The Chillingworth class is a signed stable length

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An orientation is defined on a family of curve graphs on which the Torelli group acts. It is shown that the resulting signed stable length of an element of the Torelli group is a cohomology class. This cohomology class is half the dual of the contraction of the Johnson homomorphism, the so-called "Chillingworth class".

1 Introduction

Let S_g be a closed, oriented surface with genus $g \geq 3$, and $S_{g,1}$ an oriented surface with genus $g \geq 3$ and one boundary curve. The *mapping class group* of S_g, denoted $\mathrm{Mod}(S_g)$, is the group of isotopy classes of orientation preserving homeomorphisms of S_g. The group $\mathrm{Mod}(S_{g,1})$ is defined similarly, with the added condition that the homeomorphisms act as the identity on the boundary of $S_{g,1}$. The group $\mathrm{Mod}(S_g)$ induces an action on $H_1(S_g, \mathbb{Z})$, and the *Torelli group* $\mathcal{T}_g$ is the kernel of this action. The Torelli group $\mathcal{T}_{g,1}$ is similarly defined.

Johnson homomorphisms The *Johnson homomorphisms* (see Johnson [10]) are homomorphisms

$$(1) \qquad t_{g,1} \colon \mathcal{T}_{g,1} \to \Lambda^3 H_1(S_{g,1}; \mathbb{Z}) \quad \text{and} \quad t_g \colon \mathcal{T}_g \to \Lambda^3 H_1(S_g; \mathbb{Z})/H_1(S_g; \mathbb{Z}).$$

The Johnson homomorphisms are among the most basic structures in the study of the Torelli group, for example for understanding group homology (see Johnson [11]) and 3–manifold theory (see Morita [13]). Chillingworth [5] used a notion of winding number to define the *Chillingworth class*, a map from the Torelli group to an integral first cohomology class of $S_{g,1}$ or S_g. It was shown that the vanishing of this map is a necessary condition for a mapping class in the Torelli group to be in the kernel of the Johnson homomorphism, ie the *Johnson kernel*. Johnson [10, Theorem 2] showed that the Chillingworth class is dual to a contraction of the Johnson homomorphism. For surveys on the Johnson homomorphism, see Johnson [11] and Farb and Margalit [7].

Curve graphs In this paper, we will be working with two curve graphs, $C(S_{g,1}; h)$ and $C(S_g; h)$. The graph $C(S_{g,1}; h)$ has vertices corresponding to oriented isotopy classes of simple closed curves in the primitive homology class $h \in H_1(S_{g,1}; \mathbb{Z})$. Any two vertices representing disjoint curves whose difference bounds one or more subsurfaces of $S_{g,1}$ of genus one are connected by an edge. The graph $C(S_g, h)$ is defined analogously.
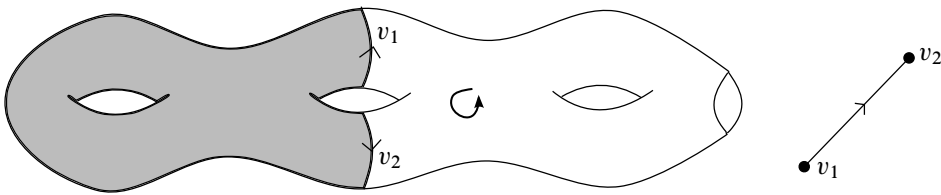

Figure 1: The shaded subsurface represents an edge of the curve graph $C(S_{3,1}, h)$.

Since there are no bounding pairs in S_2, the graph $C(S_2; h)$ does not have any edges. For this reason, and also because the Johnson homomorphism, and hence the Chillingworth class, is zero in genus two (see Johnson [10, Lemma 4A]), we will not be considering surfaces of genus two.

Signed distance There is of course the usual combinatorial distance $d(v_1, v_2)$ defined on the graphs $C(S_{g,1}; h)$ and $C(S_g; h)$. Since a path in $C(S_{g,1}; h)$ or $C(S_g; h)$ defines a surface in a 3–manifold, as described in Irmer [9], the combinatorial distance is a special case of the Thurston norm defined in [15]. To turn this norm into a cohomology class, orientation information in the form of a signed distance $d_s(v_1, v_2)$ on $C(S_{g,1}; h)$ or $C(S_g; h)$ is utilized.

