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On Kauffman bracket skein modules at roots of unity

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We reprove and expand results of Bonahon and Wong on central elements of the Kauffman bracket skein modules at roots of 1 and on the existence of the Chebyshev homomorphism, using elementary skein methods.

57N10; 57M25

0 Introduction

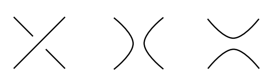
0.1 Kauffman bracket skein modules

Let us recall the definition of the Kauffman bracket skein module, which was introduced by J Przytycki [15] and V Turaev [18]. Let $R=\mathbb{C}[t^{\pm 1}]$. A *framed link* in an oriented 3–manifold M is a disjoint union of smoothly embedded circles, equipped with a nonzero normal vector field. The empty set is also considered a framed link. The Kauffman bracket skein module $\mathcal{S}(M)$ is the R–module spanned by isotopy classes of framed links in M subject to the relations

$$L = tL_+ + t^{-1}L_-, \tag{1}$$

$$L \sqcup U = -(t^2+t^{-2})L, \tag{2}$$

where in the first identity, L, L_+, L_- are identical except in a ball in which they look like they do in Figure 1, and in the second identity, the left-hand side stands for the union of a link L and the trivial framed knot U in a ball disjoint from L. If $M=\mathbb{R}^3$ then $\mathcal{S}(\mathbb{R}^3)=R$. The value of a framed link L in $\mathcal{S}(\mathbb{R}^3)=R=\mathbb{C}[t^{\pm 1}]$ is a version of the Jones polynomial; see Kauffman [10].



L $\quad L_+ \quad L_-$

Figure 1: The links L, L_+ and L_-

For a nonzero complex number ξ, let $\mathcal{S}_\xi(M)$ be the quotient $\mathcal{S}(M)/(t-\xi)$, which is a $\mathbb{C}$–vector space.

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For an oriented surface Σ, possibly with boundary, we define $\mathcal{S}(\Sigma) := \mathcal{S}(M)$, where $M = \Sigma \times [-1,1]$ is the cylinder over Σ. The skein module $\mathcal{S}(\Sigma)$ has an algebra structure induced by the operation of gluing one cylinder on top of the other.

For a framed knot K in M and a polynomial $p(z) = \sum_{j=0}^{d} a_j z^j \in \mathbb{C}[z]$, then we define $p(K)$ by

$$p(K) = \sum_{j=0}^{d} a_j K^{(j)} \in \mathcal{S}(M),$$

where $K^{(j)}$ is the link consisting of j parallels of K (using the framing of K) in a small neighborhood of K. When L is a link, define $p(L)$ by applying p to each component of L. More precisely, for a framed link $L \subset M$ with m components $L_1, \dots, L_m$, define

$$p(L) = \sum_{j_1,\dots,j_m=0}^{d} \Big(\prod_{k=1}^{m} a_{j_k}\Big)\Big(\bigsqcup_{k=1}^{m} L_k^{(j_k)}\Big).$$

Here $\bigsqcup_{k=1}^{m} L_k^{(j_k)}$ is the link which is the union, over $k \in \{1,\dots,m\}$, of j_k parallels of L_k.

Remark 0.1 Suppose $K \subset \Sigma$ is a simple closed curve on the surface Σ. Consider K as a framed knot in $\Sigma \times [-1,1]$ by identifying $\Sigma = \Sigma \times 0$ and equipping K with the vertical framing, ie the framing where the normal vector is perpendicular to Σ and has direction from -1 to 1. Then $K^{(j)} = K^j$, where K^j is the power in the algebra $\mathcal{S}(\Sigma)$. Thus, $p(K)$ has the usual meaning of applying a polynomial to an element of an algebra.

But if K is a knot in $\Sigma \times [-1,1]$, our $p(K)$ in general is not the result of applying the polynomial p to the element K using the algebra structure of $\mathcal{S}(\Sigma)$, ie $p(K) \neq \sum a_j K^j$.

0.2 Bonahon and Wong's results

Definition 1 A polynomial $p(z) \in \mathbb{C}[z]$ is called *central* at $\xi \in \mathbb{C}^\times$ if for any oriented surface Σ and any framed link L in $\Sigma \times [-1,1]$, $p(L)$ is central in the algebra $\mathcal{S}_\xi(\Sigma)$.

Bonahon and Wong [2] showed that if ξ is a root of unity of order $2N$, then $T_N(z)$ is central, where $T_N(z)$ is the Chebyshev polynomial of type 1 defined recursively by

$$T_0(z) = 2, \quad T_1(z) = 1, \quad T_n(z) = zT_{n-1}(z) - T_{n-2}(z),$$

for all $n \geq 2$. We will prove a stronger version, using a different method.

Theorem 1 *A nonconstant polynomial $p(z) \in \mathbb{C}[z]$ is central at $\xi \in \mathbb{C}^\times$ if and only if*

(i) *ξ is a root of unity,*

(ii) *$p(z) \in \mathbb{C}[T_N(z)]$, ie p is a $\mathbb{C}$–polynomial in $T_N(z)$, where N is the order of ξ^2.*

Remark 0.2 We also find a version of "skew-centrality" when $\xi^{2N} = -1$ (see Section 2), which will be useful in this paper and elsewhere.

Remark 0.3 Let us call a polynomial $p(z) \in \mathbb{C}[z]$ *weakly central* at $\xi \in \mathbb{C}^\times$ if for any oriented surface Σ and any simple closed curve K on Σ, $p(K)$ is central in the algebra $\mathcal{S}_\xi(\Sigma)$. Then our proof will also show that Theorem 1 holds true if one replaces "central" by "weakly central". It follows that being central is equivalent to being weakly central.

A remarkable result of Bonahon and Wong is the following.

Theorem 2 (Bonahon–Wong [2]) *Let M be an oriented 3–manifold, possibly with boundary. Suppose ξ^4 is a root of unity of order N. Let $\varepsilon = \xi^{N^2}$. There is a unique $\mathbb{C}$–linear map* **Ch**: *$\mathcal{S}_\varepsilon(M) \to \mathcal{S}_\xi(M)$ such that for any framed link $L \subset M$,* $\mathbf{Ch}(L) = T_n(L)$.

If $M = \Sigma \times [-1, 1]$, then the map **Ch** is an algebra homomorphism. Actually Bonahon and Wong only consider the case of $\mathcal{S}(\Sigma)$, but their proof works also in the case of skein modules of 3–manifolds. In their proof, Bonahon and Wong used the theory of quantum Teichmüller space of Chekhov and Fock [7] and Kashaev [9], and the quantum trace homomorphism developed in their earlier work [3]. Bonahon and Wong asked for a proof using elementary skein theory. We will present one here. The main idea is to use central properties (in a more general setting) and several operators and filtrations on the skein modules defined by arcs.

In general, the calculation of $\mathcal{S}(M)$ is difficult. For some results on knot and link complements in S^3, see the author [11], the author and Tran [12] and Marché [14]. Note that if $\xi^{4N} = 1$, then $\varepsilon = \xi^{N^2}$ is a 4^{th} root of 1. In this case the $\mathcal{S}_\varepsilon(M)$ is well known and is related to character varieties of M. This makes Theorem 2 interesting. At $t = -1$, $\mathcal{S}_{-1}(M)$ has an algebra structure and, modulo its nilradical, is equal to the ring of regular functions on the $\mathrm{SL}_2(\mathbb{C})$–character variety of M; see Bullock [4], Przytycki and Sikora [16] and Bullock, Frohman and Kania-Bartoszyńska [5]. For the case when ε is a primitive 4^{th} root of 1, see Sikora [17].

0.3 Plan of the paper

Section 1 is preliminaries on Chebyshev polynomials and relative skein modules. Section 2 contains the proof of Theorem 1. Section 3 introduces the filtrations and operators on skein modules, and Sections 4 and 5 contain some calculations which are used in Section 6, where the main technical lemma about the skein module of the twice-punctured torus is proved. Theorem 2 is proved in Section 7.

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1 Ground ring, Chebyshev polynomials and relative skein modules

1.1 Ground ring

Let $R=\mathbb{C}[t^{\pm 1}]$, which is a principal ideal domain. For an R–module and a nonzero complex number $\xi\in\mathbb{C}^\times$ let V_ξ be the R–module $V/(t-\xi)$. Then $R_\xi\cong\mathbb{C}$ as $\mathbb{C}$–modules, and V_ξ has a natural structure of an R_ξ–module.

