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Resolving rational cohomological dimension via a Cantor group action

MICHAEL LEVIN

By a Cantor group we mean a topological group homeomorphic to the Cantor set. We show that a compact metric space of rational cohomological dimension n can be obtained as the orbit space of a Cantor group action on a metric compact space of covering dimension n. Moreover, the action can be assumed to be free if $n=1$.

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1 Introduction

Throughout this paper we assume that maps are continuous and spaces are separable metrizable. We recall that a compactum means a compact metric space. By the dimension $\dim X$ of a space X we mean the covering dimension.

Let G be an abelian group. The cohomological dimension $\dim_G X$ of a space X is the smallest integer n such that the Čech cohomology $H^{n+1}(X, A; G)$ vanishes for every closed subset A of X. Clearly $\dim_G X \le \dim X$ for every abelian group G. Exploring connections between cohomological and covering dimensions is one of the central topics in Dimension Theory. Let us mention a few important results in this direction. By the classical result of Alexandroff $\dim X = \dim_{\mathbb{Z}} X$ if X is finite-dimensional. Solving a long-standing open problem, Dranishnikov [2] constructed an infinite-dimensional compactum X with $\dim_{\mathbb{Z}} X = 3$. Edwards' famous cell-like resolution theorem (see Edwards [10] and Walsh [16]) asserts that every compactum X with $\dim_{\mathbb{Z}} X = n$ is the image of an n–dimensional compactum under a cell-like map. A map is cell-like if its fibers are cell-like compacta and a compactum is cell-like if any map from it to a CW complex is null-homotopic. Edwards' theorem was extended in Levin [14] to rational cohomological dimension: every compactum X with $\dim_{\mathbb{Q}} X = n$, $n \ge 2$, is the image of an n–dimensional compactum under a rationally acyclic map. A map is rationally acyclic if its fibers are rationally acyclic compacta and a compactum is rationally acyclic if its reduced Čech cohomology with rational coefficients vanishes. Note that by the Begle–Vietoris theorem a cell-like map and a rationally acyclic map cannot raise the integral and the rational cohomological dimensions respectively. This

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paper is devoted to establishing another connection between rational cohomological and covering dimensions.

Theorem 1.1 *Let* X *be a compactum with* $\dim_{\mathbb{Q}} X = n$. *Then there is an* n*–dimensional compactum* Z *and an action of a Cantor group* Γ *on* Z *such that* $X = Z/\Gamma$. *Moreover, the action of* Γ *can be assumed to be free if* $n = 1$.

By a Cantor group we mean a topological group homeomorphic to the Cantor set. Since a Cantor group is a pro-finite group one can easily derive from basic properties of transformation groups (see Bredon [1, Chapter III, Theorem 7.2]) that for an action of a Cantor group Γ on a compactum Z we have $\dim_{\mathbb{Q}} Z/\Gamma \le \dim_{\mathbb{Q}} Z$ and hence $\dim_{\mathbb{Q}} Z/\Gamma \le \dim Z$. On the other hand Dranishnikov and Uspenskij proved:

Theorem 1.2 [7] *Let* $f\colon Z \to X$ *be a* 0*–dimensional map of compacta* Z *and* X. *Then* $\dim_G Z \le \dim_G X$ *for every abelian group* G.

Thus for an action of a Cantor group Γ on a compactum Z we have that $\dim_{\mathbb{Q}} Z/\Gamma = \dim_{\mathbb{Q}} Z$ and hence Theorem 1.1 provides a characterization of rational cohomological dimension in terms of Cantor group actions.

Let us also note that, in general, the action of Γ in Theorem 1.1 cannot be free for $n > 1$. Indeed, consider any compactum Y with $\dim Y > \dim_{\mathbb{Q}} Y = n-1$ and let X be the cone over Y. Then $\dim X = \dim Y + 1 > \dim_{\mathbb{Q}} Y + 1 = \dim_{\mathbb{Q}} X = n$. Now assume that X is the orbit space of a free action of a Cantor group Γ on a compactum Z. Since X is contractible, one can easily observe that $Z = X \times \Gamma$ and hence $\dim Z = \dim X > \dim_{\mathbb{Q}} X = n$.

