ERRATUM TO "LAGRANGIAN COBORDISMS VIA GENERATING FAMILIES: CONSTRUCTION AND GEOGRAPHY"

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Abstract. This erratum aims to correct three issues in the paper "Lagrangian Cobordisms via Generating Families: Constructions and Geography" by adding a missing hypothesis to the handle attachment theorem, ensuring that the hypothesis is satisfied in later applications, and replacing an erroneous explicit construction.

1. Introduction

This erratum to the paper [3] has three aims:

(1) To address a missing hypothesis in the construction of a Lagrangian handle attachment using generating families (Theorem 4.2 in the original paper),
(2) To ensure that the missing hypothesis is satisfied by the constructions in the proof of the non-classical geography theorem (Theorem 1.2), and
(3) To replace the incorrect proof of Lemma 6.10.

We note that the construction of a generating family for a Lagrangian handle attachment in Theorem 4.2 has recently been subsumed by work of Wenyuan Li [6], which asserts that a generating family at the negative end of a Lagrangian cobordism extends to a generating family for the full cobordism if a formal obstruction vanishes.

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2. Attaching Lagrangian Handles

The main result of [3, Section 4] is the Handle Attachment Theorem (Theorem 4.2). The statement of this theorem is missing a hypothesis. We will correct this oversight below by incorporating the hypothesis into the definition of a "gf-compatible q-attaching region," which strengthens the previous definition of a "gf-attaching region."

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In [3, Definition 4.1], a q-dimensional attaching region and a gf-attaching region are defined to be convenient parameterizations of the front projection of a Legendrian Λ near a cusp region. In the definition below, we make minor adjustments to Definition 4.1 to rename what we had called a "gf-attaching region" to a "q-attaching region of index j."

To set notation, we denote the front of the Legendrian Λ by Λ_{xz} and the cusps of the front of a Legendrian Λ by $\Lambda_{xz}^{\succ}$; if Λ has a generating family, then the front projection of the cusps that represent births/deaths between critical points of indices j and $j+1$ are denoted $\Lambda_{xz}^{\succ j}$. Finally, denote a k-dimensional disk of radius r by $D^k(r)$.

Definition (Modified Definition 4.1 of [3])**.** For $1 \leq q \leq n$, a smooth embedding

$$\sigma : D^q(1+\lambda) \times D^{n-q}(\lambda) \times D^1(\lambda) \to M^n \times \mathbb{R},$$

for some small λ, is a q**-attaching region** for a Lagrangian q-handle along the front of a Legendrian submanifold Λ if:

(1) $\sigma^{-1}(\Lambda_{xz}) = \left\{(u,v,w) \;:\; w^2 = (\|u\|^2 - \|v\|^2 - 1)^3\right\}$;
(2) $\sigma^{-1}(\Lambda_{xz}^{\succ}) = \left\{(u,v,0) \;:\; \|u\|^2 - \|v\|^2 = 1\right\}$, with the image of this set called the **surgery domain**; and
(3) For each fixed $(u_0, v_0) \in D^q(1+\lambda) \times D^{n-q}(\lambda)$, the image $\sigma(\{u_0\} \times \{v_0\} \times D^1(\lambda))$ is parallel to the $\mathbb{R}$ factor.

The **core disk** of the attaching region is the image of $D^q(1) \times \{0\} \times \{0\}$. A q-attaching region σ has **index** j if property (2) is strengthened to require the surgery domain to be contained in $\Lambda_{xz}^{\succ j}$, for some fixed $j \geq 0$.

We will next explain how we can associate a "stable homotopy element" to a q-attaching region of index j on a Legendrian equipped with a generating family. First recall that if a smooth function $f : \mathbb{R}^N \to \mathbb{R}$ has a birth-death singularity of index j at $p \in \mathbb{R}^N$, then there exist local coordinates, $\eta_1, \dots, \eta_N$ such that

$$f = f(p) + (-\eta_1^2 + \cdots - \eta_j^2) + (\eta_{j+1}^2 + \cdots + \eta_{N-1}^2) + \eta_N^3.$$

