Khovanov homology is a skew Howe 2–representation of categorified quantum $\mathfrak{sl}_m$

AARON D LAUDA

HOEL QUEFFELEC

DAVID E V ROSE

We show that Khovanov homology (and its $\mathfrak{sl}_3$ variant) can be understood in the context of higher representation theory. Specifically, we show that the combinatorially defined foam constructions of these theories arise as a family of 2–representations of categorified quantum $\mathfrak{sl}_m$ via categorical skew Howe duality. Utilizing Cautis–Rozansky categorified clasps we also obtain a unified construction of foam-based categorifications of Jones–Wenzl projectors and their $\mathfrak{sl}_3$ analogs purely from the higher representation theory of categorified quantum groups. In the $\mathfrak{sl}_2$ case, this work reveals the importance of a modified class of foams introduced by Christian Blanchet which in turn suggest a similar modified version of the $\mathfrak{sl}_3$ foam category introduced here.

1 Introduction

1A Categorified knot invariants and quantum groups

One of the original motivations for categorifying quantum groups was to provide a representation-theoretic explanation for the existence of Khovanov homology and other link homologies categorifying quantum link invariants. Just as the Jones polynomial is described representation theoretically by the quantum group $U_q(\mathfrak{sl}_2)$ and tensor powers of its two-dimensional representation, the categorification of the Jones polynomial via Khovanov homology should be described in terms of the 2–representation theory of the categorified quantum group associated to $U_q(\mathfrak{sl}_2)$.

Currently, the primary link between categorified quantum groups and Khovanov homology follows the indirect path through Webster's work [74; 75] on categorified tensor products. This connection utilizes an isomorphism relating Webster's categorifications of tensor products with categories associated to blocks of graded, parabolic category $\mathcal{O}$. Categorifications associated with category $\mathcal{O}$ were initiated by Bernstein, Frenkel and

Khovanov [3], further developed in work of Frenkel, Khovanov and Stroppel [68; 25], and were later shown to give rise to 2–representations of categorified quantum groups by Hill and Sussan [27] and Brundan and Stroppel [6]. The relation to the familiar picture-world, see Bar-Natan [1; 2], of Khovanov homology then relies on several technical results of Stroppel [69; 68] relating the knot homologies constructed using category $\mathcal{O}$ to Khovanov's more elementary construction [32; 33]. More generally, for link homology theories associated with fundamental $\mathfrak{sl}_n$ representations, Webster describes an isomorphism relating his construction to Sussan's category $\mathcal{O}$ based link homology theory [71], which is related via Koszul duality to a theory defined by Mazorchuk and Stroppel [57]. When $n = 3$, the latter of these link homologies can then be identified [57] with Khovanov's more elementary construction [34] of $\mathfrak{sl}_3$ link homology defined using singular cobordisms called foams.

Alternatively, there is an algebro-geometric construction of Khovanov homology and related $\mathfrak{sl}_n$ link homologies due to Cautis and Kamnitzer [12; 13]. These knot homologies arise from derived categories of coherent sheaves on algebraic varieties associated to orbits in the affine Grassmannian. In the $\mathfrak{sl}_2$ case this knot homology agrees with Khovanov homology [12, Theorem 8.2] and these geometric categories can be understood as 2–representations of categorified quantum groups; see Cautis and Lauda [20] and Cautis, Kamnitzer and Licata [16]. These link homologies are related to those of Seidel and Smith [66] and Manolescu [56] by mirror symmetry.

In this article, we provide a direct construction of foam based $\mathfrak{sl}_n$ link homology theories for $n = 2$ or $n = 3$ intrinsically in terms of categorified quantum groups. We show that all of the components involved in these knot homologies are already present within the structure of categorified quantum groups including the relations in foam categories and the complexes defining the braiding. Utilizing Cautis–Rozansky categorified clasps [10; 65] we also obtain categorified projectors lifting Jones–Wenzl idempotents and their $\mathfrak{sl}_3$ analogs purely from the higher relations of categorified quantum groups. In the $\mathfrak{sl}_2$ case this work reveals the importance of a modified class of foams introduced by Christian Blanchet [4], suggesting that this version of the foam category is most natural from the perspective of categorified quantum groups. In the $\mathfrak{sl}_3$ case these results suggest a similar modified version of the $\mathfrak{sl}_3$ foam category.

1B Categorified representation theory

Recall that in categorified representation theory, $\mathbb{C}(q)$–vector spaces V with decompositions into weight spaces $V = \bigoplus_\lambda V_\lambda$, are replaced by graded categories $\mathcal{V} = \bigoplus_\lambda \mathcal{V}_\lambda$, and instead of linear maps between spaces, Chevalley generators act by functors $\mathcal{E}_i \mathbf{1}_\lambda \colon \mathcal{V}_\lambda \to \mathcal{V}_{\lambda+\alpha_i}$, $\mathcal{F}_i \mathbf{1}_\lambda \colon \mathcal{V}_\lambda \to \mathcal{V}_{\lambda-\alpha_i}$ satisfying quantum Serre relations up to

isomorphism of functors. The higher structure of categorified representation theory appears at the level of natural transformations between these functors. In most instances when $U_q(\mathfrak{sl}_n)$ admits a categorical action of this form, the natural transformations that appear between functors are predictable and can be systematically described. A key part of this structure is that $\mathcal{F}$ is a left and right adjoint for $\mathcal{E}$ and that the endomorphisms of $\mathcal{E}^a$ are acted upon by the so-called KLR algebras developed in papers by Chuang and Rouquier [21], Khovanov and Lauda [36; 38] and Rouquier [64].

In Lauda [46] and Khovanov and Lauda [37], it was suggested that the full structure of categorical representations of $U_q(\mathfrak{sl}_n)$ is described by a 2–functor from an additive 2–category $\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)$. This 2–category categorifies Lusztig's modified version $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ of the quantum group $U_q(\mathfrak{sl}_n)$ [48]. The objects of $\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)$ are indexed by the weight lattice of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$, 1–morphisms correspond to the elements of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$, and the 2–morphisms govern the natural transformations that appear in categorical representations. However, the 2–category $\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)$ has additional relations on 2–morphisms beyond specified adjoints and KLR relations. We refer to the collection of relations on 2–morphisms as *higher relations* because they can be viewed as replacements for the quantum Serre relations. Indeed, these higher relations give rise to explicit isomorphisms lifting the defining relations in $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$, while simultaneously controlling the Grothendieck group of $\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)$, allowing for a $\mathbb{Z}[q, q^{-1}]$–algebra isomorphism between its split Grothendieck ring $K_0(\dot{\mathcal{U}}_Q(\mathfrak{sl}_n))$ and the integral version of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$. Under this isomorphism, the images of indecomposable 1–morphisms from $\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)$ map to the canonical basis of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$; see Lauda [46] and Webster [76].

Here we show that these higher relations also encode the information needed to construct *all* $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ knot homology theories in a framework where computations are accessible.

1C Braidings via skew Howe duality

The key insight for our elementary construction of knot homologies from categorified quantum groups is the fundamental observation of Cautis, Kamnitzer and Licata that the R–matrix describing the braiding in an m–fold tensor product of fundamental representations of $U_q(\mathfrak{sl}_n)$ in Reshetikhin–Turaev link invariants can be obtained from a deformed Weyl group action associated with $U_q(\mathfrak{sl}_m)$ [15].

Recall that the Weyl group W of a semi-simple Lie algebra $\mathfrak{g}$ is a finite Coxeter group associated to the root system of $\mathfrak{g}$. Passing from $U(\mathfrak{g})$ to $U_q(\mathfrak{g})$, the Weyl group deforms to a braid group of type $\mathfrak{g}$, which acts on $U_q(\mathfrak{g})$–modules. In the simplest case of $\mathfrak{g} = \mathfrak{sl}_2$, the Weyl group $W = \mathfrak{S}_2$ deforms to the braid group B_2 giving a reflection isomorphism $T\colon V_\lambda \to V_{-\lambda}$ between weight spaces of a $U_q(\mathfrak{sl}_2)$–module.

This action can be expressed in a completion of the idempotented quantum algebra $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ by the power series

(1-1)
$$T1_\lambda = \sum_{s \geq 0} (-q)^s F^{(\lambda+s)} E^{(s)} 1_\lambda \quad \lambda \geq 0,$$
$$T1_\lambda = \sum_{s \geq 0} (-q)^s E^{(-\lambda+s)} F^{(s)} 1_\lambda \quad \lambda \leq 0.$$

On any finite-dimensional representation, $T1_\lambda$ can be expressed as a finite sum. When $\mathfrak{g} = \mathfrak{sl}_m$ and $W = \mathfrak{S}_m$, there are analogous maps $T_i 1_\lambda$ for each $1 \leq i \leq m-1$ satisfying the braid relations.

Cautis, Kamnitzer and Licata related the braiding of fundamental $U_q(\mathfrak{sl}_n)$ representations to the Weyl group action using a version of Howe duality for exterior algebras they called skew Howe duality [15]. The key idea is to study quantum exterior powers. Denote by $\mathbb{C}_q^n$ the standard $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$–module with basis denoted $x_1, \ldots, x_n$. The quantum exterior algebra is the $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$–module defined as

$$\bigwedge_q^\bullet(\mathbb{C}_q^n) = \mathbb{C}(q)\langle x_1, \ldots, x_n \rangle / (x_i^2, \ x_i x_j + q x_j x_i \quad \text{for } i < j).$$

By assigning degree one to each x_i the quantum exterior algebra is a graded $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$–module whose homogeneous subspace of degree N is denoted by $\bigwedge_q^N(\mathbb{C}_q^n)$.

The space $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ admits commuting actions of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ and $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ which constitute a Howe pair. For example, when $m = 2$ the space $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^2)$ decomposes into $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ weight spaces as

$$\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^2) \cong \bigwedge_q^N(\mathbb{C}_q^n \oplus \mathbb{C}_q^n) \cong \bigoplus_{a+b=N} \bigwedge_q^a(\mathbb{C}_q^n) \otimes \bigwedge_q^b(\mathbb{C}_q^n),$$

where the weight of a summand $\bigwedge_q^a(\mathbb{C}_q^n) \otimes \bigwedge_q^b(\mathbb{C}_q^n)$ is $\lambda = b - a$. The action of $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ is given by maps

(1-2)
$$E1_\lambda \colon \bigwedge_q^a(\mathbb{C}_q^n) \otimes \bigwedge_q^b(\mathbb{C}_q^n) \to \bigwedge_q^{a-1}(\mathbb{C}_q^n) \otimes \bigwedge_q^{b+1}(\mathbb{C}_q^n),$$
$$F1_\lambda \colon \bigwedge_q^a(\mathbb{C}_q^n) \otimes \bigwedge_q^b(\mathbb{C}_q^n) \to \bigwedge_q^{a+1}(\mathbb{C}_q^n) \otimes \bigwedge_q^{b-1}(\mathbb{C}_q^n).$$

For more details on quantum skew Howe duality see Cautis, Kamnitzer and Morrison [19] and Cautis [10].