A result closely related to Theorem 1.1 can be derived from the work of Dranishnikov and West [8]. For a prime p we denote by $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ the p–cyclic group and by $\mathbb{Z}_p^{\mathbb{N}} = \prod_{i=1}^{\infty} (\mathbb{Z}_p)_i$ the product of countably many copies of $\mathbb{Z}_p$.

Theorem 1.3 [8] *Let* X *be a compactum. Then for every prime* p *there is a compactum* Y *and an action of the group* $\Gamma = \mathbb{Z}_p^{\mathbb{N}}$ *on* Y *such that* $\dim_{\mathbb{Z}_p} Y \le 1$ *and* $X = Y/\Gamma$.

Consider any compactum X. By Theorem 1.3 for every prime p there is a compactum Y_p and an action of $\Gamma_p = \mathbb{Z}_p^{\mathbb{N}}$ on Y_p such that $\dim_{\mathbb{Z}_p} Y_p \le 1$ and $X = Y_p/\Gamma_p$. Let $\gamma_p\colon Y_p \to X$ be the projection, $\mathcal{P}$ the set of prime numbers and $Y = \prod_{p\in\mathcal{P}} Y_p$. Denote by Z the pull-back of the maps γ_p, $p \in \mathcal{P}$, which is the subset of Y consisting of the points $y = (y_p) \in Y$ such that $\gamma_p(y_p) = \gamma_q(y_q)$ for every $p, q \in \mathcal{P}$. Consider the group $\Gamma = \prod_{p \in \mathcal{P}} \mathbb{Z}_p^{\mathbb{N}}$ and the pull-back action of Γ on Z defined by $gy = (g_p y_p)$ for $g = (g_p) \in \Gamma$ and $y = (y_p) \in Z$, and notice that $X = Z/\Gamma$. Since the orbits of Γ_p on Y_p are 0–dimensional, we get that the orbits of Γ on Z are 0–dimensional and the projection of Z to Y_p is 0–dimensional for every p. By Theorem 1.2 we get that $\dim_{\mathbb{Q}} Z \le \dim_{\mathbb{Q}} X$ and $\dim_{\mathbb{Z}_p} Z \le 1$ for every $p \in \mathcal{P}$. Then, by Bockstein's inequalities (see Kuz'minov [12] and Dranishnikov [5]), $\dim_{\mathbb{Z}} Z \le \max\{\dim_{\mathbb{Q}} X, 2\}$ and we obtain

Theorem 1.4 (Derived from [8]) *Let X be a compactum with $\dim_{\mathbb{Q}} X = n$. Then there is a compactum Z and an action of the group $\Gamma = \prod_{p \in \mathcal{P}} \mathbb{Z}_p^{\mathbb{N}}$ on Z such that $\dim_{\mathbb{Z}} Z \le \max\{n, 2\}$ and $X = Z/\Gamma$.*

Theorem 1.4 motivates the following problem.

Problem 1.5 Can the group Γ in Theorem 1.1 be assumed to be abelian or even the product of countably many finite cyclic groups?

Let us finally mention that the interest in Cantor group actions is inspired by the Hilbert–Smith conjecture which claims that no Cantor group acts effectively on a manifold. Smith [15] reduced the conjecture to the actions of the groups of p–adic integers A_p, $p \in \mathcal{P}$, and Yang [17] showed that if A_p acts effectively on a manifold M then $\dim_{\mathbb{Z}} M/A_p = \dim M + 2$ and if A_p acts on a finite-dimensional compactum Z then $\dim_{\mathbb{Z}} Z/A_p \le \dim Z + 3$. Theorem 1.1 shows that the dimensional restrictions imposed by A_p do not apply to general Cantor groups even in the following extreme form: There is a free action of a Cantor group on a one-dimensional compactum raising the integral cohomological dimension of the orbit space to infinity. Indeed, take an infinite-dimensional compactum X with $\dim_{\mathbb{Q}} X = 1$ and $\dim_{\mathbb{Z}} X = \infty$. Then Theorem 1.1 produces a one-dimensional compactum Z and a free action of a Cantor group Γ on Z such that $X = Z/\Gamma$. This example can be considered as a complement to the example of Dranishnikov and West in [8] of an action of $\mathbb{Z}_p^{\mathbb{N}}$ on a 2–dimensional compactum raising the dimension of the orbit space to infinity.