The **stable subspace** at p is then defined to be the subspace of $T_p\mathbb{R}^N$ spanned by $\frac{\partial}{\partial \eta_1}, \dots, \frac{\partial}{\partial \eta_j}$. To set up the definition of a stable subspace homotopy element, we begin with a Legendrian submanifold Λ_- with a tame generating family $f_- : M^n \times \mathbb{R}^N \to \mathbb{R}$ and an index j, q-attaching region σ. After an isotopy of Λ_-, we may assume that the image $\sigma(\partial D^q(1) \times \{0\} \times \{0\})$ lies in the hypersurface defined by $z = c$ for some c. We set $\sigma(u,0,0) = (x_u, c)$ and $f_{-,u}(\cdot) = f_-(x_u, \cdot) : \mathbb{R}^N \to \mathbb{R}$.

With this notation in hand, and motivated by theory developed by Hatcher and Wagoner [5], we make the following definitions.

Definition (Stable subspace homotopy element)**.** Each function $f_{-,u}$ has a unique (degenerate birth-death) critical point η_u in $\mathbb{R}^N$ whose j-dimensional stable subspace determines a point of the Grassmannian $G(j,N)$ of j-dimensional vector subspaces of $\mathbb{R}^N$. The data of these stable subspaces for all $u \in \partial D^q(1)$ determines an element $S_{\sigma,f_-} \in \pi_{q-1}(G(j,N))$, called the **stable subspace homotopy element** of the q-dimensional gf-attaching region σ for (Λ_-, f_-).

Definition (Generating Family Refinement of Definition 4.1 of [3])**.** Let Λ_- be a Legendrian submanifold of J^1M equipped with a tame generating family $f_- : M^n \times \mathbb{R}^N \to \mathbb{R}$. A **gf-compatible q-attaching region of index** j is an index j, q-attaching region σ for (Λ_-, f_-) whose stable subspace homotopy element $S_{\sigma,f_-} \in \pi_{q-1}(G(j,N))$ is trivial.

We are now ready to correct the statement of Theorem 4.2. The key difference is the addition of the *gf-compatible* condition to the attaching region, which implicitly adds the hypothesis of the triviality of the stable space homotopy element.

Theorem (Corrected Theorem 4.2 of [3])**.** *Let Λ_- be a Legendrian submanifold of J^1M equipped with a tame generating family $f_- : M^n \times \mathbb{R}^N \to \mathbb{R}$. Given an index j, gf-compatible q-attaching region σ for (Λ_-, f_-), there exists a smooth 1-parameter family of functions $f_t : M \times \mathbb{R}^N \to \mathbb{R}$ such that $F(t,x,\eta) = tf_t(x,\eta)$ is tame and generates an embedded Lagrangian cobordism $(\Lambda_-, f_-) \prec_{(\overline{L},F)} (\Lambda_+, f_+)$ satisfying:*

1. *the cobordism $\overline{L}$ has the homotopy type of a cylinder over Λ_- with a q-cell attached, and*
2. *the Legendrian Λ_+ is obtained from Λ_- by an embedded $(q-1)$-surgery along the boundary of the core disk of σ.*

The vanishing of the stable subspace homotopy element S_{σ,f_-} is necessary in the part of the proof contained in Section 4.4, page 2456, where it is written "working one x slice at a time, we modify $(f_-)_x$". This modification can indeed be done for each individual x, but in order for this modification to depend continuously on x, the collection of stable subspaces along $\partial D^q(1)$ first has to be extended continuously to $D^q(1)$ in order to serve as tangent spaces to the stable manifolds of the newly created critical points of index j. This continuous extension is possible exactly when S_{σ,f_-} vanishes in $\pi_{q-1}(G(j,N))$; see [5, pp. 180-181].

3. Application of the Handle Attachment Theorem

In the proof of the Geography Theorem ([3, Theorem 1.2]), the Handle Attachment Theorem, [3, Theorem 4.2] was applied to prove two results in [3, Section 6]. First, in the proof of the 0-surgery Lemma ([3, Lemma 6.3]) it is (implicitly) used to show that a Lagrangian cobordism corresponding to a 1-handle attachment is gf-compatible, and hence that [3, Theorem 2.1] applies. Second, it is used in the proof of the Manifold Lemma ([3, Lemma 6.5]) in the case of handle attachments along Legendrians (Λ_-, f_-) arising from a spinning construction. The following lemma shows that the extra hypothesis of having a gf-compatible attaching region is satisfied for all of the constructions.

