mathbb{C}\mathrm{P}^3) & \longrightarrow & \pi_*(\mathbb{C}\mathrm{P}^1) & \longrightarrow & \pi_*(M_\alpha(u)) & \longrightarrow & \pi_*(\mathbb{C}\mathrm{P}^3) & \longrightarrow & \pi_{*-1}(\mathbb{C}\mathrm{P}^1)
\end{array}$$

From the homotopy exact sequence of the fibration $S^1 \to S^7 \to \mathbb{C}\mathrm{P}^3$, we have $\pi_*(S^7) \cong \pi_*(\mathbb{C}\mathrm{P}^3)$ for $* \geq 3$. Therefore, by the five lemma, the proof is complete. □

Proposition 5.4 *Assume* $u(u+1)/12 \in \mathbb{Z}$.

(1) $\pi_6(P(\xi_{\alpha,u})) \cong \pi_6(M_\alpha(u)) \cong \mathbb{Z}_{12}$ *if* $\alpha \equiv u(u+1)/12 \pmod 2$.

(2) $\pi_6(P(\xi_{\beta,u})) \cong \pi_6(M_\beta(u)) \cong \mathbb{Z}_6$ *if* $\beta \not\equiv u(u+1)/12 \pmod 2$.

Proof First we prove (1). If $u(u+1)/12 \in \mathbb{Z}$ and $\alpha \equiv u(u+1)/12 \pmod 2$, then it follows from [1] that $\xi_{\alpha,u}$ is induced from the rank-2 complex vector bundle over $\mathbb{C}\mathrm{P}^4$. Namely, the following diagram commutes.

$$\begin{array}{ccccccc}
\xi_{\alpha,u} & \longrightarrow & \eta_{(\alpha,0,u)} & \longrightarrow & \widetilde{\mu}_{\alpha,u} & \longrightarrow & EU(2) \times_{U(2)} \mathbb{C}^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^7 & \xrightarrow{p} & \mathbb{C}\mathrm{P}^3 & \longrightarrow & \mathbb{C}\mathrm{P}^4 & \longrightarrow & BU(2)
\end{array} \tag{10}$$

On the other hand, $\pi_7(\mathbb{C}\mathrm{P}^4) \cong \pi_7(S^9) = \{0\}$, using the homotopy exact sequence for the fibration $S^1 \to S^9 \to \mathbb{C}\mathrm{P}^4$. This implies that $\xi_{\alpha,u}$ is the trivial $\mathbb{C}^2$–bundle over S^7. Therefore,

$$P(\xi_{\alpha,u}) = S^7 \times \mathbb{C}\mathrm{P}^1$$

when $u(u+1)/12 \in \mathbb{Z}$ and $\alpha \equiv u(u+1)/12 \pmod 2$. Hence, we also have

$$\pi_6(M_\alpha(u)) \cong \pi_6(S^7 \times \mathbb{C}\mathrm{P}^1) \cong \pi_6(\mathbb{C}\mathrm{P}^1) \cong \mathbb{Z}_{12}.$$

Next we prove the second statement. Let $\mu_{\alpha,u}\colon \mathbb{C}\mathrm{P}^3 \to BU(2)$ be a continuous map which induces the above $\eta_{(\alpha,0,u)}$, and β be the element in $\mathbb{Z}_2$ which is not equal to α. Let $x \in \mathbb{C}\mathrm{P}^3$ and $s = \mu_{\alpha,u}(x) \in BU(2)$ be base points. Take a disk neighborhood around $x \in \mathbb{C}\mathrm{P}^3$ and pinch its boundary to a point, ie the boundary of $D^6 \subset \mathbb{C}\mathrm{P}^3$ pinches to a point; then we obtain a surjective map

$$\rho\colon \mathbb{C}\mathrm{P}^3 \to \mathbb{C}\mathrm{P}^3 \vee S^6,$$

where $\mathbb{C}\mathrm{P}^3 \vee S^6$ may be regarded as the wedge sum with respect to the base points $x \in \mathbb{C}\mathrm{P}^3$ and $y \in S^6$. Due to Theorem 1.1, we have $\eta_{(\beta,0,u)} \not\equiv \eta_{(\alpha,0,u)}$. This implies that the vector bundle $\eta_{(\beta,0,u)}$ is induced from the continuous map

$$\mu_{\beta,u}\colon \mathbb{C}\mathrm{P}^3 \xrightarrow{\rho} \mathbb{C}\mathrm{P}^3 \vee S^6 \xrightarrow{\nu_\alpha} BU(2), \tag{11}$$

where $\nu_\alpha = \mu_{\alpha,u} \vee \kappa$ for the generator $\kappa \in \pi_6(BU(2), s) \cong \mathbb{Z}_2$.[1] Hence, we have the following commutative diagram.