2 Preliminaries

Let us recall basic definitions and results in extension theory and cohomological dimension that will be used in the proof of Theorem 1.1.

The extension dimension of a space X is said to be dominated by a CW complex K, written $\text{e-dim}\, X \le K$, if every map $f\colon A \to K$ from a closed subset A of X

extends over X. Note that the property e-dim $X \le K$ depends only on the homotopy type of K. The covering and cohomological dimensions can be characterized by the following extension properties: $\dim X \le n$ if and only if the extension dimension of X is dominated by the n–dimensional sphere S^n, and $\dim_G X \le n$ if and only if the extension dimension of X is dominated by the Eilenberg–MacLane complex $K(G,n)$. The extension dimension shares many properties of covering dimension. For example: if e-dim $X \le K$ then for every $A \subset X$ we have e-dim $A \le K$, and if X is a countable union of closed subsets whose extension dimension is dominated by K then e-dim $X \le K$. Let us list a few more properties.

Theorem 2.1 [9] *Let X be a space and let K and L be CW complexes. If $X = A \cup B$ is the union of subspaces A and B such that* e-dim $A \le K$ *and* e-dim $B \le L$ *then the extension dimension of X is dominated by the join $K * L$.*

Theorem 2.2 [4] *Let K and L be countable CW complexes and X a compactum such that* e-dim $X \le K * L$. *Then X decomposes into subspaces $X = A \cup B$ such that* e-dim $A \le K$ *and* e-dim $B \le L$.

Proposition 2.3 [13] *Let X be a compactum and K a simply connected CW complex such that K has only finitely many non-trivial homotopy groups and $\dim_{\pi_i(K)} X \le n$ for every $i > 0$. Then* e-dim $X \le K$.

Theorem 2.4 [3] *Let X be a compactum and K a CW complex such that* e-dim $X \le K$. *Then $\dim_{H_n(K)} X \le n$ for every $n > 0$.*

By a Moore space $M(\mathbb{Q}, n)$ we will mean the model which is the infinite telescope of a sequence of maps from S^n to S^n of all possible non-zero degrees. Note that $M(\mathbb{Q}, 1) = K(\mathbb{Q}, 1)$.

Proposition 2.5 *Let X be a compactum. Then $\dim_{\mathbb{Q}} X \le n$ if and only if* e-dim $X \le M(\mathbb{Q}, n)$.

Proof By Theorem 2.4 the condition e-dim $X \le M(\mathbb{Q}, n)$ implies $\dim_{\mathbb{Q}} X \le n$. Let us show that $\dim_{\mathbb{Q}} X \le n$ implies e-dim $X \le M(\mathbb{Q}, n)$. The case $n = 1$ is obvious since $M(\mathbb{Q}, 1) = K(\mathbb{Q}, 1)$. Assume that $n > 1$. Then $M(\mathbb{Q}, n)$ is simply connected and, since $H_i(M(\mathbb{Q}, n)) = 0$ for $i > 0, i \neq n$, and $H_n(M(\mathbb{Q}, n)) = \mathbb{Q}$, we have $\pi_i(M(\mathbb{Q}, n)) = 0$ for $0 < i < n$ and $\pi_i(M(\mathbb{Q}, n)) = \pi_i(M(\mathbb{Q}, n)) \otimes \mathbb{Q}$ for $i \ge n$ [11, Theorem 9.3]. Thus, by Bockstein's theorem [5; 12], $\dim_{\pi_i(M(\mathbb{Q},n))} X \le \dim_{\mathbb{Q}} X \le n$ for every $i \ge n$. Note that $M(\mathbb{Q}, n)$ is the direct limit of its finite subtelescopes which are homotopy equivalent to S^n. Then, since $\pi_i(S^n)$ is finite for $i \ge 2n$, we have that $\pi_i(M(\mathbb{Q}, n))$ is torsion for $i \ge 2n$ and hence $\pi_i(M(\mathbb{Q}, n)) = \pi_i(M(\mathbb{Q}, n)) \otimes \mathbb{Q} = 0$ for $i \ge 2n$. Thus, by Proposition 2.3, e-dim $X \le M(\mathbb{Q}, n)$. □