Calculating the Thurston norm, or the very closely related Gromov norm, involves finding an infimum analogous to the infimum that is a stable length of a group action on a metric space such as a graph. Let η be an endomorphism of a free group F. In [3, Section 9] the Gromov norm of the HNN-extension of F, denoted $F_{*\eta}$, was shown to be a "translation length" of η acting on $K(F, 1)$. The "translation length" in this context was defined to be a *stable* commutator length, whereas translation length (also called stable length) on $C(S_{g,1}, h)$ is a commutator length. The action of an element of the mapping class group of $S_{g,1}$ on the fundamental group of the surface gives an endomorphism of a free group. Calegari and Walker [3] emphasized that the Chillingworth class is a rotation quasi-morphism. Rotation quasi-morphisms, defined in Calegari [2, Section 2.2], are used to estimate, or sometimes calculate, stable commutator length.

Recall the convention that the boundary of a subsurface of $S_{g,1}$ or S_g is oriented in such a way that it has the subsurface to its left. A subsurface of $S_{g,1}$ or S_g is assigned $+1$ if it is oriented as a subsurface of $S_{g,1}$ or S_g, and -1 otherwise. In particular, this gives an orientation to the edges of the curve graphs, as shown in Figure 1.

Choose a path γ connecting v_1 and v_2 in $C(S_{g,1}; h)$ or $C(S_g; h)$, and let $l_s(\gamma)$ be the number of edges traversed by γ in the positive direction minus the number of edges traversed in the negative direction. In Lemma 3 it is shown that $l_s(\gamma)$ is independent of the choice of γ in $C(S_{g,1}, h)$. In this case, the *signed distance* $d_s(v_1, v_2)$ is defined to be $l_s(\gamma)$. In Lemma 2, it is also shown that for S_g the signed length of γ is only independent mod $(g-1)$ of the choice of γ. The signed distance is defined to be the equivalence class mod $(g-1)$. The surface S_3 is a special case, because as shown in Figure 2, a bounding pair cuts S_3 into two subsurfaces of genus one, so it is not clear how to orient edges of the curve graph. However, $+1$ is congruent to -1 mod $(3-1)$ so any arbitrary choice of edge orientation will determine the same signed length of paths mod 2.
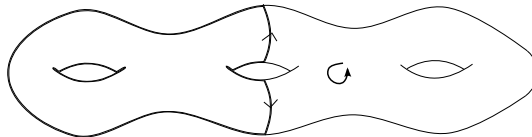

Figure 2: A bounding pair cuts S_3 into two surfaces of genus 1.

Since the respective Torelli groups preserve orientation and act on $C(S_g, h)$ and $C(S_{g,1}, h)$ by isometry, signed distance is also preserved by the Torelli group.

Signed stable length Choose a primitive homology class h and define the signed stable length $\phi_h(\tau)$ of an element τ of $\mathcal{T}_{g,1}$ acting on $C(S_{g,1}; h)$ to be

$$(2) \qquad\qquad \phi_h(\tau) = \lim_{n \to \infty} \frac{d_s(v, \tau^n v)}{n},$$

where v is any vertex of $C(S_{g,1}; h)$. It is not a priori clear that this is well-defined, however this will be shown to follow from Lemma 2.

By a construction due to Hatcher [8], $C(S_{g,1}; h)$ and $C(S_g; h)$ are seen to be connected. Since the respective Torelli groups act on the curve graphs $C(S_{g,1}; h)$ and $C(S_g; h)$ by isometry, it follows immediately that stable length, signed or otherwise, is locally independent of the choice of vertex v, and hence globally independent by connectivity.

Lemma 4 states that $\phi_h(\tau) = d_s(v, \tau v)$ for any vertex v in $C(S_{g,1}; h)$. For surfaces without boundary we have to make sense of what it means to stabilize a quantity that is

only defined mod $(g - 1)$. In analogy with surfaces with boundary, for $\tau \in \mathcal{T}_g$ and h a primitive element of $H_1(S_g; \mathbb{Z})$ we define $\phi_h(\tau) := d_s(v, \tau v)$. For both $C(S_{g,1}; h)$ and $C(S_g; h)$, if h is not primitive, ie $h = \lambda[c]$ for a primitive curve c, then we define $\phi_h(\tau) := \lambda \phi_{[c]}(\tau)$.