We will often use the constants

$$\lambda_k := -(t^{2k+2}+t^{-2k-2})\in R. \tag{3}$$

For example, λ_0 is the value of the unknot U as a skein element.

1.2 Chebyshev polynomials

Recall that the Chebyshev polynomials of type 1 $T_n(z)$ and type 2 $S_n(z)$ are given by

$$T_0=2,\quad T_1(z)=z,\quad T_n(z)=zT_{n-1}(z)-T_{n-2}(z),$$
$$S_0=1,\quad S_1(z)=z,\quad S_n(z)=zS_{n-1}(z)-S_{n-2}(z).$$

Here are some well-known facts. We drop the easy proofs.

Lemma 1.1 (i) *One has*

$$T_n(u+u^{-1})=u^n+u^{-n}, \tag{4}$$

$$T_n=S_n-S_{n-2}. \tag{5}$$

(ii) *For a fixed positive integer N, the $\mathbb{C}$–span of $\{T_{Nj}\mid j\ge 0\}$ is $\mathbb{C}[T_N(z)]$, the ring of all $\mathbb{C}$–polynomials in $T_N(z)$.*

Since $T_n(z)$ has leading term z^n, $\{T_n(z) \mid n \geq 0\}$ is a $\mathbb{C}$–basis of $\mathbb{C}[z]$.

1.3 Skein module of a surface

Suppose Σ is a compact connected orientable 2–dimensional manifold with boundary. A knot in Σ is *trivial* if it bounds a disk in Σ. Recall that $\mathcal{S}(\Sigma)$ is the skein module $\mathcal{S}(\Sigma \times [-1,1])$. If $\partial\Sigma \neq \varnothing$, then $\mathcal{S}(\Sigma)$ is a free R–module with basis the set of all links in Σ without trivial components, including the empty link; see [16]. Here a link in Σ is considered as a framed link in $\Sigma \times [-1,1]$ by identifying Σ with $\Sigma \times 0$, and the framing at every point $P \in \Sigma \times 0$ is vertical, ie given by the unit positive tangent vector of $P \times [-1,1] \subset \Sigma \times [-1,1]$.

The R–module $\mathcal{S}(\Sigma)$ has a natural R–algebra structure, where $L_1 L_2$ is obtained by placing L_1 on top of L_2.

It might happen that $\Sigma_1 \times [-1,1] \cong \Sigma_2 \times [-1,1]$ with $\Sigma_1 \ncong \Sigma_2$. In that case, $\mathcal{S}(\Sigma_1)$ and $\mathcal{S}(\Sigma_2)$ are the same as R–modules, but the algebra structures may be different.

1.4 Example: The annulus

Let $\mathbb{A} \subset \mathbb{R}^2$ be the annulus $\mathbb{A} = \{\vec{x} \in \mathbb{R}^2 \mid 1 \leq |\vec{x}| \leq 2\}$. Let $z \in \mathcal{S}(\mathbb{A})$ be the core of the annulus, $z = \{\vec{x}, |\vec{x}| = \frac{3}{2}\}$. Then $\mathcal{S}(\mathbb{A}) = R[z]$.

1.5 Relative skein modules

A *marked surface* $(\Sigma, \mathcal{P})$ is a surface Σ together with a finite set $\mathcal{P}$ of points on its boundary $\partial\Sigma$. For such a marked surface, a *relative framed link* is a 1–dimensional compact framed submanifold X in $\Sigma \times [-1,1]$ such that $\partial X = \mathcal{P} = X \cap \partial(\Sigma \times [-1,1])$, X is perpendicular to $\partial(\Sigma \times [-1,1])$ and the framing at each point $P \in \mathcal{P} = \partial X$ is vertical. The relative skein module $\mathcal{S}(\Sigma, \mathcal{P})$ is defined as the R–module spanned by the isotopy class of relative framed links modulo the same skein relations (1) and (2). We will use the following fact.

Proposition 1.2 [16, Theorem 5.2] *The R–module $\mathcal{S}(\Sigma, \mathcal{P})$ is free with basis the set of isotopy classes of relative links embedded in Σ without trivial components.*

2 Annulus with two marked points and central elements

2.1 Marked annulus

Recall that $\mathbb{A} \subset \mathbb{R}^2$ is the annulus $\mathbb{A} = \{\vec{x} \in \mathbb{R}^2 \mid 1 \leq |\vec{x}| \leq 2\}$. Let $\mathbb{A}_{io}$ be the marked surface $(\mathbb{A}, \{P_1, P_2\})$, with two marked points $P_1 = (0,1)$, $P_2 = (0,2)$, which are on different boundary components. See Figure 2, which also depicts the arcs e, $\mathfrak{u}$, $\mathfrak{u}^{-1}$.

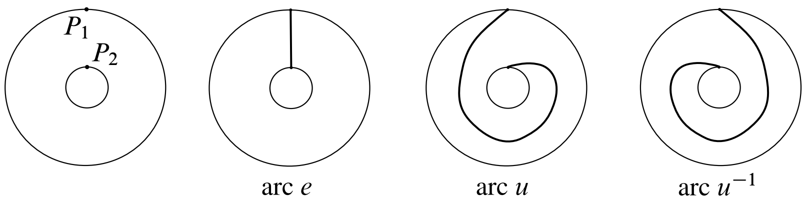


Figure 2: The marked annulus $\mathbb{A}_{io}$ and the arcs e, $\mathfrak{u}$ and $\mathfrak{u}^{-1}$

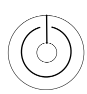
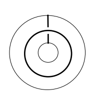
For $L_1, L_2 \in \mathcal{S}(\mathbb{A}_{io})$ define the product $L_1 L_2$ by placing L_1 inside L_2. Formally this means we first shrink $\mathbb{A}_{io} \supset L_1$ by $\frac{1}{2}$, we get $(\frac{1}{2}\mathbb{A}_{io}) \supset (\frac{1}{2}L_1)$, where $\frac{1}{2}\mathbb{A}_{io}$ is an annulus on the plane whose outer circle is the inner circle of $\mathbb{A}_{io}$. Then $L_1 L_2$ is $(\frac{1}{2}L_1) \cup L_2 \subset (\frac{1}{2}\mathbb{A}_{io}) \cup \mathbb{A}_{io}$. The identity of $\mathcal{S}(\mathbb{A}_{io})$ is presented by e, and $\mathfrak{u}^{-1}\mathfrak{u} = e = \mathfrak{u}\mathfrak{u}^{-1}$.

Proposition 2.1 *The Kauffman bracket skein modules of $\mathbb{A}_{io}$ are $\mathcal{S}(\mathbb{A}_{io}) = R[\mathfrak{u}^{\pm 1}]$, the ring of Laurent R–polynomials in one variable $\mathfrak{u}$. In particular, $\mathcal{S}(\mathbb{A}_{io})$ is commutative.*

Proof Using Proposition 1.2 one can easily show that the set $\{\mathfrak{u}^k \mid k \in \mathbb{Z}\}$ is a free R–basis of $\mathcal{S}(\mathbb{A}_{io})$. □

2.2 Passing through T_k

Recall that $\mathcal{S}(\mathbb{A}) = R[z]$. One defines a left action and a right action of $\mathcal{S}(\mathbb{A})$ on $\mathcal{S}(\mathbb{A}_{io})$ as follows. For $L \in \mathcal{S}(\mathbb{A})$, $K \in \mathcal{S}(\mathbb{A}_{io})$ let $L \bullet K$ be the element in $\mathcal{S}(\mathbb{A}_{io})$ obtained by placing L above K, and $K \bullet L \in \mathcal{S}(\mathbb{A}_{io})$ be the element in $\mathcal{S}(\mathbb{A}_{io})$ obtained by placing K above L. For example,

$e \bullet z =$  , $z \bullet e =$  .

Proposition 2.2 *One has*

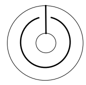
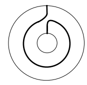
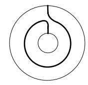
$$T_k(z) \bullet e = t^k \mathfrak{u}^k + t^{-k}\mathfrak{u}^{-k}, \tag{6}$$

$$e \bullet T_k(z) = t^k \mathfrak{u}^{-k} + t^{-k}\mathfrak{u}^k, \tag{7}$$

$$T_k(z) \bullet e - e \bullet T_k(z) = (t^k - t^{-k})(\mathfrak{u}^k - \mathfrak{u}^{-k}). \tag{8}$$

Proof It is important to note that the map $f\colon \mathcal{S}(\mathbb{A}) \to \mathcal{S}(\mathbb{A}_{io})$ given by $f(L) = L \bullet e$ is an algebra homomorphism.