Corollary 2.6 *Let X be a compactum. Then $\dim_{\mathbb{Q}} X \le n$, $n > 1$ if and only if X decomposes into $X = A \cup B$ such that $\dim_{\mathbb{Q}} A \le 1$ and $\dim B \le n-2$.*

Proof Note that $M(\mathbb{Q}, n)$ is homotopy equivalent to $\Sigma^{n-1} M(\mathbb{Q}, 1) = S^{n-2} * M(\mathbb{Q}, 1)$ and the corollary follows from Proposition 2.5 and Theorems 2.1 and 2.2. □

Proposition 2.7 *Let X be a compactum. Then $\dim_{\mathbb{Q}} X \le n$ if and only if for every map $f\colon A \to S^n$ from a closed subset A of X to a sphere S^n we have that f followed by a map of non-zero degree from S^n to S^n extends over X.*

Proof The proof follows from Proposition 2.5 and an easy adjustment of the proof of [6, Proposition 3.13]. □

By a partial map from a compactum X to a CW complex K we mean a map $f\colon F \to K$ from a closed subset F of X to K. A collection $\mathcal{F}$ of partial maps from X to K is said to be representative if for every partial map $f'\colon F' \to K$ there is a map $f\colon F \to K$ in $\mathcal{F}$ such that $F' \subset F$ and f restricted to F' is homotopic to f'. Let X be the inverse limit of a sequence of compacta X_n with bonding maps $\omega_{n+1}\colon X_{n+1} \to X_n$. We say that a partial map $f\colon F \to K$ from X_i to K is extendable in the inverse system if there is $j > i$ such that for the map $\omega^i_j = \omega_{i+1} \circ \omega_{i+2} \circ \cdots \circ \omega_j\colon X_j \to X_i$ we have that ω^i_j restricted to $(\omega^i_j)^{-1}(F)$ and followed by f extends over X_j. The proof of the following proposition is simple and left to the reader.

Proposition 2.8 *Let X be a compactum and K a CW complex.*

(1) *If K is a countable CW complex then there is a countable representative collection of partial maps from X to K.*

(2) *Let X be the inverse limit of compacta X_n and let $\mathcal{F}_n$ be a representative collection of partial maps from X_n to K such that for every n and every f in $\mathcal{F}_n$ we have that f is extendable in the inverse system. Then $\text{e-dim}\, X \le K$.*

3 Proof of Theorem 1.1

Proposition 3.1 *Let (K, L) be a pair of finite simplicial complexes with K being connected, $\psi\colon S^1 \to S^1$ a covering map, and $\phi\colon L \to S^1$ a map such that ϕ followed by ψ extends to a map $g\colon K \to S^1$. Consider a component M of the pull-back of the maps g and ψ, and the projection $\beta_M\colon M \to K$. Then β_M restricted to $\beta_M^{-1}(L)$ and followed by ϕ extends over M.*

Proof Clearly β_M is a covering map. Let $m = \deg \psi$ and assume that $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ acts freely on S^1 so that ψ is the projection to the orbit space $S^1 = S^1/\mathbb{Z}_m$. Then the action of $\mathbb{Z}_m$ on S^1 induces the corresponding free action of $\mathbb{Z}_m$ on the pull-back Z of the maps g and ψ.

Consider a component L' of L and let N' be a component of $\beta_M^{-1}(L')$. Clearly β_M restricted to N' and L' is a covering map. Let us show that β_M is one-to-one on N'. Take a map $\alpha\colon S^1 \to L'$. Note that $\phi \circ \alpha$ is a lifting of $\psi \circ \phi \circ \alpha$ via ψ and hence the map $(\alpha, \phi \circ \alpha)\colon S^1 \to Z \subset K \times S^1$ is a lifting of α via the projection of Z to K. Then $(\alpha, \phi \circ \alpha)$ followed by (the action of) an element of $\mathbb{Z}_m$ provides a lifting of α to N' and hence β_M is one-to-one on N'.

