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Relative divergence of finitely generated groups

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We generalize the concept of divergence of finitely generated groups by introducing the upper and lower relative divergence of a finitely generated group with respect to a subgroup. Upper relative divergence generalizes Gersten's notion of divergence, and lower relative divergence generalizes a definition of Cooper and Mihalik. While the lower divergence of Alonso, Brady, Cooper, Ferlini, Lustig, Mihalik, Shapiro and Short can only be linear or exponential, relative lower divergence can be any polynomial or exponential function. In this paper, we examine the relative divergence (both upper and lower) of a group with respect to a normal subgroup or a cyclic subgroup. We also explore relative divergence of CAT(0) groups and relatively hyperbolic groups with respect to various subgroups to better understand geometric properties of these groups.

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1 Introduction

Two different notions of divergence of a finitely generated group are introduced by Cooper and Mihalik [1] and Gersten [9]. We refer to the former's notion as lower divergence and Gersten's notion as upper divergence. The lower divergence of a one-ended group G is exponential if G is hyperbolic and linear otherwise (see [1] and Sisto [22]). Therefore, lower divergence only detects hyperbolicity. Upper divergence is more diverse since the upper divergence of a finitely generated group can be any polynomial or exponential function (see Macura [16] and Sisto [22]). Upper divergence has been studied by Macura [16], Behrstock and Charney [2], Duchin and Rafi [7], Druțu, Mozes and Sapir [5], Sisto [22] and others. Moreover, upper divergence is a quasi-isometry invariant, and it is therefore a useful tool to classify finitely generated groups up to quasi-isometry. Motivated by Gersten and Alonso, Brady, Cooper, Ferlini, Lustig, Mihalik, Shapiro and Short's notions, we introduce two types of relative divergence of a finitely generated group with respect to a subgroup: upper relative divergence and lower relative divergence.

We now introduce some notation and we will work on them for the concept of relative divergence. Let (X,d) be a geodesic space and A a subspace. For each positive r,

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let $d_{r,A}$ be the induced length metric on the complement of the r–neighborhood of A in X. We now define the relative divergence of the space X with respect to the subspace A (both upper relative divergence and lower relative divergence). For each $\rho \in (0,1]$ and positive integer $n \geq 2$, we define two functions δ_ρ^n and σ_ρ^n from $[0,\infty)$ to $[0,\infty]$ as follows.

For each $r \in [0,\infty)$, let $\delta_\rho^n(r) = \sup d_{\rho r,A}(x,y)$ where the supremum is taken over all x, y which lie in $\partial N_r(A)$ such that $d_{r,A}(x,y) < \infty$ and $d(x,y) \leq nr$.

Similarly, let $\sigma_\rho^n = \inf d_{\rho r,A}(x,y)$ where the infimum is taken over all x, y which lie in $\partial N_r(A)$ such that $d_{r,A}(x,y) < \infty$ and $d(x,y) \geq nr$.

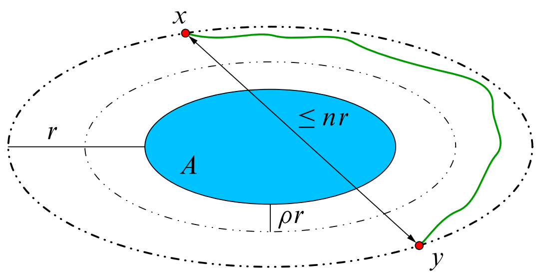

Figure 1: The picture illustrates the idea of upper relative divergence of a geodesic space X with respect to a subspace A. The picture for lower relative divergence is almost identical except the distance between x and y is greater than or equal to nr.

The family of functions $\{\delta_\rho^n\}$ is the upper relative divergence of the pair (X,A), denoted $\operatorname{Div}(X,A)$, and the family of functions $\{\sigma_\rho^n\}$ is the lower relative divergence of the pair (X,A), denoted by $\operatorname{div}(X,A)$.

In Section 4, we show that both upper relative divergence and lower relative divergence depend only on the quasi-isometry type of (X,A). Therefore, we can define both the upper and the lower relative divergence of a pair (G,H), denoted by $\operatorname{Div}(G,H)$ and $\operatorname{div}(G,H)$, where G is a finitely generated group and H is a subgroup. While upper relative divergence generalizes upper divergence introduced by Gersten [9], lower relative divergence generalizes lower divergence defined in [1]. The relative divergence of a pair (G,H) measures the distance distortion of the complement of the r–neighborhood of H in the Cayley graph of G when r increases.

1.1 Upper relative divergence

The following theorem describes the upper relative divergence of a finitely generated group with respect to a finitely generated normal subgroup.

Theorem 1.1 *Let G be a finitely generated group and H a finitely generated normal subgroup of G. Then*

$$\mathrm{Div}(G/H,e) \preceq \mathrm{Div}(G,H) \preceq \mathrm{Dist}_G^H \circ \overline{\mathrm{Div}(G/H,e)},$$

where Dist_G^H is the upper distortion of H in G and $\overline{\mathrm{Div}(G/H,e)}$ is a slight modification of $\mathrm{Div}(G/H,e)$. Moreover, if G/H is one-ended and H is undistorted in G, then $\mathrm{Div}(G,H) \sim \mathrm{Div}(G/H,e)$.

In the above theorem, we use the well-known concept of distortion of subgroups. In some sense, this measures the "upper bound" of the distance distortion of a subgroup in comparison with the distance of a whole group. However, we also need the concept of "lower bound" of the distance distortion of subgroups to better understand how a subgroup is embedded into a whole group. Therefore, we introduce the concept of lower distortion and we refer to the traditional concept of distortion as upper distortion (see Section 3). The above theorem also helps us find a pair of groups (G,H), where G is a CAT(0) group and H is a normal subgroup of G, such that $\mathrm{Div}(G,H)$ can be any polynomial or exponential function (see Remark 5.3).

The upper divergence of a one-ended relative hyperbolic group is at least exponential by Sisto [22]. The following theorem strengthens the result of Sisto.

Theorem 1.2 *Let $(G,\mathbb{P})$ be a relatively hyperbolic group and H a subgroup of G such that $0 < \tilde{e}(G,H) < \infty$, where $\tilde{e}(G,H)$ is the number of filtered ends of H in G. We assume that H is not conjugate to an infinite index subgroup of any peripheral subgroup. Then $\mathrm{Div}(G,H)$ is at least exponential.*

We refer the readers to Section 2.3 for the definition of the number of filtered ends.

1.2 Lower relative divergence

As mentioned earlier, the lower divergence of a finitely generated group is either linear or exponential. The lower relative divergence of a pair of groups, on the other hand, is more diverse.

Theorem 1.3 *Let f be any polynomial function or exponential function. There is a pair of groups (G,H), where G is a $\mathrm{CAT}(0)$ group (ie the group that acts properly and cocompactly on some $\mathrm{CAT}(0)$ space) and H is an infinite cyclic subgroup of G, such that $\mathrm{div}(G,H)$ is f.*

We compute the lower relative divergence of a pair of groups (G, H) when H is an infinite normal subgroup. The following theorem helps us find the upper bound of the relative lower divergence of a pair of groups (G, H) when H is an infinite normal subgroup of G.

Theorem 1.4 *Let G be a finitely generated group and H an infinite normal subgroup of G. Let K be any finitely generated infinite subgroup of H. Then, the relative lower divergence* $\mathrm{div}(G, H)$ *is dominated by the lower distortion of K in G. In particular, if H is finitely generated, then the relative lower divergence* $\mathrm{div}(G, H)$ *is dominated by the lower distortion of H in G.*

In order to measure the lower relative divergence of a finitely generated group with respect to a normal subgroup, we use the concept of lower distortion of a subgroup (which was mentioned earlier). Although the idea of lower distortion is implicit in works of Gromov [12], Ol'shanskii [19] and many others, the exact concept does not seem to be recorded in the literature. Applying the above theorem in the case of CAT(0) groups, we can show that the relative lower divergence of a CAT(0) group G with respect to a normal subgroup H containing at least one infinite order element is linear (see Theorem 7.8).

We also examine the lower relative divergence of a relatively hyperbolic group with respect to a subgroup. While the upper relative divergence of a finitely generated relatively hyperbolic group with respect to almost all subgroups is at least exponential (see Theorem 1.2), its lower relative divergence can be linear (see Theorem 8.25 and Theorem 8.35). Moreover, we also examine the lower relative divergence of a finitely generated relatively hyperbolic group with respect to a fully relatively quasiconvex subgroup in the following theorem.

Theorem 1.5 *Let $(G, \mathbb{P})$ be a relatively hyperbolic group and H an infinite fully relatively quasiconvex subgroup of G such that $0 < \widetilde{e}(G, H) < \infty$, where $\widetilde{e}(G, H)$ is the number of filtered ends of H in G. Then* $\mathrm{div}(G, H)$ *is at least exponential.*

In the above theorem, if we drop the condition "fully relative quasiconvexity" of the subgroup H, the conclusion of the theorem is no longer true (see Theorem 8.35).

1.3 Lower distortion

In this paper, we also introduce *lower distortion* as a new invariant for pairs (G, H) of finitely generated groups. As we mentioned earlier, upper distortion only measures the "upper bound" of the distance distortion of a subgroup in comparison with the distance of a whole group and we need the "lower bound" of the distance distortion of subgroups. We refer the readers to Definition 3.3 for the concept of lower distortion. In general, the lower and upper distortion of a pair of groups are not the same (see Example 3.17). We also show some properties of lower distortion as well as its relation with upper distortion (see Theorem 3.6 and Propositions 3.9 and 3.5). Moreover, we use one result of Ol'shanskii [19] to construct examples of pairs of groups with a large class of lower and upper distortion functions.

Theorem 1.6 *Let $f\colon [0,\infty) \to [0,\infty)$ be a strictly increasing function such that $f(0) = 0$ and f^{-1} is subadditive. Suppose that there is a positive integer C such that $f(n) \leq C^n$ for every positive n. Let H be a finitely generated group such that its growth is bounded by some polynomial function. Then there is a finitely generated group G containing H such that the upper and lower divergence of the pair (G, H) are both equivalent to f.*

1.4 Overview

In Section 2, we prepare some preliminary knowledge for the main part of the paper. This knowledge will be used to define the concept of relative divergence and compute relative divergence of certain pairs of groups.

In Section 3, we recall the concept of distortion of a subgroup, which we call upper distortion and introduce the related concept of lower distortion. Together with upper distortion, lower distortion helps us understand the connection between the geometry of a group and the geometry of its subgroups. We also carefully investigate this new concept although it is not the main part of this paper. Finally, we give the proof of Theorem 1.6 and discuss an example of Gromov to show the difference between upper and lower distortion.

In Section 4, we give precise definitions of upper and lower divergence of a pair (X, A), where X is a geodesic space and A is a subspace. We use these concepts to define the upper and lower divergence of a pair (G, H), where G is a finitely generated group and H is a subgroup. We also investigate some key properties of relative divergence.

In Sections 5 and 6, we investigate the divergence of a finitely generated group with respect to a normal subgroup or a cyclic subgroup. In Section 5, the proof of Theorems 1.1 and 1.4 are also shown.

In Section 7, we examine relative divergence of some CAT(0) groups. We also investigate a family of groups studied by Macura [16] to show that relative lower divergence can be a polynomial function with arbitrary degree. In this section, readers can find the proof of Theorem 1.3 for the case the lower divergence is polynomial.

In Section 8, we examine the relative divergence of a relatively hyperbolic group. We also investigate the lower relative divergence of a relatively hyperbolic group with respect to a fully relatively quasiconvex subgroup and use this fact to show that the lower divergence of a pair of groups can be at least exponential. In this section, we prove Theorems 1.2 and 1.5. Moreover, readers can see the proof of Theorem 1.3 for the case the lower divergence is exponential in this section.

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2 Preliminaries

In this section, we discuss some preliminary background before discussing the main part of the paper. We first construct the notions of domination and equivalence. We review some concepts in geometric group theory: geodesic spaces, quasigeodesics, quasi-isometry and quasi-isometric embedding, and the number of filtered ends of pairs of groups. We also introduce the concept of quasi-isometry between two pairs of metric spaces.

2.1 The notions of domination and equivalence

In this section, we build the notions of domination and equivalence on the set of some certain families of functions. These notions are the tool to measure the relative divergence of a finitely generated group with respect to a subgroup.

Definition 2.1 Let $\mathcal{M}$ be the collection of all functions from $[0,\infty)$ to $[0,\infty]$. Let f and g be arbitrary elements of $\mathcal{M}$. *The function f is dominated by the function g*, denoted $f \preceq g$, if there are positive constants A, B, C and D such that $f(x) \leq Ag(Bx)+Cx$ for all $x > D$. Two functions f and g are *equivalent*, denoted $f \sim g$, if $f \preceq g$ and $g \preceq f$. *The function f is strictly dominated by the function g*, denoted $f \prec g$, if f is dominated by g and they are not equivalent.

Remark 2.2 The relations $\preceq$ and $\prec$ are transitive. The relation $\sim$ is an equivalence relation on the set $\mathcal{M}$.

Let f and g be two polynomial functions in the family $\mathcal{M}$. We observe that f is dominated by g if and only if the degree of f is less than or equal to the degree of g and they are equivalent if and only if they have the same degree. All exponential functions of the form a^{bx+c}, where $a>1$, $b>0$, are equivalent. Therefore, a function f in $\mathcal{M}$ is *linear, quadratic or exponential* if f is respectively equivalent to any polynomial with degree one, two or any function of the form a^{bx+c}, where $a>1$, $b>0$.

Definition 2.3 Let $\{\delta^n_\rho\}$ and $\{\delta'^n_\rho\}$ be two families of functions of $\mathcal{M}$, indexed over $\rho \in (0,1]$ and positive integers $n \geq 2$. *The family* $\{\delta^n_\rho\}$ *is dominated by the family* $\{\delta'^n_\rho\}$, denoted $\{\delta^n_\rho\} \preceq \{\delta'^n_\rho\}$, if there exists a constant $L \in (0,1]$ and a positive integer M such that $\delta^n_{L\rho} \preceq \delta'^{Mn}_\rho$. The notions of strict domination and equivalence can be defined as above.

Remark 2.4 The relations $\preceq$ and $\prec$ are transitive. The relation $\sim$ is an equivalence relation.

If f is an element in $\mathcal{M}$, we could represent f as a family $\{\delta^n_\rho\}$ for which $\delta^n_\rho = f$ for all ρ and n. Therefore, the family $\{\delta^n_\rho\}$ is dominated by (or dominates) a function f in $\mathcal{M}$ if $\{\delta^n_\rho\}$ is dominated by (or dominates) the family $\{\delta'^n_\rho\}$ where $\delta'^n_\rho = f$ for all ρ and n. The equivalence between a family $\{\delta^n_\rho\}$ and a function f in $\mathcal{M}$ can be defined similarly. Thus a family $\{\delta^n_\rho\}$ is linear, quadratic, exponential, etc if $\{\delta^n_\rho\}$ is equivalent to the function f where f is linear, quadratic, exponential, etc.

2.2 Geodesic spaces, quasigeodesics, quasi-isometry

In this section, we review the concepts of geodesic spaces, quasigeodesics, quasi-isometry and quasi-isometric embedding, and we introduce the concept of quasi-isometry between two pair of metric spaces. These concepts play an important role in defining the concept of upper relative divergence and lower relative divergence of a finitely generated group with respect to a subgroup. Most of information in this section is cited from Ghys and de la Harpe [10].

Remark 2.5 For each path with finite length α in a geodesic space X, we denote the endpoints of α by α_+, α_- and the length of α by $\ell(\alpha)$. For each ray α in a space X, we denote the initial point of α by α_+.

Definition 2.6 Let (X,d) be a metric space.

(1) A path p in X is an (L,C)*–quasigeodesic* for some $L \geq 1$ and $C \geq 0$ if for every subpath q of p the inequality $\ell(q) \leq L\, d(q_+, q_-) + C$ holds.

(2) A path p in X is a *quasigeodesic* if it is (L,C)–quasigeodesic for some $L \geq 1$ and $C \geq 0$.

(3) A path p in X is an *L–quasigeodesic* if it is (L, L)–quasigeodesic for some $L \geq 1$.

(4) A path p in X is a *geodesic* if it is $(1, 0)$–quasigeodesic.

(5) Two quasigeodesics are *equivalent* if the Hausdorff distance between their images is finite.

(6) The metric space X is a *geodesic space* if any pair of points in X can be joined by a geodesic segment.

Definition 2.7 Let (X, d_X) and (Y, d_Y) be two metric spaces. The map Φ from X to Y is *a quasi-isometry* if there is a constant $K \geq 1$ and a function Ψ from Y to X such that the following holds:

(1)	$d_Y(\Phi(x_1), \Phi(x_2)) \leq K\, d_X(x_1, x_2) + K$	for all x_1, x_2 in X,
(2)	$d_X(\Psi(y_1), \Psi(y_2)) \leq K\, d_Y(y_1, y_2) + K$	for all y_1, y_2 in Y,
(3)	$d_Y(\Phi \circ \Psi(y), y) \leq K$	for all y in Y,
(4)	$d_X(\Psi \circ \Phi(x), x) \leq K$	for all x in X.

The proof of the following lemma is obvious, and we leave it to the reader.

Lemma 2.8 *Let (X, d_X) and (Y, d_Y) be two geodesic spaces and the map Φ from X to Y a quasi-isometry. Then there is a constant $C \geq 1$ such that the following hold.*

(1) $\frac{1}{C}\, d_X(x_1, x_2) - 1 \leq d_Y(\Phi(x_1), \Phi(x_2)) \leq C\, d_X(x_1, x_2) + C$ *for all* $x_1, x_2 \in X$.

(2) $N_C(\Phi(X)) = Y$.

(3) *If α is a path connecting two points x_1 and x_2 in X, then there is a path β connecting $\Phi(x_1)$ and $\Phi(x_2)$ in Y such that the Hausdorff distance between $\Phi(\alpha)$ and β is at most C. Moreover, $|\beta| \leq C|\alpha| + C$.*

(4) *If β is a path connecting two points $\Phi(x_1)$ and $\Phi(x_2)$ for some $x_1, x_2 \in X$, then there is a path α connecting x_1 and x_2 in X such that the Hausdorff distance between $\Phi(\alpha)$ and β is at most C. Moreover, $|\alpha| \leq C|\beta| + C$.*

Definition 2.9 Let (X, d_X) and (Y, d_Y) be two geodesic spaces and the map Φ from X to Y a quasi-isometric embedding if

$$\frac{1}{C}\, d_X(x_1, x_2) - 1 \leq d_Y(\Phi(x_1), \Phi(x_2)) \leq C\, d_X(x_1, x_2) + C$$

for all x_1, x_2 in X.