Observe that, unlike for the mapping class group acting on the curve complex, a different stable length is obtained depending on the choice of homology class defining the graph on which the element of the Torelli group acts. An illustrative example is given by the action of a bounding pair map $T_a T_b^{-1}$. Here T_a denotes a Dehn twist around the curve a, and it is assumed that the oriented multicurve $a - b$ is the boundary of a subsurface of genus 1. Since there are curves left invariant under the action of $T_a T_b^{-1}$, its orbit in Harvey's complex of curves stays within a constant distance of vertices representing these invariant curves, hence the action of $T_a T_b^{-1}$ on Harvey's complex of curves has stable length zero. The calculation in Section 3.4 gives a formula for signed stable length of $T_a T_b^{-1}$ on $C(S_{g,1}, h)$, depending on the algebraic intersection number of $[a]$ with h. For bounding pair maps, arguing as in Section 3.4, it is not hard to show that signed stable length is equal to unsigned stable length on $C(S_{g,1}, h)$.

It is finally possible to state the theorem of this paper.

Theorem 1 *The signed stable length ϕ is half the Chillingworth class.*

2 Some background on the Chillingworth class

In this section, suppose all curves are closed, simple, and oriented, with continuously varying, nowhere zero tangent vector. Let X be a nowhere zero vector field on $S_{g,1}$. In [4], Chillingworth defined the winding number of a curve γ with respect to X. When presented precisely and in all generality, the definition is quite long, so only the general idea of the special case needed for studying the mapping class group will be given here.

Intuitively, the winding number $\omega_X(\gamma)$ is the number of times the tangent vector to γ rotates relative to X as γ is traversed once in its positive direction. To relate winding numbers to cohomology classes, it was shown that $\omega_{X_1}(\gamma) - \omega_{X_2}(\gamma)$ only depends on the integral homology class of γ.

In [5] applications of winding number to the study of mapping class groups were discussed. At the end of this paper the conjecture was made that for $\tau \in \mathcal{T}_{g,1}$, if $\omega_{\tau * X}(\gamma) - \omega_X(\gamma) = 0$ for any curve γ, then τ must be in the subgroup of the Torelli group generated by Dehn twists around separating curves. In [10], Johnson showed this conjecture to be false, and started calling the cohomology class $d(\tau^* X, X)$, defined by

$$\langle d(\tau * X, X), [\gamma] \rangle = \omega_{\tau * X}(\gamma) - \omega_X(\gamma),$$

the *Chillingworth class*. This cohomology class can be shown to be independent of the choice of X. Johnson constructed a homomorphism whose kernel, the *Johnson kernel*, he showed to be the subgroup of the Torelli group generated by Dehn twists around separating curves [12].

In [4, Section 6], winding numbers on surfaces without boundary were defined. For surfaces without boundary and genus greater than or equal to two, there are no non-vanishing vector fields, so suppose X only has one zero; call this point p. When a smooth homotopy moves the curve γ over p, by the Poincaré–Hopf index theorem the winding number of γ is changed by $\pm(2 - 2g)$. It follows that for surfaces without boundary, the Chillingworth class is only defined $\mod (2g - 2)$.

3   Proof of Theorem 1

Proof   The proof of the theorem is broken up into several parts and occupies all of this section.

3.1   Signed distance

In this subsection, some basic properties of signed distance on $C(S_{g,1}; h)$ and $C(S_g; h)$ are established.

A *curve* is an oriented isotopy class of closed loops not homotopic into the boundary of $S_{g,1}$. Where there is no risk of ambiguity, the same symbol will be used for a curve and a particular representative of the isotopy class. Also, in order to show that signed length only depends on the endpoints of the path, or only on the endpoints of the path $\mod (g - 1)$, properties of surfaces in the 3–manifolds $S_g \times I$ and $S_{g,1} \times I$ will be used. In both cases, the surface S_g or $S_{g,1}$ maps into the 3–manifold. Where this does not cause confusion, the same notation will be used for a curve c on a surface (thought of as $S_{g,1} \times 0$ or $S_g \times 0$) and a representative of the isotopy class of a map of a closed loop into the 3–manifold that also contains c.

A surface invariant will now be defined that is important in the proof of the next lemma.