Let $\rho\colon M \to S^1$ be the projection to S^1. Note that ρ restricted to N' is a lifting (via the map ψ) of β_M restricted to N' and followed by $\psi \circ \phi$. Then the maps $\rho|N'$ and $\phi \circ (\beta_M|N')$ coincide up to the action of $\mathbb{Z}_m$ on S^1 and hence they are homotopic. Thus ρ extends up to homotopy the map β_M restricted to N' and followed by ϕ, and hence β_M restricted to $\beta_M^{-1}(L)$ and followed by ϕ extends over M. The proposition is proved. □

Suppose that X, Y and Y' are compacta, and Γ and Γ' are finite groups acting on compacta Y and Y' respectively, such that $Y/\Gamma = X = Y'/\Gamma'$. Let $\gamma\colon Y \to X = Y/\Gamma$ and $\gamma'\colon Y' \to X = Y'/\Gamma'$ be the projections and let $\alpha\colon \Gamma' \to \Gamma$ be an epimomorphism and $\beta\colon Y' \to Y$ a surjective map. We say that the actions of Γ and Γ' agree with α, β if $\gamma' = \gamma \circ \beta$ and $\beta(gy) = \alpha(g)\beta(y)$ for every $g \in \Gamma'$ and $y \in Y'$. Given a closed subset Z of Y and a map $f\colon F \to S^1$ from a closed subset F of Z we say that β resolves f over Z if β restricted to $\beta^{-1}(F)$ and followed by f extends over $\beta^{-1}(Z)$.

Proposition 3.2 *Suppose that X and Y are compacta, A is a closed connected subset of X with $\dim_{\mathbb{Q}} A \le 1$, and Γ is a finite group acting on Y with $X = Y/\Gamma$ such that:*

- *$\gamma^{-1}(A)$ is connected and Γ acts freely on $\gamma^{-1}(A)$, where $\gamma\colon Y \to X = Y/\Gamma$ is the projection.*

Then for every map $f\colon F \to S^1$ from a closed subset F of $\gamma^{-1}(A)$ to a circle S^1 there are a compactum Y', a finite group Γ' acting on Y' with $X = Y'/\Gamma'$, an epimorphism $\alpha\colon \Gamma' \to \Gamma$, and a surjective map $\beta\colon Y' \to Y$ such that:

- *$\gamma'^{-1}(A)$ is connected and Γ' acts freely on $\gamma'^{-1}(A)$, where $\gamma'\colon Y' \to X = Y'/\Gamma'$ is the projection.*
- *The actions of Γ and Γ' agree with α and β.*
- *β resolves f over $\gamma^{-1}(A)$.*

Proof Clearly $\dim_{\mathbb{Q}} \gamma^{-1}(A) \le 1$ and hence, by Proposition 2.7, there is a covering map $\psi\colon S^1 \to S^1$ such that $\psi \circ f$ extends over $\gamma^{-1}(A)$. Recall that Γ is finite, the action of Γ is free on $\gamma^{-1}(A)$, and $\gamma^{-1}(A)$ is compact and connected. Then one can approximate the action of Γ over a closed neighborhood B of A through a free action of Γ on a finite connected simplicial complex K and a surjective map $\mu_Z\colon Z = \gamma^{-1}(B) \to K$ such that μ_Z commutes with the actions of Γ. Taking B sufficiently close to A we may assume that $\psi \circ f$ extends over Z, and (the diameters of) the fibers of μ_Z are as small as we wish. Then, we may also assume that the map f factors up to homotopy through μ_Z restricted to F and a map $\phi\colon L \to S^1$ from a subcomplex L of K such that $\mu_Z(F) \subset L$ and ϕ followed by ψ extends to a map $g\colon K \to S^1$. Let $\mu\colon B = Z/\Gamma \to K/\Gamma$ be the map induced by μ_Z and let $\gamma_K\colon K \to K/\Gamma$ be the covering projection.