Lemma (gf-compatibility for 1-handles and spun Legendrians)**.** *The condition of gf-compatibility is automatically satisfied in the following cases:*

1. *For all j, any 1-attaching region of index j is gf-compatible.*
2. *Suppose (Λ_-, f_-) is obtained from spinning (Λ'_-, f'_-). Then, for all j and q, any q-attaching region of index j is gf-compatible.*

Proof. Statement (1) follows since $\pi_0(G(j,N))$ is trivial for $j, N \geq 1$. Statement (2) follows since spinning f'_- to produce f_- gives rise to a constant, and thus trivial, stable subspace homotopy element S_{σ,f_-} for any index j, q-attaching region. □

4. The Sphere Lemma

A key part of the proof of the Geography Theorem ([3, Theorem 1.2]) is the so-called Sphere Lemma ([3, Lemma 6.10]), which builds Legendrian spheres whose generating family homologies have a given pair of "dual" classes.

Sphere Lemma (Lemma 6.10 of [3])**.** For all integers a, there exists a Legendrian n-sphere Λ with a tame generating family f such that

$$\Gamma_f(t) = t^n + t^a + t^{n-1}t^{-a}.$$

The statement of the Sphere Lemma is correct, and hence so is its use in the proof of [3, Theorem 1.2]. The issue with the proof of the Sphere Lemma in [3] is that the spun Reidemeister move in [3, Figure 10] does not produce a generic front. As a result, the fiber indices on the right side of [3, Figure 11] are incorrect — in fact, they are off by a factor of n. See [4, §3] for a discussion of how to desingularize the fronts in [3, Figure 10].

A correct construction appears as $\tilde{\Lambda}^{n,a}$ with generating family $\tilde{f}^{n,a}$ in [7, Section 4.3], and in particular, [7, Figure 1]. We repeat the construction for the reader's convenience. Consider the spinnable Legendrian link in J^1H^1 whose front projection appears in Figure 1.[1] This link, which is isotopic to half of the Hopf link, has a generating family f such that the highest level sheets on both components are generated by critical points of the same index. Repeatedly spin the front about its central axis to obtain an n-dimensional "Hopf link" of two Legendrian spheres $\Lambda^H \subset \mathbb{R}^{n+1}$ with generating family f^H such that $\Gamma_{f^H}(t) = 2t^n + t^a + t^{n-1}t^{-a}$. Then perform a 0-surgery along the horizontal dotted 1-disk shown in Figure 1 to get a connected Legendrian sphere $\Lambda_{n,a} \subset \mathbb{R}^{2n+1}$ with generating family $f_{n,a}$. The algebra in the final paragraph of the original proof of [3, Lemma 6.10] shows that, depending on the value of a, we can either use the 0-surgery lemma [3, Lemma 6.3]

[1]See [3, §3.2] for definitions of spinnable Legendrian link and spinnable generating family.

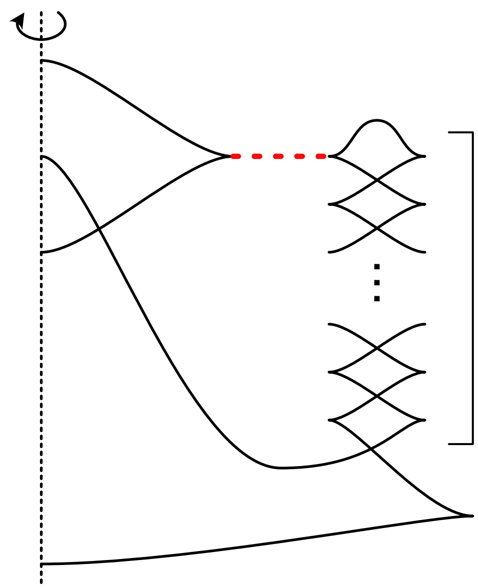

FIGURE 1. A scheme to construct a Legendrian sphere with $\Gamma_{f_{n,a}}(t) = t^n + t^a + t^{n-1}t^a$ with $a \geq \lfloor (n-1)/2 \rfloor$.

or the cobordism exact sequence [3, Theorem 2.1] to show that $\Gamma_{f_{n,a}}(t) = t^n + t^a + t^{n-1}t^{-a}$, as desired.