The pre-image function This definition is taken from [9]. Let H be a connected, immersed surface in $S_g \times I$, and let $\hat{i}(a, b)$ denote the algebraic intersection number, either in the 3–manifold or in a surface, depending on the context. Curve graph distances are related to surfaces via the *pre-image function* $p_H \colon S_g \times 0 \setminus \pi(\partial H) \to \mathbb{Z}$. Suppose $P := S_g \times I$ and B are open sets in $(S_g \times \{0\}) \setminus \pi(\partial H)$. The algebraic intersection number provides a map $H_2(P, \partial H) \times H_1(P, B) \to \mathbb{Z}$. For x in $(S_g \times \{0\}) \cap B$ set

$$(3) \qquad\qquad p_{H,B}(x) := \hat{i}(H, x \times I).$$

For all $x \subset S_g \times 0 \setminus \pi(\partial H)$ there is a choice of B such that $x \subset B$. To see that the function $p_{H,B}$ does not depend on the choice of open set B, note that if $B \subset B' \subset (S_g \times 0) \setminus \pi(\partial H)$ then it follows from the naturality of the intersection pairing with respect to inclusion [6, Proposition 1.3.4] that the diagram below commutes:

$$
\begin{array}{ccc}
B & \hookrightarrow & B' \\
& \searrow_{p_{H,B}} & \downarrow^{p_{H,B'}} \\
& & \mathbb{Z}
\end{array}
$$

If $B_1 \subset B_2 \subset B_3 \subset \cdots$ and $D_1 \subset D_2 \subset \cdots$ are any two sequences of open subsets of $(S_g \times \{0\}) \setminus \pi(\partial H)$ with direct limit $(S_g \times \{0\}) \setminus \pi(\partial H)$, it follows that they both give rise to the same function p_H as the sequence $B_1 \cap D_1 \subset B_2 \cap D_2 \subset \cdots$, which also has direct limit $(S_g \times \{0\}) \setminus \pi(\partial H)$. Hence p_H is well defined on $(S_g \times \{0\}) \setminus \pi(\partial H)$, and is extended to an upper-semicontinuous function defined on all of $S_g \times \{0\}$. An example is shown in Figure 3.

The pre-image function is defined analogously for $S_{g,1}$.

Surfaces and curve graph paths Let c_1 and c_2 be curves in $S_{g,1} \times I$ or $S_g \times I$ isotopic to their projections to $S_{g,1} \times \{0\}$ or $S_g \times \{0\}$. These projections are assumed to be in general and minimal position and represent the vertices v_1 and v_2 respectively. Suppose π is the projection of $S_g \times I$ onto $S_g \times \{0\}$, or of $S_{g,1} \times I$ onto $S_{g,1} \times \{0\}$. In [9, Section 2.1] it is explained how to construct a surface in $S_g \times I$ with boundary $c_2 - c_1$ from an oriented path in a curve graph passing from vertex v_1 to vertex v_2. A simpler case of this is sketched below. Strictly speaking, the objects so obtained are often only cell complexes, but are homotopic to immersed surfaces with boundary.

Suppose a path in $C(S_g; h)$ has j edges labeled $i = 1, \ldots, j$, where $j > 1$. Informally, each edge of $C(S_g; h)$ represents an oriented subsurface F_i of S_g which maps into $S_g \times \{i/(j+1)\} \subset S_g \times I$. Pairs of homotopic curves with opposite orientations on the boundaries of F_i and F_{i+1} are connected up by gluing in cylinders. Similarly, the boundary of the surface is put into the boundary of the 3–manifold by gluing on cylinders. The same construction works for $S_{g,1} \times I$.
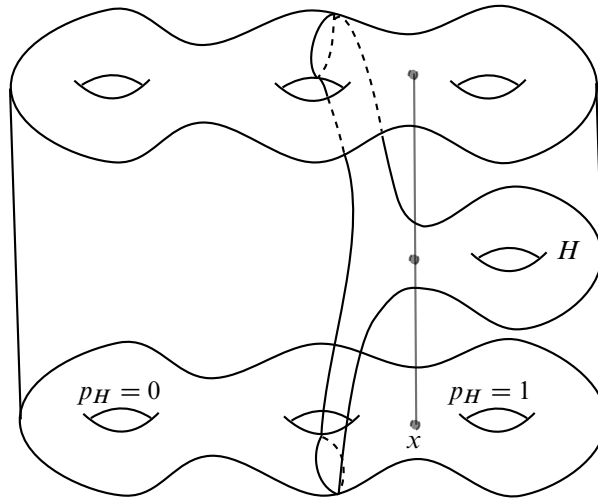
Figure 3: In this example, the pre-image function is equal to one on one side of a bounding pair, and zero elsewhere.