Resolve the only crossing point, we have

$z \bullet e =$  $= t$  $+ t^{-1}$  $= t\mathrm{u} + t^{-1}\mathrm{u}^{-1}.$

Hence

$$\begin{aligned} T_k(z) \bullet e &= T_k(t\mathrm{u} + t^{-1}\mathrm{u}^{-1}) && \text{(because } f \text{ is an algebra homomorphism)} \\ &= t^k \mathrm{u}^k + t^{-k}\mathrm{u}^{-k} && \text{(by (4)).} \end{aligned}$$

This proves (6). The proof of (7) is similar, while (8) follows from (6) and (7). □

Corollary 2.3 *Suppose* $\xi^{2N} = 1$. *Then* $T_N(z)$ *is central at* ξ.

Proof We have $\xi^N = \xi^{-N}$ since $\xi^{2N} = 1$. Then (8) shows that $T_N(z) \bullet e = e \bullet T_N(z)$, which easily implies the centrality of $T_N(z)$. □

Remark 2.4 The corollary was first proved by Bonahon and Wong [2] using another method.

2.3 Transparent elements

We say that $p(z) \in \mathbb{C}[z]$ is *transparent* at ξ if for any 3 disjoint framed knots K, K_1, K_2 in any oriented 3–manifold M, $p(K) \cup K_1 = p(K) \cup K_2$ in $\mathcal{S}_\xi(M)$, provided that K_1 and K_2 are isotopic in M. Note that in general, K_1 and K_2 are not isotopic in $M \setminus K$.

Proposition 2.5 *The following are equivalent.*

(i) $p(z) \bullet e = e \bullet p(z)$ *in* $\mathcal{S}_\xi(\mathbb{A}_{io})$.

(ii) $p(z)$ *is transparent at* ξ.

(iii) $p(z)$ *is central at* ξ.

Proof It is clear that (i) ⇒ (ii) ⇒ (iii). Let us prove (iii) ⇒ (i).

By gluing a 1–handle to $\mathbb{A}$ we get a punctured torus $\mathbb{T}_{\text{punc}}$ as in Figure 3. Here the base of the 1–handle is glued to a small neighborhood of $\{P_1 \cup P_2\}$ in $\partial\mathbb{A}$, and the core of the 1–handle is an arc β connecting P_1 and P_2. Let ι: $\mathcal{S}(\mathbb{A}_{io}) \to \mathcal{S}(\mathbb{T}_{\text{punc}})$ be the R–map which is the closure by β, ie $\iota(K) = K \cup \beta$. Then $\iota(\mathrm{u}^k)$ is a knot in $\mathbb{T}_{\text{punc}}$ for every $k \in \mathbb{Z}$, and $\iota(\mathrm{u}^k)$ is not isotopic to $\iota(\mathrm{u}^l)$ if $k \neq l$. Since $\{\mathrm{u}^k \mid k \in \mathbb{Z}\}$ is an R–basis of $\mathcal{S}(\mathbb{A}_{io})$ and the isotopy classes of links in $\mathbb{T}_{\text{punc}}$ form an R–basis of $\mathcal{S}(\mathbb{T}_{\text{punc}})$, ι is injective.

Assume (iii). Then $p(z)\iota(e) = \iota(e)p(z)$, or $\iota(p(z) \bullet e) = \iota(e \bullet p(z))$. Since ι is injective, we have $p(z) \bullet e = e \bullet p(z)$. □

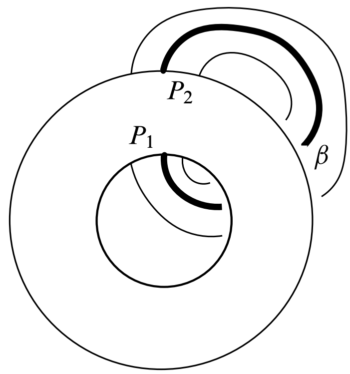


Figure 3: The core β connects P_1 and P_2 in $\mathbb{T}_{\text{punc}}$.

2.4 Proof of Theorem 1

The "if" part has been proved; see Corollary 2.3. Let us prove the "only if" part. Assume that $p(z)$ is central at ξ and has degree $k \geq 1$. Since $\{T_j(z) \mid j \geq 0\}$ is a basis of $\mathbb{C}[z]$, we can write

$$p(z) = \sum_{j=0}^{k} c_j T_j(z), \quad c_j \in \mathbb{C}, c_k \neq 0. \tag{9}$$

By Proposition 2.5, $p(z) \bullet e - e \bullet p(z) = 0$. Using expression (9) for $p(z)$ and (8), we get

$$0 = p(z) \bullet e - e \bullet p(z) = \sum_{j=0}^{k} c_j (\xi^j - \xi^{-j})(\mathfrak{u}^j - \mathfrak{u}^{-j}).$$

Because $\{\mathfrak{u}^j \mid j \in \mathbb{Z}\}$ is a basis of $\mathcal{S}_\xi(\mathbb{A}_{io})$, the coefficient of each $\mathfrak{u}^j$ on the right-hand side is 0. This means

$$c_j = 0 \quad \text{or} \quad \xi^{2j} = 1 \quad \text{for all } j. \tag{10}$$

Since $c_k \neq 0$, we have $\xi^{2k} = 1$. Since $k \geq 1$, this shows ξ^2 is a root of unity of some order N. Then (10) shows that $c_j = 0$ unless $N | j$. Thus, $p(z)$ is a $\mathbb{C}$–linear combination of T_j with $N | j$. This completes the proof of Theorem 1.

2.5 Skew transparency

One more consequence of Proposition 2.2 is the following.

Corollary 2.6 *Suppose $\xi^{2N} = -1$. Then in $\mathcal{S}_\xi(\mathbb{A}_{io})$,*

$$T_N(z) \bullet e = -e \bullet T_N(z).$$

This means every time we pass $T_N(K)$ through a component of a link L, the value of the skein gets multiplied by -1. Following is a precise statement.

Suppose K_1 and K_2 are knots in a 3–manifold M. Recall that an isotopy between K_1 and K_2 is a smooth map $H\colon S^1 \times [1,2] \to M$ such that for each $t \in [1,2]$, the map $H_t\colon S^1 \to M$ is an embedding, and the image of H_i is K_i for $i = 1,2$. Here $H_t(x) = H(x,t)$. For a knot $K \subset M$ let $I_2(H,K)$ be the mod 2 intersection number of H and K. Thus, if H is transversal to K then $I_2(H,K)$ is the number of points in the finite set $H^{-1}(K)$ modulo 2.

Definition 2 Suppose $\mu = \pm 1$. A polynomial $p(z) \in \mathbb{C}[z]$ is called μ–transparent at $\xi \in \mathbb{C}^\times$ if for any 3 disjoint framed knots K, K_1, K_2 in any oriented 3–manifold M, with K_1 and K_2 connected by an isotopy H, one has the following equality in $\mathcal{S}_\xi(M)$:

$$p(K) \cup K_1 = \mu^{I_2(H,K)}[p(K) \cup K_2].$$

From Corollary 2.6 we have:

Corollary 2.7 *Assume $\xi^{4N} = 1$. Then $\mu := \xi^{2N} = \pm 1$, and $T_N(z)$ is μ–transparent.*

A special case is the following. Suppose $D \subset M$ is a disk in M with $\partial D = K$, and a framed link $L \subset M$ is disjoint from K. Then, if $\xi^{2N} = \mu = \pm 1$, one has

$$K \cup T_N(L) = \mu^{I_2(D,L)} \lambda_0 T_N(L) \quad \text{in } \mathcal{S}_\xi(M). \tag{11}$$

Here $\lambda_0 = -(\xi^2 + \xi^{-2})$ is the value of trivial knot in $\mathcal{S}_\xi(M)$.

3 Filtrations of skein modules

Suppose Φ is a link in ∂M. We define an R–map $\Phi\colon \mathcal{S}(M) \to \mathcal{S}(M)$ by $\Phi(L) = \Phi \cup L$.

3.1 Filtration by an arc

Suppose α is an arc properly embedded in a marked surface $(\Sigma, \mathcal{P})$ with $\partial\Sigma \neq \varnothing$. Assume the two boundary points of α, which are on the boundary of Σ, are disjoint from the marked points. Then $\mathcal{D}_\alpha := \alpha \times [-1,1]$ is a disk properly embedded in $\Sigma \times [-1,1]$, with boundary $\Phi_\alpha = \partial(\alpha \times [-1,1]) = (\alpha \times \{-1,1\}) \cup (\partial\alpha \times [-1,1])$.