The Weyl group action gives an isomorphism between the λth and $-\lambda^{\text{th}}$ weight spaces of $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^2)$.
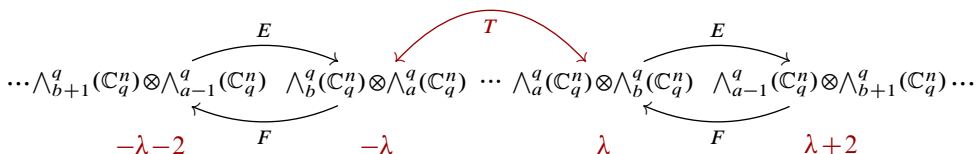
Since $\mathbb{C}_q^n$ is the defining representation of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$, the quantum exterior powers $\bigwedge_q^a(\mathbb{C}_q^n) = V_{\omega_a}$ correspond to fundamental $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$–representations, where ω_a for $1 \leq a \leq n-1$ are the fundamental weights of $\mathfrak{sl}_n$. The deformed reflection isomorphism

$$V_{\omega_a} \otimes V_{\omega_b} \cong \bigwedge_q^a(\mathbb{C}_q^n) \otimes \bigwedge_q^b(\mathbb{C}_q^n) \xrightarrow{T} \bigwedge_q^b(\mathbb{C}_q^n) \otimes \bigwedge_q^a(\mathbb{C}_q^n) \cong V_{\omega_b} \otimes V_{\omega_a}.$$

gives a braiding of fundamental representations that agrees with the R–matrix in the Reshetikhin–Turaev construction [15] (up to a power of $\pm q$). The key advantage of this realization of the R–matrix in terms of skew Howe duality is that it suggests a procedure for categorification.

1D   Knot homology from categorical skew Howe duality

Following the ideas of Chuang and Rouquier [21] (see also Cautis and Kamnitzer [14]), one can define a categorification of the reflection isomorphism $T1_\lambda\colon V_\lambda \to V_{-\lambda}$ using the 2–category $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$ categorifying $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$. Passing to the category of complexes $\text{Kom}(\dot{\mathcal{U}}_Q(\mathfrak{sl}_2))$, it is possible to define a complex $\mathcal{T}1_\lambda$ of 1–morphisms

$$(1\text{-}3) \quad \mathcal{E}^{(-\lambda)}1_\lambda \xrightarrow{d_1} \mathcal{E}^{(-\lambda+1)}\mathcal{F}1_\lambda\{1\} \xrightarrow{d_2} \mathcal{E}^{(-\lambda+2)}\mathcal{F}^{(2)}1_\lambda\{2\} \longrightarrow \cdots$$

$$\longrightarrow \mathcal{E}^{(-\lambda+k)}\mathcal{F}^{(k)}1_\lambda\{k\} \xrightarrow{d_{k+1}} \cdots$$

for $\lambda \leq 0$ and a similar complex for $\lambda \geq 0$ (compare with (1-1)). The differentials in this complex can be explicitly defined using the 2–morphisms in $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$. Verification that $d^2 = 0$ follows from the relations in the 2–category $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$; the enhanced graphical calculus from Khovanov, Lauda, Mackaay and Stošić [40] is useful for this computation.

Given a 2–representation $\mathcal{V}$ of the 2–category $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$ with weight decomposition into abelian categories $\mathcal{V}_\lambda$, the functor of tensoring with the complex $\mathcal{T}1_\lambda$ gives rise to derived equivalences $\mathcal{T}1_\lambda\colon D(\mathcal{V}_\lambda) \to D(\mathcal{V}_{-\lambda})$. The resulting derived equivalences are highly non-trivial and have led to the resolution of several important conjectures [21; 16; 18]. Our interest in these equivalences stems from their application to knot homology theory. Given a categorification of $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^2)$ with commuting categorical actions of $\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)$ and $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$, the categorified braid group action gives a categorification of the R–matrix. More generally, one can categorify the braid group action on an m–fold tensor product of $U_q(\mathfrak{sl}_n)$ representation using the categorified braid group action coming from the deformed Weyl group action of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ [14].

In fact, Cautis and Kamnitzer's algebro-geometric construction of Khovanov homology [12] and $\mathfrak{sl}_n$ link homology [13] can be understood in this framework. Their invariants arise from a categorification of $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ using derived categories of coherent sheaves on varieties related to orbits in the affine Grassmannian [10, Theorem 2.6].

1E Reinterpreting $\mathfrak{sl}_n$ skein theory using skew Howe duality

While categorifications of $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ defined via derived categories of coherent sheaves are far from elementary, it turns out that this story has a more combinatorial description. In the decategorified case, the usual skein theory description of $\mathfrak{sl}_n$ link invariants in terms of MOY calculus [60], can also be understood in terms of skew Howe duality.

Recall that an $\mathfrak{sl}_n$ web is a graphical presentation of intertwiners between tensor products of fundamental representations of $U_q(\mathfrak{sl}_n)$. When $n = 2$, the calculus of $\mathfrak{sl}_2$ webs is described by the Temperley–Lieb algebra; Kuperberg described the $n = 3$ case using a graphical calculus of oriented trivalent graphs [45] which depict the morphisms in a combinatorially defined pivotal category called the $\mathfrak{sl}_3$ spider. These descriptions have recently been generalized by Cautis, Kamnitzer and Morrison [19] to general n, building on earlier work of Kim [44] and Morrison [58]. We briefly summarize this construction, referring the reader to their work for the details.

The category $n\mathbf{Web}$ is the pivotal category whose objects are sequences in the symbols $\{1^{\pm}, \ldots, (n-1)^{\pm}\}$. Morphisms are oriented graphs with edges labeled by $\{1, \ldots, n-1\}$ generated by the following:

(1-4)
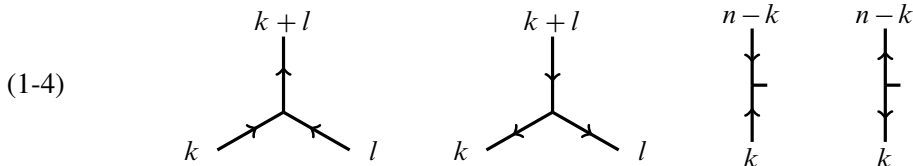
where a strand labeled by k is directed out from the label k^+ and into the label k^- in the domain, and vice versa in the codomain. These graphs, called $\mathfrak{sl}_n$ webs, are considered up to isotopy (relative to their boundary) and local relations. The category $n\mathbf{Web}$ can be identified with the full subcategory of $U_q(\mathfrak{sl}_n)$ representations generated (as a pivotal category) by the fundamental representations by identifying the symbol k^+ with $\bigwedge_q^k(\mathbb{C}_q^n)$ and identifying k^- with its dual. Sequences correspond to tensor products of the corresponding representations.

The connection to skew Howe duality is given by considering a related family of m–sheeted web categories. Let $n\mathbf{Web}_m(N)$ denote the category whose objects are sequences $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ with $0 \le a_i \le n$ and $\sum_{i=1}^m a_i = N$. Note that here we allow the symbols 0 and n in the object sequences, but none of the dual symbols k^-. As above, these labels should be interpreted as representations $\bigwedge_q^k(\mathbb{C}_q^n)$ for $0 \le k \le n$ with $\bigwedge_q^0(\mathbb{C}_q^n) = \bigwedge_q^n(\mathbb{C}_q^n) = \mathbb{C}(q)$ corresponding to the trivial representation.

Morphisms in $n\mathbf{Web}_m(N)$ are $\mathfrak{sl}_n$ webs mapping between the symbols $a_i \neq 0, n$ in each sequence.

Via skew Howe duality, the action of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ on $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ gives morphisms between tensor products of fundamental representations. This map has a graphical interpretation described in [19] using "ladder diagrams" to represent webs:

$$1_\lambda \mapsto \quad \begin{array}{c} a_m \\ \vdots \\ a_1 \end{array}$$

$$E_i 1_\lambda \mapsto \quad \begin{array}{cc} a_{i+1}+1 & a_{i+1} \\ a_i - 1 & a_i \end{array} \qquad F_i 1_\lambda \mapsto \quad \begin{array}{cc} a_{i+1}-1 & a_{i+1} \\ a_i + 1 & a_i \end{array}$$
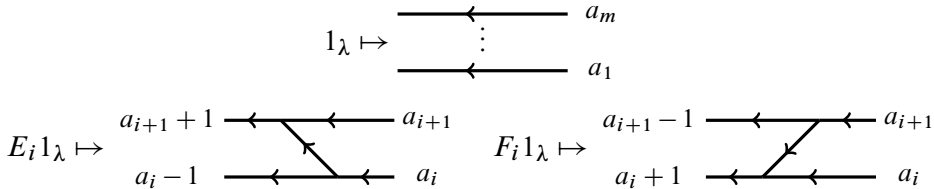
These diagrams should be read from right to left and we omit $m-2$ lines in each of the latter two diagrams (compare with (1-2)). The sequences on the right are determined by the $\mathfrak{sl}_m$ weight $\lambda = (\lambda_1, \ldots, \lambda_{m-1})$ by $\lambda_i = a_{i+1} - a_i$; edges connected to the label 0 should be deleted and those connected to the label n should be truncated to the "tags" depicted in the last two diagrams in (1-4).

In this paper, we categorify Cautis, Kamnitzer and Morrison's construction for the cases $n = 2$ and $n = 3$. In fact, for the $\mathfrak{sl}_2$ case we work with related categories $2\mathbf{BWeb}_m(N)$ where we allow strands labeled by 2 and no longer require the tag morphisms.[1] For example in $2\mathbf{BWeb}_2(2)$ we have the morphism

$$\begin{array}{cc} 1 & 2 \\ 1 & 0 \end{array}$$
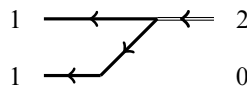
where $\rightarrow\!\!-$ depicts a 1–labeled edge and $\Rightarrow\!\!-$ depicts a 2–labeled edge.

In the $\mathfrak{sl}_3$ case, we continue to work with $3\mathbf{Web}_m(N)$, although this category has a simpler description than the one given above. Since $\bigwedge_q^2(\mathbb{C}_q^3)$ can be canonically identified with the dual of $\bigwedge_q^1(\mathbb{C}_q^3)$ we can replace 2–labeled edges with 1–labeled edges oriented in the opposite direction and do away with the tag morphisms. For example, the diagram

$$\begin{array}{cc} 2 & 2 \\ 0 & 1 \\ 1 & 1 \\ 3 & 2 \end{array}$$

---
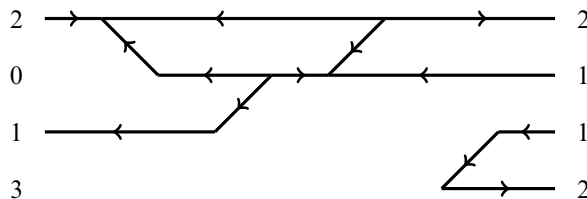
[1] Here the "B" stands for Blanchet; this category is related to a decategorification of his work [4].

depicts a morphism in $3\mathbf{Web}_4(6)$. We will later also consider a (categorified) version of this category in which we retain 3–labeled edges.