A surface constructed from the path γ will be called a *trace surface* of γ. The trace surface of an *oriented* path in $C(S_g; h)$ or $C(S_{g,1}; h)$ is unique up to homotopy.

Remark The vertices v_1 and v_2 are connected by paths of finite length. The correct intuition is that each edge of a given path represents a surface with boundary that contributes at most ± 1 to the pre-image function on any given subsurface. It follows that the pre-image function is finite, and each level set has a finite number of connected components. A simple means of calculating the pre-image function and hence verifying finiteness is given in [9, Section 4].

Euler integrals Throughout this paper we will want to compute a "signed genus", coming from a "signed Euler characteristic". The *signed genus* of the trace surface of γ is equal to the signed length of γ. To calculate signed Euler characteristic in such a way that the signs work out automatically, a very convenient notation makes use of integration with respect to Euler characteristic. The simple formulation given here is [1, Definition 3].

A collection $\mathcal{A}$ of subsets of a topological space X is said to be *tame* if $\mathcal{A}$ is closed with respect to the operations of finite intersection, finite union and complement, and all elements of $\mathcal{A}$ possess well-defined Euler characteristics.

An indicator function 1_U is the function that is equal to 1 on the set U and zero elsewhere.

Let $\mathcal{A}$ be a tame collection and $f = \sum \lambda_\alpha 1_{U_\alpha}$ a finite $\mathbb{Z}$–linear combination of indicator functions of elements $U_\alpha \in \mathcal{A}$. Then the integral of f is defined to be

$$\text{(4)} \qquad\qquad \int_X f \, d\chi := \sum_\alpha \lambda_\alpha \chi(U_\alpha).$$

Lemma 2 [16, Statement 1A]   *The integral in* (4) *is independent of the way in which f is written as a linear combination of indicator functions.*

Proof   In this paper we can give $S_{g,1}$ or S_g the structure of a simplicial complex, such that all the U_α can be chosen to be closed simplices of the simplicial complex. It then follows from a simple argument (see [1, Lemma 4]) that the integral is independent of the way f is written as a finite sum of indicator functions. These assumptions are, however, not necessary, and the lemma follows from the sheaf-theoretic approach in [14].   □

Remark   Pre-image functions were defined to be upper-semicontinuous because we will want to relate their Euler integrals to signed genus of trace surfaces. Although the boundary of a trace surface has Euler characteristic zero, its projection to $S_g \times \{0\}$ or $S_{g,1} \times \{0\}$ does not, so it is necessary to be careful what value to assign it. Upper semicontinuity ensures that a union of surfaces gives rise to a pre-image function that is the sum of the pre-image functions of the surfaces.

Lemma 3   *Let v_1 and v_2 be two vertices of $C(S_{g,1}; h)$. All paths in $C(S_{g,1}; h)$ connecting v_1 to v_2 have the same signed length. In $C(S_g; h)$, all paths connecting v_1 to v_2 have the same signed length* $\mathrm{mod}\,(g-1)$.

Proof   By [9, Lemma 10] any surface in $S_g \times I$ with boundary $c_2 - c_1$ has the same pre-image function up to an additive constant. The same argument applies to $S_{g,1} \times I$, although it is not explicitly stated.

We first prove the statement for surfaces with boundary, and then explain why surfaces without boundary are different. Let γ_1 and γ_2 be paths connecting v_1 to v_2 in $C(S_{g,1}; h)$, where H_1 is the trace surface of γ_1 and H_2 the trace surface of γ_2. As before, $\partial H_1 = \partial H_2 = c_2 - c_1$. In this context, c_2 is a curve in $S_{g,1} \times \{1\}$ and c_1 is a curve in $S_{g,1} \times \{0\}$. Both curves are identified with vertices in the curve graph by projecting isotopy classes of $S_{g,1} \times I$ onto isotopy classes of $S_{g,1} \times \{0\}$. By assumption, neither πc_2 nor c_1 is homotopic to $\partial S_{g,1} \times \{0\}$. From the definition of pre-image function, it follows that p_H must be zero on some neighborhood of $\partial S_{g,1}$. Since $p_{H_1} - p_{H_2}$ is a constant, it follows that $p_{H_1} = p_{H_2}$.