Let $\mathcal{F}_k^\alpha = \mathcal{F}_k^\alpha(\mathcal{S}(\Sigma))$ be the R–submodule of $\mathcal{S}(\Sigma)$ spanned by all relative links which intersect with $\mathcal{D}_\alpha$ at less than or equal to k points. For $L \in \mathcal{S}(\Sigma)$, we define $\mathrm{fil}_\alpha(L) = k$ if $L \in \mathcal{F}_k^\alpha \setminus \mathcal{F}_{k-1}^\alpha$. The filtration is compatible with the algebra structure, ie

$$\mathrm{fil}_\alpha(L_1 L_2) \le \mathrm{fil}_\alpha(L_1) + \mathrm{fil}_\alpha(L_2).$$

Remark 3.1 A similar filtration was used in [14] to calculate the skein module of torus knot complements.

A convenient way to count the number of intersection points of a link L with $\mathcal{D}_\alpha$ is to count the intersection points of the diagram of L with α. Let D be the vertical projection of L onto Σ. In general position D has only singular points of type double points, and we assume further that D is transversal to α. In that case, the number of intersection points of L with $\mathcal{D}_\alpha$ is equal to the number of intersections of D with α, where each intersection point of α and D at a double point of D is counted twice.

Recall that $\Phi_\alpha(L) = L \cup \Phi_\alpha$, where Φ_α is the boundary of the disk $\mathcal{D}_\alpha = \alpha \times [-1, 1]$. It is clear that $\mathcal{F}_k^\alpha$ is Φ_α–invariant, ie $\Phi_\alpha(\mathcal{F}_k^\alpha) \subset \mathcal{F}_k^\alpha$. It turns out that the action of Φ_α on the quotient $\mathcal{F}_k^\alpha/\mathcal{F}_{k-1}^\alpha$ is very simple. Recall that $\lambda_k = -(t^{2k+2} + t^{-2k-2})$.

Proposition 3.2 *For $k \geq 0$, the action of Φ_α on $\mathcal{F}_k^\alpha/\mathcal{F}_{k-1}^\alpha$ is λ_k times the identity.*

This is a consequence of Proposition 3.3, proved in the next subsection.

3.2 The Temperley–Lieb algebra and the operator Φ

The well-known Temperley–Lieb algebra TL_k is the skein module of the disk with $2k$ marked points on the boundary. We will present the disk as the square $\mathrm{Sq} = [0, 1] \times [0, 1]$ on the standard plane, with k marked points on the top side and k marked points on the bottom side. The product $L_1 L_2$ in TL_k is defined as the result of placing T_1 on top of T_2. The unit $\widetilde{e}_k$ of TL_k is presented by k vertical straight arcs; see Figure 4.

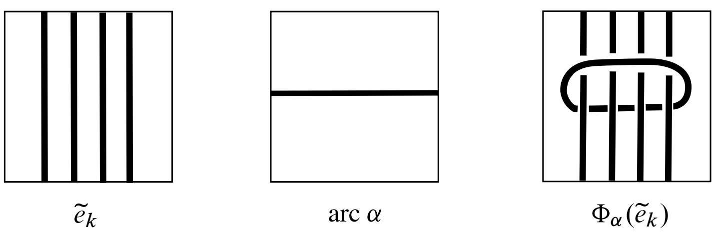


Figure 4: The unit $\widetilde{e}_k$, the arc α and $\Phi_\alpha(\widetilde{e}_k)$: here $k = 4$

Let $\alpha \subset \mathrm{Sq}$ be the horizontal arc $[0, 1] \times \frac{1}{2}$. The element $\Phi_\alpha(\widetilde{e}_k)$ is depicted in Figure 4. In general, $\Phi_\alpha(L)$ is L encircled by one simple closed curve.

Proposition 3.3 *With the above notation, one has*

$$\Phi_\alpha(\widetilde{e}_k) = \lambda_k \widetilde{e}_k \pmod{\mathcal{F}_{k-1}^\alpha}. \tag{12}$$

Proof A direct proof can be carried out as follows. Using the skein relation (1) one resolves all the crossings of the diagram of $\Phi_\alpha(\widetilde{e}_k)$, and finds that only a few terms are not in $\mathcal{F}^\alpha_{k-1}$, and the sum of these terms is equal to $\lambda_k \widetilde{e}_k$. This is a good exercise for the dedicated reader.

Here is another proof using more advanced knowledge of the Temperley–Lieb algebra. First we extend the ground ring to the field of fractions $\mathbb{C}(t)$. Then the Temperley–Lieb algebra contains a special element called the Jones–Wenzl idempotent f_k (see eg Lickorish [13, Chapter 13]). We have $f_k = \widetilde{e}_k \pmod{\mathcal{F}^\alpha_{k-1}}$, and f_k is an eigenvector of Φ_α with eigenvalue λ_k. Hence, we have (12). □

4 Another annulus with two marked points

4.1 Annulus with two marked points on the same boundary

Let $\mathbb{A}_{oo}$ be the annulus A with two marked points Q_1, Q_2 on the outer boundary as in Figure 5. Let u_0, u_1 be arcs connecting Q_1 and Q_2 in $\mathbb{A}_{oo}$ as in Figure 5.

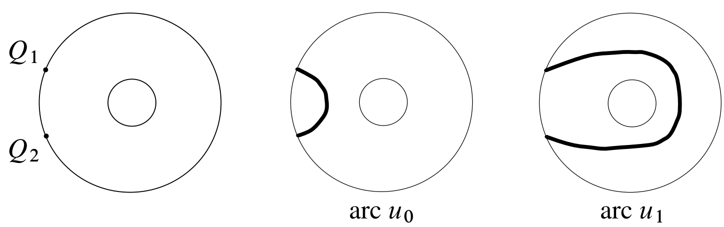


Figure 5: The marked annulus $\mathbb{A}_{oo}$ and arcs u_0, u_1

Define a left $\mathcal{S}(\mathbb{A})$–module and a right $\mathcal{S}(\mathbb{A})$–module on $\mathcal{S}(\mathbb{A}_{oo})$ as follows. For $K \in \mathcal{S}(\mathbb{A}_{oo})$ and $L \in \mathcal{S}(\mathbb{A})$ let KL be the skein in $\mathcal{S}(\mathbb{A}_{oo})$ obtained by placing K on top of L, and $LK \in \mathcal{S}(\mathbb{A}_{oo})$ obtained by placing L on top of K. It is easy to see that $KL = LK$. Recall that $\mathcal{S}(\mathbb{A}) = R[z]$.

Proposition 4.1 *The module $\mathcal{S}(\mathbb{A}_{oo})$ is a free $\mathcal{S}(\mathbb{A})$–module with basis $\{u_0, u_1\}$:*

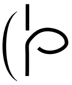
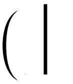
$$\mathcal{S}(\mathbb{A}_{oo}) = R[z]u_0 \oplus R[z]u_1.$$

Proof Any relative link in $\mathbb{A}_{oo}$ is of the form $u_i z^m$ with $i = 0, 1$ and $m \in \mathbb{Z}$. The proposition now follows from Proposition 1.2. □

4.2 Framing change and the unknot

Recall that S_k is the k^{th} Chebyshev polynomial of type 2. The values of the unknot colored by S_k and the framing change are well known (see eg Blanchet, Habegger, Masbaum and Vogel [1]): in $\mathcal{S}(M)$, where M is an oriented 3–manifold, one has

$$L \sqcup S_k(U) = (-1)^k \frac{t^{2k+2} - t^{-2k-2}}{t^2 - t^{-2}} L, \tag{13}$$

(14) $S_k\Big($  $\Big) = (-1)^k t^{k^2+2k} S_k\Big($  $\Big).$

Here in (13), U is the trivial knot lying in a ball disjoint from L.

4.3 Some elements of $\mathcal{S}(\mathbb{A}_{oo})$

Let $u_k, k \geq 0$ are arcs in $\mathbb{A}_{oo}$ depicted in Figure 6. The elements u_1 and u_0 are the same as the ones defined in Figure 5. Let $v_0 = u_0$ and $v_k, k \geq 1$ be arcs in $\mathbb{A}_{oo}$ depicted in Figure 6.