We will now exhibit the power of the skew Howe approach to diagrammatic representation theory (which hints at the utility of its categorified counterpart) in an example. The decomposition of $\bigwedge_q^3(\mathbb{C}_q^3 \otimes \mathbb{C}_q^2)$ into $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ weight spaces gives

$$
\underset{F1_{-1}}{\overset{E1_{-3}}{\bigwedge_q^3(\mathbb{C}_q^3) \otimes \bigwedge_q^0(\mathbb{C}_q^3)}}
\quad
\underset{F1_1}{\overset{E1_{-1}}{\bigwedge_q^2(\mathbb{C}_q^3) \otimes \bigwedge_q^1(\mathbb{C}_q^3)}}
\quad
\underset{F1_3}{\overset{E1_1}{\bigwedge_q^1(\mathbb{C}_q^3) \otimes \bigwedge_q^2(\mathbb{C}_q^3)}}
\quad
\bigwedge_q^0(\mathbb{C}_q^3) \otimes \bigwedge_q^3(\mathbb{C}_q^3)
$$

or diagrammatically
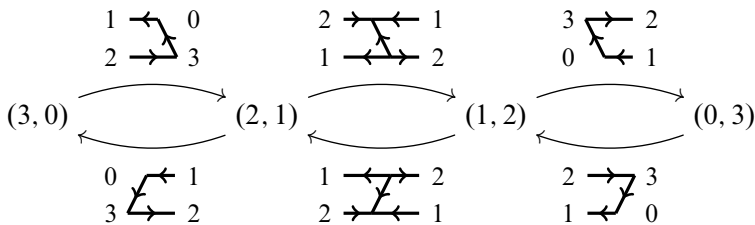
where we again read the webs from right to left in the above.

The local relations for $\mathfrak{sl}_3$ webs from Kupperburg [45] can be deduced from the fact that the above is an $\mathfrak{sl}_2$ representation. Indeed, the $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ relation $EF1_3 = FE1_3 + [3]1_3$ gives the circle relation

and the square relation

follows from the relation $EF1_1 = FE1_1 + [1]1_1$. The above diagrammatics extends to a description of the action by the integral version $_A\dot{\mathbf{U}}(\mathfrak{sl}_2)$ on $\bigwedge_q^3(\mathbb{C}_q^3 \otimes \mathbb{C}_q^2)$ where divided powers $E^{(k)} := E^k/[k]!$ act by ladder web diagrams with diagonal lines labeled by k. In the above example, the divided power relation $E^2 1_{-1} = [2]E^{(2)}1_{-1}$ gives the remaining bigon relation

from [45].

The skein-theoretic definition of the braiding can also be constructed from skew Howe duality using the deformed Weyl group action. For example, the braiding for edges labeled by the standard $\mathfrak{sl}_3$ representation can be recovered from the action of $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ on $\bigwedge_q^2(\mathbb{C}_q^3 \otimes \mathbb{C}_q^2)$. The decomposition into weight spaces is given diagrammatically by

$$(2,0) \rightleftarrows (1,1) \rightleftarrows (0,2)$$
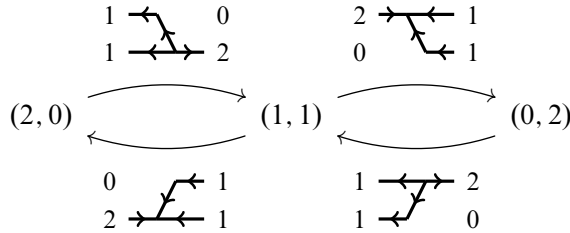
and the Weyl group action (1-1) on the 0–weight space gives the braiding

since $T1_0 = 1_0 - qFE1_0$. Up to a power of q, this recovers the formula for the positive crossing from [45]; the negative crossing can be recovered by considering $T^{-1}1_0 = 1_0 - q^{-1}EF1_0$.

In a similar manner, one can recover the $\mathfrak{sl}_2$ skein theory (ie the Kauffman bracket) from the action of $_A\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ on $2\mathbf{Web}_m(N)$. In fact, Cautis, Kamnitzer and Morrison use this approach to deduce the $\mathfrak{sl}_n$ web relations for $n \geq 4$. One can use their setup to give a combinatorial description of $\mathfrak{sl}_n$ link invariants labeled by any fundamental representation of $U_q(\mathfrak{sl}_n)$.

Moreover, one may realize the invariant of a link (or tangle) as the image of an element in $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ under the (appropriate) skew Howe map. For example, the $\mathfrak{sl}_3$ invariant of the Hopf link
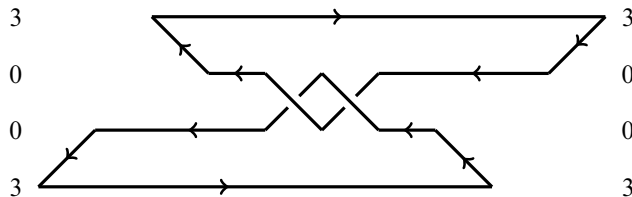
is the element in

$$\mathrm{End}_{\mathfrak{sl}_3}\left(\bigwedge_q^3(\mathbb{C}_q^3) \otimes \bigwedge_q^0(\mathbb{C}_q^3) \otimes \bigwedge_q^0(\mathbb{C}_q^3) \otimes \bigwedge_q^3(\mathbb{C}_q^3)\right) \cong \mathbb{C}(q)$$

given by the action of $F_1 E_3 T_2^2 E_1 F_3 1_{(-3,0,3)} \in \dot{\mathbf{U}}_q(\mathfrak{sl}_4)$ on $\bigwedge_q^6(\mathbb{C}_q^3 \otimes \mathbb{C}_q^4)$.

1F   Foamation functors for knot homologies

The observations from the previous section suggest an approach to obtaining diagrammatic $\mathfrak{sl}_n$ link homologies using categorical skew Howe duality. In his work categorifying the $\mathfrak{sl}_3$ polynomial, Khovanov utilized certain singular web cobordisms called foams [34]. In Mackaay, Stošić and Vaz [52] these singular surfaces were generalized to the $\mathfrak{sl}_n$ case to supply a diagrammatic counterpart of Khovanov–Rozansky homology [42; 43; 55]. These foams also appear to be connected with category $\mathcal{O}$, see Mazorchuk and Stroppel [57], and with Soergel bimodules; see Vaz [73]. However, unlike Khovanov's construction for $\mathfrak{sl}_3$, there is no known finite set of relations on $\mathfrak{sl}_n$ foams for $n > 3$ that guarantee any closed foam can be evaluated to an element of the ground ring. For general $\mathfrak{sl}_n$, matrix factorizations become the primary computation tool [77; 78; 79], and the only way to evaluate a closed foam is through the mysterious Kapustin–Li formula [52], Dyckerhoff and Murfet [24]. For foams this formula was discovered by Khovanov and Rozansky [41] generalizing work of the physicists Vafa [72], Kapustin and Li [30]. It arises from the topological Landau–Ginzburg model associated to components of the foam. A purely combinatorial foam construction of $\mathfrak{sl}_n$ link homology remains an important open problem.

Foams can be viewed as a categorification of webs. Indeed, this point of view motivates our approach to constructing $\mathfrak{sl}_n$ link homologies for $n = 2$ and $n = 3$. In Section 3 we describe 2–categories of m–sheeted $\mathfrak{sl}_n$ foams categorifying the above web categories. We define 2–functors

$$\Phi_n \colon \dot{\mathcal{U}}_Q(\mathfrak{sl}_m) \to n\mathbf{Foam}_m(N)$$

for $n = 2$ and $n = 3$. The existence of such functors was predicted by Khovanov and previously defined by Mackaay in the $n = 3$ case working in the restrictive setting of $\mathbb{Z}/2\mathbb{Z}$ coefficients in [50] where he called them "foamation" 2–functors.

Here we reinterpret Mackaay's work (and extend it to the $\mathfrak{sl}_2$ case) using skew Howe duality, defining foamation functors for $n = 2, 3$ with integer coefficients. Working over $\mathbb{Z}$, it is not obvious for $n = 2$ how to connect categorified quantum groups with the Bar-Natan's foam description of Khovanov homology. For example, one of the most basic relations for categorified quantum groups $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$ is the nilHecke relation:

which should correspond to the following neck-cutting relation
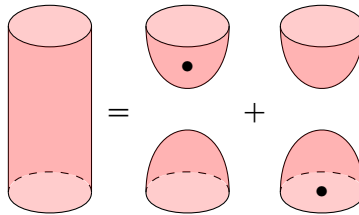
via the foamation 2–functor. However, the signs under this assignment do not match. One can try to rescale the foamation functors, but one quickly finds that there is no way to fix the signs under this assignment.

The difficulty in matching the neck-cutting relation with the nilHecke relation is closely related to the solution of another famous problem related to Khovanov homology. As originally defined, Khovanov homology is a projective functorial invariant, meaning that to a cobordism $f\colon T \to T'$ between two tangles one can assign a map $Kh(f)\colon Kh(T) \to Kh(T')$ between the respective homologies well defined only up to a ± 1 sign; see Khovanov [35; 33], Bar-Natan [2] and Jacobsson [28].

Clark, Morrison and Walker [22], and independently Caprau [7; 8], showed that the functoriality of Khovanov homology could be fixed by considering modified foam categories. From the representation-theoretic point of view, these foam categories keep track of the fact that the defining representation of $U_q(\mathfrak{sl}_2)$ is non-canonically isomorphic to its dual. Keeping track of this information gives rise to a fully functorial tangle invariant. For both of these fixes to Khovanov homology one must work with foams defined over the Gaussian integers $\mathbb{Z}[i]$.

Christian Blanchet proposed yet another construction fixing the functoriality of Khovanov homology [4]. He works with an enhanced version of the foam category where one labels facets by elements of the set $\{1, 2\}$. The 2–labeled facets are the primary difference from the previous two constructions. The presence of these 2–labeled facets introduces additional signs that are not present in the CMW or Caprau approaches to functoriality. Blanchet's approach gives rise to a functorial version of Khovanov homology defined over the integers [4].