Recall that the surface H_1 is homotopic to a union of genus one subsurfaces F_i connected along boundary curves. Each of the F_i projects one to one onto an embedded, incompressible subsurface of $S_{g,1} \times 0$ of genus 1. In order to define the pre-image function of F_i for some i, strictly speaking the boundary of F_i should be put in the boundary of the 3–manifold. This detail will be ignored, because it is not hard to see that a homotopy taking F_i to F_i' can be found, such that $p_{F_i'}$ is one on the interior of the subsurface $\pi(F_i)$ of S_g or $S_{g,1}$ and zero on $S_g \setminus \pi(F_i)$ or $S_{g,1} \setminus \pi(F_i)$. The pre-image function p_H is the sum of the pre-image functions of the F_i, except along the curves where annuli are glued in to join the F_i together, over which p_H varies continuously.

The signed genus of F_i is obtained by

$$-\frac{1}{2} \int_{S_g} p_{F_i} \, d\chi.$$

Depending on the orientation of F_i, this will be plus or minus one. Since integrals preserve finite linear combinations, it follows that the signed length of γ_1 is equal to

(5)
$$-\frac{1}{2} \int_{S_{g,1}} p_{H_1} \, d\chi,$$

which is independent of the choice of γ_1 because p_{H_1} is.

The proof is the same for surfaces without boundary, except that paths can be chosen such that the pre-image functions of the trace surfaces differ by any integer. To see this, we show how to increase or decrease the pre-image function by one. Suppose v is a vertex on γ_1 corresponding to the curve c. Cut H_1 along the curve c, and glue in a copy of a surface homotopic in $S_g \times I$ to $(S_g \times \{\frac{1}{2}\}) \setminus (c \times \{\frac{1}{2}\})$. The orientation of the surface glued in will determine whether the pre-image function increases or decreases by one.

To see that we need to take $\mathrm{mod}\,(g-1)$ instead of $\mathrm{mod}\, g$, note that when S_g is cut into two pieces by a pair of disjoint, homologous curves, the genus of these two pieces sums to $g - 1$, not g, as illustrated in Figure 2. □

Remark The ambiguity in signed distance on $C(S_g; h)$ will be related to the observation that Chillingworth's winding numbers are only defined $\mathrm{mod}\,(2g-2)$ for S_g.

When defining signed stable length on $C(S_g, h)$, it is necessary to make sense of the limit. If the limit exists at all, is it independent of the choice of path $\mathrm{mod}\,(g-1)$? The next lemma will be used to resolve these problems.

Lemma 4 *Let v and w be any two vertices of $C(S_g, h)$, and τ an element of $\mathcal{T}_g$. Then for any $n \in \mathbb{N}$,*

$$d_s(w, \tau^n w) \equiv d_s(v, \tau^n v) \equiv n d_s(v, \tau v) \mod (g-1).$$

Similarly, when w and v are vertices of $C(S_{g,1}; h)$ and $\tau \in \mathcal{T}_{g,1}$,

$$d_s(w, \tau^n w) = d_s(v, \tau^n v) = n d_s(v, \tau v).$$

Proof From Lemma 3, any path connecting w to $\tau^n w$ can be used to calculate $d_s(w, \tau^n w)$. So suppose γ is a path connecting w to $\tau^n w$ passing through v and $\tau^n v$, in that order. Suppose also that the unoriented subpath of γ connecting $\tau^n v$ to $\tau^n w$ is the image under τ^n of the subpath connecting w to v. Reversing the orientation of a path changes the sign of its signed length, and the action of τ preserves signed distance, so the signed length of the subpath connecting w to v cancels out the signed length of the subpath connecting $\tau^n v$ to $\tau^n w$. It follows that $d_s(w, \tau^n w) \equiv d_s(v, \tau^n v)$.

To calculate $d_s(v, \tau^n v)$, we now suppose that γ is the union of oriented subpaths δ, $\tau\delta, \ldots, \tau^n \delta$, where δ connects v to τv. Since τ preserves signed distance, the second equality follows.

The proof is identical for $S_{g,1}$. $\qquad \square$

3.2 Signed stable length and cohomology

Choose an element τ from $\mathcal{T}_g$ or $\mathcal{T}_{g,1}$, and consider it fixed throughout this section. A similar argument to that in the previous section is used to show that the map from $H_1(S_{g,1}; \mathbb{Z})$ or $H_1(S_g; \mathbb{Z})$ into $\mathbb{Z}$ defined by ϕ is a homomorphism.