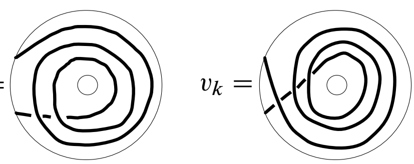


Figure 6: The arcs u_k and v_k, with $k = 3$

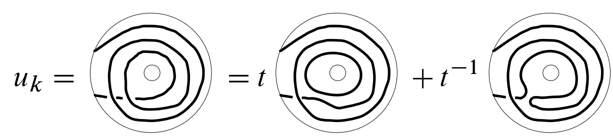
Proposition 4.2 *One has*

$$u_k = t^{k-1} S_{k-1}(z) u_1 + t^{k-3} S_{k-2}(z) u_0, \tag{15}$$

$$v_k = t^{2-k} S_{k-1}(z) u_1 + t^{-k} S_k(z) u_0, \tag{16}$$

for all $k \geq 1$, *for all* $k \geq 0$, *respectively.*

Proof Suppose $k \geq 3$. Applying the skein relation to the innermost crossing of u_k, we get



which, after an isotopy and removing a framing crossing, is

$$u_k = t u_{k-1} z - t^2 u_{k-2},$$

from which one can easily prove (15) by induction.

Similarly, using the skein relation to resolve the innermost crossing point of v_k, we get

$$v_k = t^{-1} v_{k-1} z - t^{-2} v_{k-2} \quad \text{for } k \geq 2,$$

from which one can prove (16) by induction. □

Remark 4.3 Identity (15) does not hold for $k=0$. This is due to a framing change.

4.4 Operator Ψ

Let Ψ be the arc in $\partial\mathbb{A}\times[-1,1]$ beginning at Q_1 and ending at Q_2, as depicted in Figure 7. Here we draw $\mathbb{A}\times[-1,1]$ as a handlebody. For any element $\alpha\in\mathcal{S}(\mathbb{A})$ let

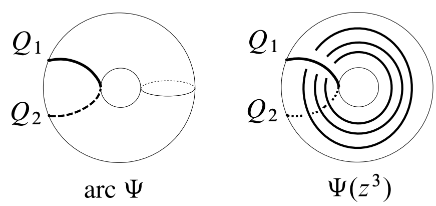


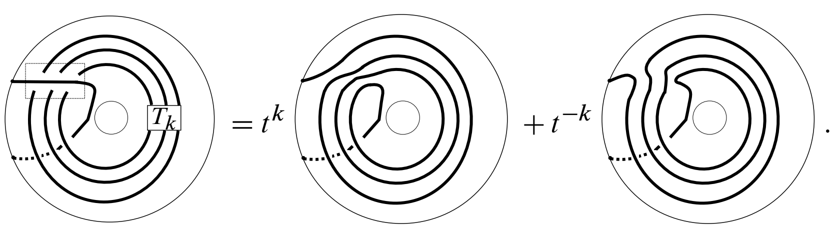
Figure 7: Arc Ψ connecting Q_1 and Q_2 and $\Psi(z^3)$

$\Psi(\alpha)\in\mathcal{S}(\mathbb{A}_{oo})$ be the skein $\Psi\cup\alpha$. For example, $\Psi(z^3)$ is given in Figure 7.

Proposition 4.4 *For $k\geq 1$, one has*

$$\Psi(T_k(z)) = u_1[t^2(t^{-2k}-t^{2k})S_{k-1}(z)] + u_0[t^{-2k}S_k(z) - t^{2k}S_{k-2}(z)]. \tag{17}$$

Proof Applying Proposition 2.2 to the part in the left rectangle box, we get



The positive framing crossing in the first term gives a factor $-t^3$. Thus

$$\Psi(T_k(z)) = -t^{k+3}u_k + t^{-k}v_k.$$

Plugging in the values of u_k, v_k given by Proposition 4.2, we get the result. □

Remark 4.5 One can use Proposition 4.4 to establish product-to-sum formulas similar to the ones in Frohman and Gelca [8].

5 Twice-punctured disk

5.1 Skein module of twice-punctured disk

Let $\mathcal{D} \subset \mathbb{R}^2$ be the disk of radius 4 centered at the origin, $\mathcal{D}_1 \subset \mathbb{R}^2$ the disk of radius 1 centered at $(-2, 0)$, and $\mathcal{D}_2$ the disk of radius 1 centered at $(2, 0)$. We define $\mathbb{D}$ to

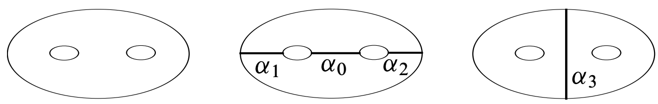


Figure 8: The twice-punctured disk $\mathbb{D}$ and the arcs $\alpha_1, \alpha_0, \alpha_2, \alpha_3$

be $\mathcal{D}$ with the interiors of $\mathcal{D}_1$ and $\mathcal{D}_2$ removed. The horizontal axis intersects $\mathbb{D}$ at 3 arcs denoted from left to right by $\alpha_1, \alpha_0, \alpha_2$; see Figure 8. The vertical axis of $\mathbb{R}^2$ intersects $\mathbb{D}$ at an arc denoted by α_3. The corresponding curve Φ_{α_i} on $\partial\mathbb{D} \times [-1, 1]$ will be denoted simply by Φ_i for $i = 0, 1, 2, 3$. If $\mathbb{D} \times [-1, 1]$ is presented as the handlebody $\mathcal{H}$, which is a thickening of $\mathbb{D}$ in $\mathbb{R}^3$, then the curves $\Phi_1, \Phi_0, \Phi_2, \Phi_3$ are shown in Figure 9.

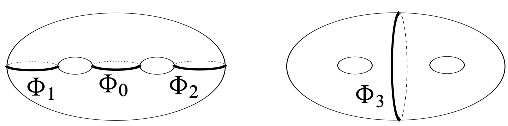
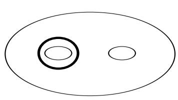
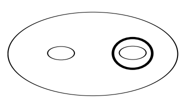
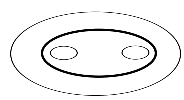


Figure 9: The curves $\Phi_1, \Phi_0, \Phi_2, \Phi_3$ on the boundary of the handlebody

Let x_1, x_2, and y be the closed curves in $\mathbb{D}$:

$x_1 =$  $x_2 =$ , $y =$ .

It is known that we have the equality $\mathcal{S}(\mathbb{D}) = R[x_1, x_2, y]$, the R–polynomial in the variables x_1, x_2, y; see Bullock and Przytycki [6]. In particular, $\mathcal{S}(\mathbb{D})$ is commutative.

Let σ be the rotation about the origin of $\mathbb{R}^2$ by $180°$. Then $\sigma(\mathbb{D}) = \mathbb{D}$. Hence σ induces an automorphism of $\mathcal{S}(\mathbb{D}) = R[x_1, x_2, y]$, which is an algebra automorphism. One has $\sigma(y) = y, \sigma(x_1) = x_2, \sigma(x_2) = x_1$.

5.2 Degrees on $\mathcal{S}(\mathbb{D}) = R[x_1, x_2, y]$

Define the left degree, right degree and double degree on $R[x_1, y, x_2]$ as follows. For a monomial $\mathfrak{m} = x_1^{a_1} y^b x_2^{a_2}$ define its left degree $\deg_l(\mathfrak{m}) = a_1 + b$, right degree

$\deg_r(\mathfrak{m}) = a_2 + b$, double degree $\deg_{lr}(\mathfrak{m}) = \deg_l(\mathfrak{m}) + \deg_r(\mathfrak{m}) = a_1 + a_2 + 2b$. One readily finds that

$$\deg_l(\mathfrak{m}) = \mathrm{fil}_{\alpha_1}(\mathfrak{m}), \quad \deg_r(\mathfrak{m}) = \mathrm{fil}_{\alpha_2}(\mathfrak{m}),$$

where fil_α is defined in Section 3.1. Using the definition of fil_α involving the numbers of intersection points we get the following.

Lemma 5.1 *Suppose L is an embedded link in $\mathbb{D}$ and L intersects transversally the arc α_i at k_i points for $i = 1, 2, 3$. Then, as an element of $\mathcal{S}(\mathbb{D})$, $L = x_1^{a_1} x_2^{a_2} y^b$, where $2b \le k_3$ and*

$$\deg_l(L) \le k_1, \quad \deg_l(L) \equiv k_1 \pmod 2,$$
$$\deg_r(L) \le k_2, \quad \deg_r(L) \equiv k_2 \pmod 2.$$

Consequently, $\deg_{lr}(L) \le k_1 + k_2$ *and* $\deg_{lr}(L) \equiv k_1 + k_2 \pmod 2$.