These modified foam categories are quite natural from the representation-theoretic viewpoint. In the skew Howe framework, foams naturally provide a representation of $\mathcal{U}_Q(\mathfrak{gl}_n)$. Seen from this perspective, the foams introduced by Blanchet keep track of the difference between the trivial representation $\bigwedge_q^0(\mathbb{C}_q^2)$ and the determinant representation $\bigwedge_q^2(\mathbb{C}_q^2)$. As $U_q(\mathfrak{sl}_2)$ representations there is of course no difference between these two representations, but it appears that Blanchet's approach has additional information that contributes additional signs coming from the 2–labeled facets corresponding to the determinant representation $\bigwedge_q^2(\mathbb{C}_q^2)$.

In this article, we construct foamation functors into both the CMW foam categories as well as the foam categories of Blanchet. To define the functors into the CMW foam categories one must continue working with complex coefficients, while Blanchet's foam categories naturally admit foamation functors defined over the integers. This suggests that Blanchet's approach is the most natural from the perspective of categorified representation theory. It is also interesting to note that the $\mathfrak{sl}_2$ knot homology most closely related to categorified quantum groups is integral and functorial.

It turns out that in the $n = 3$ case it is possible to modify Mackaay's definition of the foamation functors to work over $\mathbb{Z}$, although this requires rather complicated and unnatural sign assignments. Motivated by the $\mathfrak{sl}_2$ case, we consider a modified $\mathfrak{sl}_3$ foam category that incorporates additional 3–labeled facets. To distinguish these foams from the usual $\mathfrak{sl}_3$ foams we call them Blanchet $\mathfrak{sl}_3$ foams. We show that there are 2–functors into Blanchet $\mathfrak{sl}_3$ foams with much more natural sign assignments for the generating 2–morphisms in $\mathcal{U}_Q(\mathfrak{sl}_m)$. There is also a natural construction of a forgetful 2–functor into the usual $\mathfrak{sl}_3$ foams defined intrinsically in terms of the topology of the Blanchet foams. Taking the composite of these 2–functors provides an explanation for the complicated signs occurring in the standard $\mathfrak{sl}_3$ foamation functors.

Checking the relations for the 2–category $\mathcal{U}_Q(\mathfrak{sl}_m)$ needed to define foamation functors is a laborious task. Here we utilize recent results of the first author with Cautis showing that in a 2–representation with finitely many nonzero weight spaces many of the relations come for free [20].

An independent construction of the integral foamation functors into the usual $\mathfrak{sl}_3$ foam 2–category was given by Mackaay, Pan and Tubbenhauer in a recent update to their work in [51]. They utilize the foamation functors for a different application related to a generalization of Khovanov's arc algebra to the $\mathfrak{sl}_3$ setting.

1G Comparing knot homologies

A careful analysis of Cautis' arguments in [10] reveals that the skew Howe duality approach also supplies a mechanism for equating different constructions of $\mathfrak{sl}_n$ link homologies. Indeed, given any 2–representation of $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$ whose objects are indexed by the nonzero weights in $\bigwedge_N^q(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$, and whose endomorphisms of the highest weight object are one-dimensional in degree zero and zero-dimensional otherwise, one obtains a unique knot homology theory that is formally determined by the relations imposed by the 2–representation. In Section 4B, we show that Khovanov's $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ link homology theories fit into this framework. We also sketch a proof of the $\mathfrak{sl}_3$ case of Cautis and Kamnitzer's [13, Conjecture 6.4] relating Khovanov–Rozansky link homology to the geometrically defined Cautis–Kamnitzer link homology, contingent on results to appear in Cautis [11].

1H   Cautis–Rozansky categorified clasps

Categorifying $\mathfrak{sl}_n$ link invariants labeled by arbitrary (non-fundamental) representations appears to be a much more difficult problem; see Webster [75]. In the $n = 2$ case, there are several approaches to defining categorifications of the coloured Jones polynomial by categorifying Jones–Wenzl projectors. The approach of Cooper and Krushkal uses foam based methods [23], while another approach of Frenkel, Stroppel and Sussan uses Lie-theoretic methods [26] based on category $\mathcal{O}$ for $\mathfrak{gl}_n$. These two approaches are compared and related via Koszul duality in Stroppel and Sussan [70]. Rozansky defined yet another approach to categorifying Jones–Wenzl projectors using complexes in Bar-Natan's foam category [65]. These complexes are presented as the stable limit of the complexes assigned to k–twist torus braids as $k \to \infty$, or *infinite twists*. This construction also agrees with the Cooper–Krushkal $\mathfrak{sl}_2$ projectors.

There are analogs of Jones–Wenzl projectors for $\mathfrak{sl}_n$. Given a tensor product of fundamental $U_q(\mathfrak{sl}_n)$ representations, there is a corresponding idempotent

$$P \colon V_{\omega_{i_1}} \otimes V_{\omega_{i_2}} \otimes \cdots \otimes V_{\omega_{i_m}} \to V_{\sum_k i_k},$$

called a *clasp* following Kuperberg's terminology from the $\mathfrak{sl}_3$ case. For $n = 3$ these clasps were categorified by the third authors using an $\mathfrak{sl}_3$ foam based construction and a generalization of Rozansky's infinite twist approach to projectors [63].

A related, but more general, approach using infinite twists was independently considered by Cautis who showed that $\mathfrak{sl}_n$ clasps can be categorified explicitly using the higher structure of categorified quantum groups [10]. His approach utilizes an infinite twist construction together with the categorified braid group action described above. Given a reduced decomposition of $w = s_{i_1} \cdots s_{i_k}$ of the longest braid word w in the Weyl group for $\mathfrak{sl}_m$, Cautis defines a complex $\mathcal{T}_w \mathbf{1}_\lambda := \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_k} \mathbf{1}_\lambda$ in $\mathrm{Kom}(\dot{\mathcal{U}}_Q(\mathfrak{sl}_m))$. He shows that the infinite twist $\lim_{\ell \to \infty} \mathcal{T}_w^{2\ell} \mathbf{1}_\lambda$ converges and categorifies the clasp P in any appropriate 2–representation.

Cautis' categorified clasps are formulated explicitly using the 2–morphisms in $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$. Given appropriate families of 2–representations of $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$ with nonzero weight spaces matching the vector space $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ where N and m vary, Cautis' framework gives rise to $\mathfrak{sl}_n$ knot homology theories and categorifications of $\mathfrak{sl}_n$ clasps. Cautis describes such 2–representations using derived categories of coherent sheaves. In Section 4A we show that foamation functors allow Cautis' categorified clasps to be utilized in the foam setting. In the $\mathfrak{sl}_2$ case this gives categorified projectors which can be viewed as an extension of the Cooper–Krushkal and Rozansky projectors to the functorial foam categories of Clark–Morrison–Walker and Blanchet. In the $\mathfrak{sl}_3$ case the resulting projectors agree with those constructed by the third author [63].

1I Recovering relations from categorified quantum groups

Foams can be thought of as a categorification of webs. This perspective suggests that new insights into foam categories can be achieved through categorical skew Howe duality. In [19], Cautis, Kamnitzer and Morrison use skew Howe duality to deduce the $\mathfrak{sl}_n$ web relations. In Section 4C, we show that this holds at the categorified level as well, namely that relations for $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ foams can be deduced from the categorified quantum group.

This suggests that one may gain further insight to $\mathfrak{sl}_n$ foams for $n \geq 4$ using categorical skew Howe duality. In a follow-up paper the second and third author give a foam-based construction of $\mathfrak{sl}_n$ link homologies for $n \geq 4$ which avoids the use of the Kapustin–Li formula [62].

Note that the relations we derive use graded parameters that are usually set to zero in the literature. These relations are similar to the ones of [54] in the $\mathfrak{sl}_3$ case, but in the $\mathfrak{sl}_2$ case we obtain relations that slightly extend both Blanchet's [4] and Clark–Morrison–Walker models [22].

2 Categorified quantum groups

In this section we recall the relevant background information on categorified quantum groups and higher representation theory.

2A The 2–category $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$

Fix a base field $\Bbbk$. We will always work over this field which is not assumed to be of characteristic 0, nor algebraically closed.

2A1 The Cartan datum Let $I = \{1, 2, \ldots, m-1\}$ consist of the set of vertices of the Dynkin diagram of type A_{m-1}

$$\underset{1}{\circ}\!\!-\!\!-\!\!\underset{2}{\circ}\!\!-\!\!-\!\!\underset{3}{\circ}\!\!-\!\!-\!\!-\quad \cdots \quad \!\!-\!\!-\!\!\underset{m-1}{\circ}$$

enumerated from left to right. Let $X = \mathbb{Z}^{m-1}$ denote the weight lattice for $\mathfrak{sl}_m$ and $\{\alpha_i\}_{i \in I} \subset X$ and $\{\Lambda_i\}_{i \in I} \subset X$ denote the collection of simple roots and fundamental

weights, respectively. There is a symmetric bilinear form on X defined by $(\alpha_i, \alpha_j) = a_{ij}$, where

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i - j| = 1, \\ 0 & \text{if } |i - j| > 1, \end{cases}$$

is the (symmetric) Cartan matrix associated to $\mathfrak{sl}_m$. For $i \in I$ denote the simple coroots by $h_i \in X^\vee = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$. Write $\langle \cdot, \cdot \rangle \colon X^\vee \times X \to \mathbb{Z}$ for the canonical pairing $\langle i, \lambda \rangle := \langle h_i, \lambda \rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$ for $i \in I$ and $\lambda \in X$ that satisfies $\langle h_i, \Lambda_i \rangle = \delta_{i,j}$. Any weight $\lambda \in X$ can be written as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m-1})$, where $\lambda_i = \langle h_i, \lambda \rangle$.

We let $X^+ \subset X$ denote the dominant weights, which are those of the form $\sum_i \lambda_i \Lambda_i$ with $\lambda_i \geq 0$. Finally, let $[n] = (q^n - q^{-n})/(q - q^{-1})$ and $[n]! = [n][n-1] \cdots [1]$.