In this proof we always consider a covering map as a pointed map and identify the fundamental group of the covering space with the subgroup of the fundamental group of the base space obtained under the induced monomorphism of the fundamental groups. Thus $\pi_1(K)$ is a normal subgroup of $\pi_1(K/\Gamma)$ with $\pi_1(K/\Gamma)/\pi_1(K) = \Gamma$. Let M be a connected component of the pull-back of g and ψ and let $\beta_M\colon M \to K$ be the projection. Clearly β_M is a covering map. Then $\pi_1(M)$ is a subgroup of $\pi_1(K)$ of finite index and hence there is a normal subgroup G of $\pi_1(K/\Gamma)$ of finite index such that G is contained in $\pi_1(M)$. Consider covering maps $\gamma'_K\colon K' \to K/\Gamma$ and $\beta_K\colon K' \to K$ from a covering space K' with $\pi_1(K') = G$. Then β_K factors through β_M and $\Gamma' = \pi_1(K/\Gamma)/G$ acts on K' so that $K'/\Gamma' = K/\Gamma$ and for the induced epimorphism $\alpha\colon \Gamma' \to \Gamma$ we have that the actions of Γ and Γ' on K and K' respectively agree with β_K and α. Denote by Z' the pull-back of γ'_K and μ, consider the pull-back action of Γ' on Z', and let $\gamma'_Z\colon Z' \to B = Z/\Gamma = Z'/\Gamma'$ and $\beta_Z\colon Z' \to Z$ be the projections induced by γ'_K and β_K respectively.

Take an open neighborhood V of A in X such that $V \subset B$, and write $U = \gamma^{-1}(V)$ and $U' = \gamma_Z'^{-1}(V) = \beta_Z^{-1}(U)$. Set Y' to be the disjoint union of $Y \setminus U$ and U' and $\beta\colon Y' \to Y$ to be the function defined by the identity map on $Y \setminus U$ and by the map β_Z on U'. Turn Y' into a compactum by preserving the topologies of $Y \setminus U$ and U' and declaring the function β to be the quotient map. Extend the action of Γ' on U' over Y' by setting $gy = \alpha(g)y$ for $y \in Y' \setminus U' = Y \setminus U$ and $g \in \Gamma'$. It is easy to see that the action of Γ' is well-defined, $X = Y'/\Gamma'$, and the actions of Γ and Γ' on Y and Y' respectively agree with α and β.

By Proposition 3.1 the map β_M restricted to $\beta_M^{-1}(L)$ and followed by ϕ extends over M. Recall that β_K factors through β_M and hence β_K restricted to $\beta_K^{-1}(L)$ and followed by ϕ extends over K'. Then β_Z restricted to $\beta_Z^{-1}(F)$ and followed by f

extends over Z' and hence β restricted to $\beta^{-1}(F)$ and followed by f extends over $\beta^{-1}(\gamma^{-1}(A))$; that is, β resolves f over $\gamma^{-1}(A)$.

The only conclusion of the proposition that is not obtained yet is the connectedness of $\gamma'^{-1}(A)$. This can be achieved as follows. Denote $A_Y = \gamma^{-1}(A)$ and $A'_Y = \gamma'^{-1}(A)$. Take a closed neighborhood W_X of A such that the actions of Γ and Γ' are free on $W_Y = \gamma^{-1}(W_X)$ and $W'_Y = \gamma'^{-1}(W_X)$ respectively. Then $\ker\alpha$ acts freely on W'_Y so that $W_Y = W'_Y/\ker\alpha$ and β restricted to W'_Y and W_Y is the projection to the orbit space. Thus both β restricted to W'_Y and W_Y and β restricted to A'_Y and A_Y are open maps. Then for every non-empty clopen (in the subspace topology) set $\Omega \subset A'_Y$ we have that $\beta(\Omega)$ is clopen in A_Y and, since A_Y is connected, $\beta(\Omega) = A_Y$. Fix a component C of A'_Y. Then the fact that a component of a compactum is the intersection of clopen sets implies that $\beta(C) = A_Y$. Consider the global stabilizer $\Gamma'_C = \{g \in \Gamma' : gC = C\}$ of C. It is easy to see that the facts that C is a component of A'_Y and $\beta(C) = A_Y$ imply that $\alpha(\Gamma'_C) = \Gamma$. Also note that, since $\beta(C) = A_Y$ and Γ' is finite, there are only finitely many components of A'_Y. Take an open neighborhood U'_C of C in W'_Y so that U'_C is invariant under Γ'_C, and U'_C is so close to C that U'_C is also open in Y' and gU'_C does not intersect U'_C if $g \in \Gamma \setminus \Gamma'_C$. Then $U_C = \beta(U'_C)$ is open in Y and U'_C is closed in $\beta^{-1}(U_C)$.