Remark 2.10 Throughout this paper, we denote (X, A) to be a pair of metric spaces, where X is a geodesic space and A is a subspace of X.

Definition 2.11 Two pairs of spaces (X, A) and (Y, B) are *quasi-isometric* if there is a quasi-isometry Φ from X to Y such that the Hausdorff distance between $\Phi(A)$ and B is finite.

It is not hard to prove the following proposition and we leave it to the reader.

Proposition 2.12 *A quasi-isometry of pairs of metric spaces is an equivalence relation.*

2.3 Filtered ends of pairs of groups

In this section, we review the concepts of the number of ends of groups and the number of filtered ends of pairs of groups. We refer the readers to Geoghegan [8, Chapter 14] for the proof of all the statements on these concepts. We also prove the lemma on the existence of subgroup perpendicular ray which is defined below.

We now define the concept of the number of filtered ends of a pair of groups and we will see that this concept generalizes the concept of the number of ends of a group.

Definition 2.13 Let G be a group with a finite generating set S and H a subgroup of G. For each positive r we denote $C_r(H)$ to be the complement of the r–neighborhood of H in the Cayley graph $\Gamma(G, S)$. A connected component U of $C_r(H)$ is *deep* if U does not lie in the s–neighborhood of H for any positive s. Let $\tilde{e}_r(G, H)$ be the number of deep components of $C_r(H)$. We note that $\tilde{e}_r(G, H) \geq \tilde{e}_s(G, H)$ if $r > s$. The number of *filtered ends of the pair* (G, H), denoted $\tilde{e}(G, H)$, is the supremum of the set $\{\tilde{e}_r(G, H) \mid r > 0\}$.

Remark 2.14 Let G be a finitely generated group and H a subgroup.

(1) The number $\tilde{e}(G, H)$ does not depend on the choice of finite generating set S of G and $\tilde{e}(G, H) = 0$ if and only if H is a finite index subgroup of G.

(2) If $\tilde{e}(G, H) = m < \infty$, then there is a positive number r_0 such that $C_r(H)$ has exactly m deep components for each $r > r_0$.

(3) When H is the trivial subgroup, $\tilde{e}(G, H)$ is *the number of ends of* G, denoted $\tilde{e}(G)$. A finitely generated group is *one-ended* if $\tilde{e}(G) = 1$.

Theorem 2.15 [8, Proposition 14.5.9] *If H is a finitely generated normal subgroup of G then $\tilde{e}(G, H)$ equals the number of ends of G/H.*

Definition 2.16 Let G be a group with a finite generating set S and H an infinite index subgroup of G. A geodesic ray γ in the Cayley graph $\Gamma(G,S)$ is *H–perpendicular* if the initial point h of γ lies in H and $d_S(\gamma(r),H)=r$ for all positive r.

The following lemma shows the existence of many H–perpendicular geodesic rays.

Lemma 2.17 *Let G be a group with a finite generating set S and H an infinite index subgroup of G. Then for each element h in H, there is an H–perpendicular geodesic ray with the initial point h.*

Proof For each positive integer n, there is a vertex g_n in $C_n(H)$. Let k_n be an element in H and α_n a geodesic segment connecting g_n and k_n such that the length of α_n is equal to the distance between g_n and H. We define $\gamma_n=(hk_n^{-1})\alpha_n$, then γ_n is a geodesic segment with the initial point h and $d_S(\gamma_n(r),H)=r$ for all positive r less than the length of γ_n. By the Arzela–Ascoli theorem, there is a geodesic ray γ with the initial point h such that $d_S(\gamma(r),H)=r$ for all positive r. □

3 Distortion of subgroups

In this section, we will review the concept of distortion of a subgroup, which we call upper distortion. This concept of distortion will later help us compute relative divergence of a large class of pairs of groups. We also introduce the concept of lower distortion of a subgroup. This new concept is also a tool to compute relative divergence. We investigate some key properties of lower distortion and the relation between lower distortion and upper distortion.

First of all, we will review the concept of upper distortion.

Definition 3.1 Let G be a group with a finite generating set S and H a subgroup of G with a finite generating set T. The *upper subgroup distortion* of H in G is the function $\mathrm{Dist}_G^H\colon (0,\infty)\to(0,\infty)$ defined by

$$\mathrm{Dist}_G^H(r)=\max\{|h|_T \mid h\in H, |h|_S\le r\}.$$

Remark 3.2 It is well known that the concept of upper distortion does not depend on the choice of finite generating sets S and T. More precisely, the functions Dist_G^H are equivalent for all pairs of finite sets (S,T) generating (G,H) respectively.

The function Dist_G^H is nondecreasing, and dominates a linear function.

A finitely generated subgroup H of G is *undistorted* if Dist_G^H is linear.

We now introduce the concept of lower distortion.

Definition 3.3 Let G be a group with a finite generating set S and H a subgroup of G with a finite generating set T. The *lower distortion* of H in G is the function $\mathrm{dist}_G^H\colon (0,\infty) \to (0,\infty)$ defined as

$$\mathrm{dist}_G^H(r) = \min\{|h|_T \mid h \in H, |h|_S \geq r\}.$$

We use the convention that the minimum of the empty set is 0.

Remark 3.4 Similar to the concept of upper distortion, the concept of lower distortion also does not depend on the choice of generating sets. When H is an infinite subgroup, the function dist_G^H is nondecreasing and dominates a linear function.

The following proposition shows a relation between upper and lower distortion.

Proposition 3.5 *Let G be a finitely generated group and H a finitely generated subgroup of G. Then $\mathrm{dist}_G^H \preceq \mathrm{Dist}_G^H$.*

Proof Let S be a finite generating set of G and we assume that S contains the finite generating set T of the subgroup H. Thus we could consider $\Gamma(H,T)$ as a subgraph of $\Gamma(G,S)$. If H is a finite subgroup then dist_G^H is a bounded function and the proof follows easily. Thus we assume H is an infinite subgroup.

For each $r > 1$, we could chose an element k in H such that $|k|_S \geq 2r$. We connect the identity element e and k by a geodesic α in $\Gamma(H,T)$. Thus we can choose h to be an element in α such that $r \leq |h|_S \leq 2r$. Since h is also an element of H, then $\mathrm{dist}_G^H(r) \leq |h|_T \leq \mathrm{Dist}_G^H(2r)$. Thus $\mathrm{dist}_G^H \preceq \mathrm{Dist}_G^H$. □

We now investigate some key properties of lower distortion:

Theorem 3.6 *Suppose that G, H, K are all infinite finitely generated groups and $K \leq H \leq G$.*

(1) $\mathrm{dist}_H^K \circ \mathrm{dist}_G^H \preceq \mathrm{dist}_G^K$.

(2) $\mathrm{dist}_H^K \preceq \mathrm{dist}_G^K$.

(3) $\mathrm{dist}_G^H \preceq \mathrm{dist}_G^K$.

(4) *If $|G:H| < \infty$, then $\mathrm{dist}_G^K \sim \mathrm{dist}_H^K$.*

(5) *If $|H:K| < \infty$, then $\mathrm{dist}_G^K \sim \mathrm{dist}_G^H$.*

(6) *If H_1, H_2 are two commensurable finitely generated subgroups, then $\mathrm{dist}_G^{H_1} \sim \mathrm{dist}_G^{H_2}$.*

Proof We call S_1, S_2 and S_3 finite generating sets of G, H and K respectively. We can assume that $S_3 \subset S_2 \subset S_1$. We now prove that

$$\operatorname{dist}_H^K \circ \operatorname{dist}_G^H(n) \leq \operatorname{dist}_G^K(n) \quad \text{for all } n.$$

For any positive number n, choose $k_0 \in K$ such that $|k_0|_{S_1} \geq n$ and $|k_0|_{S_3} = \operatorname{dist}_G^K(n)$. Since $k_0 \in H$ and $|k_0|_{S_1} \geq n$, then $|k_0|_{S_2} \geq \operatorname{dist}_G^H(n)$. Therefore, we have $|k_0|_{S_3} \geq \operatorname{dist}_H^K(\operatorname{dist}_G^H(n))$. Thus

$$\operatorname{dist}_H^K \circ \operatorname{dist}_G^H(n) \leq \operatorname{dist}_G^K(n) \quad \text{for all } n.$$

Statements (2) and (3) are immediate results of (1) since the lower distortion functions dist_G^H and dist_H^K are nondecreasing and at least linear.

We now prove statement (4). Since $\operatorname{dist}_H^K \preceq \operatorname{dist}_G^K$, then we only need to prove $\operatorname{dist}_G^K \preceq \operatorname{dist}_H^K$. Since $|G : H| < \infty$, then there is a positive integer C such that

$$d_{S_2}(h_1, h_2) \leq C\, d_{S_1}(h_1, h_2) + C \quad \text{for all } h_1 \text{ and } h_2 \text{ in } H.$$

We now prove that

$$\operatorname{dist}_G^K(n) \leq \operatorname{dist}_H^K(2Cn) \quad \text{for all } n.$$

For any positive number $n > 1$, we choose $k_0 \in K$ such that $|k_0|_{S_2} \geq 2Cn$ and $|k_0|_{S_3} = \operatorname{dist}_H^K(2Cn)$. Thus

$$|k_0|_{S_1} \geq \frac{|k_0|_{S_2} - C}{C} \geq 2n - 1 \geq n.$$

Therefore, $\operatorname{dist}_G^K(n) \leq \operatorname{dist}_H^K(2Cn)$. In particular, $\operatorname{dist}_G^K \preceq \operatorname{dist}_H^K$.

We now prove statement (5). Since $\operatorname{dist}_G^H \preceq \operatorname{dist}_G^K$, then we only need to prove $\operatorname{dist}_G^K \preceq \operatorname{dist}_G^H$. Since $|H : K| < \infty$, then there is a positive integer C such that

$$d_{S_3}(k_1, k_2) \leq C\, d_{S_2}(k_1, k_2) + C \quad \text{for all } k_1 \text{ and } k_2 \text{ in } K,$$

and $H \subset N_C(K)$ with respect to metric d_{S_2}. We now show that

$$\operatorname{dist}_G^K(n) \leq C \operatorname{dist}_G^H(2n) + C^2 + C \quad \text{for all } n \geq C.$$

For any positive number n greater than C, we choose $h_0 \in H$ such that $|h_0|_{S_1} \geq 2n$ and $|h_0|_{S_2} = \operatorname{dist}_G^H(2n)$. Since $H \subset N_C(K)$ with respect to metric d_{S_2}, then there is $k_0 \in K$ such that $d_{S_2}(k_0, h_0) \leq C$. In particular, $d_{S_1}(k_0, h_0) \leq C$. Thus

$$|k_0|_{S_1} \geq |h_0|_{S_1} - C \geq 2n - C \geq n.$$

Thus $|k_0|_{S_3} \geq \operatorname{dist}_G^K(n)$.

Also

$$|k_0|_{S_3} \le C|k_0|_{S_2} + C \le C(|h_0|_{S_2} + C) + C,$$
$$|h_0|_{S_2} = \mathrm{dist}_G^H(2n).$$

Therefore, $\mathrm{dist}_G^K(n) \le C\,\mathrm{dist}_G^H(2n) + C^2 + C$. In particular, $\mathrm{dist}_G^K \preceq \mathrm{dist}_G^H$.

We easily obtain (6) from (5) by observing that

$$|H_1 : (H_1 \cap H_2)| < \infty \quad \text{and} \quad |H_2 : (H_1 \cap H_2)| < \infty. \qquad \square$$

We now explain the relationship between the lower distortion and the growth of a finitely generated group. We will see that the growth function will be an upper bound of the lower distortion. Before showing this fact, we need to review the concept of growth of groups.

Definition 3.7 Let G be a group with a finite set of generators S. *The growth* of G, denoted by Growth_G, is a function $f\colon [0,\infty) \to [0,\infty)$ to itself defined by letting $f(r)$ be the number of elements of G that lie in the ball $B(e,r)$ for each $r \ge 0$.

Remark 3.8 It is well known that the growth of a finitely generated group does not depend on the choice of finite generating set (the proof is almost identical to the case of upper distortion). More precisely, the functions Growth_G are equivalent for all finite sets S of generators of G. Moreover, the function Growth_G is dominated by the exponential function.

Proposition 3.9 *Let G be a finitely generated group and H a finitely generated subgroup of G. Then the lower distortion dist_G^H is dominated by the growth function Growth_G of G. In particular, the lower distortion dist_G^H is dominated by the exponential function.*

Proof Let S be a finite generating set of G. We will assume that S contains the finite generating set T of the subgroup H. Thus we could consider $\Gamma(H,T)$ as a subgraph of $\Gamma(G,S)$. If H is finite, then dist_G^H is bounded and the proof follows easily. Thus we assume H is an infinite subgroup.

For each $r > 1$, we could chose an element h in H such that $|h|_S \ge r$. We connect the identity element e and h by a geodesic α in $\Gamma(H,T)$. Let h' be a vertex in α such that $|h'|_S \ge r$ and the subpath α' of α connecting e and h' must lie in the closed ball with center e and radius $2r$ of $\Gamma(G,S)$. Thus the length of α' is bounded by the number of vertices in this ball. Therefore, $|h'|_T$ is bounded by the number of vertices of the closed ball with center e and radius $2r$ in $\Gamma(G,S)$. Thus $\mathrm{dist}_G^H(r) \le \mathrm{Growth}_G(2r)$. Therefore, $\mathrm{dist}_G^H \preceq \mathrm{Growth}_G$. $\square$

We now find some examples of finitely generated groups and its finitely generated subgroups to see their lower distortion. The following theorem can be deduced from the work of Milnor (see the proof of [18, Lemma 4]). We just use the new concept of lower distortion to interpret a part of Milnor's work.

Theorem 3.10 *Let $G = \langle a, b, c \mid bab^{-1}a^{-1} = c, ac = ca, bc = cb\rangle$ be the Heisenberg group and H the cyclic group generated by c. Then dist_G^H and Dist_G^H are both quadratic.*

Remark 3.11 In [23], Tits investigates the growth of a finitely generated virtually nilpotent group. We can use a part of his work to find a pair (G, H), where G is a finitely generated nilpotent group and H is a finitely generated subgroup, such that dist_G^H and Dist_G^H can be equivalent to the same polynomial with arbitrary degree.

In [20], Osin also gives a formula to compute upper distortion of arbitrary subgroups of nilpotent groups.

Before studying more examples about lower distortion, we need to review the concept of length functions and a key theorem.

Definition 3.12 Let G be a group with a finite generating set S and H a subgroup of G. The *length function* ℓ of H inside G is the function from the group H to the set of natural numbers defined as

$$\ell(h) = |h|_S \quad \text{for } h \in H .$$

Remark 3.13 In some sense, the concept of length function can give us more information than the concepts of upper and lower distortion when we investigate an embedding of a subgroup.

Theorem 3.14 [19] *Let ℓ be the length function of group H inside some finitely generated group G. Then the following conditions hold.*

(1) *$\ell(h) = \ell(h^{-1})$ for every $h \in H$; $\ell(h) = 0$ if and only if $h = e$.*

(2) *$\ell(h_1h_2) \le \ell(h_1) + \ell(h_2)$ for every $h_1, h_2 \in H$.*

(3) *There is a positive integer C such that the cardinality of the set $\{h \in H \mid \ell(h) \le n\}$ does not exceed C^n for every natural number n.*

Conversely for every group H and every function ℓ from H to the set of natural numbers satisfying (1)–(3), there exists an embedding of H into a 2–generated group G with a finite generating set $S = \{g_1, g_2\}$ such that the length function ℓ_1 of H inside G is equivalent to ℓ (ie there exists a positive integer B such that $(1/B)\ell(h) \le \ell_1(h) \le B\ell(h)$).

Definition 3.15 A function $f\colon [0,\infty) \to [0,\infty)$ is subadditive if $f(i+j) \le f(i) + f(j)$ for every positive numbers i and j.

We now apply Theorem 3.14 to show that any finitely generated group H can be a subgroup of a finitely generated group G such that the lower distortion and the upper distortion of H in G can be both equivalent to any element of some large class of functions.

Theorem 3.16 *Let $f\colon [0,\infty) \to [0,\infty)$ be a strictly increasing function such that $f(0)=0$ and f^{-1} is subadditive. Suppose that there is a positive integer C such that $f(n) \le C^n$ for every positive n. Let H be a finitely generated group such that its growth is bounded by some polynomial function. Then there is a finitely generated group G containing H such that $\mathrm{dist}_G^H \sim \mathrm{Dist}_G^H \sim f$.*

Proof We fix some finite generating set T for H. Let A and m be a positive integers such that the number of group elements in a ball with radius n is bounded by An^m for every positive integer n. For each nonnegative number x, we define $\lceil x \rceil$ to be the smallest integer that is greater than or equal to x. We now define the length function $\ell\colon H \to \mathbb{N}$ by

$$\ell(h) = \lceil f^{-1}(|h|_T) \rceil \quad \text{for every } h \in H .$$

We will check ℓ satisfies conditions (1)–(3) in Theorem 3.14. Obviously, $\ell(h) = \ell(h^{-1})$ for every $h \in H$ and $\ell(h) = 0$ if and only if $h = e$. We now check ℓ satisfies condition (2). Indeed, for every $h_1, h_2 \in H$,

$$\begin{aligned}\ell(h_1h_2) = \lceil f^{-1}(|h_1h_2|_T) \rceil &\le \lceil f^{-1}(|h_1|_T + |h_2|_T) \rceil \\ &\le \lceil f^{-1}(|h_1|_T) + f^{-1}(|h_2|_T) \rceil \\ &\le \lceil f^{-1}(|h_1|_T) \rceil + \lceil f^{-1}(|h_2|_T) \rceil \le \ell(h_1) + \ell(h_2).\end{aligned}$$

Finally, we need to check ℓ satisfies condition (3). Since for each nonnegative integer n

$$\begin{aligned}\{h \in H \mid \ell(h) \le n\} &= \{h \in H \mid \lceil f^{-1}(|h|_T) \rceil \le n\} \\ &= \{h \in H \mid f^{-1}(|h|_T) \le n\} \\ &= \{h \in H \mid |h|_T \le f(n)\} \subset \{h \in H \mid |h|_T \le C^n\}\end{aligned}$$

and the cardinality of the set $\{h \in H \mid |h|_T \le C^n\}$ is bounded by $A(C^m)^n$, then the cardinality of the set $\{h \in H \mid \ell(h) \le n\}$ is bounded by $A(C^m)^n$.

By Theorem 3.14, the group H is a subgroup of some finitely generated group G with a finite generating set S such that the function ℓ is equivalent to ℓ_1, where

$\ell_1(h) = |h|_S$ for every $h \in H$. Therefore, there is a positive integer B such that $(1/B)\ell(h) \le \ell_1(h) \le B\ell(h)$ for every $h \in H$.