2A2 The algebra $\mathbf{U}_q(\mathfrak{sl}_m)$ The algebra $\mathbf{U}_q(\mathfrak{sl}_m)$ is the $\mathbb{Q}(q)$–algebra with unit generated by the elements E_i, F_i and $K_i^{\pm 1}$ for $i = 1, 2, \ldots, m-1$, with the defining relations

$$(2\text{-}1) \qquad K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i,$$

$$(2\text{-}2) \qquad K_i E_j K_i^{-1} = q^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j,$$

$$(2\text{-}3) \qquad E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$(2\text{-}4) \qquad E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad \text{if } j = i \pm 1,$$

$$(2\text{-}5) \qquad F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0 \quad \text{if } j = i \pm 1,$$

$$(2\text{-}6) \qquad E_i E_j = E_j E_i \quad F_i F_j = F_j F_i \quad \text{if } |i - j| > 1.$$

Recall that $\dot{\mathbf{U}}(\mathfrak{sl}_m)$ is the modified version of $\mathbf{U}_q(\mathfrak{sl}_m)$ where the unit is replaced by a collection of orthogonal idempotents 1_λ indexed by the weight lattice X of $\mathfrak{sl}_m$,

$$(2\text{-}7) \qquad 1_\lambda 1_{\lambda'} = \delta_{\lambda \lambda'} 1_\lambda,$$

such that if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m-1})$, then

$$(2\text{-}8) \qquad K_i 1_\lambda = 1_\lambda K_i = q^{\lambda_i} 1_\lambda, \quad E_i 1_\lambda = 1_{\lambda + \alpha_i} E_i, \quad F_i 1_\lambda = 1_{\lambda - \alpha_i} F_i,$$

where

$$(2\text{-}9) \quad \lambda + \alpha_i = \begin{cases} (\lambda_1 + 2, \lambda_2 - 1, \lambda_3, \ldots, \lambda_{m-2}, \lambda_{m-1}) & \text{if } i = 1, \\ (\lambda_1, \lambda_2, \ldots, \lambda_{m-2}, \lambda_{m-1} - 1, \lambda_{m-1} + 2) & \text{if } i = m - 1, \\ (\lambda_1, \ldots, \lambda_{i-1} - 1, \lambda_i + 2, \lambda_{i+1} - 1, \ldots, \lambda_{m-1}) & \text{otherwise.} \end{cases}$$

Let $\mathcal{A} := \mathbb{Z}[q, q^{-1}]$; the $\mathcal{A}$–algebra ${}_{\mathcal{A}}\dot{\mathbf{U}}(\mathfrak{sl}_m)$ is the integral form of $\dot{\mathbf{U}}(\mathfrak{sl}_m)$ generated by products of divided powers $E_i^{(a)} \mathbf{1}_\lambda := (E_i^a/[a]!)\mathbf{1}_\lambda$, $F_i^{(a)} \mathbf{1}_\lambda := (F_i^a/[a]!)\mathbf{1}_\lambda$ for $\lambda \in X$ and $i = 1, 2, \ldots, m-1$.

2A3 Choice of scalars Q Associated to the Cartan datum for $\mathfrak{sl}_m$ we also fix a choice of scalars Q consisting of t_{ij} for all $i, j \in I$, such that

- $t_{ii} = 1$ for all $i \in I$ and $t_{ij} \in \Bbbk^\times$ for $i \neq j$,

- $t_{ij} = t_{ji}$ when $a_{ij} = 0$.

2A4 The definition We now recall the general version of the 2–category categorifying $\dot{\mathbf{U}}(\mathfrak{sl}_m)$ given in [20]. There, a 2–category $\mathcal{U}_Q(\mathfrak{g})$ was defined associated to any root datum and choice of scalars Q. This 2–category is a modest generalization of the 2–category originally defined in [37] for the choice of scalars Q, where all $t_{ij} = 1$. It follows from [38, page 15] and [37; 39] that $\mathcal{U}_Q(\mathfrak{sl}_m)$ is independent of the choice of scalars Q up to isomorphism. Here we present the general definition; in later sections we will choose a convenient choice of scalars.

Definition 2.1 The 2–category $\mathcal{U}_Q(\mathfrak{sl}_m)$ is the graded additive $\Bbbk$–linear 2–category consisting of:

- Objects λ for $\lambda \in X$.

- 1–morphisms are formal direct sums of (shifts of) compositions of

$$\mathbf{1}_\lambda, \quad \mathbf{1}_{\lambda+\alpha_i}\mathcal{E}_i = \mathbf{1}_{\lambda+\alpha_i}\mathcal{E}_i\mathbf{1}_\lambda = \mathcal{E}_i\mathbf{1}_\lambda, \quad \mathbf{1}_{\lambda-\alpha_i}\mathcal{F}_i = \mathbf{1}_{\lambda-\alpha_i}\mathcal{F}_i\mathbf{1}_\lambda = \mathcal{F}_i\mathbf{1}_\lambda$$

  for $i \in I$ and $\lambda \in X$.

- 2–morphisms are $\Bbbk$–vector spaces spanned by compositions of (decorated) tangle-like diagrams illustrated below.
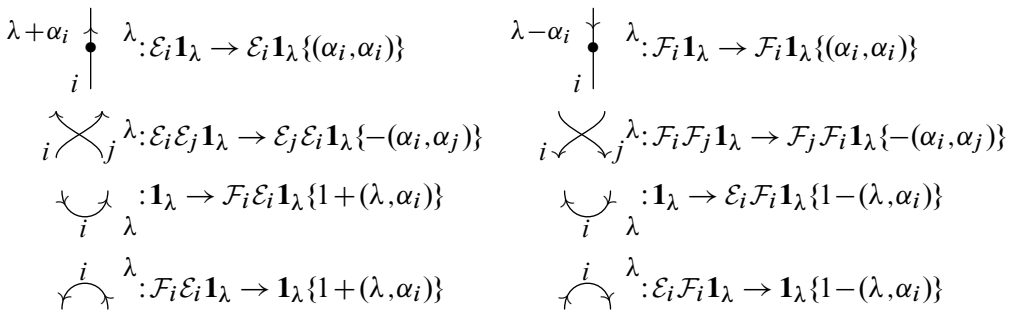
Here we follow the grading conventions in [20] which are opposite to those from [37] but line up nicely with the gradings on foams used later in the paper. For example, a dot

$$\lambda + \alpha_i \underset{i}{\overset{}{\bullet}} \lambda$$

in [37] is a degree-zero map from $\mathcal{E}_i \mathbf{1}_\lambda$ to $\mathcal{E}_i \mathbf{1}_\lambda \{-2\}$, while for us it is a degree-zero map from $\mathcal{E}_i \mathbf{1}_\lambda$ to $\mathcal{E}_i \mathbf{1}_\lambda \{2\}$. In this 2–category (and those throughout the paper) we read diagrams from right to left and bottom to top. The identity 2–morphism of the 1–morphism $\mathcal{E}_i \mathbf{1}_\lambda$ is represented by an upward oriented line labeled by i and the identity 2–morphism of $\mathcal{F}_i \mathbf{1}_\lambda$ is represented by a downward such line.

The 2–morphisms satisfy the following relations:

(1) The 1–morphisms $\mathcal{E}_i \mathbf{1}_\lambda$ and $\mathcal{F}_i \mathbf{1}_\lambda$ are biadjoint (up to a specified degree shift). These conditions are expressed diagrammatically as the following equations.

(2-10)

(2-11)

(2) The 2–morphisms are Q–cyclic with respect to this biadjoint structure.

(2-12)

The Q–cyclic relations for crossings are given below:

(2-13)

The Q–cyclic condition for sideways crossings is given by the equalities

(2-14)
$$\lambda = \lambda = t_{ij} \lambda$$

(2-15)
$$\lambda = = t_{ji} \lambda$$

where the second equality in (2-14) and (2-15) follow from (2-13).

(3) The $\mathcal{E}$'s carry an action of the KLR algebra associated to Q. The KLR algebra $R = R_Q$ associated to Q is defined by finite $\Bbbk$–linear combinations of braid-like diagrams in the plane, where each strand is labeled by a vertex $i \in I$. Strands can intersect and can carry dots but triple intersections are not allowed. Diagrams are considered up to planar isotopy that do not change the combinatorial type of the diagram. We recall the local relations:

(i) If all strands are labeled by the same $i \in I$ then the nilHecke algebra axioms hold:

(2-16)
$$\lambda = 0 \qquad \lambda = \lambda$$

(2-17)
$$\uparrow \; \uparrow \; = \; - \; = \; -$$

(ii) For $i \neq j$

(2-18)
$$\lambda = \begin{cases} t_{ij} \quad i \quad j & \text{if } (\alpha_i, \alpha_j) = 0, \\[2em] t_{ij} \quad i \quad j \; + \; t_{ji} \quad i \quad j & \text{if } (\alpha_i, \alpha_j) \neq 0. \end{cases}$$

(iii) For $i \neq j$ the dot sliding relations

(2-19)
$$\overset{i}{\diagup} \overset{j}{\diagdown} = \overset{i}{\diagup} \overset{j}{\diagdown} \qquad \overset{i}{\diagup} \overset{j}{\diagdown} = \overset{i}{\diagup} \overset{j}{\diagdown}$$

hold.

(iv) Unless $i = k$ and $(\alpha_i, \alpha_j) < 0$ the relation

(2-20)
$$
\vcenter{\hbox{}}\ \lambda \ = \ \vcenter{\hbox{}}\ \lambda
$$

holds. Otherwise, $(\alpha_i, \alpha_j) = -1$ and we have the following.

(2-21)
$$
\vcenter{\hbox{}}\ \lambda \ - \ \vcenter{\hbox{}}\ \lambda \ = \ t_{ij}\ \vcenter{\hbox{}}
$$

(4) When $i \neq j$ one has the mixed relations relating $\mathcal{E}_i \mathcal{F}_j$ and $\mathcal{F}_j \mathcal{E}_i$.

(2-22)
$$
\vcenter{\hbox{}}\ \lambda \ = \ t_{ji}\ \vcenter{\hbox{}}\ \lambda \qquad \vcenter{\hbox{}}\ \lambda \ = \ t_{ij}\ \vcenter{\hbox{}}\ \lambda
$$

(5) Negative degree bubbles are zero. That is, for all $m \in \mathbb{Z}_+$ one has

(2-23)
$$
\vcenter{\hbox{}} = 0 \quad \text{if } m < \lambda_i - 1, \qquad \vcenter{\hbox{}} = 0 \quad \text{if } m < -\lambda_i - 1.
$$

Note that a dotted bubble of degree zero is just the identity 2–morphism:[2]

$$
\vcenter{\hbox{}} = \mathrm{Id}_{\mathbf{1}_\lambda} \quad \text{for } \lambda_i \geq 1, \qquad \vcenter{\hbox{}} = \mathrm{Id}_{\mathbf{1}_\lambda} \quad \text{for } \lambda_i \leq -1.
$$

(6) For any $i \in I$ one has the extended $\mathfrak{sl}_2$–relations. In order to describe certain extended $\mathfrak{sl}_2$–relations it is convenient to use a shorthand notation from [46] called fake bubbles. These are diagrams for dotted bubbles where the labels of the number of dots is negative, but the total degree of the dotted bubble taken with these negative dots is still positive. They allow us to write these extended $\mathfrak{sl}_2$ relations more uniformly (ie independent on whether the weight λ_i is positive or negative).

---

[2]One can define the 2–category so that degree-zero bubbles are multiplication by arbitrary scalars at the cost of modifying some of the other relations; see for example [47; 53]. However, it is shown in [20] that the resulting 2–categories are all isomorphic.