Define the compactum Y'_C to be the quotient space of $\beta^{-1}(Y \setminus U_C) \cup U'_C$ by identifying the points of $\beta^{-1}(Y \setminus U_C)$ with $Y \setminus U_C$ according to the map β. Extend the action of Γ'_C on U'_C to Y'_C by setting $gy = \alpha(g)y$ for $y \in Y'_C \setminus U'_C = Y \setminus U_C$ and $g \in \Gamma'_C$. Let $\beta_C \colon Y'_C \to Y$ be the map induced by β and let α_C be the restriction of α to Γ'_C. Now we replace Y', Γ', β and α by Y'_C, Γ'_C, β_C and α_C respectively and get that $\gamma'^{-1}(A) = C$ is connected. One can easily verify that the other properties required in the proposition are preserved under this replacement. □

Proposition 3.3 *Let X be a compactum and A a closed subset of X with $\dim_{\mathbb{Q}} A \le 1$. Then there is a compactum Y and an action of a Cantor group Γ on Y such that $X = Y/\Gamma$ and for the projection $\gamma \colon Y \to X$ we have that $\dim \gamma^{-1}(A) \le 1$ and the action of Γ is free on $\gamma^{-1}(A)$.*

Proof Note that without loss of generality one can replace X by any larger compactum and A by any larger closed subset of $\dim_{\mathbb{Q}} \le 1$. Also note that by adding to A a set of $\dim \le 1$ we still preserve $\dim_{\mathbb{Q}} \le 1$. Take a Cantor set C embedded into an interval I and a surjective map $\phi \colon C \to A$, and consider the compactum obtained by attaching to I the mapping cylinder of ϕ. Then attaching this compactum to X through the set A we enlarge A to a connected set by adding a subset of $\dim = 1$ and hence we may assume that A is connected.

Set $Y_1 = X$ and $\Gamma_1 = \{1\}$. We will construct for every n a compactum Y_n and a finite group Γ_n acting on Y_n such that $X = Y_n/\Gamma_n$, $\gamma_n^{-1}(A)$ is connected and Γ_n freely acts on $\gamma_n^{-1}(A)$ where $\gamma_n\colon Y_n \to X = Y_n/\Gamma_n$ is the projection. The construction is carried out by induction in the following manner. We take a map (that will be specified later) $f_n\colon F_n \to S^1$ from a closed subset F_n of $\gamma_n^{-1}(A)$, and apply Proposition 3.2 to construct Y_{n+1}, Γ_{n+1}, an epimorphism $\alpha_{n+1}\colon \Gamma_{n+1} \to \Gamma_n$, and a surjective map $\beta_{n+1}\colon Y_{n+1} \to Y_n$ such that the actions of Γ_n and Γ_{n+1} agree with α_{n+1} and β_{n+1}, and β_{n+1} resolves f_n over $\gamma_n^{-1}(A)$.

Denote $Y = \varprojlim(Y_n, \beta_n)$ and $\Gamma = \varprojlim(\Gamma_n, \alpha_n)$. Clearly Γ is a compact 0–dimensional group, Γ acts on Y such that $X = Y/\Gamma$, and Γ acts freely on $\gamma^{-1}(A)$, where $\gamma\colon Y \to X = Y/\Gamma$ is the projection. Let us show that at each step of the construction the map f_n can be chosen in a way that guarantees that $\dim \gamma^{-1}(A) \le 1$.