We now show that the upper distortion Dist_G^H is dominated by f. For each positive number n and any $h \in H$ such that $|h|_S \le n$, we see that

$$f^{-1}(|h|_T) \le \ell(h) \le B\ell_1(h) \le Bn.$$

Thus $|h|_T \le f(Bn)$. Therefore, $\mathrm{Dist}_G^H(n) \le f(Bn)$. In particular, the upper distortion Dist_G^H is dominated by f.

We finish the proof of the theorem by showing that the lower distortion dist_G^H dominates f. For each positive number n and any $h \in H$ such that $|h|_S \ge Bn + B$, we see that

$$f^{-1}(|h|_T) \ge \ell(h) - 1 \ge \tfrac{1}{B}\ell_1(h) - 1 \ge n.$$

Thus $|h|_T \ge f(n)$. Therefore, $\mathrm{dist}_G^H(Bn + B) \ge f(n)$. In particular, the lower distortion dist_G^H dominates f. □

We know show one pair of groups (G, H) such that dist_G^H and Dist_G^H are not equivalent. The following example is defined by Gromov [12].

Example 3.17 Let $G = \langle a, b, c \mid bab^{-1} = a^2, cbc^{-1} = b^2 \rangle$ and let H be the cyclic subgroup generated by a. Observe that

$$a^{2^{2^n}} = b^{2^n} a b^{-2^n} = c^n b c^{-n} a c^n b^{-1} c^{-n}.$$

Thus $\mathrm{Dist}_G^H(4n+2) \ge 2^{2^n}$ for each positive number n. Therefore, the upper distortion Dist_G^H is superexponential. However, the lower distortion dist_G^H is at most exponential by Proposition 3.9. Thus two functions dist_G^H and Dist_G^H are not equivalent.

4 Relative divergence of geodesic spaces and finitely generated groups

4.1 Relative upper divergence

In this section, we introduce the concept of relative upper divergence of geodesic spaces as well as finitely generated groups. We also prove that upper relative divergence is a quasi-isometry invariant.

Definition 4.1 Let X be a geodesic space and A a subspace of X. Let r be any positive number.

(1) $N_r(A)=\{x\in X \mid d_X(x,A)<r\}$.

(2) $\partial N_r(A)=\{x\in X \mid d_X(x,A)=r\}$.

(3) $C_r(A)=X-N_r(A)$.

(4) Let $d_{r,A}$ be the induced length metric on the complement of the r–neighborhood of A in X. If the subspace A is clear from context, we can use the notation d_r instead of using $d_{r,A}$.

Definition 4.2 Let (X,A) be a pair of metric spaces. For each $\rho\in(0,1]$ and positive integer $n\geq 2$, we define a function $\delta^n_\rho\colon [0,\infty)\to[0,\infty]$ as follows.

For each r, let $\delta^n_\rho(r)=\sup d_{\rho r}(x_1,x_2)$ where the supremum is taken over all $x_1,x_2\in \partial N_r(A)$ such that $d_r(x_1,x_2)<\infty$ and $d(x_1,x_2)\leq nr$.

The family of functions $\{\delta^n_\rho\}$ is *the relative upper divergence* of X with respect A, denoted $\mathrm{Div}(X,A)$.

Before defining the upper relative divergence of a finitely generated group with respect to a subgroup, we need the following proposition.

Proposition 4.3 *If two pairs of spaces (X,A) and (Y,B) are quasi-isometric, then* $\mathrm{Div}(X,A)\sim\mathrm{Div}(Y,B)$.

Before proving the above proposition, we need the following lemmas.

Lemma 4.4 *Let X, Y be geodesic spaces and A a subspace of X. Let Φ be a quasi-isometry from X to Y. Then* $\mathrm{Div}(X,A)\preceq\mathrm{Div}(Y,\Phi(A))$.

Proof Let $B=\Phi(A)$. Let $\mathrm{Div}(X,A)=\{\delta^n_\rho\}$ and $\mathrm{Div}(Y,B)=\{\delta'^n_\rho\}$. Let K be the number provided by Lemma 2.8. Let $L=1/8K^2$ and $M=[2K(2K+1)+1]+1$. We will prove that $\delta^n_{L\rho}\preceq\delta'^{Mn}_\rho$. More precisely, we define $r_0=3K(1+K)+8K^2/\rho$ and we are going to show that for each $r>r_0$

$$\delta^n_{L\rho}(r)\leq K\delta'^{Mn}_\rho\Big(\frac{r}{2K}\Big)+(2K^2+1)r.$$

Indeed, let x_1 and x_2 be arbitrary points in $\partial N_r(A)$ such that $d_X(x_1,x_2)\leq nr$ and $d_{r,A}(x_1,x_2)<\infty$. Thus there is a path α in $C_r(A)$ connecting x_1 and x_2. By

Lemma 2.8, there is a path β connecting $\Phi(x_1)$, $\Phi(x_2)$ such that the Hausdorff distance between $\Phi(\alpha)$ and β is at most K. Thus

$$d_Y(\beta, B) \geq d_Y(\Phi(\alpha), B) - K \geq \frac{1}{K} d_X(\alpha, A) - 1 - K \geq \frac{r}{K} - 1 - K \geq \frac{r}{2K}.$$

Thus we could choose y_1 in $\partial N_{r/2K}(B)$ and a geodesic β_1 in $C_{r/2K}(B)$ connecting $\Phi(x_1)$ and y_1 such that the length of β_1 is bounded above by the distance between $\Phi(x_1)$ and B. Also, $d_Y(\Phi(x_1), B) \leq K\, d_X(x_1, A) + K \leq Kr + K$. Therefore, the length of β_1 is at most $Kr + K$. Similarly, we could choose y_2 in $\partial N_{r/2K}(B)$ and a geodesic β_2 in $C_{r/2K}(B)$ connecting $\Phi(x_2)$ and y_2 such that the length of β_2 is bounded above by $Kr + K$.

We define $\beta_3 = \beta_1 \cup \beta \cup \beta_2$, then β_3 is a path in $C_{r/2K}(B)$ connecting y_1 and y_2. Thus $d_{r/2K,B}(y_1, y_2) < \infty$.

Also

$$\begin{aligned} d_Y(y_1, y_2) &\leq d_Y(y_1, \Phi(x_1)) + d_Y(\Phi(x_1), \Phi(x_2)) + d_Y(\Phi(x_2), y_2) \\ &\leq (Kr + K) + (K\, d_X(x_1, x_2) + K) + (Kr + K) \\ &\leq 2Kr + 3K + Knr \leq (2K+1)nr \leq Mn\Big(\frac{r}{2K}\Big). \end{aligned}$$

We are now going to show that

$$d_{L\rho r,A}(x_1, x_2) \leq K d_{\rho(r/2K),B}(y_1, y_2) + (2K^2 + 1)r.$$

Indeed, let β' be an arbitrary path in $C_{\rho(r/2K)}(B)$ connecting y_1 and y_2. We define $\gamma = \beta_1 \cup \beta' \cup \beta_2$; then γ is a path in $C_{\rho(r/2K)}(B)$ connecting $\Phi(x_1)$, $\Phi(x_2)$ and the length of γ is bounded above by $2Kr + 2K + |\beta'|$.

By Lemma 2.8, there is a path α' connecting x_1 and x_2 in X such that the Hausdorff distance between $\Phi(\alpha')$ and γ is at most K. Moreover, $|\alpha'| \leq K|\gamma| + K$. Since

$$d_Y(\Phi(\alpha'), B) \geq d_Y(\gamma, B) - K \geq \frac{\rho r}{2K} - K \geq \frac{\rho r}{4K},$$

then

$$d_X(\alpha', A) \geq \frac{1}{K} d_Y(\Phi(\alpha'), B) - 1 \geq \frac{\rho r}{4K^2} - 1 \geq \frac{\rho r}{8K^2} \geq L\rho r.$$

Thus α' is a path in $C_{L\rho r}(A)$ connecting x_1 and x_2. Therefore, the distance in $C_{L\rho r}(A)$ between x_1 and x_2 is bounded above by the length of α'.

Also

$$|\alpha'| \leq K|\gamma| + K \leq K(2Kr + 2K + |\beta'|) + K \leq K|\beta'| + (2K^2 + 1)r,$$

and β' is an arbitrary path in $C_{\rho(r/2K)}(B)$ connecting y_1 and y_2.

Thus
$$d_{L\rho r,A}(x_1,x_2) \leq K d_{\rho(r/2K),B}(y_1,y_2) + (2K^2+1)r.$$
Therefore
$$\delta^n_{L\rho}(r) \leq K\delta'^{Mn}_{\rho}\Big(\frac{r}{2K}\Big) + (2K^2+1)r.$$
Thus $\delta^n_{L\rho} \preceq \delta'^{Mn}_{\rho}$. □

Lemma 4.5 *Let X be a geodesic space. Let A and B be two subspaces such that the Hausdorff distance between them is finite. Then* $\mathrm{Div}(X,A) \sim \mathrm{Div}(X,B)$.

Proof We only need to prove $\mathrm{Div}(X,A) \preceq \mathrm{Div}(X,B)$ since the argument for the other direction is almost identical. There is a positive number r_0 such that A lies in the r_0–neighborhood of B and B also lies in the r_0–neighborhood of A. Thus $\overline{N_r(A)} \subset N_{r+r_0}(B)$ and $\overline{N_r(B)} \subset N_{r+r_0}(A)$ for each positive r. Let $\mathrm{Div}(X,A) = \{\delta^n_\rho\}$ and $\mathrm{Div}(X,B) = \{\delta'^n_\rho\}$. We will show $\delta^n_{\rho/4} \preceq \delta'^{6n}_\rho$. More precisely, we are going to prove that for each $r > 4r_0/\rho$,
$$\delta^n_{\rho/4}(r) \leq \delta'^{6n}_\rho\Big(\frac{r}{2}\Big) + 4r.$$
Let x_1, x_2 be arbitrary points in $\partial N_r(A)$ such that $d_X(x_1,x_2) \leq nr$ and $d_{r,A}(x_1,x_2) < \infty$. Thus there is a path α in $C_r(A)$ connecting x_1 and x_2. Therefore, α lies in $C_{r-r_0}(B)$. Thus α also lies in $C_{r/2}(B)$ because $r/2 > r_0$. Moreover, x_1 and x_2 lies in $N_{r+r_0}(B)$. Therefore, we could choose y_1, y_2 in $\partial N_{r/2}(B)$ and two geodesics β_1, β_2 in $C_{r/2}(B)$ connecting x_1, y_1 and x_2, y_2 respectively such that the length of β_1 and β_2 are at most $r+r_0$. Since the distance between x_1 and x_2 is bounded above by nr, then the distance between y_1 and y_2 is at most $nr+2r+2r_0$. Thus $d_X(y_1,y_2) \leq (n+4)r \leq 3nr \leq 6n(r/2)$. We define $\alpha' = \beta_1 \cup \alpha \cup \beta_2$, then α' is a path in $C_{r/2}(B)$ connecting y_1 and y_2. Thus $d_{r/2,B}(y_1,y_2) < \infty$.

We are now going to show that
$$d_{\rho r/4,A}(x_1,x_2) \leq d_{\rho(r/2),B}(y_1,y_2) + 4r.$$
Indeed, let γ be an arbitrary path in $C_{\rho(r/2)}(B)$ connecting y_1 and y_2. Then γ also lies in $C_{\rho(r/2)-r_0}(A)$. Therefore, γ lies in $C_{\rho r/4}(A)$. Since β_1 and β_2 lies in $C_{r/2}(B)$, then they also lies in $C_{r/2-r_0}(A)$. Thus β_1 and β_2 lies in $C_{\rho r/4}(A)$. We define $\gamma' = \beta_1 \cup \gamma \cup \beta_2$, then γ' is a path in $C_{\rho r/4}(A)$ connecting x_1 and x_2. Thus $d_{\rho r/4,A}(x_1,x_2) \leq |\gamma'|$.

Also
$$|\gamma'| \leq |\beta_1| + |\gamma| + |\beta_2| \leq (r+r_0) + |\gamma| + (r+r_0) \leq |\gamma| + 4r$$
and γ is an arbitrary path in $C_{\rho(r/2)}(B)$ connecting y_1, y_2.

Thus

$$d_{\rho r/4,A}(x_1,x_2) \le d_{\rho(r/2),B}(y_1,y_2) + 4r.$$

Therefore

$$\delta^n_{\rho/4}(r) \le \delta'^{6n}_{\rho}\left(\frac{r}{2}\right) + 4r.$$

Thus $\delta^n_{\rho/4} \preceq \delta'^{6n}_{\rho}$. □

We now finish the proof of Proposition 4.3.

Proof Let Φ be a map from X to Y such that the Hausdorff distance between $\Phi(A)$ and B is finite. Then $\mathrm{Div}(X,A) \preceq \mathrm{Div}(Y,\Phi(A))$ by Lemma 4.4 and $\mathrm{Div}(Y,\Phi(A)) \sim \mathrm{Div}(Y,B)$ by Lemma 4.5. Thus $\mathrm{Div}(X,A) \preceq \mathrm{Div}(Y,B)$. Similarly, $\mathrm{Div}(Y,B) \preceq \mathrm{Div}(X,A)$. Therefore, $\mathrm{Div}(X,A) \sim \mathrm{Div}(Y,B)$. □

We are now ready to define the concept of relative upper divergence of a finitely generated group with respect to a subgroup.

Definition 4.6 Let G be a finitely generated group and H its subgroup. We define *the relative upper divergence* of G with respect to H, denoted $\mathrm{Div}(G,H)$ to be the relative upper divergence of the Cayley graph $\Gamma(G,S)$ with respect to H for some finite generating set S.

Remark 4.7 If H is the trivial subgroup, then $\delta^n_\rho = \delta^2_\rho$ for all $n \ge 2$. Thus we can ignore the parameter n in the family $\{\delta^n_\rho\}$ and consider that $\mathrm{Div}(G,e)$ is characterized by the one-parametrized family of functions $\{\delta_\rho\}$. By this way, the upper relative divergence $\mathrm{Div}(G,e)$ is the same as the upper divergence $\mathrm{Div}(G)$ of the group G in terms of Gersten [9].

4.2 Relative lower divergence

In this section, we introduce the concept of relative lower divergence of geodesic spaces as well as finitely generated groups. Similar to upper divergence, this concept is also a quasi-isometry invariant.

Definition 4.8 Let (X,A) be a pair of spaces. For each $\rho \in (0,1]$ and positive integer $n \ge 2$, we define a function $\sigma^n_\rho\colon [0,\infty) \to [0,\infty]$ as follows.

For each positive r, if there is no pair of $x_1, x_2 \in \partial N_r(A)$ such that $d_X(x_1,x_2) \ge nr$ and $d_r(x_1,x_2) < \infty$, we define $\sigma^n_\rho(r) = \infty$.

Otherwise, we define $\sigma^n_\rho(r) = \inf d_{\rho r}(x_1, x_2)$ where the infimum is taken over all $x_1, x_2 \in \partial N_r(A)$ such that $d_r(x_1, x_2) < \infty$ and $d(x_1, x_2) \geq nr$.

The family of functions $\{\sigma^n_\rho\}$ is *the relative lower divergence* of X with respect A, denoted $\mathrm{div}(X, A)$.

By using the same argument from the previous section, we have the following proposition.

Proposition 4.9 *If two pairs of spaces (X, A) and (Y, B) are quasi-isometric, then* $\mathrm{div}(X, A) \sim \mathrm{div}(Y, B)$.

We now define the concept of relative lower divergence of a finitely generated group with respect to a subgroup.

Definition 4.10 Let G be a finitely generated group and H its subgroup. We define *the relative lower divergence* of G with respect to H, denoted $\mathrm{div}(G, H)$, to be the relative lower divergence of the Cayley graph $\Gamma(G, S)$ with respect to H for some finite generating set S.

Before moving on to another section, we would like to discuss the concept of lower divergence of a geodesic ray in Charney and Sultan [4], and the connection between this concept and the concept of lower relative divergence. We first recall the concept of lower divergence of a geodesic ray in Charney and Sultan [4].

Definition 4.11 Let γ be a geodesic ray in a geodesic space X. For any $t > r > 0$, let $\rho_\gamma(r, t)$ denote the infimum of the lengths of all paths from $\gamma(t-r)$ to $\gamma(t+r)$ which lie outside the open ball of radius r about $\gamma(t)$. Define *the lower divergence of* γ to be the growth rate of the following function:

$$\mathrm{ldiv}_\gamma(r) = \inf_{t>r} \rho_\gamma(r, t).$$

The following theorem shows the concept of lower relative divergence generalizes the concept of lower divergence of a geodesic ray.

Theorem 4.12 *Let γ be a geodesic ray in a geodesic space X. Then*

$$\mathrm{div}(X, \gamma) \sim \mathrm{ldiv}_\gamma .$$

The proof of the above theorem is similar to the proof we are going to give for Proposition 6.6 and we leave it to the reader.

4.3 Some properties of relative divergence of finitely generated groups

In this section, we examine some key properties of relative divergence and we compare upper and lower relative divergence.

Theorem 4.13 *Let G be a finitely generated group and H a subgroup of G. Suppose that* $\operatorname{Div}(G,H)=\{\delta^n_\rho\}$ *and* $\operatorname{div}(G,H)=\{\sigma^n_\rho\}$.

(1) *If H is an infinite index subgroup of G, then $\delta^n_\rho(r)<\infty$ for every $r>0$.*

(2) *If H is infinite and $0<\widetilde{e}(G,H)<\infty$, then $\sigma^n_\rho(r)<\infty$ for every $r>0$.*

Proof Fix a finite set S of generators of G.

First, we will prove that $\delta^n_\rho(r)<\infty$ for every $r>0$. We define

$$A=S(e,r)\cap\partial N_r(H).$$

Obviously, A is a nonempty finite set. We define

$$B=\{(x,y)\mid x\in A,\ y\in\partial N_r(H),\ d_r(x,y)<\infty \text{ and } d_S(x,y)\le nr\}.$$

Therefore, B is also a nonempty finite set. Define $M=\{d_{\rho r}(x,y)\mid (x,y)\in B\}$ and we will show $\delta^n_\rho(r)\le M$.

Indeed, let x, y be arbitrary points in $\partial N_r(H)$ such that $d_r(x,y)<\infty$ and $d_S(x,y)\le nr$. Let h be an element in H such that $d_S(x,H)=d_S(x,h)=r$. Therefore, $(h^{-1}x,h^{-1}y)\in B$ and $d_{\rho r}(x,y)=d_{\rho r}(h^{-1}x,h^{-1}y)$. Thus $d_{\rho r}(x,y)\le M$. It follows that $\delta^n_\rho(r)\le M$.