- Degree zero fake bubbles are equal to the identity 2–morphisms

$$\bigcirc_{\lambda_i-1}^{i\quad\lambda} = \mathrm{Id}_{\mathbf{1}_\lambda} \quad \text{if } \lambda_i \leq 0, \qquad \bigcirc_{-\lambda_i-1}^{i\quad\lambda} = \mathrm{Id}_{\mathbf{1}_\lambda} \quad \text{if } \lambda_i \geq 0.$$

- Higher degree fake bubbles for $\lambda_i < 0$ are defined inductively as

$$(2\text{-}24) \qquad \bigcirc_{\lambda_i-1+j}^{i\quad\lambda} = \begin{cases} -\displaystyle\sum_{\substack{a+b=j \\ b\geq 1}} \bigcirc_{\lambda_i-1+a}^{\lambda} \bigcirc_{\lambda-1+b} & \text{if } 0 \leq j < -\lambda_i+1, \\[2em] 0 & \text{if } j < 0. \end{cases}$$

- Higher degree fake bubbles for $\lambda_i > 0$ are defined inductively as

$$(2\text{-}25) \qquad \bigcirc_{-\lambda_i-1+j}^{i\quad\lambda} = \begin{cases} -\displaystyle\sum_{\substack{a+b=j \\ a\geq 1}} \bigcirc_{\lambda_i-1+a}^{\lambda} \bigcirc_{\lambda-1+b} & \text{if } 0 \leq j < \lambda_i+1, \\[2em] 0 & \text{if } j < 0. \end{cases}$$

These equations arise from the homogeneous terms in t of the 'infinite Grassmannian' equation

$$(2\text{-}26) \qquad \left(\bigcirc_{-\lambda_i-1}^{i\quad\lambda} + \bigcirc_{-\lambda_i-1+1}^{i\quad\lambda} t + \cdots + \bigcirc_{-\lambda_i-1+\alpha}^{i\quad\lambda} t^\alpha + \cdots \right)$$

$$\times \left(\bigcirc_{\lambda_i-1}^{i\quad\lambda} + \bigcirc_{\lambda_i-1+1}^{i\quad\lambda} t + \cdots + \bigcirc_{\lambda_i-1+\alpha}^{i\quad\lambda} t^\alpha + \cdots \right) = \mathrm{Id}_{\mathbf{1}_\lambda}.$$

Now we can define the extended $\mathfrak{sl}_2$ relations. Note that in [20] additional curl relations were provided that can be derived from those above. Here we provide a minimal set of relations.

(i) $\lambda_i > 0$:

$$(2\text{-}27) \qquad \bigotimes_{i}^{\lambda} = 0 \qquad \Big|_{i}\ \Big|_{i}^{\lambda} = -\bigotimes_{i\quad i}^{\lambda}$$

$$(2\text{-}28) \qquad \Big|_{i}\ \Big|_{i}^{\lambda} = -\bigotimes_{i\quad i}^{\lambda} + \sum_{\substack{f_1+f_2+f_3 \\ =\lambda_i-1}} \bigcirc_{-\lambda_i-1+f_2}^{\lambda}$$

(ii) $\lambda_i < 0$:

(2-29) $$\text{(diagram)} = 0 \qquad \text{(diagram)} \; \lambda = - \; \text{(diagram)}$$

(2-30) $$\lambda \; \text{(diagram)} \; \lambda = - \; \text{(diagram)} \; \lambda + \sum_{\substack{g_1+g_2+g_3 \\ =-\lambda_i-1}} \text{(diagram)}$$

(iii) $\lambda_i = 0$:

(2-31) $$\text{(diagram)} = - \; \text{(diagram)} \qquad \text{(diagram)} = \; \text{(diagram)}$$

(2-32) $$\text{(diagram)} \; \lambda = - \; \text{(diagram)} \qquad \text{(diagram)} \; \lambda = - \; \text{(diagram)}$$

2A5 Karoubi completions Recall that an idempotent $e\colon b \to b$ in a category $\mathcal{C}$ is a morphism such that $e^2 = e$. The idempotent is said to split if there exist morphisms $b \xrightarrow{g} b' \xrightarrow{h} b$ such that $e = hg$ and $gh = \mathrm{id}_{b'}$. The Karoubi envelope $\mathrm{Kar}(\mathcal{C})$ (also called the idempotent completion or Cauchy completion) of a category $\mathcal{C}$ is a minimal enlargement of the category $\mathcal{C}$ in which all idempotents split. More precisely, the category $\mathrm{Kar}(\mathcal{C})$ has:

• Objects of $\mathrm{Kar}(\mathcal{C})$: pairs (b, e) where $e\colon b \to b$ is an idempotent of $\mathcal{C}$.

• Morphisms $(e, f, e')\colon (b, e) \to (b', e')$, where $f\colon b \to b'$ in $\mathcal{C}$ makes the diagram

(2-33)

commute; ie $ef = f = fe'$.

• Identity 1–morphisms: $(e, e, e)\colon (b, e) \to (b, e)$.

When $\mathcal{C}$ is an additive category we write $(b, e) \in \mathrm{Kar}(\mathcal{C})$ as $\mathrm{im}\, e$ and we have $b \cong \mathrm{im}\, e \oplus \mathrm{im}(1 - e)$ in $\mathrm{Kar}(\mathcal{C})$.

The Karoubi envelope $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m) := \mathrm{Kar}(\mathcal{U}_Q(\mathfrak{sl}_m))$ of the 2–category $\mathcal{U}_Q(\mathfrak{sl}_m)$ is the 2–category with the same objects as $\mathcal{U}_Q(\mathfrak{sl}_m)$ whose Hom categories are given by

$$\dot{\mathcal{U}}_Q(\mathbf{1}_\lambda, \mathbf{1}_{\lambda'}) := \mathrm{Kar}(\mathcal{U}_Q(\mathbf{1}_\lambda, \mathbf{1}_{\lambda'})).$$

In particular, all idempotent 2–morphisms split in $\dot{\mathcal{U}}_Q(\mathbf{1}_\lambda, \mathbf{1}_{\lambda'})$. It was shown in [37] that there is an isomorphism of $\mathcal{A}$–algebras

(2-34)
$$\gamma \colon K_0(\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)) \to {}_{\mathcal{A}}\dot{\mathbf{U}}(\mathfrak{sl}_m)$$

between the split Grothendieck ring $K_0(\dot{\mathcal{U}}_Q(\mathfrak{sl}_m))$ and the integral form ${}_{\mathcal{A}}\dot{\mathbf{U}}(\mathfrak{sl}_m)$ of the idempotent modified quantum enveloping algebra. Recent results of Webster have generalized this statement to arbitrary type [74]. Furthermore, the images of the indecomposable 1–morphisms in $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$ in $K_0(\mathcal{U}_Q(\mathfrak{sl}_m))$ agree with the Lusztig canonical basis in ${}_{\mathcal{A}}\dot{\mathbf{U}}(\mathfrak{sl}_m)$ [76].

Typically the passage from a diagrammatically defined category to its Karoubi envelope results in the loss of a completely diagrammatic description of the resulting category. However, the Karoubi envelope $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$ of the 2–category $\mathcal{U}_Q(\mathfrak{sl}_2)$ still admits a completely diagrammatic description [40]. In this case, one defines idempotent 2–morphisms $e_a \colon \mathcal{E}^a \mathbf{1}_\lambda \to \mathcal{E}^a \mathbf{1}_\lambda$ given by the composite of any reduced presentation of the longest braid word on a strands together with a specific pattern of dots starting with $a - 1$ dots on the left-most strand, $a - 2$ on the next strand, and ending with no dots on the last of the a strands. An example is shown below for $a = 4$.
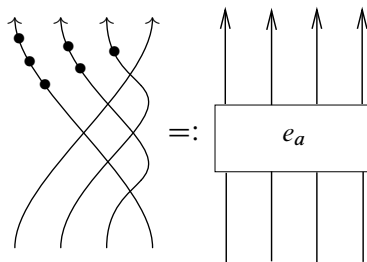
It is convenient to introduce a box notation for this composite 2–morphism.

The divided power $\mathcal{E}^{(a)} \mathbf{1}_\lambda$ is defined in the Karoubi envelope $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$ as the pair

$$\mathcal{E}^{(a)} \mathbf{1}_\lambda := (\mathcal{E}^a \mathbf{1}_\lambda \{a(a-1)/2\}, e_a),$$

where the grading shift is necessary to get an isomorphism $\mathcal{E}^a \mathbf{1}_\lambda \cong \bigoplus_{[a]!} \mathcal{E}^{(a)} \mathbf{1}_\lambda$. The divided power $\mathbf{1}_\lambda \mathcal{F}^{(a)}$ is then defined as the adjoint of $\mathcal{E}^{(a)} \mathbf{1}_\lambda$. It was shown in [40]

that splitting the idempotents e_a by adding $\mathcal{E}^{(a)}\mathbf{1}_\lambda$ and $\mathcal{F}^{(b)}\mathbf{1}_\lambda$ gives rise to explicit decompositions of arbitrary 1–morphisms into indecomposable 1–morphisms using only the relations from $\mathcal{U}_Q(\mathfrak{sl}_2)$. This allows for a strengthening of the categorification result to the case when we define $\mathcal{U}_Q(\mathfrak{sl}_2)$ by taking $\mathbb{Z}$–linear combinations of 2–morphisms, rather than $\mathbb{k}$–linear combinations for a field $\mathbb{k}$.

It is possible to represent the 1–morphisms $\mathcal{E}^{(a)}\mathbf{1}_\lambda$ in $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$ by introducing an augmented graphical calculus of thickened strands. For example, the identity 2–morphism for $\mathcal{E}^{(a)}\mathbf{1}_\lambda$ is given by the following triple, where we think of the label a placed next to the green line as describing the thickness of the strand:

(2-35)
$$(e_a, e_a, e_a) = \left(e_a, \boxed{\quad e_a \quad}_\lambda , e_a \right) =: $$

A downward oriented line of thickness b conveniently describes the 1–morphism $\mathcal{F}^{(b)}\mathbf{1}_\lambda$ in $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$.

One can introduce further notation to describe natural 2–morphisms in $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$. For example, using the shorthand

there are 2–morphisms in $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$ given by

$$:= \left(e_{a+b}, \boxed{e_a}\,\boxed{e_b}_\lambda , e_a e_b \right) : \mathcal{E}^{(a+b)}\mathbf{1}_\lambda\{t\} \to \mathcal{E}^{(a)}\mathcal{E}^{(b)}\mathbf{1}_\lambda\{t-ab\}$$

$$:= \left(e_a e_b, \boxed{e_{a+b}}_\lambda , e_{a+b} \right) : \mathcal{E}^{(a)}\mathcal{E}^{(b)}\mathbf{1}_\lambda\{t\} \to \mathcal{E}^{(a+b)}\mathbf{1}_\lambda\{t-ab\}.$$

To compute the degree of the above diagrams one must account for the shift in the definition of divided powers. For example, in the first diagram the degree shift in the divided power for $\mathcal{E}^{(a+b)}\mathbf{1}_\lambda$ is $\frac{1}{2}(a+b)(a+b-1)$, while the degree shift in the composite $\mathcal{E}^{(a)}\mathcal{E}^{(b)}\mathbf{1}_\lambda$ is $\frac{1}{2}a(a-1)+\frac{1}{2}b(b-1)$, so that the net difference is $\frac{1}{2}2ab = ab$. Both of the above diagrams in the thick calculus have degree $-ab$.