We note that there are other possible constructions of Legendrian spheres with the generating family homology specified in [3, Lemma 6.10]. For example, we summarize another construction as follows: start with a standard n-dimensional Hopf link, stretch out the bottom component, modify a neighborhood in the top sheet of this bottom component by adding a stack of "slugs", and then perform a 0-surgery between the top of the stack of slugs and the top component of the Hopf link. Modifying a neighborhood in a sheet by a "slug" means that we replace a neighborhood of the sheet with a codimension 2 sphere's worth of swallowtails, bounding a codimension 1 disk of crossings of the deformed sheet. Since the top of the slug is again a sheet, we can repeatedly stack slugs on slugs. See the top right of Figure 2 for a 2-dimensional picture. Slicing the 2-dimensional slug between its two swallowtails and spinning half of what remains produces the 3-dimensional slug, which in turn can be sliced and rotated repeatedly to produce slugs of arbitrary dimensions. The codimension 1 disks of crossings and the singularities are now generic; see, for example, [1, Section 1.3] or [2, Chapter 21]. That the resulting Legendrian sphere has the desired algebraic properties follows from the same reasoning as in the previous construction.

A disadvantage of the construction of stacking slugs is that it involves swallowtail singularities, not just cusp singularities. Another version of the

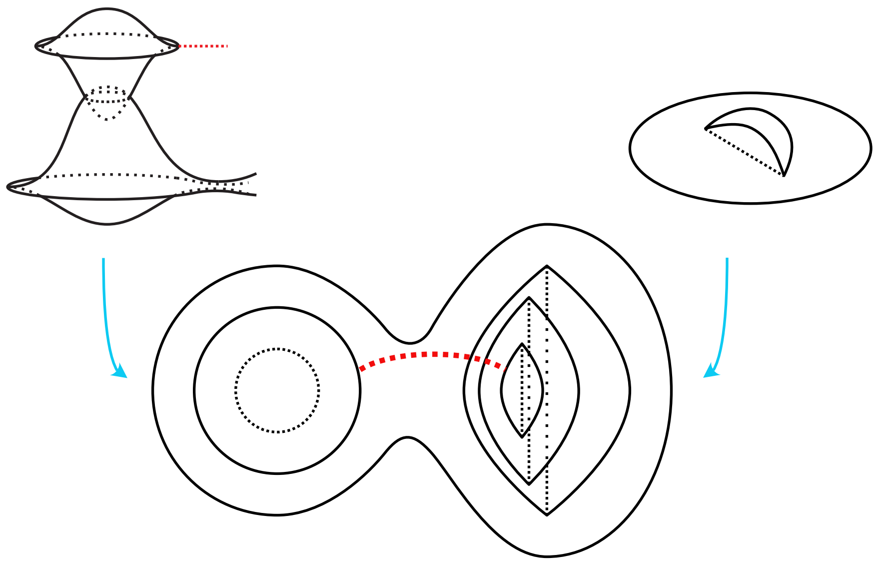

FIGURE 2. An illustration of the stacked slug construction of a 2-sphere in the proof of [3, Lemma 6.10], with the main figure showing a projection to the x_1x_2-plane and insets showing (partial) elevations of the projection. The 2-dimensional slug is represented by the region containing the cusped components of the projection; considering the intersection of the slug projection with $\{x_2 \geq 0\}$ produces a front that can be spun to obtain a 3-dimensional slug.

stacking construction comes from first spinning the lower component in Figure 1 in order to produce an n-dimensional sphere with a stack of "rings", having only cusp singularities, and then performing two connected sum operations of this sphere with a stack of rings to a standard n-dimensional Hopf link. One connect sum joins the lower component of the Hopf link with the bottom cusp of the stack, while the second connect sum joins the upper component of the Hopf link with the upper cusp of the stack.

An advantage of these two stacking constructions is that there is no need to alter the original Hopf link, as it suffices to make two connected sums with an appropriate type of stack, which can be constructed independently of the Hopf link. There are undoubtedly even more methods.

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