Proof If $L = L_1 \sqcup L_2$ is the union of 2 disjoint sublinks, and the statement holds for each of L_i, then it holds for L. Hence we assume L has one component, ie L is an embedded loop in $\mathbb{D} \subset \mathbb{R}^2$. Then L is isotopic to either a trivial loop, x_1, x_2 or y. In each case, the statement can be verified easily. For example, suppose $L = x_1$. For the mod 2 intersection numbers, $I_2(L, \alpha_1) = I_2(x_1, \alpha_1) = 1$. Hence k_1, the geometric intersection number between L and α_1, must be odd and bigger than or equal to 1. Hence, we have $\deg_l(L) \le k_1$ and $\deg_l(L) \equiv k_1 \pmod 2$. □

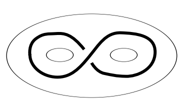
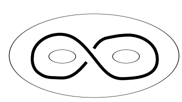
Corollary 5.2 *Suppose L is a link diagram on $\mathbb{D}$ which intersects transversally the arc α_i at k_i points for $i = 1, 2, 3$. Then, as an element in $\mathcal{S}(\mathbb{D})$,*

$$\deg_l(L) \le k_1, \quad \deg_r(L) \le k_2, \quad 2\deg_y(L) \le k_3,$$

and L is a linear R–combination of monomials whose double degrees are equal to $k_1 + k_2$ modulo 2.

5.3 The R–module V_n and the skein γ

Let γ and $\bar{\gamma}$ be the following link diagrams on $\mathbb{D}$:

(18) $\gamma =$  , $\bar{\gamma} =$  .

Let

$$V_n = \{p \in R[x_1, x_2, y] \mid \deg_l(p) \le n, \deg_r(p) \le n, \deg_{lr}(p) \text{ even}\}.$$

In other words, $V_n \subset R[x_1, x_2, y]$ is the R–submodule spanned by $x_1^{a_1} x_2^{a_2} y^b$, with $a_i + b \le n$ for $i = 1, 2$ and $a_1 + a_2$ even.

Lemma 5.3 *One has $T_n(\gamma), T_n(\overline{\gamma}) \in V_n$.*

Proof The diagram γ^k has k intersection points with each of α_1 and α_2. By Corollary 5.2, we have $\deg_l(\gamma^k) \leq k, \deg_r(\gamma^k) \leq k$, and each monomial of γ^k has double degree $\equiv k + k \equiv 0 \pmod 2$. This means $\gamma^k \in V_k$ for every $k \geq 0$. Because $T_n(\gamma)$ is $\mathbb{Z}$–linear combination of γ^k with $k \leq n$, we have $T_n(\gamma) \in V_n$. The proof for $\overline{\gamma}$ is similar. □

Remark 5.4 It is an easy exercise to show that $T_N(\overline{\gamma}) = T_N(\gamma)|_{t \to t^{-1}}$.

6 Skein module of twice-punctured disk at root of 1

Recall that γ and $\overline{\gamma}$ are knot diagrams on $\mathbb{D}$ defined by (18). The following was proved by Bonahon and Wong, using quantum Teichmüller algebras and their representations.

Proposition 6.1 *Suppose ξ^4 is a root of 1 of order N. Then in $\mathcal{S}_\xi(\mathbb{D})$ one has*

$$T_N(\gamma) = \xi^{-N^2} T_N(y) + \xi^{N^2} T_N(x_1) T_N(x_2), \tag{19}$$

$$T_N(\overline{\gamma}) = \xi^{N^2} T_N(y) + \xi^{-N^2} T_N(x_1) T_N(x_2). \tag{20}$$

As mentioned above, there was an urge to find a proof using elementary skein theory; one such proof is presented here. Our proof roughly goes as follows. Using the transparent property of $T_N(\gamma)$, we show that $T_N(\gamma)$ is a common eigenvector of several operators. We then prove that the space of common eigenvectors has dimension at most 3, with a simple basis. We then fix coefficients of $T_N(\gamma)$ in this basis using calculations in highest order. Then the result turns out to be the right-hand side of (19).

Throughout this section we fix a complex number ξ such that ξ^4 is a root of unity of order N. Define $\varepsilon = \xi^{N^2}$. We will write $V_{N,\xi}$ simply by V_N and λ_k for $\lambda_k(\xi)$. Thus, in the whole section,

$$\lambda_k = -(\xi^{2k+2} + \xi^{-2k-2}).$$

6.1 Properties of ξ and λ_k

Recall that ξ^4 is a root of 1 of order N.

Lemma 6.2 *Suppose $1 \leq k \leq N-1$. Then:*

(i) $\lambda_{2k} = \lambda_0$ *if and only if* $k = N-1$.

(ii) $\lambda_k = \xi^{2N} \lambda_0$ *implies that* $k = N-2$.

(iii) *If* N *is even then* $\xi^{2N} = -1$.

(iv) *One has*

$$\xi^{2N^2+2N} = (-1)^{N+1}. \tag{21}$$

Proof (i) With $\lambda_k = -(\xi^{2k+2} + \xi^{-2k-2})$, we have

$$\lambda_{2k} - \lambda_0 = -\xi^{-2-4k}(\xi^{4k} - 1)(\xi^{4k+4} - 1).$$

Hence, $\lambda_{2k} - \lambda_0 = 0$ if and only if either $N \mid k$ or $N \mid (k+1)$. With $1 \le k \le N-1$, this is equivalent to $k = N-1$.

(ii) We have

$$\lambda_k - \xi^{2N}\lambda_0 = -\xi^{-2N-2}(\xi^{2N-2k} - 1)(\xi^{2N+2k+4} - 1).$$

Either $\xi^{2N-2k} = 1$ or $\xi^{2N+2k+4} = 1$. Taking the squares of both identities, we see that either $N \mid (N-k)$ or $N \mid (k+2)$. With $1 \le k \le N-1$, we conclude that $k = N-2$.

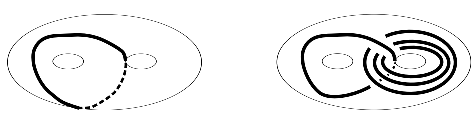
(iii) Suppose N is even. Since ξ^4 has order N, one has $(\xi^4)^{N/2} = -1$. Then $\xi^{2N} = (\xi^4)^{N/2} = -1$.

(iv) The proof is left for the reader. □

6.2 Operators Φ_i and the vector space W

Recall that $\Phi_i := \Phi_{\alpha_i}$, $i = 0, 1, 2, 3$, is defined in Section 5.1. Then $\Phi_i(V_N) \subset V_N$ for $i = 0, 1, 2, 3$.

Let Φ_4 be the curve on $\partial\mathbb{D} \times [-1, 1]$ depicted in Figure 10. Here we draw $\mathcal{H} = \mathbb{D} \times [-1, 1]$ as a handlebody. We also depict $\Phi_4(x_2^3)$.



Φ_4 $\Phi_4(x_2^3)$

Figure 10: The curve Φ_4 and $\Phi_4(x_2^3)$

We do not have $\Phi_4(V_N) \subset V_N$, since Φ_4 in general increases the double degree. By counting the intersection points with α_1 and α_2, we have, for every $E \in \mathcal{S}_\xi(\mathbb{D}) = \mathbb{C}[x_1, x_2, y]$,

$$\deg_{lr}(\Phi_4(E)) \le \deg_{lr}(E) + 1. \tag{22}$$

Proposition 6.3 *If E is one of $\{T_N(\gamma), T_N(\overline{\gamma}), T_N(y), T_N(x_1)T_N(x_2)\}$, then*

$$\sigma(E) = E, \tag{23}$$

$$\Phi_1(E) = \xi^{2N}\lambda_0 E, \tag{24}$$

$$\Phi_i(E) = \lambda_0 E \quad \text{for } i = 0, 3, \tag{25}$$

$$\Phi_4(E) = \xi^{2N} x_1 E. \tag{26}$$

Proof The first identity follows from the fact that each of $\gamma, \overline{\gamma}, y, x_1 \cup x_2$ is invariant under σ. The remaining identities follow from the ξ^{2N}–transparent property of $T_N(z)$, Corollary 2.7. □

Remark 6.4 Note that Φ_4 is $\mathbb{C}[x_1]$–linear and (26) says E is a $\xi^{2N}x_1$–eigenvector of Φ_4.