For general m there is no completely diagrammatic description of the Karoubi envelope of $\mathcal{U}_Q(\mathfrak{sl}_m)$. In this case one lacks a set of diagrammatic relations needed to decompose arbitrary 1–morphisms into indecomposables, though explicit isomorphisms giving higher Serre relations were defined by Stošić [67]. It will nevertheless be convenient to introduce a version of the 2–category $\mathcal{U}_Q(\mathfrak{sl}_m)$ where we have split the idempotents needed to define divided powers, but where we have not passed to the full Karoubi completion. Diagrammatically this 2–category can be defined using thick strands carrying two labels, one indicating the thickness of the strand, and one indicated the label $i \in I$ of the a strands. Since the thick strands are defined in terms of idempotents in thin strands, all the 2–morphisms can be studied using only the relations from $\mathcal{U}_Q(\mathfrak{sl}_m)$.

Definition 2.2 Let $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$ denote the full sub-2–category of $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$ with the same objects $\lambda \in X$ as $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$ and with 1–morphisms generated as a graded additive $\Bbbk$–linear category by the 1–morphisms $\mathcal{E}_i^{(a)}\mathbf{1}_\lambda := (\mathcal{E}_i^a\mathbf{1}_\lambda\{\frac{1}{2}a(a-1)\}, e_a)$ and their adjoints.

2A6 2–representations Let $\mathcal{U}_Q$ denote any of the 2–categories $\mathcal{U}_Q(\mathfrak{sl}_m)$, $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$ or $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$.

Definition 2.3 A 2–representation of $\mathcal{U}_Q$ is a graded additive $\Bbbk$–linear 2–functor $\mathcal{U}_Q \to \mathcal{K}$ for some graded, additive 2–category $\mathcal{K}$.

When all of the Hom categories $\mathcal{K}(x, y)$ between objects x and y of $\mathcal{K}$ are idempotent complete, in other words $\mathrm{Kar}(\mathcal{K}) \cong \mathcal{K}$, then any graded additive $\Bbbk$–linear 2–functor $\mathcal{U}_Q(\mathfrak{g}) \to \mathcal{K}$ extends uniquely to a 2–representation of $\dot{\mathcal{U}}_Q(\mathfrak{g})$.

Remark 2.4 For each $i \in I$ there is a sub 2–category $\mathcal{U}_Q(\mathfrak{sl}_2)_i$ of $\mathcal{U}_Q(\mathfrak{sl}_m)$ where we restrict to diagrams where all strands are labeled i. For general 2–representations $\mathcal{F}\colon \mathcal{U}_Q(\mathfrak{sl}_m) \to \mathcal{K}$ it may happen that $\mathcal{K}$ is not Karoubi complete. However, there are many instances when the images of divided powers $\mathcal{E}_i^{(a)}\mathbf{1}_\lambda$ and $\mathcal{F}_i^{(b)}\mathbf{1}_\lambda$ exist in $\mathcal{K}$. In this case, the composite 2–functors $\mathcal{F}_i\colon \mathcal{U}_Q(\mathfrak{sl}_2)_i \to \mathcal{U}_Q(\mathfrak{sl}_m) \to \mathcal{K}$ extend to give 2–representations from the Karoubi envelope of the $\mathfrak{sl}_2$ subcategories $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)_i \to \mathcal{K}$. In this case, the 2–representation $\mathcal{F}$ extends to a 2–representation $\mathcal{F}\colon \check{\mathcal{U}}_Q(\mathfrak{sl}_m) \to \mathcal{K}$.

2A7 Minimal relations and defining 2–functors In [20], it is shown that a 2–representation of $\mathcal{U}_Q(\mathfrak{sl}_m)$ can be specified by defining a 2–category satisfying a small number of axioms. The following is a slightly stronger statement of the main theorem from that work.

Theorem 2.5 [20, Theorem 1.1] *A map $\mathcal{R}$ from the set of weights X of $\mathfrak{sl}_m$ to the objects of a graded additive $\Bbbk$–linear 2–category $\mathcal{K}$ extends to a 2–representation $\mathcal{U}_Q(\mathfrak{sl}_m) \to \mathcal{K}$ provided the following conditions are satisfied:*

(1) *The object $\mathcal{R}(\lambda + r\alpha_i)$ is (isomorphic to) the zero object for $r \gg 0$ or $r \ll 0$.*

(2) *$\mathrm{Hom}_{\mathcal{K}}(\mathbb{1}_\lambda, \mathbb{1}_\lambda\{l\})$ is zero if $l < 0$ and one-dimensional if $l = 0$, where $\mathbb{1}_\lambda$ denotes the identity endomorphism of $\mathcal{R}(\lambda)$. Moreover, the space of 2–morphisms between any two 1–morphisms in $\mathcal{K}$ is finite-dimensional.*

(3) *There exist 1–morphisms $\mathsf{E}_i\mathbb{1}_\lambda\colon \mathcal{R}(\lambda) \to \mathcal{R}(\lambda + \alpha_i)$ in $\mathcal{K}$ which possess both right and left adjoints.*

(4) *Defining 1–morphisms $\mathsf{F}_i\mathbb{1}_\lambda\colon \mathcal{R}(\lambda) \to \mathcal{R}(\lambda - \alpha_i)$ for all $\lambda \in X$ via*

$$\mathsf{F}_i\mathbb{1}_{\lambda+\alpha_i} := (\mathsf{E}_i\mathbb{1}_\lambda)_R\{-\lambda_i - 1\}$$

we have the following isomorphisms in $\mathcal{K}$:

$$\mathsf{F}_i\mathbb{1}_{\lambda+\alpha_i}\mathsf{E}_i\mathbb{1}_\lambda \cong \mathsf{E}_i\mathbb{1}_{\lambda-\alpha_i}\mathsf{F}_i\mathbb{1}_\lambda \oplus \bigoplus_{[-\langle i,\lambda\rangle]} \mathbb{1}_\lambda \quad \text{if } \langle i, \lambda\rangle \leq 0,$$

$$\mathsf{E}_i\mathbb{1}_{\lambda-\alpha_i}\mathsf{F}_i\mathbb{1}_\lambda \cong \mathsf{F}_i\mathbb{1}_{\lambda+\alpha_i}\mathsf{E}_i\mathbb{1}_\lambda \oplus \bigoplus_{[\langle i,\lambda\rangle]} \mathbb{1}_\lambda \quad \text{if } \langle i, \lambda\rangle \geq 0.$$

(5) *The E's carry an action of the KLR algebra associated to Q.*

(6) *If $i \neq j \in I$ then $\mathsf{F}_j\mathbb{1}_{\lambda+\alpha_i}\mathsf{E}_i\mathbb{1}_\lambda \cong \mathsf{E}_i\mathbb{1}_{\lambda-\alpha_j}\mathsf{F}_j\mathbb{1}_\lambda$ in $\mathcal{K}$.*

In the above, we set

$$\bigoplus_{f(q)} M := \bigoplus_{i=-l}^{k} (M\{i\})^{\oplus r_i}$$

when $f(q) = \sum_{i=-l}^{k} r_i q^i$ is a Laurent polynomial with $r_i \geq 0$.

2B Categorified Weyl group action

The Weyl group for $\mathfrak{sl}_m$ is the symmetric group $\mathfrak{S}_m$ generated by transpositions s_i associated to the roots $\alpha_1, \ldots, \alpha_{m-1}$. The action of the Weyl group on the weights lifts

to a braid group action on representations of the associated quantum group $U_q(\mathfrak{sl}_m)$; see for example [49; 29; 15].

The action of a simple transposition is described by an element of the completion $\tilde{U}_q(\mathfrak{sl}_m)$ of $U_q(\mathfrak{sl}_m)$. This ring is defined as a quotient of the ring of series $\sum_{k=1}^{\infty} X_k$ of elements of $U_q(\mathfrak{sl}_m)$ acting on each irreducible representation V_λ of highest weight λ by zero but for finitely many terms X_k; see [29]. To s_i, we associate the braiding map $T_i \in U_q(\tilde{\mathfrak{sl}}_m)$:

$$(2\text{-}36) \qquad T_i 1_\lambda := \sum_{s \geq 0} (-q)^s E_i^{(-\lambda_i+s)} F_i^{(s)} 1_\lambda \quad \text{if } \lambda_i \leq 0,$$

$$(2\text{-}37) \qquad T_i 1_\lambda := \sum_{s \geq 0} (-q)^s F_i^{(\lambda_i+s)} E_i^{(s)} 1_\lambda \quad \text{if } \lambda_i \geq 0.$$

This definition differs from the one given in [49, Section 5.2.1] but is equivalent up to rescaling; see [10, Remark 6.4]. With this definition, $T_i = \sum_{\lambda \in X} T_i 1_\lambda$ gives an endomorphism of any finite-dimensional representation. Note that if v is a weight vector of weight λ, $T_i(v)$ is a weight vector of weight $s_i(\lambda)$.

For $\mathfrak{sl}_2$ the deformed Weyl group action on a $U_q(\mathfrak{sl}_2)$–representation V gives a reflection isomorphism from the λ weight space of V to the $-\lambda$ weight space. This reflection isomorphism was categorified by Chuang and Rouquier in the context of abelian categories [21]. Their work is closely related to a variant of the 2–category $\mathcal{U}_Q(\mathfrak{sl}_2)$ where the nilHecke algebra is replaced by the affine Hecke algebra and there is no grading. Cautis, Kamnitzer and Licata later developed analogous complexes in the context of $\mathcal{U}_Q(\mathfrak{sl}_2)$ and generalized Chuang and Rouquier's results to triangulated categories [18].

To categorify the reflection isomorphism $T_i 1_\lambda$ it is clear from (2-36) and (2-37) that we will need to work in $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$ so that we have lifts of divided powers. Also, the minus signs in the definition of the braid group generators suggests that we will have to pass to the 2–category $\text{Kom}(\check{\mathcal{U}}_Q(\mathfrak{sl}_m))$ of bounded complexes over the 2–category $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$ whose objects are weights $\lambda \in X$, 1–morphisms are bounded complexes of 1–morphisms in $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$, and 2–morphisms are chain maps constructed from the 2–morphisms in $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$.