We now assume that $0<\widetilde{e}(G,H)<\infty$ and we will prove $\sigma^n_\rho(r)<\infty$ for all $r>0$. Let $m=\widetilde{e}(G,H)$. For each $i\in\{0,1,2,\dots,m\}$ we could choose h_i in H such that the distance between h_i and h_j is at least $(n+2)r$ whenever $i\ne j$. By Lemma 2.17, for each $i\in\{0,1,2,\dots,m\}$ we could choose an H–perpendicular ray γ_i with the initial point h_i. Thus there are at least two different rays γ_i and γ_j such that $\gamma_i\cap C_r(H)$ and $\gamma_i\cap C_r(H)$ lie in the same component of $C_r(H)$. We define $u=\gamma_i(r)$ and $v=\gamma_j(r)$. Then u, v lie in $\partial N_r(H)$, the distance $d_r(u,v)<\infty$ and $d_S(u,v)\ge nr$. Thus

$$\sigma^n_\rho(r)\le d_{\rho r}(x,y)<\infty. \qquad \square$$

Theorem 4.14 *Let G be an infinite finitely generated group and H an infinite finitely generated subgroup of G. If $0<\widetilde{e}(G,H)<\infty$, then* $\operatorname{div}(G,H)\preceq\operatorname{Div}(G,H)$.

Proof Fix a finite generating set S of G such that $T = S \cap H$ generates H. We could consider $\Gamma(H,T)$ as a subgraph of $\Gamma(G,S)$. We denote $\mathrm{Div}(G,H)=\{\delta^n_\rho\}$ and $\mathrm{div}(G,H)=\{\sigma^n_\rho\}$. Let $m=\tilde{e}(G,H)$ and $M=4(2m+1)$. We will show $\sigma^n_\rho \preceq \delta^{Mn}_\rho$. More precisely, we are going to prove that for each $r>2$,

$$\sigma^n_\rho(r) \le \delta^{Mn}_\rho(r).$$

For each $i \in \{0,1,2,\dots,m\}$ we choose h_i in H such that $4nir \le |h_i|_S < 4nir+1$ and γ_i to be an H–perpendicular geodesic ray with the initial point h_i. Since $m = \tilde{e}(G,H)$, there are two different geodesics γ_i and γ_j $(i<j)$ such that $\gamma_i \cap C_r(H)$ and $\gamma_j \cap C_r(H)$ lie in the same component of $C_r(H)$. We define $x=\gamma_i(r)$ and $y=\gamma_j(r)$; then x and y lie in $\partial N_r(H)$ and $d_r(x,y)<\infty$. Also,

$$\begin{aligned}
d_S(x,y) &\le d_S(x,h_i)+d_S(h_i,h_j)+d_S(h_j,y)\\
&\le r+4n(i+j)r+2+r \le 8mnr+4r \le (Mn)r,\\
d_S(x,y) &\ge d_S(h_i,h_j)-d_S(h_i,x)-d_S(h_j,y)\\
&\ge 4njr-4nir-1-r-r \ge 4nr-3r \ge nr.
\end{aligned}$$

Thus

$$\sigma^n_\rho(r) \le d_{\rho r}(x,y) \le \sigma^{Mn}_\rho(r).$$

Therefore $\sigma^n_\rho \preceq \delta^{Mn}_\rho$. □

Theorem 4.15 (Commensurability) *Let G be a finitely generated group.*

(1) *If $K \le H \le G$ and $[H:K]<\infty$, then*

$$\mathrm{Div}(G,H) \sim \mathrm{Div}(G,K) \quad \textit{and} \quad \mathrm{div}(G,H) \sim \mathrm{div}(G,K).$$

(2) *If H_1 and H_2 are two commensurable subgroups of G, then*

$$\mathrm{Div}(G,H_1) \sim \mathrm{Div}(G,H_2) \quad \textit{and} \quad \mathrm{div}(G,H_1) \sim \mathrm{div}(G,H_2).$$

(3) *If $K \le H \le G$ and $[G:H]<\infty$, then*

$$\mathrm{Div}(G,K) \sim \mathrm{Div}(H,K) \quad \textit{and} \quad \mathrm{div}(G,K) \sim \mathrm{div}(H,K).$$

(4) *For any conjugate gHg^{-1} of H, we have*

$$\mathrm{Div}(G,gHg^{-1}) \sim \mathrm{Div}(G,H) \quad \textit{and} \quad \mathrm{div}(G,gHg^{-1}) \sim \mathrm{div}(G,H).$$

Proof The theorem follows immediately from Propositions 4.3 and 4.9. □

5 Relative divergence of finitely generated groups with respect to their normal subgroups

In this section, we investigate the upper and lower divergence of a finitely generated group relative to a normal subgroup.

Lemma 5.1 *Let G be a group with a finite generating set S and H a normal subgroup of G. Suppose g_1H, g_2H are arbitrary left cosets of H and the distance between them is n. Then for any element g_1h in g_1H the distance between g_1h and g_2H is also n.*

Proof Obviously, the distance between g_1h and g_2H is at least n. Thus we only need to show this distance is bounded above by n. Choose g_1h_1 in g_1H and g_2h_2 in g_2H such that the distance between them is n. Define $g = g_1hh_1^{-1}g_1^{-1}$. Since H is a normal subgroup, then g lies in H and $g' = g(g_2h_2)$ is an element in g_2H. Also, $d_S(g_1h, g') = d_S(gg_1h_1, gg_2h_2) = d_S(g_1h_1, g_2h_2) = n$. Therefore, the distance between g_1h and g_2H is at most n. □

Theorem 5.2 *Let G be a finitely generated group and H a finitely generated normal subgroup of G. Suppose that $\mathrm{Div}(G, H) = \{\delta_\rho^n\}$ and $\mathrm{Div}(G/H, e) = \{\delta_\rho\}$. Let*

$$\overline{\delta_\rho^n}(r) = \delta_\rho(r) + nr$$

for each positive r and $\overline{\mathrm{Div}(G/H, e)} = \{\overline{\delta_\rho^n}\}$. Then

$$\mathrm{Div}(G/H, e) \preceq \mathrm{Div}(G, H) \preceq \mathrm{Dist}_G^H \circ \overline{\mathrm{Div}(G/H, e)}.$$

Moreover, if G/H is one-ended and H is undistorted in G, then $\mathrm{Div}(G, H) \sim \mathrm{Div}(G/H, e)$.

Proof Let S be a finite generating set of G and assume that $T = G \cap S$ generates H. Moreover, the image $\overline{S}$ of S under the quotient map is a finite generating set of the quotient group G/H. We see that the Cayley graph $\Gamma(G/H, \overline{S})$ is the quotient graph of the Cayley graph $\Gamma(G, S)$ under the action of H.

We will first show that $\delta_\rho^n \preceq \mathrm{Dist}_G^H \circ \overline{\delta_\rho^n}$. More precisely, we will show that $\delta_\rho^n(r) \leq 2\,\mathrm{Dist}_G^H \circ \overline{\delta_\rho^n}(r)$ for all positive r.

Indeed, let x, y be arbitrary points in $\partial N_r(H)$ such that $d_{r,H}(x, y) < \infty$ and $d_S(x, y) \leq nr$. We assume that r is an integer and x, y are vertices. Thus there is a path in $C_r(H)$ connecting x and y. Let $\overline{x}$ and $\overline{y}$ be the associated points of x and y

respectively in $\Gamma(G/H,\overline{S})$. Thus $\overline{x}$ and $\overline{y}$ lie in the sphere $S_r(\overline{e})$ and there is a path outside the ball $B_r(\overline{e})$ connecting them.

Since $d_{\rho r,\overline{e}}(\overline{x},\overline{y}) \leq \delta_\rho(r)$, then there is a path α in $C_{\rho r}(\overline{e})$ connecting $\overline{x}$, $\overline{y}$ such that the length of α is bounded above by $\delta_\rho(r)$. Thus there is a path β in $C_{\rho r}(H)$ connecting x and some point y' in $\partial N_r(H)$. Moreover, $y' = hy$ for some h, and α, β have the same length. Thus the length of β is also bounded above by $\delta_\rho(r)$. Thus the distance between x and y' is also bounded above by $\delta_\rho(r)$ with respect to the metric d_S. Therefore, the distance between y and y' is bounded above by $\delta_\rho(r)+nr$ with respect to the metric d_S. Since y and y' lie in the same left coset gH, then there is a path γ with vertices in gH connecting y and y'. Thus the path γ must lie in $C_r(H)$ by Lemma 5.1. Moreover, the path γ can be chosen with the length bounded above by $\mathrm{Dist}_G^H(\delta_\rho(r)+nr)$. We define $\beta' = \beta \cup \gamma$ then β' is a path in $C_{\rho r}(H)$ connecting x, y and the length of β' is bounded above by $\mathrm{Dist}_G^H(\delta_\rho(r)+nr)+\delta_\rho(r)$. Thus

$$d_{\rho r,H}(x,y) \leq \mathrm{Dist}_G^H(\delta_\rho(r)+nr)+\delta_\rho(r) \leq 2\,\mathrm{Dist}_G^H \circ \overline{\delta_\rho^n}(r).$$

Therefore

$$\delta_\rho^n(r) \leq 2\,\mathrm{Dist}_G^H \circ \overline{\delta_\rho^n}(r).$$

Thus

$$\delta_\rho^n \preceq \mathrm{Dist}_G^H \circ \overline{\delta_\rho^n}.$$

We now show $\delta_\rho \preceq \delta_\rho^n$. More precisely, we are going to show that $\delta_\rho(r) \leq \delta_\rho^n(r)$ for all positive r.

Indeed, let u and v be arbitrary points in $S_r(\overline{e})$ of $\Gamma(G/H,\overline{S})$ and $d_{r,\overline{e}}(u,v) < \infty$. We assume that r is an integer and u, v are vertices. Choose x_1 and y_1 be lifting points of u and v respectively such that $d_S(x_1,y_1) = d_{\overline{S}}(u,v) \leq 2r \leq nr$. Obviously, x_1 and y_1 lie in $\partial N_r(H)$. We will show $d_{r,H}(x_1,y_1) < \infty$.

Indeed, since there is a path in $C_r(\overline{e})$ connecting u and v, then there is a path α_1 in $C_r(H)$ connecting two points x_1 and some point y_1', where $y_1' = h'y_1$ for some h' in H. Since y_1 and y_1' lie in the same left coset $g'H$, then there is a path α_2 with vertices in $g'H$ connecting y_1 and y_1'. By Lemma 5.1, the path α_2 also lies in $C_r(H)$. By concatenating α_1 and α_2, we have a path in $C_r(H)$ connecting x_1 and y_1. Thus $d_{r,H}(x_1,y_1) < \infty$.

We now prove that $d_{\rho r,\overline{e}}(u,v) \leq d_{\rho r,H}(x_1,y_1)$. Indeed, for any path γ' in $C_{\rho r}(H)$ connecting x_1 and y_1, there is a path $\overline{\gamma}'$ connecting u, v such that the length of $\overline{\gamma}'$ is less than or equal to the length of γ'. Thus $d_{\rho r,\overline{e}}(u,v) \leq d_{\rho r,H}(x_1,y_1)$. Therefore, $\delta_\rho(r) \leq \delta_\rho^n(r)$. Thus $\delta_\rho \preceq \delta_\rho^n$.

If a quotient group G/H is one-ended, then $\delta_\rho(r) \geq 2r$ for each $r > 0$. Thus

$$\overline{\delta_\rho^n}(r) = \delta_\rho(r) + nr \leq (n+1)\delta_\rho(r).$$

Therefore

$$\delta_\rho^n(r) \leq 2\,\mathrm{Dist}_G^H \circ \overline{\delta_\rho^n}(r) \leq 2\,\mathrm{Dist}_G^H((n+1)\delta_\rho(r)).$$

So $\delta_\rho^n \preceq \delta_\rho$ if Dist_G^H is dominated by a linear function.

Thus $\mathrm{Div}(G,H) \sim \mathrm{Div}(G/H,e)$ if G/H is one-ended and H is undistorted in G. □

Remark 5.3 If $G = H \times K$ and K is a one-ended group, then $\mathrm{Div}(G,H) \sim \mathrm{Div}(K,e)$. Thus we could have any desired relative upper divergence $\mathrm{Div}(G,H)$ by controlling the divergence $\mathrm{Div}(K,e)$. In particular, any finitely generated group H could be embedded as a subgroup of a larger finitely generated group G such that $\mathrm{Div}(G,H)$ is any polynomial function or exponential function. Indeed, we only need to choose K to be a one-ended hyperbolic group to have the upper relative divergence $\mathrm{Div}(G,H)$ as the exponential function. Similarly, we can choose a one-ended group K such that $\mathrm{Div}(K,e)$ is equivalent to a desired polynomial (for example, see [16]) and $\mathrm{Div}(G,H)$ is also equivalent to this desired polynomial.

Theorem 5.4 *Let G be a finitely generated group and H an infinite normal subgroup of G. Let K be any finitely generated infinite subgroup of H. Then,* $\mathrm{div}(G,H) \preceq \mathrm{dist}_G^K$. *In particular, if H is finitely generated, then* $\mathrm{div}(G,H) \preceq \mathrm{dist}_G^H$.

Proof Let S be a finite generating set of G and assume that $T = K \cap S$ generates K. Thus $\Gamma(K,T)$ is a subgraph of $\Gamma(G,S)$. Denote $\mathrm{div}(G,H) = \{\sigma_\rho^n\}$. We will prove that $\sigma_\rho^n \preceq \mathrm{dist}_G^K$. More precisely, $\sigma_\rho^n(r) \leq \mathrm{dist}_G^K(nr)$.

For each $r > 0$, we assume that r is an integer. Since $\mathrm{dist}_G^K(nr) = \min\{|k|_T \mid |k|_S \geq nr\}$, then there is an element k_0 in K such that $|k_0|_S \geq nr$ and $|k_0|_T \leq \mathrm{dist}_G^K(nr)$. Let α be a geodesic in $\Gamma(K,T)$ connecting the identity element e and k_0. Thus all vertices of α lie in H, and the length of α is bounded above by $\mathrm{dist}_G^K(nr)$. Choose any element g in G such that $d_S(g,H) = r$ and define $x = g$ and $y = gk_0$. By Lemma 5.1, the points x and y lie in $\partial N_r(H)$ and $g\alpha$ is a path in $C_r(H)$ connecting x and y. Moreover, $d_S(x,y) = |k_0|_S \geq nr$. Thus

$$\sigma_\rho^n(r) \leq d_{\rho r}(x,y) \leq \ell(g\alpha) \leq \ell(\alpha) \leq \mathrm{dist}_G^K(nr).$$

Therefore $\sigma_\rho^n \preceq \mathrm{dist}_G^K$. □

Corollary 5.5 *Let G be a finitely generated group and H an infinite normal subgroup of G. If H contains some infinite finitely generated subgroup, then* $\mathrm{div}(G,H)$ *is dominated by the growth of G. In particular,* $\mathrm{div}(G,H)$ *is at most exponential.*

Remark 5.6 In Corollary 5.5, it is unknown whether or not $\mathrm{div}(G, H)$ is dominated by the exponential function when every finitely generated subgroup of H is finite.

In Theorem 5.4, the relative lower divergence $\underline{\mathrm{div}(G, H)}$ can be strictly dominated by dist^H_G. Similarly, $\mathrm{Div}(G, H)$ could be strictly dominated by $\mathrm{Dist}^H_G \circ \overline{\mathrm{Div}(G/H, e)}$ in Theorem 5.2. We now compute the relative divergence of the Heisenberg group with respect to some cyclic subgroup to show these facts.

Before computing the relative divergence of the Heisenberg group with respect to some cyclic subgroup, we need some results about this group.

Lemma 5.7 *Let $G = \langle a, b, c \mid bab^{-1}a^{-1} = c, ac = ca, bc = cb\rangle$ be the Heisenberg group and H the cyclic subgroup generated by c. Then we have the following.*

(1) *Each element of G can be written uniquely in the form $a^k b^\ell c^p$, where k, ℓ, p are integers.*

(2) *We have*

$$(a^k b^\ell c^p)a = a^{k+1} b^\ell c^{p+l},$$
$$(a^k b^\ell c^p)b = a^k b^{\ell+1} c^p,$$
$$(a^k b^\ell c^p)c = a^k b^\ell c^{p+1}.$$

(3) *H is a normal subgroup of G, and $G/H = \mathbb{Z}^2$ is one-ended.*

(4) *If $|a^k b^\ell c^p| \le N$, then $|k| \le N$, $|\ell| \le N$, $|p| \le N^2$.*

(5) *$d_S(a^k b^\ell c^p, H) = |k| + |\ell|$.*

Proof For facts (1), (2), (3) and (4), we refer the reader to [10, Examples 1.5 and 1.18]. We now prove fact (5).

First we observe that c commutes with every element of G. Since $d_S(a^k b^\ell c^p, c^p) = d_S(c^p a^k b^\ell, c^p) = |a^k b^\ell|_S \le |k| + |\ell|$ and $c^p \in H$, then $d_S(a^k b^\ell c^p, H) \le |k| + |\ell|$. Let $c^{p'}$ be an element in H such that $d_S(a^k b^\ell c^p, H) = d_S(a^k b^\ell c^p, c^{p'})$. Thus $d_S(a^k b^\ell c^p, H) = |c^{-p'} a^k b^\ell c^p|_S = |a^k b^\ell c^{p-p'}|_S$. Let w be the shortest word such that $a^k b^\ell c^{p-p'} \equiv_G w$. Write w in the form $w = a^{k_1} b^{\ell_1} c^{p_1} a^{k_2} b^{\ell_2} c^{p_2} \cdots a^{k_n} b^{\ell_n} c^{p_n}$ and $|w|_S = \sum_{i=1}^n (|k_i| + |\ell_i| + |p_i|)$. We note that the values of k_i, ℓ_i, p_i can be zero. Thus

$$d_S(a^k b^\ell c^p, H) = \sum_{i=1}^n (|k_i| + |\ell_i| + |p_i|).$$

Also, there is p'' such that $w \equiv_G a^{k_1+k_2+\cdots+k_n} b^{\ell_1+\ell_2+\cdots+\ell_n} c^{p''}$.

Thus $a^k b^\ell c^{p-p'} \equiv_G a^{k_1+k_2+\cdots+k_n} b^{\ell_1+\ell_2+\cdots+\ell_n} c^{p''}$.

By (1), it implies that $k = k_1 + k_2 + \cdots + k_n$ and $\ell = \ell_1 + \ell_2 + \cdots + \ell_n$.