$$\begin{array}{ccccc}
P(\xi_{\beta,u}) & \longrightarrow & M_\beta(u) & \longrightarrow & EU(2) \times_{U(2)} \mathbb{C}\mathrm{P}^1 \\
\downarrow & & \downarrow & & \downarrow \\
S^7 & \xrightarrow{p} & \mathbb{C}\mathrm{P}^3 & \xrightarrow{\mu_{\beta,u}} & BU(2) \\
& \searrow & \downarrow \rho & \nearrow \nu_\alpha & \\
& & \mathbb{C}\mathrm{P}^3 \vee S^6 & &
\end{array} \tag{12}$$

From the $\mathbb{C}\mathrm{P}^1$–fibrations $\mathbb{C}\mathrm{P}^1 \to P(\xi_{\beta,u}) \to S^7$ and $\mathbb{C}\mathrm{P}^1 \to EU(2) \times_{U(2)} \mathbb{C}\mathrm{P}^1 \cong BT^2 \to BU(2)$ in the diagram (12), we get the following commutative diagram.

$$\begin{array}{ccccccc}
\pi_7(S^7) \cong \mathbb{Z} & \longrightarrow & \pi_6(\mathbb{C}\mathrm{P}^1) & \longrightarrow & \pi_6(P(\xi_{\beta,u})) & \longrightarrow & \pi_6(S^7) = \{0\} \\
\downarrow & & \downarrow \cong & & \downarrow & & \downarrow \\
\pi_7(BU(2)) \cong \mathbb{Z}_{12} & \xrightarrow{\cong} & \pi_6(\mathbb{C}\mathrm{P}^1) & \longrightarrow & \pi_6(BT^2) = \{0\} & \longrightarrow & \pi_6(BU(2)) \cong \mathbb{Z}_2
\end{array}$$

This diagram shows that the following sequence is exact:

$$\mathbb{Z} \cong \pi_7(S^7) \to \pi_7(BU(2))(\cong \mathbb{Z}_{12}) \to \pi_6(P(\xi_{\beta,u})) \to \{0\}. \tag{13}$$

In this diagram, the left homomorphism is induced from $\widetilde{\mu} := \mu_{\beta,u} \circ p\colon S^7 \to BU(2)$, say $\widetilde{\mu}_\#\colon \mathbb{Z} \to \mathbb{Z}_{12}$. We claim $\widetilde{\mu}_\#(1) = [6]_{12} \in \mathbb{Z}_{12}$. Because the diagram (12) is commutative, we can think of $\widetilde{\mu} := \mu_{\beta,u} \circ p\colon S^7 \to BU(2)$ as being defined by passing through the map $\nu_\alpha\colon \mathbb{C}\mathrm{P}^3 \vee S^6 \to BU(2)$; ie $\widetilde{\mu} = \nu_\alpha \circ \rho \circ p$. Because $\nu_\alpha = \mu_{\alpha,u} \vee \kappa$, we also have

$$\widetilde{\mu} = (\mu_{\alpha,u} \vee \kappa) \circ \rho \circ p = (\mu_{\alpha,u} \circ \rho \circ p) \vee (\kappa \circ \rho \circ p).$$

By the argument we used while proving the first statement, we see that $\mu_{\alpha,u} \circ \rho \circ p$ induces the trivial bundle over S^7; ie $\mu_{\alpha,u} \circ \rho \circ p$ is homotopic to the trivial map. This

[1] This construction induces the free $\pi_6(BU(2)) \cong \pi_5(U(2)) \cong \mathbb{Z}_2$ action on $\widetilde{K\mathrm{Sp}}(\mathbb{C}\mathrm{P}^3) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$; see [1].