Lemma 5 *For fixed τ, the signed stable length ϕ defines an element of $H^1(S_{g,1}, \mathbb{Z})$ or $H^1(S_g, \mathbb{Z}/\langle g-1\rangle)$.*

Proof This lemma needs to show that ϕ is a homomorphism on homology. Let $([\alpha_1], \ldots, [\alpha_g], [\beta_1], \ldots, [\beta_g])$ be a set of generators for $H_1(S_{g,1}, \mathbb{Z})$ or $H_1(S_g, \mathbb{Z})$. A union of oriented curves consisting of a_1 curves in the homotopy class α_1, a_2 curves in the homotopy class α_2, etc will be denoted by $a_1\alpha_1 + a_2\alpha_2 + \cdots$, where a minus sign denotes reversed orientation. Suppose c is a curve representing the homology class $[c]$, where

$$[c] = \sum_{i=1}^{g} a_i[\alpha_i] + b_i[\beta_i].$$

For any n, a cell complex homotopic to an immersed surface in $S_g \times I$ or $S_{g,1} \times I$ with boundary $\tau c - c$ can be constructed by taking a surface H with boundary

$$\tau \left(\sum_{i=1}^{g} a_i \alpha_i + \sum_{i=1}^{g} b_i \beta_i \right) - \sum_{i=1}^{g} a_i \alpha_i - \sum_{i=1}^{g} b_i \beta_i$$

and attaching two surfaces F_1 and F_2 with boundaries $\tau c - \tau (\sum_{i=1}^{g} a_i \alpha_i + \sum_{i=1}^{g} b_i \beta_i)$ and $\sum_{i=1}^{g} a_i \alpha_i + \sum_{i=1}^{g} b_i \beta_i - c$ respectively.

Since ∂F_1 is $-\tau(\partial F_2)$, the contribution of F_1 and F_2 to the signed genus of a surface with boundary $\tau c - c$ cancel out, either absolutely or $\mod(g-1)$, depending on whether the surface is in $S_{g,1} \times I$ or $S_g \times I$. This implies that

$$\phi_{[c]}(\tau) = -\frac{1}{2} \int p_H d\chi.$$

For $S_{g,1} \times I$ it was shown in Lemma 3 that the pre-image function only depends on the boundary of the surface, so H can be chosen to be a union of surfaces $\{A_i\}$ and $\{B_i\}$, where A_i is a union of a_i copies of a surface with boundary $\tau \alpha_i - \alpha_i$ and B_i is a union of b_i copies of a surface with boundary $\tau \beta_i - \beta_i$. It follows that

$$p_H = \sum_{i=1}^{g} p_{A_i} + p_{B_i}$$

For $S_{g,1}$ the lemma follows from expression (5). Similarly for S_g. $\qquad\square$

3.3 Signed stable length is a homomorphism on Torelli

The proof that ϕ_h is a homomorphism on the Torelli group for any h is very similar to the proof of Lemma 4.

Lemma 6 *The map ϕ_h is a homomorphism.*

Proof Choose a primitive element h of $H_1(S_{g,1}, \mathbb{Z})$ and fix it for the duration of this proof. The quantity $\phi_h(\tau) = d_s(v, \tau v)$ is then a function of one variable τ, because it was shown in Lemma 4 that ϕ does not depend on the choice of vertex v in $C(S_{g,1}, h)$. Suppose that τ, τ_1 and $\tau_2 \in \mathcal{T}_{g,1}$, where $\tau = \tau_2 \tau_1$. By the path independence of signed distance,

$$d_s(c, \tau c) = d_s(c, \tau_2 \tau_1 c) = d_s(c, \tau_1 c) + d_s(\tau_1 c, \tau_2 \tau_1 c).$$

Since the action of the Torelli group preserves signed distances, it follows that

$$d_s(c, \tau_1 c) + d_s(\tau_1 c, \tau_2 \tau_1 c) = d_s(c, \tau_1 c) + d_s(c, \tau_2 c) = \phi_h(\tau_1) + \phi_h(\tau_2).$$

The proof is identical for τ_1 and τ_2 in $\mathcal{T}_g$. $\qquad\square$

3.4 Signed stable length is zero on the Johnson kernel

Recall that the Johnson kernel is the kernel of the Johnson homomorphism, and is generated by Dehn twists around separating curves [12]. Once we have evaluated signed stable length on bounding pair maps, it would be possible to use the lantern relation to write a Dehn twist around a separating curve as a product of bounding pair maps, and show that the signed distance between any two vertices of $C(S_{g,1}; h)$ or $C(S_g; h)$ in the same orbit of the Johnson kernel is zero. However it is also not hard to show this directly from the definition, which is what will be done in this subsection. The basic idea is that a separating curve s has zero algebraic intersection number with any other curve, so Dehn twisting around s essentially adds as many copies of s as it does $-s$.