Then

$$d_S(a^k b^\ell c^p, H) = \sum_{i=1}^{n} (|k_i| + |\ell_i| + |p_i|) \geq |k| + |\ell|.$$

Therefore $d_S(a^k b^\ell c^p, H) = |k| + |\ell|$. □

Theorem 5.8 *Let $G = \langle a, b, c \mid bab^{-1}a^{-1} = c, ac = ca, bc = cb \rangle$ be the Heisenberg group and H the cyclic group generated by c. Then*

(1) dist_G^H *and* Dist_G^H *are both quadratic;*

(2) $\mathrm{div}(G, H)$ *and* $\mathrm{Div}(G, H)$ *are both linear.*

Proof The fact that dist_G^H and Dist_G^H are both quadratic could be seen in Theorem 3.10. We see that $\widetilde{e}(G, H) = e(G/H) = 1$ by Theorem 2.15. Thus $\mathrm{div}(G, H) \preceq \mathrm{Div}(G, H)$ by Theorem 4.14. Therefore, it is sufficient to show $\mathrm{Div}(G, H)$ is linear.

Denote $\mathrm{Div}(G, H) = \{\delta_\rho^n\}$. We will show that $\delta_\rho^n \preceq r$. More precisely, we are going to show that $\delta_\rho^n(r) \leq 50nr$ for all positive r.

Indeed, let x and y be arbitrary points in $\partial N_r(H)$ such that $d_r(x, y) < \infty$ and $d_S(x, y) \leq nr$. Assume that r is an integer and x, y are vertices. Write $x = a^k b^\ell c^p$ and $y = a^{k'} b^{\ell'} c^{p'}$. Thus $|k| + |\ell| = r$ and $|k'| + |\ell'| = r$ by Lemma 5.7(5).

By Lemma 5.7(2) and the fact that c commutes with any element of group G, we compute

$$x^{-1} y = a^{k'-k} b^{\ell'-\ell} c^{(p'-p)-\ell(k'-k)}.$$

Also,

$$|x^{-1} y|_S = d_S(x, y) \leq nr.$$

Thus $|k' - k| \leq nr$, $|\ell' - \ell| \leq nr$ and $|(p' - p) - \ell(k' - k)| \leq n^2 r^2$.

Therefore,

$$|p' - p| \leq |(p' - p) - \ell(k' - k)| + |\ell(k' - k)| \leq n^2 r^2 + nr^2 \leq 2n^2 r^2.$$

Let ℓ_1 be a number such that $\ell\ell_1 \geq 0$ and $|\ell_1| = r$. Let $x_1 = xb^{\ell_1 - \ell}$; $x_2 = x_1 a^{r-k}$ and $x_3 = x_2 b^{13nr - \ell_1}$. By Lemma 5.7(2), we see that $x_3 = a^r b^{13nr} c^{p + \ell_1(r-k)}$.

Since $x_1 = xb^{\ell_1 - \ell}$ and $|\ell_1 - \ell| \leq r$, there is a path α_1 with edges labeled by b connecting x and x_1 such that the length of α_1 is less than or equal to r. Similarly, there is a path α_2 with edges labeled by a connecting x_1, x_2 such that the length of α_2 is less than $2r$ and a path α_3 with edges labeled by b connecting x_2, x_3 such that the length of α_3 is less than $14nr$. Let $\alpha = \alpha_1 \cup \alpha_2 \cup \alpha_3$. We see that each vertex of α is of the form $x = a^{k_1} b^{\ell_1} c^{p_1}$ where $|k_1| + |\ell_1| \geq r$. Therefore, α is a path in $C_r(H)$ by Lemma 5.7(5) and α connects x and x_3, where $x_3 = a^r b^{13nr} c^{p+\ell_1(r-k)}$ and $|\ell_1| = r$. Moreover, the length of α is bounded above by $17nr$.

By a similar argument, there is a path β in $C_r(H)$ connecting y and y_3, where $y_3 = a^r b^{13nr} c^{p'+\ell'_1(r-k')}$ and $|\ell'_1| = r$. Moreover, the length of β is bounded above by $17nr$.

We now try to connect x_3 and y_3 by a path γ in $C_r(H)$ with length bounded above by $14nr$. Indeed, let $p_1 = p + \ell_1(r-k)$ and $p'_1 = p' + \ell'_1(r-k')$. If $p_1 = p'_1$ (ie $x_3 = y_3$), then we can consider γ is a trivial path connecting x_3 and y_3 with length 0. If $p_1 \neq p'_1$, then we assume that $p_1 < p'_1$. Thus

$$|p'_1 - p_1| \leq |p' - p| + |\ell_1(r-k)| + |\ell'_1(r-k')| \leq 2n^2r^2 + 2r^2 + 2r^2 \leq 4n^2r^2.$$

Thus $0 < p'_1 - p_1 \leq 4n^2r^2$.

Let t be a positive number such that $t^2 \leq (p'_1 - p_1) < (t+1)^2$ and let $t_1 = (p'_1 - p_1) - t^2$. Then $t \leq 2nr$ and $t_1 \leq (t+1)^2 - t^2 \leq 2t + 1 \leq 5nr$. Also, $c^{p'_1 - p_1} = c^{t^2} c^{t_1} = b^t a^t b^{-t} a^{-t} c^{t_1}$ and $y_3 = x_3 c^{p'_1 - p_1}$. Thus we could connect x_3, y_3 by a path γ such that the length of γ is bounded above by $4t + t_1$. Therefore, this length is bounded above by $13nr$. Also, the distance between x_3 and H is $(13n+1)r$. Thus γ must lie in $C_r(H)$. Let $\overline{\gamma} = \alpha \cup \gamma \cup \beta$ then $\overline{\gamma}$ is a path in $C_r(H)$ connecting x, y and the length of $\overline{\gamma}$ is bounded above by $50nr$. Thus $d_{\rho r}(x,y) < 50nr$. Therefore, $\delta^n_\rho(r) \leq 50nr$. Thus $\delta^n_\rho \preceq r$. □

6 Relative divergence of finitely generated groups with respect to their cyclic subgroups

In this section, we investigate the upper and lower divergence of a finitely generated group relative to an infinite cyclic subgroup.

Definition 6.1 Let G be a group with finite generating set S and H an infinite cyclic subgroup of G generated by some element h in S. Let e_h be the edge with the identity vertex as the initial point and labeled by h in $\Gamma(G,S)$. A bi-infinite arc $\alpha = \bigcup_{n \in \mathbb{Z}} h^n e_h$ is *the axis* of H.

Suppose G is a finitely generated one-ended group and H is an infinite cyclic subgroup of G in this section. Let h be a generator of H and assume that the finite generating set S of G contains h. Let α be the axis of H. Thus α is a bi-infinite arc with all vertices in H.

We now define the concept of divergence of a bi-infinite arc in a one-ended geodesic space. This concept will play an important role for investigating the lower divergence of a one-ended group G with respect to an infinite cyclic subgroup.

Definition 6.2 Let X be a one-ended geodesic space and β a proper bi-infinite arc. Let c be one point on β. *The divergence of* (β, c), denoted $\operatorname{div}(\beta, c)$, is the function $f\colon (0,\infty) \to (0,\infty)$ defined as follows.

For each positive r, we define

$$f(r) = \inf\{|\gamma| \mid \gamma \text{ is a path in } X - B(c,r) \text{ with endpoints on } \beta \text{ and on different sides of } c\}.$$

Remark 6.3 Observe that $\operatorname{div}(\beta, c)$ is a nondecreasing function.

Let α be the axis of the infinite cyclic subgroup H, which is defined in Definition 6.1. Then $\operatorname{div}(\alpha, h^i) = \operatorname{div}(\alpha, e)$ in the Cayley graph $\Gamma(G,S)$ for any element h^i in H and let $\operatorname{div}_\alpha = \operatorname{div}(\alpha, e)$.

For each x in $\Gamma(G,S) - \alpha$ and u a point in α such that $d_S(x,\alpha) = d_S(x,u)$, the point u must be a vertex of $\Gamma(G,S)$. Thus $N_r(\alpha) = N_r(H)$ for each $r > 1$. Therefore, $\partial N_r(\alpha) = \partial N_r(H)$ and $C_r(\alpha) = C_r(H)$ for each $r > 1$.

Definition 6.4 Let c be an arc in $\Gamma(G,S)$. If c_0 is any subset of c, the *Hull* of c_0 in c, denoted $\operatorname{Hull}_c(c_0)$, is the smallest connected subspace of c containing c_0.

Lemma 6.5 *Choose $r > 1$ and let n be a positive integer. Choose s such that $s \geq 3\operatorname{Dist}_G^H((n+2)r)$. Let a, b, c be three different points in α such that c lies between a, b. Assume that a, b lie outside the ball $B(c,s)$. Let γ be an arc outside $B(c,s)$ connecting a and b. Then there are two points x, y in $\gamma \cap \partial N_r(\alpha)$ such that $d_S(x,y) \geq nr$ and the segment of γ connecting x and y lies in $C_r(\alpha)$.*

Proof First, we will show that γ does not lie in the r–neighborhood of α. Assume by way of contradiction that γ lies in the r–neighborhood of α. For each G–vertex v of γ, let

$$c_v = \operatorname{Hull}_\alpha(\alpha \cap \overline{B(v,r)}).$$

For each edge e of γ with G–endpoints v and w, let

$$c_e = \mathrm{Hull}_\alpha(c_v \cup c_w).$$

We see that the subsegment $[a,b]$ of α is covered by the sets c_e for all edges e of γ. In particular, c lies in some c_e, where e is an edge of γ. Therefore, c lies between two vertices u_1 and v_1 of α whose distance from vertices of e is at most r. Thus the distance between u_1 and v_1 is less than $2r+1$. Therefore, the length of the subsegment $[u_1, v_1]$ of α is less than $\mathrm{Dist}_G^H(2r+1)$. Thus

$$d_S(c,\gamma) \le \mathrm{Dist}_G^H(2r+1) + r < 2\,\mathrm{Dist}_G^H((n+2)r) < s,$$

which is a contradiction. Thus γ does not lie in the r–neighborhood of α.

Let $M = \{x_i \mid i \in \{0,1,2,\dots,n\}\}$ be the set of points of γ that satisfies the following conditions.

(1) We have $x_0 = a$ and $x_n = b$.

(2) For each $i \in \{1,2,\dots,n-1\}$, the distance between x_i and α is r.

(3) For each $i \in \{0,1,2,\dots,n-1\}$, the open segment (x_i, x_{i+1}) does not contain any point in $\partial N_r(\alpha)$.

For each $i \in \{1,2,\dots,n-1\}$, let x_i' be a vertex of α such that $d_S(x_i, x_i') = r$. We again assign $x_0' = a$ and $x_n' = b$. For each $i \in \{0,1,2,\dots,n-1\}$, define d_i to be the subsegment of α that connects x_i' and x_{i+1}'. Therefore, c must lie in some d_{i_0}. Since $(x_{i_0}, x_{i_0+1}) \cap \partial N_r(\alpha) = \varnothing$, then either $(x_{i_0}, x_{i_0+1}) \subset N_r(\alpha)$ or $(x_{i_0}, x_{i_0+1}) \cap N_r(\alpha) = \varnothing$.

If $(x_{i_0}, x_{i_0+1}) \subset N_r(\alpha)$, we can use the same argument as above to show $d_S(c,\gamma) < s$, which is a contradiction. Thus $(x_{i_0}, x_{i_0+1}) \cap N_r(\alpha) = \varnothing$ or $(x_{i_0}, x_{i_0+1}) \subset C_r(\alpha)$.

Since the distance between x_{i_0} and c is at least s and the distance between x_{i_0}' and x_{i_0} is r, then the distance between x_{i_0}' and c is at least $s-r$. Thus the length of the segment of α connecting x_{i_0}' and c is at least $s-r$. Similarly, the length of the segment of α connecting x_{i_0+1}' and c is also at least $s-r$. Thus the length of the segment of α connecting x_{i_0}' and x_{i_0+1}' is also at least $2s-2r$. Therefore, this length is strictly bounded below by

$$\mathrm{Dist}_G^H((n+2)r).$$

Thus the distance in H between x_{i_0}' and x_{i_0+1}' is strictly greater than $\mathrm{Dist}_G^H((n+2)r)$. Therefore, the distance in G between x_{i_0}' and x_{i_0+1}' is at least $(n+2)r$. Also, the distances $d_S(x_{i_0}', x_{i_0})$ and $d_S(x_{i_0+1}', x_{i_0+1})$ are both r. Thus the distance between x_{i_0} and x_{i_0+1} is at least nr. We let $x = x_{i_0}$ and $y = x_{i_0+1}$. □

Proposition 6.6 *Let G be a one-ended group with a finite generating set S. Let H be an infinite cyclic subgroup generated by some element in S and α the axis of H. Then*

$$\operatorname{div}_\alpha \preceq \operatorname{div}(G,H) \preceq \operatorname{div}_\alpha \circ (3\operatorname{Dist}_G^H).$$

Proof Denote $\operatorname{div}(G,H)=\{\sigma_\rho^n\}$.

We will first show that $\sigma_\rho^n \preceq \operatorname{div}_\alpha \circ (3\operatorname{Dist}_G^H)$. More precisely, we are going to show that $\sigma_\rho^n(r) \leq \operatorname{div}_\alpha \circ (3\operatorname{Dist}_G^H)((n+2)r)$ for all numbers $r>1$.

Indeed, let $s=3\operatorname{Dist}_G^H((n+2)r)$. Let γ be any arc outside the ball $B(e,s)$ connecting two points u and v on α such that e lies between u and v. By Lemma 6.5, there are two points x and y in $\gamma \cap \partial N_r(\alpha)$ such that $d_S(x,y)\geq nr$ and the segment of γ connecting x and y lies in $C_r(\alpha)$. By Remark 6.3, two points x and y also lie in $\partial N_r(H)$. Then $d_{\rho r}(x,y)$ is bounded above by the length of γ. Therefore, $\sigma_\rho^n(r)$ is bounded above by the length of γ. Thus

$$\sigma_\rho^n(r) \leq \operatorname{div}_\alpha(s).$$

Therefore,

$$\sigma_\rho^n(r) \leq \operatorname{div}_\alpha \circ (3\operatorname{Dist}_G^H)((n+2)r).$$

We now will show that $\operatorname{div}_\alpha \preceq \sigma_\rho^n$ for each $n\geq 20$. More precisely, we are going to show that for each $r>3$,

$$\operatorname{div}_\alpha(\rho r) \leq \sigma_\rho^n(r)+2r.$$

Indeed, let x_1 and y_1 be arbitrary points in $\partial N_r(H)$ such that $d_X(x_1,y_1)\geq nr$ and $d_r(x_1,y_1)<\infty$. Let β be any arc in $C_{\rho r}(H)$ connecting x_1 and y_1. Let x_2 and y_2 be vertices in α such that $d_S(x_1,\alpha)=d_S(x_1,x_2)=r$ and $d_S(y_1,\alpha)=d_S(y_1,y_2)=r$. Let β_1 be a geodesic connecting x_1 and x_2 and β_2 a geodesic connecting y_1 and y_2. Since the distance between x_1 and y_1 is bounded below by nr, the distance between x_2 and y_2 is bounded below by $(n-2)r$. Let h^i be a vertex of α such that h^i lies between x_2, y_2 such that x_2, y_2 do not lie in the ball of center h^i with radius $5r$. Let $\overline{\beta}=\beta_1\cup\beta\cup\beta_2$. Thus $\overline{\beta}$ is a path outside the ball $B(h^i,\rho r)$ connecting the two points x_2, y_2 in α and h^i lies between x_2, y_2. Therefore, we could have an arc β' from $\overline{\beta}$ connecting two points x_2 and y_2. Thus $\operatorname{div}_\alpha(\rho r)$ is bounded above by the length of $\overline{\beta}$. Therefore, $\operatorname{div}_\alpha(\rho r)$ is bounded above by $|\beta|+2r$. Therefore, $\operatorname{div}_\alpha(\rho r)$ is bounded above by $d_{\rho r}(x_1,y_1)+2r$. Thus

$$\operatorname{div}_\alpha(\rho r) \leq \sigma_\rho^n(r)+2r.$$

Therefore,

$$\operatorname{div}_\alpha \preceq \sigma_\rho^n. \qquad \square$$

Theorem 6.7 *Let G be a one-ended finitely generated group and H an infinite cyclic subgroup of G. Suppose that* $\operatorname{div}(G,H)=\{\sigma_\rho^n\}$ *and* $\operatorname{Div}(G,e)=\{\delta_\rho\}$. *Then* $\sigma_\rho^n \preceq \delta_\rho \circ ((3/\rho)\operatorname{Dist}_G^H)$. *In particular,* $\operatorname{div}(G,H) \preceq \operatorname{Div}(G,e)$ *if H is an undistorted subgroup.*

Proof We will show that $\sigma_\rho^n(r) \le \delta_\rho \circ ((3/\rho)\operatorname{Dist}_G^H)((n+2)r)$ for all $r>1$.

Indeed, let $s=(3/\rho)\operatorname{Dist}_G^H((n+2)r)$. Choose x and y in $\alpha \cap S(e,s)$ such that e lies between x and y. Let γ be an arbitrary arc outside $B_{\rho s}(e)$ connecting x and y. Since $\rho s = 3\operatorname{Dist}_G^H((n+2)r)$, then there are two points x_1 and y_1 in $\gamma \cap \partial N_r(\alpha)$ such that $d_S(x_1,y_1) \ge nr$ and the segment of γ connecting x_1 and y_1 lies in $C_r(\alpha)$ by Lemma 6.5. Thus the two points x_1 and y_1 also lie in $\partial N_r(H)$ and the segment of γ connecting x_1 and y_1 also lies in $C_r(H)$ by Remark 6.3. Thus the distance $d_{\rho r}(x_1,y_1)$ is bounded above by the length of γ. Therefore, $\sigma_\rho^n(r)$ is also bounded above by the length of γ. Thus

$$\sigma_\rho^n(r) \le \delta_\rho(s).$$

Therefore,

$$\sigma_\rho^n(r) \le \delta_\rho \circ \left(\frac{3}{\rho}\operatorname{Dist}_G^H\right)((n+2)r).$$

Thus $\sigma_\rho^n \preceq \delta_\rho \circ ((3/\rho)\operatorname{Dist}_G^H)$. □

Remark 6.8 In Theorem 6.7, we could not replace $\operatorname{div}(G,H)$ by $\operatorname{Div}(G,H)$. For example, let $H=\mathbb{Z}$ and K be any one-ended finitely generated group such that $\operatorname{Div}(K,e)$ is superlinear. We define $G=H\times K$. Thus G is a one-ended finitely generated group and H is an infinite cyclic subgroup of G. Then, Dist_G^H is linear and $\operatorname{Div}(G,H)=\operatorname{Div}(K,e)$ is superlinear. Also the divergence $\operatorname{Div}(G,e)$ is linear (see [9, Theorem 4.1]). Thus Theorem 6.7 is no longer true if we replace $\operatorname{div}(G,H)$ by $\operatorname{Div}(G,H)$.