also implies that the decomposition

$$\widetilde{\mu}\colon S^7 \xrightarrow{p} \mathbb{C}\mathrm{P}^3 \xrightarrow{\rho} \mathbb{C}\mathrm{P}^3 \vee S^6 \xrightarrow{\pi} S^6 \xrightarrow{\kappa} BU(2)$$

exists up to homotopy, where π is the collapsing map of $\mathbb{C}\mathrm{P}^3$ to a point. Therefore, we have the following decomposition for the induced map:

$$\widetilde{\mu}_{\#}\colon \pi_7(S^7) \xrightarrow{\Psi_{\#}} \pi_7(S^6) \cong \mathbb{Z}_2 \xrightarrow{\kappa_{\#}} \pi_7(BU(2)) \cong \mathbb{Z}_{12},$$

where the first map is induced from the surjective map $\Psi = \pi \circ \rho \circ p$. Because Ψ is surjective, ie not homotopic to the trivial map, we have $\Psi_{\#}(1) = [1]_2$ (the generator of $\pi_7(S^6) \cong \mathbb{Z}_2$). Moreover, because $\kappa \in \pi_6(BU(2)) \cong \mathbb{Z}_2$ is the generator, ie non-trivial map, we have $\kappa_{\#}([1]_2) = [6]_{12} \in \mathbb{Z}_{12}$. This shows that $\widetilde{\mu}_{\#}(1) = [6]_{12}$; therefore $\widetilde{\mu}_{\#}(\pi_7(S^7)) = \{[0]_{12}, [6]_{12}\} \subset \mathbb{Z}_{12}$.

Consequently, by the exact sequence (13),

$$\pi_6(P(\xi_{\beta,u})) \cong \pi_7(BU(2))/\widetilde{\mu}_{\#}(\pi_7(S^7)) \cong \mathbb{Z}_{12}/\{[0]_{12}, [6]_{12}\} \cong \mathbb{Z}_6.$$

By Lemma 5.3, we have the statement. □

Remark 2 For example, the condition $u(u+1)/12 \in \mathbb{Z}$ is satisfied when $u = 0$ and $u = 3$. In these cases, using Proposition 5.4, we have

$$\pi_6(M_\alpha(0)) \cong \begin{cases} \mathbb{Z}_{12} & \text{for } \alpha \equiv 0 \\ \mathbb{Z}_6 & \text{for } \alpha \equiv 1 \end{cases} \quad \text{and} \quad \pi_6(M_\alpha(3)) \cong \begin{cases} \mathbb{Z}_6 & \text{for } \alpha \equiv 0 \\ \mathbb{Z}_{12} & \text{for } \alpha \equiv 1. \end{cases}$$

On the other hand, the case when $u = 1$ does not satisfy the condition $u(u+1)/12 \in \mathbb{Z}$. It follows from the cohomology ring of the flag manifold of type C (see for example [2] or [6]) that the flag manifold $\mathrm{Sp}(2)/T^2$ is one for which $u = 1$; ie $M_0(1)$ or $M_1(1)$. However, using the homotopy exact sequence for the fibration $T^2 \to \mathrm{Sp}(2) \to \mathrm{Sp}(2)/T^2$ and the computation in [8],

$$\pi_6(\mathrm{Sp}(2)/T^2) \cong \pi_6(\mathrm{Sp}(2)) = 0.$$

Therefore, Proposition 5.4 is not true in the case where $u(u+1)/12 \notin \mathbb{Z}$.

Proof of Theorem 5.2 $(2) \Rightarrow (1)$ is trivial, as is $(1) \Rightarrow (3)$. We claim $(3) \Rightarrow (2)$. Assume $M_\alpha(u)$ and $M_\beta(u')$ are homotopy equivalent. Then $H^*(M_\alpha(u)) \cong H^*(M_\beta(u'))$. Therefore, it follows from Lemma 5.1 that $u = u'$. Moreover, in this case, $\pi_6(M_\alpha(u)) \cong \pi_6(M_\beta(u))$. If $\alpha \not\equiv \beta \mod 2$, then this gives a contradiction to Proposition 5.4. Hence, $\alpha \equiv \beta \mod 2$. We have $(3) \Rightarrow (2)$. This establishes Theorem 5.2. □

Lemma 5.1 and Theorem 5.2 imply the following corollary, establishing Theorem 1.2(2).

Corollary 5.5 *The set of* 8*–dimensional* $\mathbb{C}\mathrm{P}$ *manifolds is not cohomologically rigid.*

Note that if we restrict the class of 8–dimensional $\mathbb{C}$P manifolds to the 8–dimensional generalized Bott manifolds with height 2, then cohomological rigidity holds [4]. On the other hand, the following seems to be more natural to ask of the class of $\mathbb{C}$P manifolds $\mathbb{C}\mathrm{P}\mathcal{M}$ than the cohomological rigidity problem.

Problem 2 Is the class $\mathbb{C}\mathrm{P}\mathcal{M}$ of $\mathbb{C}$P manifolds (up to diffeomorphism) determined by homotopy types? More precisely, are $M_1, M_2 \in \mathbb{C}\mathrm{P}\mathcal{M}$ diffeomorphic if they have the same homotopy type?

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