Let W be the subspace of V_N consisting of elements satisfying (23)–(26). This means $W \subset V_N$ consists of elements which are at the same time 1–eigenvector of σ, $\xi^{2N}\lambda_0$–eigenvector of Φ_1, λ_0–eigenvector of Φ_0 and Φ_3, and $\xi^{2N}x_1$–eigenvector of Φ_4.

We will show that W is spanned by $T_N(y), T_N(x_1)T_N(x_2)$, and possibly 1.

6.3 Action of Φ_3, Φ_0 and Φ_1

For an element $F \in \mathbb{C}[x_1, x_2, y]$ and a monomial $\mathfrak{m} = x_1^{a_1} x_2^{a_2} y^b$ let $\operatorname{coeff}(F, \mathfrak{m})$ be the coefficient of $\mathfrak{m}$ in F.

Lemma 6.5 *Suppose $E \in W$ and $\operatorname{coeff}(E, y^N) = 0$. Then $E \in \mathbb{C}[x_1, x_2]$.*

Proof Let k be the y–degree of E. Since $E \in W$ and $\operatorname{coeff}(E, y^N) = 0$, one has $k \le N - 1$.

We need to show that $k = 0$. Suppose to the contrary that $1 \le k$. Then $1 \le k \le N - 1$.

First we will prove $k = N - 1$, using the fact that E is a λ_0–eigenvector of Φ_3 by (25).

Recall that $\operatorname{fil}_{\alpha_3}$ is twice the y–degree. One has $\operatorname{fil}_{\alpha_3}(E) = 2k$. Thus $E \ne 0 \in \mathcal{F}^{\alpha_3}_{2k}/\mathcal{F}^{\alpha_3}_{2k-1}$. By Proposition 3.2, any nonzero element in $\mathcal{F}^{\alpha_3}_{2k}/\mathcal{F}^{\alpha_3}_{2k-1}$ is an eigenvector of Φ_3 with eigenvalue λ_{2k}. But E is an eigenvector of Φ_3 with eigenvalue λ_0. It follows that $\lambda_{2k} = \lambda_0$. By Lemma 6.2, we have $k = N - 1$.

Because $\deg_{lr}(E)$ is even and less than or equal to $2N$, we must have

$$E = y^{N-1}(c_1 x_1 x_2 + c_2) + O(y^{N-2}), \quad c_1, c_2 \in \mathbb{C}.$$

We will prove $c_1 = 0$ by showing that otherwise, Φ_0 will increase the y–degree. Note that Φ_0 can increase the y–degree by at most 1, and Φ_0 is $\mathbb{C}[y]$–linear. We have

$$\Phi_0(E) = y^{N-1}(c_1\Phi_0(x_1x_2) + c_2\Phi_0(1)) + O(y^{N-1}). \tag{27}$$

The diagram of $\Phi_0(x_1x_2)$ has 4 crossings; see Figure 11.

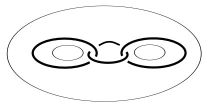


Figure 11: The diagram of $\Phi_0(x_1x_2)$

A simple calculation shows

$$\Phi_0(x_1x_2) = (1-t^4)(1-t^{-4})y + O(y^0).$$

Plugging this value in (27), with $\Phi_0(1) = \lambda_0 \in \mathbb{C}$,

$$\Phi_0(E) = y^N c_1(1-t^4)(1-t^{-4}) + O(y^{N-1}). \tag{28}$$

If $c_1 \neq 0$, then the y–degree of $\Phi_0(E)$ is N, strictly bigger than that of E and E cannot be an eigenvector of Φ_0. Thus $c_1 = 0$.

One has now

$$E = c_2 y^{N-1} + O(y^{N-2}). \tag{29}$$

Since the y–degree of E is $N-1$, one must have $c_2 \neq 0$. By counting the intersections with α_3, we see that Φ_1 does not increase the y–degree. We have

$$\begin{aligned}\Phi_1(E) &= c_2\Phi_1(y^{N-1}) + O(y^{N-2})\\ &= c_2\lambda_{N-1}y^{N-1} + O(y^{N-2}) \quad \text{by Proposition 3.2.}\end{aligned}$$

Comparing the above identity with (29) and using the fact that E is a $\xi^{2N}\lambda_0$–eigenvector of Φ_1, we have

$$\lambda_{N-1} = \xi^{2N}\lambda_0,$$

which is impossible since Lemma 6.2 says that $\lambda_k = \xi^{2N}\lambda_0$ only when $k = N-2$. This completes the proof of the lemma. □

6.4 Action of Φ_4

Recall that Φ_4 is the curve on the boundary of the handlebody $\mathcal{H}$ (see Figure 10) which acts on $\mathcal{S}_\xi(\mathbb{D}) = \mathbb{C}[x_1, x_2, y]$. The action of Φ_4 is $\mathbb{C}[x_1]$–linear, and every element of W is a $\xi^{2N}x_1$–eigenvector of Φ_4.

Recall that $\deg_r = \mathrm{fil}_{\alpha_2}$ and $\deg_r(x_1^{a_1}x_2^{a_2}y^b) = a_2 + b$. Note that for $F \in \mathbb{C}[x_1, x_2]$, $\deg_r(F)$ is exactly the x_2–degree of F. By looking at the intersection with α_2, we see that Φ_4 preserves the α_2–filtration, ie $\deg_r \Phi_4(F) \le \deg_r(F)$. We will study actions of Φ_4 on the associated graded spaces.

We will use the notation $F + \deg_r - \mathrm{lot}$ to mean $F + F_1$, where $\deg_r(F_1) < \deg_r(F)$.

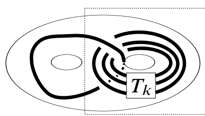
Lemma 6.6 *Suppose* $1 \le k \le N-1$. *One has*

$$\Phi_4(a(x_1)T_N(x_2)) = \xi^{2N} x_1[a(x_1)T_N(x_2)], \tag{30}$$

$$\Phi_4(T_k(x_2)) = y[\xi^2(\xi^{-2k} - \xi^{2k})x_2^{k-1}] + \deg_r - \mathrm{lot} \pmod{\mathbb{C}[x_1, x_2]}. \tag{31}$$

Proof Identity (30) follows from the ξ^{2N}–transparency of $T_N(z)$.

Let us prove (31). Applying identity (17) to the dashed box below, we have

$$\Phi_4(T_k(x_2)) =$$ 

$$= y[\xi^2(\xi^{-2k} - \xi^{2k})S_{k-1}(x_2)] + x_1[\xi^{-2k}S_k(x_2) - \xi^{2k}S_{k-2}(x_2)],$$

which implies (31). □

6.5 The space $W \cap \mathbb{C}[x_1, x_2]$

Lemma 6.7 *Suppose* $E \in W \cap \mathbb{C}[x_1, x_2]$ *and the coefficient of* $x_1^N x_2^N$ *in* E *is 0. Then* $E \in \mathbb{C}$.

Proof Since $T_k(x_2)$ is a basis of $\mathbb{C}[x_2]$, we can write E uniquely as

$$E = \sum_{k=0}^{N} a_k(x_1)T_k(x_2), \quad a_k(x_1) \in \mathbb{C}[x_1].$$

Let j be the x_2–degree of $E' := E - a_N(x_1)T_N(x_2)$. Then $j \le N-1$.

First we will show $j = 0$. Assume to the contrary that $j \ge 1$. Thus $1 \le j \le N-1$. Note that E, by assumption, and $a_N(x_1)T_N(x_2)$, by (30), are eigenvectors of Φ_4 with eigenvalue $\xi^{2N}x_1$. It follows that E' is also an eigenvector of Φ_4 with eigenvalue $\xi^{2N}x_1$. We have

$$E' = \sum_{k=0}^{j} a_k(x_1)\Phi_4(T_k(x_2)) = a_j(x_1)T_j(x_2) + \deg_r - \mathrm{lot}.$$

Using (31) and the fact that Φ_4 does not increase $\deg_r$, we have

$$\Phi_4(E') = y[a_j(x_1)\xi^2(\xi^{-2k} - \xi^{2k})x_2^{k-1}] + \deg_r - \mathrm{lot} \pmod{\mathbb{C}[x_1, x_2]}.$$

When $1 \le j \le n-1$, the coefficient of y, which is the element in the square bracket, is nonzero. Thus $\Phi_4(E') \notin \mathbb{C}[x_1, x_2]$, while $E' \in \mathbb{C}[x_1, x_2]$. This means E' cannot be an eigenvector of Φ_4, a contradiction. This proves $j = 0$.