Moreover, the two functions σ_ρ^n and $\delta_\rho \circ ((3/\rho)\operatorname{Dist}_G^H)$ in Theorem 6.7 can be equivalent in some cases (for example: $G=\mathbb{Z}^2$ and H any cyclic subgroup of G), and σ_ρ^n can be strictly dominated by $\delta_\rho \circ ((3/\rho)\operatorname{Dist}_G^H)$ in some other cases (see Theorem 5.8).

7 Relative divergence of CAT(0) groups

In this section, we investigate the relative divergence of (G,H) where G is a CAT(0) group. We use Theorem 5.2 to build CAT(0) groups with arbitrary polynomial upper relative divergences with respect to some subgroup (see Theorem 7.7). We also examine the class of groups defined by Macura [16] to obtain arbitrary polynomial lower relative divergence (see Corollary 7.12).

We now review some concepts and some basic properties of a $\mathrm{CAT}(0)$ group. We refer the reader to Bridson and Haefliger [3] for studying more on $\mathrm{CAT}(0)$ groups.

Definition 7.1 Let X be a geodesic space. A *geodesic triangle* Δ in X consists of three points p, q, r in X and three geodesic segments $[p,q], [q,r], [r,p]$. A *comparison triangle* for Δ in $\mathbb{E}^2$ is a geodesic triangle $\overline{\Delta}$ in $\mathbb{E}^2$ with vertices $\overline{p}, \overline{q}, \overline{r}$ such that $d(p,q) = d(\overline{p},\overline{q}), d(q,r) = d(\overline{q},\overline{r})$ and $d(r,p) = d(\overline{r},\overline{p})$. A point $\overline{x}$ in $[\overline{q},\overline{r}]$ is called a *comparison point* for x in $[q,r]$ if $d(q,x) = d(\overline{q},\overline{x})$. Comparison points on $[p,q]$ and $[p,r]$ are defined in the same way.

Definition 7.2 A geodesic triangle Δ in a geodesic space X satisfies *the* $\mathrm{CAT}(0)$ *inequality* if $d(x,y) \leq d(\overline{x},\overline{y})$ for all points x and y on Δ and corresponding points $\overline{x}, \overline{y}$ on the comparison triangle $\overline{\Delta}$ in Euclidean space $\mathbb{E}^2$.

Definition 7.3 A geodesic space X is $\mathrm{CAT}(0)$ *space* if every triangle in X satisfies the $\mathrm{CAT}(0)$ inequality.

A group is $\mathrm{CAT}(0)$ if it acts properly and cocompactly on some proper $\mathrm{CAT}(0)$ space.

The proof of the following proposition can be found in [3].

Proposition 7.4 *Let (X_1, d_1) and (X_2, d_2) be* $\mathrm{CAT}(0)$ *spaces. Then the Cartesian product $X_1 \times X_2$ endowed with the metric d defined by $d^2 = d_1^2 + d_2^2$ is a* $\mathrm{CAT}(0)$ *space.*

The following corollary is an immediate result of the above proposition.

Corollary 7.5 *The direct product of two* $\mathrm{CAT}(0)$ *groups is a* $\mathrm{CAT}(0)$ *group.*

The following theorem is a direct result from [3, Corollary III.Γ.4.8 and Theorem III.Γ.4.10].

Theorem 7.6 *Every finitely generated abelian subgroup of a* $\mathrm{CAT}(0)$ *group is undistorted.*

Theorem 7.7 *Let f be any polynomial function or exponential function. There is a pair of groups (G,H), where G is a* $\mathrm{CAT}(0)$ *group and H is a normal infinite cyclic subgroup of G such that* $\mathrm{Div}(G,H) \sim f$.

Proof We will build the group G of the form $G = K \times \mathbb{Z}$ and we choose a suitable one-ended $\mathrm{CAT}(0)$ groups K. We choose H to be the $\mathbb{Z}$ factor of G. Thus we observe that $\mathrm{Div}(G,H) = \mathrm{Div}(G/H, e) = \mathrm{Div}(K, e)$ by Theorem 5.2.

If f is a polynomial of degree d, then we choose a subgroup K such that $\mathrm{Div}(K, e)$ is equivalent to f (see [16] for example). If f is the exponential function, we choose K to be a surface group of genus $g \geq 2$. Since a surface group of genus $g \geq 2$ is a CAT(0) group, then the group G is also a CAT(0) group by Corollary 7.5. Moreover, K is a one-ended hyperbolic group, then the upper divergence of K is exponential. Thus the relative upper divergence $\mathrm{Div}(G, H)$ is also exponential. □

Theorem 7.8 *Let G be a* CAT(0) *group and H a normal subgroup of G that contains at least one infinite order element. Then* $\mathrm{div}(G, H)$ *is linear.*

Proof By Theorem 7.6, there is an undistorted cyclic subgroup K in H. By Theorem 5.4, we observe that $\mathrm{div}(G, H)$ is linear. □

We now investigate relative lower divergence of a class of CAT(0) groups introduced by Macura in [16]. First, we will review this class of groups.

For each integer $d \geq 2$, we define

$$G_d = \langle a_0, a_1, \ldots, a_d \mid a_0 a_1 = a_1 a_0, a_i^{-1} a_0 a_i = a_{i-1}, \text{for } 2 \leq i \leq d \rangle$$

and H_d to be the cyclic subgroup generated by a_d.

Let X_d be the presentation complex of G_d and $\widetilde{X}_d$ is the universal cover of X_d. The space $\widetilde{X}_d$ is a CAT(0) square complex (see Macura [16]). Moreover, G_d is one-ended and we could consider the 1–skeleton $\widetilde{X}_d^{(1)}$ of $\widetilde{X}_d$ as the Cayley graph of G_d. Let α be the axis of the infinite cyclic subgroup of H_d as in Definition 6.1. By Proposition 6.6 and Theorem 7.6, we can investigate the divergence div_α of α in $\widetilde{X}_d$ to understand the lower divergence $\mathrm{div}(G_d, H_d)$. Before computing div_α, we need to review some results from [16].

Proposition 7.9 [16, Proposition 4.4] *There is a polynomial q_d, of degree d, such that for any point O in $\widetilde{X}_d$ and any two points P, Q on the sphere $S(O, r) \subset \widetilde{X}_d$, there is a path γ in $\widetilde{X}_d - B(O, r)$ connecting P and Q such that the length of γ is at most $q_d(r)$.*

Proposition 7.10 [16, Theorem 5.3] *There is a polynomial p_d, of degree d, such that the following holds. Let T be any vertex on $\widetilde{X}_d$. Let γ_0 be a geodesic ray which is the infinite concatenation of edges a_0, and γ_d a geodesic ray which is the infinite concatenation of edges a_d. We assume that γ_0 and γ_d have the same initial point T. For each path β outside the ball $B(T, r)$ connecting $P \in \gamma_d$ and $Q \in \gamma_0$, the length of β is bounded below by $p_d(r)$.*

Proposition 7.11 *The divergence* div_α *is polynomial of degree d.*

Proof By Proposition 7.9, there is a polynomial q_d, of degree d such that the following holds. Let r be any positive number and u, v two points in $S(e,r)\cap\alpha$ such that e lies between u, v. There is a path outside $B(e,r)$ of length at most $q_d(r)$ connecting u and v. Therefore, div_α is bounded above by q_d.

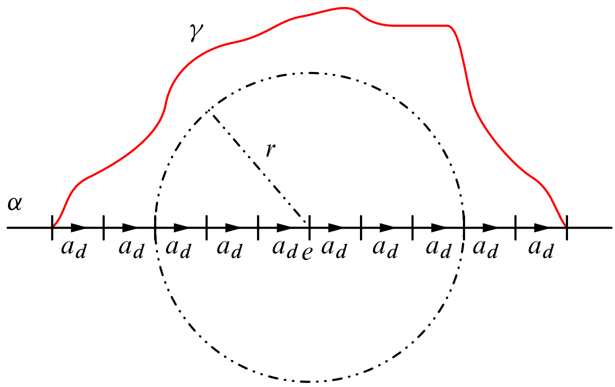

Figure 2: The path γ lies outside $B(e,r)$ with endpoints on α and on different sides of e.

We now prove that div_α has some polynomial of degree d as a lower bound. Let p_d be the polynomial of degree d in Proposition 7.10. We will show div_α is bounded below by this polynomial. Indeed, for each positive r, let γ be any path outside $B(e,r)$ with endpoints on α and on different sides of e (see Figure 2).

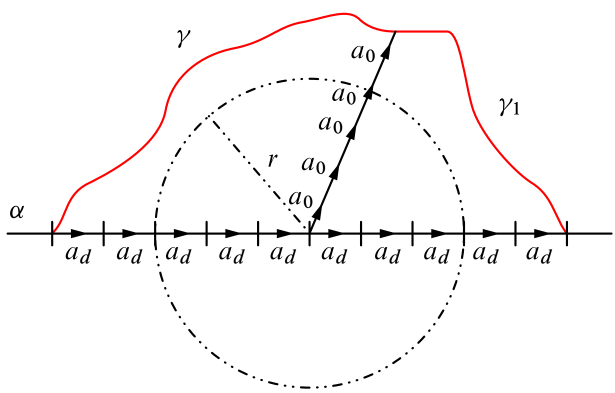

Figure 3: The subsegment γ_1 of γ connecting two points of γ_0 and γ_d, where γ_0 and γ_d are two geodesic rays issuing from e such that they are infinite concatenations of edges a_0 and a_d respectively

We are going to show that there exists a subsegment γ_1 of γ connecting two points of γ_0 and γ_d, where γ_0 and γ_d are two geodesic rays issuing from e such that they are infinite concatenations of edges a_0 and a_d respectively (see Figure 3).

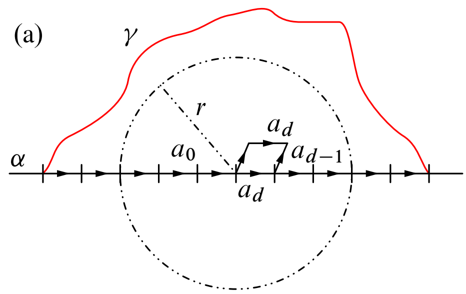

Figure 4: The position of the 2–cell c_1 in the diagram D

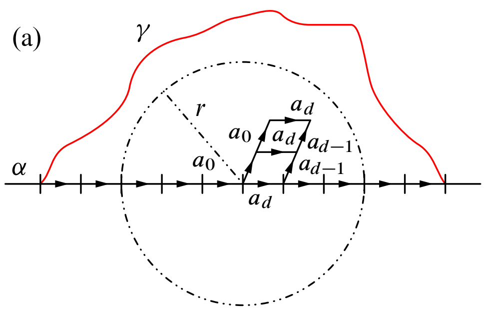

Figure 5: The position of the 2–cell c_2 in the diagram D

We will use the same technique as in [9] for this argument. We observe that the path γ and the subsegment of α between two endpoints of γ form a loop in $\widetilde{X}_d$ which may fill in with a reduced van Kampen diagram D (see Lyndon and Schupp [15]). Since the path γ lies outside the ball $B(e,r)$, the edge $a_d^{(1)}$ of α with the initial point e must lie in some 2–cell of D. By the presentation of G_d, the edge $a_d^{(1)}$ must lie in a 2–cell c_1 labeled by $a_d^{-1}a_0a_da_{d-1}^{-1}$. There are two cases for c_1 depending on its orientation in D (see Figure 4).

We now only argue on the first case (see Figure 4(a)) and the argument of the second case (see Figure 4(b)) is almost identical. If the edge $a_d^{(2)}$ that is opposite to $a_d^{(1)}$ in c_1 lies in the path γ, it is obvious that there exist a subsegment γ_1 of γ connecting two points of γ_0 and γ_d. Otherwise, $a_d^{(2)}$ must lie in some 2–cell c_2 labeled by $a_d^{-1}a_0a_da_{d-1}^{-1}$ of D. Again, there are two possibilities for c_2 depending on the orientation of c_2 in D (see Figure 5).

In the second case (see Figure 5(b)), we see that the two 2–cells c_1 and c_2 form a cancellable pair in D. This is impossible since the diagram D is reduced. Thus the second possibility is ruled out. By arguing inductively, we obtain a corridor that is a concatenation of 2–cells labeled by $a_d^{-1}a_0a_da_{d-1}^{-1}$ such that one edge $a_d^{(n)}$ labeled by a_d of the last 2–cell in the corridor must lie in the boundary of D. If $a_d^{(n)}$ is an edge of α, the diagram D would not be planar topologically. Thus $a_d^{(n)}$ must be an edge of γ (see Figure 6).

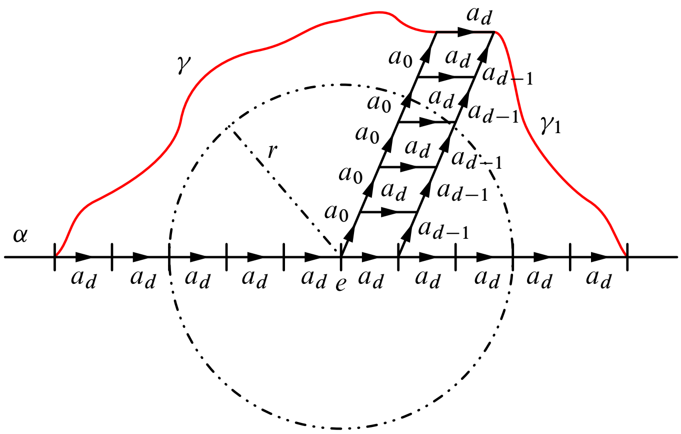

Figure 6: The corridor that is a concatenation of 2–cells labeled by $a_d^{-1}a_0a_da_{d-1}^{-1}$ in the diagram D

Therefore, there exists a subsegment γ_1 of γ connecting two points of γ_0 and γ_d. Since the length of γ_1 is bounded below by $p_d(r)$ by Proposition 7.10, then the length of γ is also bounded below by $p_d(r)$. Therefore, the divergence div_α must be dominated the polynomial $p_d(r)$. □

Corollary 7.12 *Let H_d be a cyclic subgroup of G_d generated by a_d. Then the relative lower divergence* $\mathrm{div}(G_d, H_d)$ *is polynomial function of degree d.*

Proof This is an immediate consequence of Propositions 6.6 and 7.11. □

8 Relative divergence of relatively hyperbolic groups

We now investigate the relative divergence of a relatively hyperbolic group with respect to a subgroup.

Definition 8.1 A geodesic metric space (X, d) is *δ–hyperbolic* if every geodesic triangle with vertices in X is *δ–thin* in the sense that each side lies in the δ–neighborhood of the union of other sides.

A finitely generated group G is *hyperbolic* if the Cayley graph $\Gamma(G, S)$ is a hyperbolic space for some finite set of generators S.

Definition 8.2 A subspace Y of a geodesic metric space X is *quasiconvex* when there exists some $k > 0$ such that every geodesic in X that connects a pair of points in Y lies inside the k–neighborhood of Y.

Suppose G is a hyperbolic group with a finite generating set S. A subgroup H of a group G is quasiconvex if it is quasiconvex in the Cayley graph $\Gamma(G, S)$.

Remark 8.3 The concepts of hyperbolic groups and quasiconvex subgroups do not depend on the choice of finite set of generators (see [10; 1]).

We now discuss a generalization of the concepts of hyperbolic groups and quasiconvex subgroups. They are relatively hyperbolic groups and relatively quasiconvex subgroups.

Definition 8.4 Given a finitely generated group G with Cayley graph $\Gamma(G, S)$ which is equipped with the path metric and a finite collection $\mathbb{P}$ of subgroups of G, one can construct the *coned off Cayley graph* $\widehat{\Gamma}(G, S, \mathbb{P})$ as follows. For each left coset gP where $P \in \mathbb{P}$, add a vertex v_{gP}, called a *peripheral vertex*, to the Cayley graph $\Gamma(G, S)$ and for each element x of gP, add an edge $e(x, gP)$ of length 1/2 from x to the vertex v_{gP}. This results in a metric space that may not be proper (ie closed balls need not be compact).

Remark 8.5 Throughout this section, we denote the metric in $\Gamma(G, S)$ by d_S and the metric in $\widehat{\Gamma}(G, S, \mathbb{P})$ by d.

Definition 8.6 (Relatively hyperbolic group) A finitely generated group G is *hyperbolic relative to a finite collection $\mathbb{P}$ of subgroups of G* if the coned off Cayley graph is δ–hyperbolic and *fine* (ie for each positive number n, each edge of the coned off Cayley graph is contained in only finitely many circuits of length n).

Each group $P \in \mathbb{P}$ is a *peripheral subgroup* and its left cosets are *peripheral left cosets* and we denote the collection of all peripheral left cosets by Π.

An element g of G is *hyperbolic* if g is not conjugate to any element of any peripheral subgroups.

Lemma 8.7 (Hruska [14, Proposition 9.4]) *Let G be a group with a finite generating set S. Suppose xH and yK are arbitrary left cosets of subgroups of G. For each constant L there is a constant $L' = L'(G, S, xH, yK)$ so that in the metric space $(\Gamma(G,S), d_S)$ we have*

$$N_L(xH) \cap N_L(yK) \subset N_{L'}(xHx^{-1} \cap yKy^{-1}).$$

Definition 8.8 Let $(G, \mathbb{P})$ be a relatively hyperbolic group. A subgroup H of G is *relatively quasiconvex* if the following holds. Let S be some (any) finite generating set for G. Then there is a constant $\kappa = \kappa(S)$ such that for each geodesic $\overline{c}$ in $\widehat{\Gamma}(G, S, \mathbb{P})$ connecting two points of H, every G–vertex of $\overline{c}$ lies within a d_S–distance κ of H.

Remark 8.9 We note that the concepts of relative hyperbolicity and relative quasiconvexity do not depend on the choice of finite set of generators (see Osin [21]).

Definition 8.10 Let $(G, \mathbb{P})$ be a relatively hyperbolic group.

(1) A relatively quasiconvex subgroup H of G is *strongly relatively quasiconvex* if for each conjugate $g^{-1}Pg$ of any peripheral subgroup P and $H \cap g^{-1}Pg$ is a finite subgroup of $g^{-1}Pg$.

(2) A relatively quasiconvex subgroup H of G is *fully relatively quasiconvex* if for each conjugate $g^{-1}Pg$ of any peripheral subgroup P, $H \cap g^{-1}Pg$ is a finite subgroup or finite index subgroup of $g^{-1}Pg$.