Let $i(a, b)$ denote the geometric intersection number of the curve a with the curve b, ie the minimum possible number of points of intersection between two curves, one of which is isotopic to a and the other to b.

Let c be a curve representing a vertex in $C(S_{g,1}; h)$ or $C(S_g; h)$. Up to homotopy, a curve $T_s(c)$ can be obtained by taking the union of c with $i(c, s)/2$ copies of the curve s and $i(c, s)/2$ copies of $-s$, and performing 1–surgeries to resolve the points of intersection, as illustrated in Figure 4. The null-homologous union of curves

$$T_s c - c - \frac{i(c, s)}{2} s + \frac{i(c, s)}{2} s$$

is the boundary of an incompressible surface F in $S_{g,1} \times I$ or $S_g \times I$. The support of p_F is contained within the union of tubular neighborhoods of c and tubular neighborhoods of $i(c, s)$ disjoint curves freely homotopic to s. Since the union of tubular neighborhoods has genus zero, so does F. Since

$$-\frac{i(c, s)}{2} s + \frac{i(c, s)}{2} s$$

is the boundary of a surface with zero signed genus, the same is true for a surface with boundary $T_s c - c$.

3.5 Evaluation on the generators

Now that ϕ has been shown to be a homomorphism both on the Torelli group (Lemma 6) and on $H_1(S_{g,1}, \mathbb{Z})$ and $H_1(S_g, \mathbb{Z}/\langle g - 1 \rangle)$ (Lemma 5), all that remains to prove Theorem 1 is to verify it on a choice of generating sets.

The Torelli group is generated by bounding pair maps. Let (a, b) be a bounding pair. Choose curves representing a basis B for $H_1(S_{g,1}, \mathbb{Z})$, so that only one of them, α,
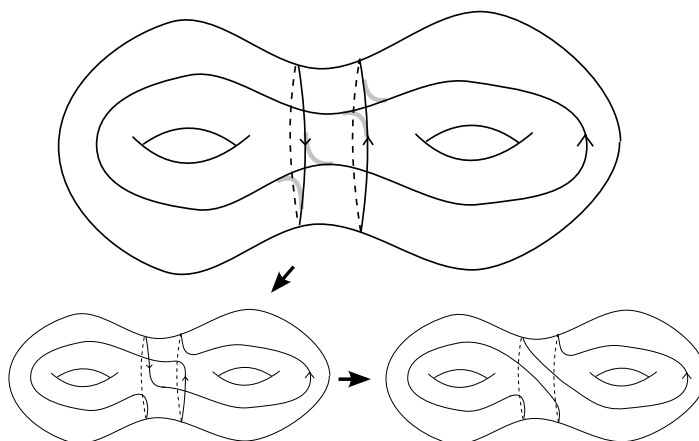
Figure 4: The surgeries to resolve the points of intersection are shown in grey in the top figure. The horizontal arrow denotes a homotopy.

intersects a and b, and $i(a, \alpha) = i(b, \alpha) = 1$. Clearly, $d_s(\beta, (T_a T_b^{-1})\beta) = 0$ whenever $[\beta]$ is any generator other than $[\alpha]$. The same argument as in Section 3.4 shows that $d_s(\alpha, (T_a T_b^{-1})\alpha)$ is equal to the signed genus of the surface with boundary $a - b$. So $\phi_{[\alpha]}(T_a T_b^{-1})$ is equal to the signed genus of the subsurface bounded by $a - b$ and $\phi(T_a T_b^{-1}, [\beta]) = 0$. This is consistent with half the Chillingworth class evaluated on this homology basis, as computed in [10, Corollary 1].

For a surface without boundary the calculation is identical. Let B be a homology basis chosen as in the previous paragraph. It follows from [10, Section 6] that this is again equal to half of the Chillingworth class evaluated on the basis B.

This concludes the proof of Theorem 1. □

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