So we have

$$E = a_N(x_1)T_N(x_2) + a_0(x_1).$$

Because $\deg_{lr}(E) < 2N$, the x_1–degree of $a_N(x_1)$ is less than n. Using the invariance under σ, one sees that E must be of the form

$$E = c_1(T_N(x_1) + T_N(x_2)) + c_2, \quad c_1, c_2 \in \mathbb{C}. \tag{32}$$

To finish the proof of the lemma, we need to show that $c_1 = 0$. Assume that $c_1 \neq 0$. Since E has even double degree, N is even. By Lemma 6.2(iii), $\xi^{2N} = -1$.

Recall that E is a λ_0–eigenvector of Φ_0. Applying Φ_0 to (32),

$$\lambda_0[c_1(T_N(x_1) + T_N(x_2)) + c_2] = \Phi_0(c_1(T_N(x_1) + T_N(x_2)) + c_2).$$

Both $T_N(x_1)$ and $T_N(x_2)$ are eigenvectors of Φ_0 with eigenvalues $\xi^{2N}\lambda_0 = -\lambda_0$, while $\Phi_0(1) = \lambda_0$. Hence we have

$$\lambda_0[c_1(T_N(x_1) + T_N(x_2)) + c_2] = \lambda_0[-c_1(T_N(x_1) - T_N(x_2)) + c_2],$$

which is impossible since $c_1\lambda_0 \neq 0$. Hence, we have $c_1 = 0$ and $E \in \mathbb{C}$. □

6.6 Some maximal degree parts of $T_N(\gamma)$

Lemma 6.8 *One has*

$$\operatorname{coeff}(T_N(\gamma), y^N) = \xi^{-N^2}, \tag{33}$$

$$\operatorname{coeff}(T_N(\gamma), x_1^N x_2^N) = \xi^{N^2}. \tag{34}$$

Proof Since $T_N(\gamma) = \gamma^N + \deg_{lr} - \text{lot}$, we have

$$\operatorname{coeff}(T_N(\gamma), y^N) = \operatorname{coeff}(\gamma^N, y^N), \quad \operatorname{coeff}(T_N(\gamma), x_1^N x_2^N) = \operatorname{coeff}(\gamma^N, x_1^N x_2^N).$$

There are N^2 crossing points in the diagram of γ^N. Each crossing can be smoothed in two ways. The positive smoothing acquires a factor t in the skein relation, and the negative smoothing acquires a factor t^{-1}. The are 2^{N^2} smoothings of γ^N. Each smoothing s of all the N^2 crossings gives rise to a link L_s embedded in $\mathbb{D}$. Then γ^N is a linear combination of all L_s. We will show that the only s for which $L_s = y^N$ is the all negative smoothing.

Consider a crossing point C of γ^N. The vertical line passing through C intersects $\mathbb{D}$ in an interval α_3' which is isotopic to α_3, and $\operatorname{fil}_{\alpha_3} = \operatorname{fil}_{\alpha_3'}$. For an embedded link L in $\mathbb{D}$, as an element of $\mathcal{S}(\mathbb{D}) = R[x_1, x_2, y]$, L is a monomial whose y–degree is bounded above by half the number of intersection points of L with α_3'. The diagram γ^N has exactly $2N$ intersection points with α_3', with C contributing two (of the $2N$ intersection points). If we positively smooth γ^N at C, the result is a link diagram with $2N-2$ intersection points with α_3', and no matter how we smooth other crossings, the resulting link will have less than or equal to $2N-2$ intersection points with α_3'. Thus we cannot get y^N if any of the crossing is smoothed positively. The only smoothing which results in y^N is the all negative smoothing. The coefficient of this smoothing is ξ^{-N^2}.

Similarly, one can prove that the only smoothing which results in $x_1^N x_2^N$ is the all positive smoothing, whose coefficient is ξ^{N^2}. □

6.7 Proof of Proposition 6.1

Let

$$E = T_N(\gamma) - \xi^{N^2} T_N(x_1) T_N(x_2) - \xi^{-N^2} T_N(y).$$

Then $E \in W$. Lemma 6.8 shows that $\mathrm{coeff}(E, y^N) = 0 = \mathrm{coeff}(E, x_1^N x_2^N)$. By Lemma 6.5, $E \in \mathbb{C}[x_1, x_2]$. Then by Lemma 6.7, we have $E \in \mathbb{C}$, ie E is a constant.

We will show that $E = 0$. This is done by using the inclusion of $\mathcal{H}$ into $\mathbb{R}^3$, which gives a $\mathbb{C}$–linear map $\iota\colon \mathcal{S}_\xi(\mathbb{D}) \to \mathcal{S}_\xi(\mathbb{R}^3) = \mathbb{C}$. Under ι, we have

$$E = \iota(T_N(\gamma)) - \iota(\xi^{N^2} T_N(x_1) T_N(x_2)) - \iota(\xi^{-N^2} T_N(y)). \tag{35}$$

The right-hand side involves the trivial knot and the trivial knot with framing 1, and can be calculated explicitly as follows. Note that $\iota(\gamma)$ is the unknot with framing 1, while $\iota(x_1) = \iota(x_2) = \iota(y) = U$, the trivial knot. With $T_N = S_N - S_{N-2}$, and the framing change given by (14), we find

(36) $T_N\Big($  $\Big) = (-1)^N \xi^{N^2+2N} T_N\Big(\,|\, \Big) = -\xi^{-N^2} T_N\Big(\,|\, \Big),$

where the second identity follows from (21). Similarly, using (13), we have

$$T_N(L \sqcup U) = 2(-1)^N \xi^{2N} T_N(L) = -(\xi^{2N^2} + \xi^{-2N^2}) T_N(L). \tag{37}$$

From (36) and (37), we calculate the right-hand side of (35), and find that $E = 0$. This proves (19).

The proof of (20) is similar. Alternatively, one can get (20) from (19) by noticing that the mirror image map on $R[x_1, x_2, y]$ is the $\mathbb{C}$–algebra map sending t to t^{-1}, leaving each of x_1, x_2, y fixed.

This completes the proof of Proposition 6.1.

7 Proof of Theorem 2

Recall that $\varepsilon = \xi^{N^2}$, where ξ^4 is a root of 1 of order N. Then $\varepsilon^4 = 1$. The map $\mathcal{S}_\varepsilon(M) \to \mathcal{S}_\xi(M)$, defined for framed links by $L \to T_N(L)$, is well defined if and only if it preserves the skein relations (1) and (2), ie in $\mathcal{S}_\xi(M)$,

$$T_N(L) = \varepsilon T_N(L_+) + \varepsilon^{-1} T_N(L_-), \tag{38}$$

$$T_N(L \sqcup U) = -(\varepsilon^2 + \varepsilon^{-2}) T_N(L). \tag{39}$$

Here, in (38), L, L_+, L_- are links appearing in the original skein relation (1), they are identical everywhere, except in a ball B, where they appear as in Figure 12.

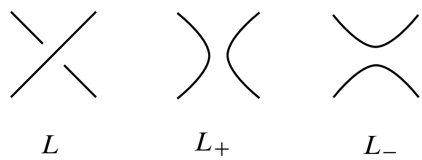


Figure 12: The links L, L_+ and L_-

Identity (39) follows from (37). Let us prove (38).

Case 1 The two strands of L in the ball B belong to the same component. Then (38) follows from Proposition 6.1, applied to the handlebody which is the union of B and a tubular neighborhood of L.

Case 2 The two strands of L in B belong to different components. Then the two strands of L_+ belong to the same component, and we can apply (38) to the case when the left-hand side is L_+. We have

$T_N(L_+) = T_N\Big($  $\Big) = T_N\Big($  $\Big)$

(40) $= \varepsilon^{-1} T_N\Big($  $\Big) + \varepsilon T_N\Big($  $\Big)$

(41) $= \varepsilon^{-1} T_N\Big($  $\Big) + \varepsilon(-\varepsilon) T_N\Big($  $\Big)$

$$= \varepsilon^{-1} T_N(L) - \varepsilon^2 T_N(L_-), \tag{42}$$

where (40) follows from Case 1 and (41) follows from the framing factor formula (36). Multiplying (42) by ε and using $\varepsilon^3 = \varepsilon^{-1}$, we get (38) in this case. This completes the proof of Theorem 2.

Remark 7.1 In [2], in order to prove Theorem 2, the authors proved in addition to Proposition 6.1 a similar statement for links in the cylinder over a punctured torus.

Here we bypass this extra statement by reducing the extra statement to Proposition 6.1. Essentially this is due to the fact that the cylinder over a punctured torus is the same as the cylinder over a twice-punctured disk.

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