Lemma 8.11 [21, Theorem 4.13] *Let $(G, \mathbb{P})$ be a relatively hyperbolic group. Let H be a subgroup of G. Then the following conditions are equivalent.*

(1) *H is strongly relatively quasiconvex.*

(2) *H is generated by a finite set T such that the natural map $(H, d_T) \to \widehat{\Gamma}(G, S, \mathbb{P})$ is a quasi-isometric embedding.*

Lemma 8.12 [21, Theorem 1.14] *Let $(G, \mathbb{P})$ be a relatively hyperbolic group with a finite generating set S. Then for any hyperbolic element $h \in G$ of infinite order, there exist $\lambda > 0$ and $c \geq 0$ such that $d(e, h^n) > \lambda|n| - c$. In particular, the cyclic subgroup H generated by h is undistorted with respect to (G, d_S) and strongly relatively quasiconvex.*

The following lemma is an immediate result of [21, Proposition 2.36].

Lemma 8.13 *Let $(G,\mathbb{P})$ be a relatively hyperbolic group. Then the following conditions hold.*

(1) $g_1P_1g_1^{-1}\cap g_2P_2g_2^{-1}$ *is finite, where* P_1 *and* P_2 *are two different peripheral subgroups.*

(2) $gPg^{-1}\cap P$ *is finite, where* P *is a peripheral subgroup and* $g\notin P$.

Theorem 8.14 (Gromov [11, Section 8.2]) *Let $(G,\mathbb{P})$ be a relatively hyperbolic group and H an infinite subgroup of G. If H is not conjugate to a subgroup of any peripheral subgroup, H contains a hyperbolic element.*

Lemma 8.15 *Let $(G,\mathbb{P})$ be a relatively hyperbolic group and H an infinite index, infinite normal subgroup of G. Then H contains at least one infinite-order hyperbolic element.*

Proof If H is not conjugate to a subgroup of any peripheral subgroup, H contains a hyperbolic element by Theorem 8.14. Suppose that H is a subgroup of some conjugate gPg^{-1} of some peripheral subgroup P, then $H=g^{-1}Hg$ is a subgroup of P. Let g_1 be an element in $G-P$, then $H=g_1^{-1}Hg_1$ is also a subgroup of $g_1^{-1}Pg_1$. Then, $|P\cap g_1^{-1}Pg_1|=\infty$, which is contradicts Lemma 8.13. Therefore, H is not a subgroup of any conjugate of any peripheral subgroup. □

Lemma 8.16 [21, Theorem 3.26] *There is a positive constant σ such that the following holds. Let $\Delta=pqr$ be a triangle whose sides p,q,r are geodesic in $\widehat{\Gamma}(G,S,\mathbb{P})$. Then for each G–vertex v on p, there is a G–vertex u in the union $q\cup r$ such that $d_S(u,v)\leq\sigma$.*

The following lemma is an immediate result of Lemma 8.16.

Lemma 8.17 *There is a positive constant σ such that the following holds. Let $pqrs$ be a quadrilateral whose sides p,q,r,s are geodesic in $\widehat{\Gamma}(G,S,\mathbb{P})$. Then for each G–vertex v on p, there is a G–vertex u in the union $q\cup r\cup s$ such that $d_S(u,v)\leq 2\sigma$.*

Lemma 8.18 (Druțu and Sapir [6, Lemma A.3]) *Let $(G,\mathbb{P})$ be a relatively hyperbolic group with a finite generating set S. Then there is a constant $K>1$ such that the following holds. Let p and q be paths in $\widehat{\Gamma}(G,S,\mathbb{P})$ such that $p_-=q_-$, $p_+=q_+$ and q is geodesic in $\widehat{\Gamma}(G,S,\mathbb{P})$. Then for any vertex $v\in q$, there exists a vertex $w\in p$ such that $d_S(w,v)\leq K\log_2|p|$.*

Lemma 8.19 [6, Lemma 4.15] *Let $(G, \mathbb{P})$ be a relatively hyperbolic group with a finite generating set S. For each A_0 there is a constant $A_1 = A_1(A_0)$ such that the following holds in* $\mathrm{Cayley}(G, \mathcal{S})$. *Let c be a geodesic segment whose endpoints lie in the A_0–neighborhood of a peripheral left coset gP. Then c lies in the A_1–neighborhood of gP.*

Lemma 8.20 [6, Theorem 4.1] *Suppose $(G, \mathbb{P})$ is relatively hyperbolic with a finite generating set $\mathcal{S}$. For each $M, M' < \infty$ there is a constant $\iota = \iota(M, M') < \infty$ so that for any two peripheral cosets $gP \neq g'P'$ we have*

$$\mathrm{diam}(\mathcal{N}_M(gP) \cap \mathcal{N}_{M'}(g'P')) < \iota$$

with respect to the metric d_S.

The following concepts are introduced by Hruska (see [14, Definition 8.9]) and he used it to describe the connection between geodesics in $\Gamma(G, S)$ and geodesics in $\widehat{\Gamma}(G, S, \mathbb{P})$.

Definition 8.21 Let c be a geodesic of $\Gamma(G, S)$, and let ϵ, R be positive constants. A point $x \in c$ is (ϵ, R)*–deep* in a peripheral left coset gP (with respect to c) if x is not within a distance R of an endpoint of c and $B(x, R) \cap c$ lies in $\mathcal{N}_\epsilon(gP)$. A point $x \in c$ is (ϵ, R)*–deep* if x is (ϵ, R)–deep in some peripheral left coset gP. If x is not (ϵ, R)–deep in any peripheral left coset gP then x is an (ϵ, R)*–transition point* of c.

Lemma 8.22 [14, Lemma 8.10] *Let $(G, \mathbb{P})$ be relatively hyperbolic with a finite generating set S. For each ϵ there is a constant $R = R(\epsilon)$ such that the following holds. Let c be any geodesic of $\Gamma(G, S)$, and let $\overline{c}$ be a connected component of the set of all (ϵ, R)–deep points of c. Then there is a peripheral left coset gP such that each $x \in \overline{c}$ is (ϵ, R)–deep in gP and is not (ϵ, R)–deep in any other peripheral left coset.*

Lemma 8.23 [14, Proposition 8.13] *Let $(G, \mathbb{P})$ be relatively hyperbolic with a finite generating set S. There exist constants ϵ, R and L such that the following holds. Let c be any geodesic of $\Gamma(G, S)$ with endpoints in G, and let $\hat{c}$ be a geodesic of $\widehat{\Gamma}(G, S, \mathbb{P})$ with the same endpoints as c. Then in the metric d_S, the set of G–vertices of $\hat{c}$ is at a Hausdorff distance at most L from the set of (ϵ, R)–transition points of c. Furthermore, the constants ϵ and R satisfy the conclusion of Lemma 8.22.*

Lemma 8.24 [6, Lemma 4.12] *Let $(G, \mathbb{P})$ be relatively hyperbolic with a finite generating set S. Then for each $\theta \in [0, \frac{1}{2})$ there exist a number $M = M(\theta) > 0$ such that for every geodesic q of length ℓ and every peripheral left coset gP with $q(0), q(\ell) \in N_{\theta\ell}(gP)$ we have $q \cap N_M(gP) \neq \varnothing$.*

Theorem 8.25 *Let $(G,\mathbb{P})$ be a relatively hyperbolic group and H an infinite index, infinite normal subgroup of G. Then $\operatorname{div}(G,H)$ is linear.*

Proof The proof follows from Theorem 5.4 and Lemmas 8.12 and 8.15. □

Proposition 8.26 *Let $(G,\mathbb{P})$ be a relatively hyperbolic group and H a subgroup of G for which H contains at least one infinite order hyperbolic element. If $0<\widetilde{e}(G,H)<\infty$, then $\operatorname{Div}(G,H)$ is at least exponential.*

Proof Suppose that H contains an infinite order hyperbolic element h and assume that h is an element of the finite generating set S of G. By Lemma 8.12, there is a positive integer L such that $d(1,h^n)\geq (n/L)-L$. Moreover, the subgroup H_1 generated by h is strongly relatively quasiconvex. Thus there is a constant $A>1$ such that the set of G–vertices of any geodesic β in $\widehat{\Gamma}(G,S,\mathbb{P})$ connecting two elements of H_1 must lie in the A–neighborhood of H_1 with respect to the metric d_S.

We define $m=\widetilde{e}(G,H)$ and $M=L(12m+2L+2)$. Let $K>1$ be the constant in Lemma 8.18 and let σ the constant in Lemma 8.17. Denote $\operatorname{Div}(G,H)=\{\delta^n_\rho\}$. We will prove that $e^r\preceq \delta^{Mn}_\rho$. More precisely, we define $r_0=2\sigma+(2/\rho)(A+2\sigma)+L+1$ and we will prove $2^{\rho r/2K}\leq \delta^{Mn}_\rho(r)$ for each $r>r_0$. We assume r is an integer.

Indeed, for each $i\in\{0,1,2,\dots,m\}$ we define γ_i to be an H–perpendicular geodesic ray with the initial point $k_i=h^{L(6inr+L)}$. Since $m=\widetilde{e}(G,H)$, there are two different geodesics γ_i and γ_j $(i<j)$ such that $\gamma_i\cap C_r(H)$ and $\gamma_j\cap C_r(H)$ lie in the same component of $C_r(H)$. We define $x=\gamma_i(r)$ and $y=\gamma_j(r)$; then x, y lie in $\partial N_r(H)$ and $d_r(x,y)<\infty$. Also,

$$\begin{aligned}
d_S(x,y)&\leq d_S(x,k_i)+d_S(k_i,e)+d_s(e,k_j)+d_S(h_j,y)\\
&\leq r+L(6inr+L)+L(6jnr+L)+r\\
&\leq L(12mnr+2L)+2r\leq L(12m+2L+2)nr\leq (Mn)r,\\
d(k_i,k_j)&=d(e,h^{6L(j-i)nr})\geq 6(j-i)nr-L\geq 12r-L\geq 6r.
\end{aligned}$$

Let α_1 be a geodesic in $\widehat{\Gamma}(G,S,\mathbb{P})$ connecting k_i, k_j and let α_2 a geodesic in $\widehat{\Gamma}(G,S,\mathbb{P})$ connecting x, y. Let β_1 be a geodesic in $\widehat{\Gamma}(G,S,\mathbb{P})$ connecting x, k_i and β_2 a geodesic in $\widehat{\Gamma}(G,S,\mathbb{P})$ connecting y and k_j. Let u be a point in α_1 such that $d(u,k_i)>2r$ and $d(u,k_j)>2r$. Thus there is a G–vertex v in $\beta_1\cup\alpha_2\cup\beta_2$ such that $d_S(u,v)\leq 2\sigma$.

If v lies in β_1, then the distance in $\widehat{\Gamma}(G,S,\mathbb{P})$ between u and k_i is bounded above by $r+2\sigma$. Thus this distance is at most $2r$ which contradicts the choice of u. Thus v does not lie in β_1. Similarly, v does not lie in β_2. Thus v must lie in α_2. Also, u lies in the A–neighborhood of the subgroup H_1 with respect to the metric d_S. Thus v lies in the $(A+2\sigma)$–neighborhood of H_1 with respect to the metric d_S. Therefore, the distance in $\Gamma(G,S)$ between v and H is bounded above by $(A+2\sigma)$.

We now prove that $d_{\rho r}(x,y) \geq 2^{\rho r/2K}$. Indeed, let γ be any path in $C_{\rho r}(H)$ connecting x and y. By Lemma 8.18, there exists a vertex $w \in \gamma$ such that $d_S(w,v) \leq K\log_2|\gamma|$. Since

$$d_S(w,v) \geq d_S(w,H) - d_S(v,H) \geq \rho r - A - 2\sigma \geq \frac{\rho r}{2},$$

then

$$K\log_2|\gamma| \geq \frac{\rho r}{2}.$$

Thus $|\gamma| \geq 2^{\rho r/2K}$. Therefore, $d_{\rho r}(x,y) \geq 2^{\rho r/2K}$. Therefore, $2^{\rho r/2K} \leq \delta_\rho^{Mn}(r)$. Thus $e^r \preceq \delta_\rho^{Mn}$. □

The following is a key lemma we are going to use to investigate the lower divergence of a relatively hyperbolic group with respect to a fully relatively quasiconvex subgroup.

Lemma 8.27 *Let $(G,\mathbb{P})$ be relatively hyperbolic with a finite generating set S. There exist constants ϵ, R, σ, K and A such that the following hold.*

(1) *A subgroup H is relatively quasiconvex if and only if there is a constant κ such that for each geodesic c in $\Gamma(G,S)$ joining points in H, the set of (ϵ,R)–transition points of c lies in the κ–neighborhood of H.*

(2) *Let $\Delta = pqr$ be a triangle whose sides p,q,r are geodesic in $\Gamma(G,S)$. Then for each (ϵ,R)–transition point v on p, there is an (ϵ,R)–transition point u in the union $q \cup r$ such that $d_S(u,v) \leq \sigma$.*

(3) *Let p and q be paths in $\Gamma(G,S)$ such that $p- = q-$, $p+ = q+$ and q is geodesic in $\Gamma(G,S)$. For any (ϵ,R)–transition point $v \in q$, there exists a vertex $w \in p$ such that $d_S(w,v) \leq K\log_2|p| + K$.*

(4) *For each peripheral left coset gP and any geodesic c with endpoints outside $N_A(gP)$. If $\ell(c) > 9\max\{d_S(c^+,gP); d_S(c^-,gP)\}$, then the path c contains an (ϵ,R)–transition point w which lies in the A–neighborhood of gP.*

Furthermore, the constants ϵ and R satisfy the conclusion of Lemma 8.22.

We now give the proof for the above lemma. The reader can also find the proof of statement (1) in [14].

Proof Let ϵ and R be constants in Lemma 8.23. Statements (1), (2) and (3) are immediate results of Definition 8.8 and Lemmas 8.16, 8.18 and 8.23. We now focus on proving statement (4).

Let

$$\begin{aligned} A_0 &= A_0(\tfrac{1}{3}) \text{ be the constant in Lemma 8.24,} \\ A_1 &= A_1(A_0) \text{ be the constant in Lemma 8.19,} \\ A_2 &= A_2(A_1, \epsilon) \text{ be the constant in Lemma 8.20,} \\ A &= A_0 + A_1 + A_2 + \epsilon + 1. \end{aligned}$$

Let gP be any peripheral left coset. Let c be any geodesic with endpoints outside $N_A(gP)$ such that $\ell(c) > 9\max\{d_S(c^+, gP), d_S(c^-, gP)\}$. Let

$$r = \max\{d_S(c^+, gP), d_S(c^-, gP)\}.$$

Thus the length of c is greater than $9r$ and $r > A$. Since

$$\ell(c) > 9\max\{d_S(c^+, gP), d_S(c^-, gP)\},$$

$c \cap N_{A_0}(u_1 P) \neq \varnothing$ by Lemma 8.24. Let a_1 and a_2 be the first vertex and the last vertex in $c \cap N_{A_0}(gP)$. Thus the subsegment $[a_1, a_2]$ of c connecting a_1 and a_2 must lie in the A_1–neighborhood of gP. Let a_1' and a_2' the vertices in $c - [a_1, a_2]$ such that $d_S(a_1, a_1') \leq 1$ and $d_S(a_2, a_2') \leq 1$. We assume that a_1' lies between c^+, a_1 and that a_2' lies between c^-, a_2. Obviously, a_1' and a_2' must lie in the $(A_0 + 1)$–neighborhood of gP. In particular, they lie in the r–neighborhood of gP. If the distance between c^+ and a_1 is greater than $4r$, then the distance in between c^+ and a_1' is greater than $3r$. Thus the subsegment of c connecting x^+ and a_1' must intersect the A_0–neighborhood of gP which contradicts to the choice of a_1. Thus $d_S(c^+, a_1) \leq 4r$. Similarly, $d_S(c^-, a_2) \leq 4r$. Also, the length of c is at least $9r$. Thus the length of $[a_1, a_2]$ is at least r. In particular, this length is bounded below by A_2.

We now show that c contains an (ϵ, R)–transition point w in the A–neighborhood of gP. Indeed, if $[a_1, a_2]$ contains an (ϵ, R)–transition point w, then w must lie in the A_1–neighborhood of gP. In particular, w lies in the A–neighborhood of gP and we are done.

We now consider the case that $[a_1, a_2]$ contains only (ϵ, R)–deep points. Therefore, $[a_1, a_2]$ lies in some ϵ–neighborhood of some peripheral left coset $g'P'$. Thus

$$[a_1, a_2] \subset N_{A_1}(gP) \cap N_\epsilon(g'P').$$

Also, the length of $[a_1, a_2]$ is at least r. Thus the length of $[a_1, a_2]$ is bounded below by A_2. Therefore, $\operatorname{diam}(N_{A_1}(gP) \cap N_\epsilon(g'P'))$ is strictly greater than A_2. Thus $gP = g'P'$. It follows that $[a_1, a_2]$ lies in the ϵ–neighborhood of gP. Also, the endpoints of c both lie outside the ϵ–neighborhood of gP. Thus we could find an (ϵ, R)–transition point w in c such that $d_S(w, gP) \leq \epsilon + 1$. In particular, w lies in the A–neighborhood of gP. $\square$

Theorem 8.28 *Let $(G, \mathbb{P})$ be a relatively hyperbolic group and H an infinite fully relatively quasiconvex subgroup of G. If $0 < \widetilde{e}(G, H) < \infty$, then $\operatorname{div}(G, H)$ is at least exponential.*

Remark 8.29 Before giving the proof of the theorem, we would like to discuss a large class of groups and their subgroups to which the theorem applies. More precisely, we are going to discuss different pairs of groups (G, H), where G is a relatively hyperbolic group and H is an infinite fully relatively quasiconvex subgroup of G with $0 < \widetilde{e}(G, H) < \infty$.

Let G be the fundamental group of some hyperbolic surface and H an infinite cyclic subgroup of G. Thus G is a hyperbolic group and H is an infinite malnormal quasiconvex subgroup of G. In particular, G is a relatively hyperbolic group and H is an infinite fully relatively quasiconvex subgroup. Obviously, the number of filtered ends $\widetilde{e}(G, H) = 2$.

We now come up with other example. Let G be the fundamental group of some hyperbolic finite volume three manifold with cusps. Therefore, G is relatively hyperbolic with respect to the collection of its cusp subgroups. Let H be any cusp subgroup of G. We can see that H is an infinite fully relatively quasiconvex subgroup of G and $\widetilde{e}(G, H) = 1$.