Once Y_i is constructed, choose, by (1) of Proposition 2.8, a countable representative collection $\mathcal{F}_i$ of partial maps from $A_i = \gamma_i^{-1}(A)$ to S^1 and fix a surjection $\tau_i\colon \mathbb{N} \to \mathcal{F}_i$. Take any bijection $\tau\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ such that for every n and $\tau(n) = (i, j)$ we have $i \le n$. Consider the inductive step of the construction from n to $n+1$ and take $f = \tau_i(j) \in \mathcal{F}_i$ with $(i, j) = \tau(n)$. Recall that $i \le n$ and hence the collection $\mathcal{F}_i$ is already defined. Denote $\beta_n^i = \beta_{i+1} \circ \cdots \circ \beta_n\colon Y_n \to Y_i$ for $i < n$, $\beta_n^n = \mathrm{id}\colon Y_n \to Y_n$, and $F_n = (\beta_n^i)^{-1}(F)$, where F is the domain of f, and let $f_n = f \circ \beta_n^i | F_n\colon F_n \to S^1$ be the map we use at the inductive step of the construction. Then f_n is extendable in the inverse system $\gamma^{-1}(A) = \varprojlim(A_n, \beta_n | A_n)$ and hence f is extendable in this inverse system as well. Thus, by (2) of Proposition 2.8, $\text{e-dim}\, \gamma^{-1}(A) \le S^1$ and hence $\dim \gamma^{-1}(A) \le 1$.

If Γ is a finite group then replacing both Y and Γ by the products $Y \times C$ and $\Gamma \times C$ with any Cantor group C we may assume that Γ is a Cantor group. □

Proof of Theorem 1.1 The case $n = 0$ is trivial. The case $n = 1$ follows from Proposition 3.3. Assume that $n \ge 2$. By Corollary 2.6 decompose X into $X = A \cup B$ so that $\dim_{\mathbb{Q}} A \le 1$ and $\dim B \le n-2$. Enlarging B to a G_δ–subset with the same covering dimension and replacing A by the smaller set $X \setminus B$ we may assume that A is σ–compact. Represent A as the union $A = \bigcup_{i=1}^{\infty} A_i$ of compacta A_i. By Proposition 3.3 there is a compactum Y_i and a Cantor group Γ_i acting on Y_i such that $X = Y_i/\Gamma_i$ and for the projection $\gamma_i\colon Y_i \to X$ we have that $\dim \gamma_i^{-1}(A_i) \le 1$. Let $Y = \prod_{i=1}^{\infty} Y_i$ and let $Z \subset Y$ be the pull-back of the maps γ_i. Then $\Gamma = \prod_{i=1}^{\infty} \Gamma_i$ is a Cantor group and for the pull-back action of Γ on Z we have that $X = Z/\Gamma$. Let $\gamma\colon Z \to X$ be the projection. Note that γ is a 0–dimensional map. Also note that for every i the map γ factors through the 0–dimensional projection $p_i\colon Z \to Y_i$ and the map $\gamma_i\colon Y_i \to X$. Thus, by Hurewicz's theorem, we have that $\dim \gamma^{-1}(B) \le \dim B \le n-2$.

Since $\gamma^{-1}(A_i)=p_i^{-1}(\gamma_i^{-1}(A_i))$, we also have $\dim\gamma^{-1}(A_i)\le\dim\gamma_i^{-1}(A_i)\le 1$ and hence $\dim\gamma^{-1}(A)\le 1$. Thus $\dim Z\le\dim\gamma^{-1}(A)+\dim\gamma^{-1}(B)+1=n$. Recall that $\dim_{\mathbb{Q}} Z=\dim_{\mathbb{Q}} X=n$ and hence $\dim Z=n$. □

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Department of Mathematics, Ben Gurion University of the Negev
PO Box 653, Be'er Sheva 84105, Israel

mlevine@math.bgu.ac.il

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