We now discuss the case where H is a strongly relatively quasiconvex subgroup with finite number of filtered ends $\widetilde{e}(G, H)$. We can choose G be the fundamental group of some hyperbolic finite volume three manifold with cusps as above. Again, G is relatively hyperbolic with respect to the collection of its cusp subgroups. Let H be a cyclic subgroup generated by a hyperbolic element. It is obvious that H is a strongly relatively quasiconvex subgroup and the number of filtered ends $\widetilde{e}(G, H) = 1$

Now, we come up with a pair of groups (G, H) satisfying all conditions in Theorem 8.28 and H is neither strongly relative quasiconvex nor a subgroup of some peripheral subgroup. Let G be the fundamental group of some hyperbolic finite volume three manifold with more than one cusp. We can pick up any cusp subgroup P and any cyclic subgroup K of G generated by some hyperbolic element. By Martínez-Pedroza and

Sisto [17, Theorem 2], it is obvious that we can choose some finite index subgroup P_1 of P and some finite index subgroup K_1 of K such that the subgroup H generated by P_1 and K_1 is isomorphic to their free product and H is also a fully relatively quasiconvex subgroup. It is not hard to see that the number of filtered ends $\widetilde{e}(G,H)=1$.

Proof Let ϵ, R, σ, K and A be the constants in Lemma 8.27.

Let κ be the constant such that for each geodesic c in $\Gamma(G,S)$ joining points in H, the set of (ϵ,R)–transition points of c lies in the κ–neighborhood of H.

By Lemma 8.7, we observe that the diameter of the set $(N_\kappa(H)\cap N_\epsilon(tP))$ is finite whenever $|tPt^{-1}\cap H|<\infty$. Also, the number of peripheral left cosets tP, where $|t|_S\leq \kappa+\epsilon$ and $P\in\mathbb{P}$, is finite. Thus the number $B=\max\{\operatorname{diam}(N_\kappa(H)\cap N_\epsilon(tP) \mid |t|_S\leq\kappa+\epsilon,\ P\in\mathbb{P}$ and $|tPt^{-1}\cap H|<\infty\}$ is finite. Similarly, we could choose a finite number C such that the C–neighborhood of H contains all peripheral left cosets tP where $|t|_S\leq\kappa+\epsilon$ and $|tPt^{-1}:(tPt^{-1}\cap H)|<\infty$.

Denote $\operatorname{div}(G,H)=\{\sigma^n_\rho\}$. We will prove that $e^r\preceq\sigma^{27n}_\rho$. More precisely, we define

$$r_0=\frac{4C}{\rho}(\kappa+K+A+B+C+2\sigma)$$

and we will prove $2^{\rho r/4K}\leq\sigma^{27n}_\rho(r)$ for each $r>r_0$. We assume r is an integer.

Let x and y be arbitrary points in $\partial N_r(H)$ such that $d_S(x,y)\geq(27n)r$ and $d_r(x,y)<\infty$. (The existence of x and y is guaranteed by the condition $0<\widetilde{e}(G,H)<\infty$.) Let x_1 and y_1 be points in H such that $d_S(x,x_1)=d_S(x,H)=r$ and $d_S(y,y_1)=d_S(y,H)=r$.

Let γ be any path in $C_{\rho r}(H)$ connecting x and y. Let c be a geodesic in $\Gamma(G,S)$ connecting x and y and c_1 a geodesic in $\Gamma(G,S)$ connecting x_1 and y_1. Let β_1 be a geodesic in $\Gamma(G,S)$ connecting x and x_1 and β_2 a geodesic in $\Gamma(G,S)$ connecting y and y_1.

By Lemma 8.27, for each (ϵ,R)–transition point u in c_1 there is an (ϵ,R)–transition point v_u in $\beta_1\cup c\cup\beta_2$ such that $d_S(u,v_u)\leq 2\sigma$. We have two main cases.

Case 1 Suppose that v_u lies in c for some u in c_1.

Since u lies in the κ–neighborhood of H, v_u lies in the $(\kappa+2\sigma)$–neighborhood of H. By Lemma 8.27, there exists a vertex $w\in\gamma$ such that $d_S(w,v_u)\leq K\log_2|\gamma|+K$. Since w lies outside $N_{\rho r}(H)$, the distance $d_S(w,v_u)$ is bounded below by $\rho r-\kappa-2\sigma$. Thus $K\log_2|\gamma|\geq\rho r-\kappa-2\sigma-K\geq\rho r/4$ by the choice of r. Thus the length of γ is bounded below by $2^{\rho r/4K}$.

Case 2 Suppose that v_u lies in $\beta_1\cup\beta_2$ for all (ϵ,R)–transition point u in c_1.

We could choose u_1 and u_2 in c_1 such that $v_{u_1} \in \beta_1$, $v_{u_2} \in \beta_2$ and all points in the geodesic c_1 lies between u_1 and u_2 are (ϵ, R)–deep points with respect to some peripheral left coset gP. In particular, the two points u_1, u_2 lie in the ϵ–neighborhood gP. Since v_{u_1} lies in β_1 and the length of β_1 is r, the distance between u_1 and x_1 is bounded above by $r+2\sigma$. Thus the distance between u_1 and x_1 is bounded above by $2r$ by the choice of r. Similarly, the distance between u_2 and y_1 is bounded above by $2r$ with respect to the metric d_S. By the same argument, the distances $d_S(u_1, x)$ and $d_S(u_2, y)$ are also bounded above by $2r$. Also, the distance between x and y is at least $(27n)r$. Thus the distance between u_1 and u_2 is bounded below by $(27n-4)r$. Therefore, this distance is bounded below by $(23)r$ by the choice of n.

Since the distance $d_S(H, gP) \leq d_S(H, u_1) + d_S(u_1, gP) \leq \kappa + \epsilon$, there are some h_1 in H and t in G such that $|t|_S \leq \kappa + \epsilon$ and $gP = h_1 tP$. Thus

$$\begin{aligned}\operatorname{diam}(N_\epsilon(tP) \cap N_\kappa(H)) &= \operatorname{diam}(N_\epsilon(h_1 tP) \cap N_\kappa(h_1 H)) \\ &= \operatorname{diam}(N_\epsilon(gP) \cap N_\kappa(H)).\end{aligned}$$

Since u_1 and u_2 lie in $N_\epsilon(gP) \cap N_\kappa(H)$, then

$$\operatorname{diam}(N_\epsilon(gP) \cap N_\kappa(H)) \geq d_S(u_1, u_2) \geq (23)r > 23r > r_0 > B.$$

Thus

$$\operatorname{diam}(N_\epsilon(tP) \cap N_\kappa(H)) > B.$$

Therefore, $|tPt^{-1} \cap H| = \infty$ by the choice of B. It follows that

$$|tPt^{-1} : (tPt^{-1} \cap H)| < \infty$$

since H is a fully relatively quasiconvex subgroup. Therefore, $tP \subset N_C(H)$. Thus

$$gP = h_1 tP \subset h_1 N_C(H) = N_C(H).$$

Therefore, γ lies outside the $(\rho r - C)$–neighborhood of gP. Thus γ lies outside the $(\rho r/2)$–neighborhood of gP by the choice of r.

We now show that there is an (ϵ, R)–transition point w in c such that $d_S(w, gP) \leq A$. Since gP lies in the C–neighborhood of H and the distance between x and H is r, then x lies outside the $(r-C)$–neighborhood of gP. In particular, x lies outside the A–neighborhood of gP. Similarly, y also lies outside the A–neighborhood of gP. Since the distance between x and u_1 is bounded above by $2r$ and u_1 lies in the ϵ–neighborhood of gP, then x lies in the $(2r+\epsilon)$–neighborhood of gP. In particular, x lies in the $3r$–neighborhood of gP. Similarly, y also lies in the $3r$–neighborhood of gP. Since x and y lies in the $3r$–neighborhood of gP and the distance between x

and y is greater than $27r$, then $\ell(c) > 9\max\{d_S(c^+, gP), d_S(c^-, gP)\}$, then c contains an (ϵ, R)–transition point w in the A–neighborhood of gP by Lemma 8.27.

We now prove that the length of γ is bounded below by $2^{\rho r/4K}$. Indeed, by Lemma 8.27, there exists a vertex $v \in \gamma$ such that $d_S(v, w) \leq K\log_2|\gamma| + K$ Also

$$d_S(v, w) \geq d_S(v, gP) - d_S(gP, w) \geq \frac{\rho r}{2} - A.$$

Thus

$$K\log_2|\gamma| \geq \frac{\rho r}{2} - A - K \geq \frac{\rho r}{4}.$$

Therefore, the length of γ is bounded below by $2^{\rho r/4K}$. Thus $d_{\rho r}(x, y) \geq 2^{\rho r/4K}$. Thus $2^{\rho r/4K} \leq \sigma_\rho^{27n}$. Therefore, $e^r \preceq \sigma_\rho^{27n}$. □

Question 8.30 For the pair (G, H) as in Theorem 8.28, is the relative lower divergence $\operatorname{div}(G, H)$ exactly exponential? What conditions do we need to put on the pair (G, H) to force the lower relative divergence $\operatorname{div}(G, H)$ to be exactly exponential?

Corollary 8.31 *Let G be a hyperbolic group and H an infinite quasiconvex subgroup of G. If $0 < \widetilde{e}(G, H) < \infty$, then $\operatorname{div}(G, H)$ is at least exponential.*

Corollary 8.32 *Let $(G, \mathbb{P})$ be a relatively hyperbolic group and P an infinite peripheral subgroup. If $0 < \widetilde{e}(G, P) < \infty$, then $\operatorname{div}(G, P)$ is at least exponential.*

Corollary 8.33 *Let $(G, \mathbb{P})$ be a relatively hyperbolic group and H an infinite strongly relatively quasiconvex subgroup. If $0 < \widetilde{e}(G, H) < \infty$, then $\operatorname{div}(G, H)$ is at least exponential.*

Remark 8.34 From the results of Corollary 8.31 and Theorem 6.7, we could extend the result of Corollary 7.12. More precisely, there is a pair of groups (G, H), where G is a one-ended CAT(0) group and H is an infinite cyclic subgroup of G such that $\operatorname{div}(G, H)$ is exponential. For example, let G be the fundamental group of a hyperbolic surface M and H the fundamental group of a closed essential curve C of M. Then G is a one-ended CAT(0) group and it is also hyperbolic. Since the infinite cyclic subgroup H is also quasiconvex, then $\operatorname{div}(G, H)$ is at least exponential. Also, $\operatorname{div}(G, H)$ is dominated by the upper divergence of G (see Theorem 6.7) and the upper divergence of one-ended finitely presented group is at most exponential (see [22, Lemma 6.15]). Thus $\operatorname{div}(G, H)$ is at most exponential. Therefore, $\operatorname{div}(G, H)$ is exactly exponential.

In Theorem 8.28, we could not replace the condition "fully relative quasiconvexity" of the subgroup H by the condition "relative quasiconvexity". Readers could look at the following theorem as a counterexample.

Theorem 8.35 *Let $G = \langle a_1, a_2, a_3, b, c \mid [a_1, a_2][a_3, b] = e, [b, c] = e\rangle$ and H be the cyclic subgroup of G generated by c. Let P be the subgroup generated by b and c. Then, G is a relatively hyperbolic group with respect to the subgroup P, $0 < \widetilde{e}(G, H) < \infty$, H is a relatively quasiconvex subgroup and $\operatorname{div}(G, H)$ is linear.*

Before giving the proof of Theorem 8.35, we need to review a result in Hruska [13].

Definition 8.36 [13, Definition 5.1] A CAT(0) 2–complex X has the *isolated flats property* if there is a function $\Phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for every pair of distinct flat planes $F_1 \neq F_2$ in X and for every $k \geq 0$, the intersection $N_k(F_1) \cap N_k(F_1)$ of k–neighborhoods of F_1 and F_2 has diameter at most $\Phi(k)$.

Theorem 8.37 [13, Theorem 1.6] *Suppose a group G acts properly and cocompactly by isometry on a* CAT(0) *2–complex with the isolated flats property. Then G is hyperbolic relative to the collection of maximal virtually abelian subgroups of rank two.*

We now give the proof for Theorem 8.35.

Proof We are going to show that G acts properly and cocompactly by isometry on a CAT(0) 2–complex with the isolated flats property. It is obvious that

$$G = G_1 \underset{<b_1>=<b_2>}{*} P,$$

where $G_1 = \langle a_1, a_2, a_3, b_1 \mid [a_1, a_2][a_3, b_1] = e\rangle$ and $P = \langle b_2, c \mid [b_2, c] = e\rangle$. Let X_1 be the presentation 2–complex of G_1 and X_2 the presentation 2–complex of P. We build the 2–complex presentation for G by identifying the 1–cell b_1 of X_1 and the 1–cell b_2 of X_2 into one 1–cell called b. Let $\widetilde{X}_1$ and $\widetilde{X}_2$ be the universal covers of X_1 and X_2 respectively. It is well known that we can put a metric on $\widetilde{X}_1$ such that $\widetilde{X}_1$ becomes the 2–dimensional hyperbolic plane and G_1 acts properly and cocompactly on $\widetilde{X}_1$ by isometry. Similarly, we can put a metric on $\widetilde{X}_2$ such that $\widetilde{X}_2$ becomes the 2–dimensional flat and P acts properly and cocompactly on $\widetilde{X}_2$ by isometry. It is obvious that the universal cover $\widetilde{X}$ of X is the union of copies of $\widetilde{X}_1$ and $\widetilde{X}_2$ such that a copy of $\widetilde{X}_1$ intersects a copy of $\widetilde{X}_2$ in a bi-infinite arc labeled by b. Thus $\widetilde{X}$ is a CAT(0) 2–complex with the isolated flats property. Moreover, the group G acts properly and cocompactly by isometry on $\widetilde{X}$. Therefore, G is a relatively hyperbolic group with respect to the subgroup P by Theorem 8.37.

By examining the construction of $\widetilde{X}$, we can see that $\widetilde{e}(G, H) = 1$. Moreover, H is a relatively quasiconvex subgroup since it is a subgroup of peripheral subgroup P. We now show that the relative lower divergence $\operatorname{div}(G, H)$ is linear.

First we show that $|b^n|_S = |n|$. Let $m = |b^n|_S$. Obviously, $m \leq |n|$. There is a homomorphism Φ from G to $\mathbb{Z}$ that maps every element in S to the generator of $\mathbb{Z}$. Since $m = |b^n|_S$, then there is a word w in $S \cup S^{-1}$ with the length m such that $b^n \equiv_G w$. Therefore

$$b^n \equiv_G s_1 s_2 \cdots s_m, \quad \text{where } s_i \in S \cup S^{-1}.$$

Thus

$$\Phi(b^n) = \Phi(s_1) + \Phi(s_2) + \cdots + \Phi(s_m).$$

Since $\Phi(b^n) = n$ and $\Phi(s_i) = -1$ or 1, then $|n| \leq m$. Thus $|b^n|_S = m = |n|$. Similarly, $|c^n|_S = |n|$.

We now show that $d_S(b^m c^n, H) = |m|$. Denote $d_S(b^m c^n, H) = \ell$. Obviously, $\ell \leq |m|$. There is a group homomorphism Ψ from G to $\mathbb{Z}$ that maps b to the generators of $\mathbb{Z}$ and the remaining elements in S to 0. Suppose that $d_S(b^m c^n, H) = d_S(b^m c^n, c^{n'})$ for some $c^{n'}$ in H. Thus there is a word w' with the length ℓ such that $b^m c^n \equiv_G c^{n'} w'$. Therefore,

$$b^m c^n \equiv_G c^{n'} s'_1 s'_2 \cdots s'_\ell, \quad \text{where } s'_i \in S \cup S^{-1}.$$

Thus

$$\Psi(b^m) + \Psi(c^n) = \Psi(c^{n'}) + \Psi(s'_1) + \Psi(s'_2) + \cdots + \Psi(s'_\ell).$$

Since $\Psi(b^m) = m$, $\Psi(c^n) = \Psi(c^{n'}) = 0$ and $\Psi(s_i) = -1$, 0 or 1, then $|m| \leq \ell$. Thus $d_S(b^m c^n, H) = |m|$.

Denote $\operatorname{div}(G, H) = \{\sigma^n_\rho\}$. We will prove that σ^n_ρ is bounded above by a linear function. More precisely, we will show $\sigma^n_\rho \leq nr$ for each $r > 0$.

We assume r is an integer. Let $x = b^r$ and $y = b^r c^{nr}$. Then x and y lie in $\partial N_r(H)$ and $d_S(x, y) \geq nr$. Let γ be the path with vertices $\{b^r, b^r c, b^r c^2, \ldots, b^r c^{nr}\}$. Then, γ is a path in $C_r(H)$ connecting x and y. Thus $d_r(x, y) < \infty$. Moreover, $d_{\rho r}(x, y) \leq nr$ since the length of γ is nr. Thus $\sigma^n_\rho \leq nr$. Therefore, σ^n_ρ is bounded above by a linear function. □

Theorem 8.38 *Let $(G, \mathbb{P})$ be a relatively hyperbolic group and H a subgroup of G such that $0 < \widetilde{e}(G, H) < \infty$. We assume that H is not conjugate to any infinite-index subgroup of any peripheral subgroup. Then $\operatorname{Div}(G, H)$ is at least exponential.*

Proof If H is a finite subgroup, then the relative upper divergence $\operatorname{Div}(G, H)$ is equivalent to the upper divergence of G by Theorem 4.15 and Remark 4.7. Also, the upper divergence of G is at least exponential by Sisto [22]. Thus $\operatorname{Div}(G, H)$ is at least exponential.

In the case that H is conjugate to a finite index subgroup of some peripheral subgroup, we assume that H is a finite index subgroup of some peripheral subgroup by Theorem 4.15. Thus $\operatorname{div}(G, H)$ is at least exponential by Theorem 8.28. Also, $\operatorname{div}(G, H)$ is dominated by $\operatorname{Div}(G, H)$ by Theorem 4.14. Therefore, the upper relative divergence $\operatorname{Div}(G, H)$ is also at least exponential.

If H is an infinite subgroup that is not conjugate to any subgroup of any peripheral subgroup, H contains a hyperbolic element by Theorem 8.14. Thus $\operatorname{Div}(G, H)$ is at least exponential by Proposition 8.26. □

Remark 8.39 In Theorem 8.38, if the group G is finitely presented, then the upper divergence of G is exactly exponential. Therefore, the upper relative divergence $\operatorname{Div}(G, H)$ is also exponential when the subgroup H is finite. However, it is still unknown whether the upper relative divergence $\operatorname{Div}(G, H)$ is exactly exponential in general, or what conditions we need to put on the pair (G, H) to make the relative upper divergence $\operatorname{Div}(G, H)$ to be exactly exponential.

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