The braid group generator $T_i 1_\lambda$ lifts to a complex $\mathcal{T}_i 1_\lambda$ in $\text{Kom}(\check{\mathcal{U}}_Q(\mathfrak{sl}_m))$ of the form,[3]

$$(2\text{-}38) \quad \mathcal{T}_i \mathbf{1}_\lambda = \mathcal{E}_i^{(-\lambda_i)} \mathbf{1}_\lambda \xrightarrow{d_1} \mathcal{E}_i^{(-\lambda_i+1)} \mathcal{F}_i \mathbf{1}_\lambda\{1\} \xrightarrow{d_2} \cdots \xrightarrow{d_s} \mathcal{E}_i^{(-\lambda_i+s)} \mathcal{F}_i^{(s)} \mathbf{1}_\lambda\{s\} \xrightarrow{d_{s+1}} \cdots$$

---

[3]Note that we take the mirror of Cautis's definition of these complexes in [10], in order to better fit with usual definition of Khovanov homology. This also reverses the decategorification process so that a shift by k will decategorify to q^k, while it decategorifies to q^{-k} in [10].

when $\lambda_i \leq 0$ and

$$(2\text{-}39) \quad \mathcal{T}_i \mathbf{1}_\lambda = \mathcal{F}_i^{(\lambda_i)} \mathbf{1}_\lambda \xrightarrow{d_1} \mathcal{F}_i^{(\lambda_i+1)} \mathcal{E}_i \mathbf{1}_\lambda \{1\} \xrightarrow{d_2} \cdots \xrightarrow{d_s} \mathcal{F}_i^{(\lambda_i+s)} \mathcal{E}_i^{(s)} \mathbf{1}_\lambda \{s\} \xrightarrow{d_{s+1}} \cdots$$

when $\lambda_i \geq 0$, where in the above formulae the leftmost term is in homological degree 0. The differential d_k that appears in the first complex is conveniently expressed in the extended graphical calculus from [40] as

$$d_k =$$

where all strands are colored by the index $i \in I$ and the labels indicate the thickness of strands. The differential in the second complex is defined similarly. Using the extended calculus it is easy to see that $d^2 = 0$. Results of Cautis and Kamnitzer show the images of the complexes $\mathcal{T}_i \mathbf{1}_\lambda$ under any integrable 2–representation $\check{\mathcal{U}}(\mathfrak{sl}_m) \to \mathcal{K}$ satisfy braid relations up to homotopy in $\mathrm{Kom}(\mathcal{K})$ [14, Section 6].

The complexes $\mathcal{T}_i \mathbf{1}_\lambda$ are invertible, up to homotopy, with inverses given by taking the left adjoint of the complex $\mathcal{T}_i \mathbf{1}_\lambda$ in the 2–category $\mathrm{Kom}(\check{\mathcal{U}}_Q(\mathfrak{sl}_m))$. More explicitly, the inverses are given by

$$\mathbf{1}_\lambda \mathcal{T}_i^{-1} = \cdots \xrightarrow{d_{s+1}^*} \mathbf{1}_\lambda \mathcal{E}_i^{(s)} \mathcal{F}_i^{(-\lambda_i+s)} \{-s\} \xrightarrow{d_s^*} \cdots \xrightarrow{d_2^*} \mathbf{1}_\lambda \mathcal{E}_i \mathcal{F}_i^{(-\lambda_i+1)} \{-1\} \xrightarrow{d_1^*} \mathbf{1}_\lambda \mathcal{F}_i^{(-\lambda_i)}$$

when $\lambda_i \leq 0$ and

$$\mathbf{1}_\lambda \mathcal{T}_i^{-1} = \cdots \xrightarrow{d_{s+1}^*} \mathbf{1}_\lambda \mathcal{F}_i^{(s)} \mathcal{E}_i^{(\lambda_i+s)} \{-s\} \xrightarrow{d_s^*} \cdots \xrightarrow{d_2^*} \mathbf{1}_\lambda \mathcal{F}_i \mathcal{E}_i^{(\lambda_i+1)} \{-1\} \xrightarrow{d_1^*} \mathbf{1}_\lambda \mathcal{E}_i^{(\lambda_i)}$$

when $\lambda_i \geq 0$, where in these formulae the rightmost term is in homological degree zero.

Given a 2–representation $\mathcal{F}: \check{\mathcal{U}}_Q(\mathfrak{sl}_m) \to \mathcal{K}$, the braiding $T_i 1_\lambda$ is lifted to a complex $\mathcal{T}_i \mathbf{1}_\lambda$ that gives an equivalence between $\mathcal{F}(\lambda)$ and $\mathcal{F}(s_i(\lambda))$ in the homotopy 2–category of complexes over $\mathcal{K}$. For $\mathfrak{sl}_2$ there are no interesting braid relations to check. The content of a categorification of the reflection isomorphism is that the complex $\mathcal{T}_i \mathbf{1}_\lambda$ has a homotopy inverse, so that a 2–representation $\check{\mathcal{U}}_Q(\mathfrak{sl}_2) \to \mathcal{K}$ induces an equivalence in the homotopy category of complexes over $\mathcal{K}$ [21, Theorem 6.4]. The resulting equivalences are highly nontrivial and have been applied to a variety of contexts ranging from the representation theory of the symmetric group [21] to coherent sheaves on cotangent bundles [16; 18]. Cautis and Kamnitzer later showed that given an integrable 2–representation $\mathcal{U}_Q(\mathfrak{sl}_m) \to \mathcal{K}$ the complexes $\mathcal{T}_i \mathbf{1}_\lambda$ defined for each $i \in I$ satisfy the braid relations [14, Section 6]; see also [10, Section 4.1]. This is a crucial

observation for Cautis's construction of knot homology theories from the 2–category $\mathcal{U}_Q(\mathfrak{sl}_m)$.

3 Foams and foamation

We now aim to define families of foamation 2–functors from $\mathcal{U}_Q(\mathfrak{sl}_m)$ to certain 2–categories of $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ foams. We use the particular choice of scalars Q given by $t_{i,i+1} = 1$, $t_{i,i-1} = -1$ and $t_{i,j} = 1$ when $a_{i,j} = 0$.

3A $\mathfrak{sl}_2$ foam 2–categories

In this section, we define a family of 2–functors from $\mathcal{U}_Q(\mathfrak{sl}_m)$ to suitable categories of $\mathfrak{sl}_2$ foams. We first review Bar-Natan's cobordism-based construction of ($\mathfrak{sl}_2$) Khovanov homology [2] as well as a functorial enhancement of this theory due to Blanchet [4] which encodes additional representation-theoretic information. We will define our foamation 2–functors into a family of related 2–categories which are natural to consider from the perspective of skew Howe duality. We also construct such 2–functors into the Clark–Morrison–Walker functorial formulation of Khovanov homology [22].

3A1 Standard $\mathfrak{sl}_2$ foams In [2], Bar Natan gave a construction of Khovanov homology as a quotient of the cobordism category of planar tangles and surfaces. This work gives a categorification of (a version of) the category 2**Web**. We summarize this construction, which can be understood as a 2–category defined as follows:

- Objects are sequences of points in the interval $[0, 1]$, together with a zero object.

- 1–morphisms are formal direct sums of $\mathbb{Z}$–graded shifts[4] of planar tangles, with boundary corresponding to the sequences of points in the domain and codomain.

- 2–morphisms are formal matrices of $\mathbb{k}$–linear combinations of degree-zero dotted cobordisms between such planar curves, modulo isotopy (relative to the boundary) and local relations.
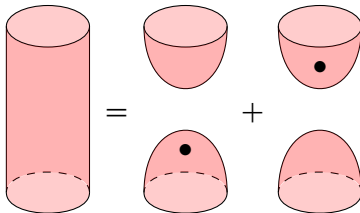
Let $q^k T$ denote the shift of a planar tangle T by k in $\mathbb{Z}$–grading. The degree of a cobordism $C\colon q^{t_1}T_1 \to q^{t_2}T_2$ is then defined by the formula

$$(3\text{-}1) \qquad \deg(C) = \chi(C) - 2\#D - \tfrac{1}{2}\#\partial + t_2 - t_1,$$

---
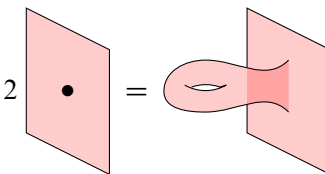
[4]In other words, a web is a priori of degree zero, and can be shifted into a different $\mathbb{Z}$–degree to yield a different 1–morphism.

where #D is the number of dots and #∂ is the number of boundary points in either T_1 or T_2 (they agree!). The local relations are then given as follows.

(3-2)   = 0   = 1

(3-3)

The neck-cutting relation (3-3) gives the formula

(3-4)

which allows for a completely topological description of the 2–category when 2 is invertible in $\Bbbk$.

The version of the Bar-Natan cobordism category presented here is the generic one, as defined in [2, Section 4]. In this formulation, not all closed surfaces of genus $g > 1$ equal zero. To be precise, the 3–dotted sphere (which is a multiple of a genus 3 surface) survives as a degree-4 parameter and all other closed surfaces can be expressed in terms of this (the fact that a 2–dotted sphere equals zero follows from relation (3-3)). Specializing this parameter to zero recovers Khovanov's original homology theory, and to one yields the Lee degeneration [2, Section 9]. In our context, it makes sense to keep this parameter, as the same phenomenon will appear on the categorified quantum group side of skew Howe duality. See Section 4 for more details.

As mentioned in Section 1, the $+$ sign in the neck-cutting relation prevents us from defining a 2–functor from $\mathcal{U}_Q(\mathfrak{sl}_m)$ to this 2–category since it is incompatible with the sign in the nilHecke relation. We hence consider related versions of this construction.

3A2 Enhanced foams Bar-Natan formulates Khovanov homology in the homotopy category of complexes in the above 2–category, giving an invariant which is functorial only up to a $\pm$ sign under tangle cobordism. This functoriality issue was fixed by Clark, Morrison and Walker [22] working in a related 2–category of disoriented curves

and cobordisms defined over the Gaussian integers;[5] see also the work of Caprau [7; 8; 9] for a related construction. Blanchet [4] later gave another functorial construction of Khovanov homology in a related 2–category defined over the integers.

It turns out that in addition to fixing functoriality, these later constructions also fix the incompatibility of the neck-cutting and nilHecke relations. We will work in Blanchet's enhanced foam model since it is more natural to consider from the perspective of skew Howe duality and it avoids the introduction of complex coefficients. We return to the Clark–Morrison–Walker (CMW) construction in the following section.

We begin by defining a family of 2–categories related to Blanchet's construction which should be viewed as categorifications of the categories $2\mathbf{BWeb}_m(N)$.

Definition 3.1 $2\mathbf{BFoam}_m(N)$ is the 2–category defined as follows:

- Objects are sequences $(a_1, \ldots, a_m)$ labeling points in the interval $[0, 1]$ with $a_i \in \{0, 1, 2\}$ and $N = \sum_{i=1}^m a_i$, together with a zero object.

- 1–morphisms are formal direct sums of $\mathbb{Z}$–graded shifts of enhanced $\mathfrak{sl}_2$ webs: directed planar graphs with boundary with two types of edges — 1–labeled edges ⟶◀— and 2–labeled edges ⟹◀— — where all vertices are trivalent and take the following two forms:

(3-5)
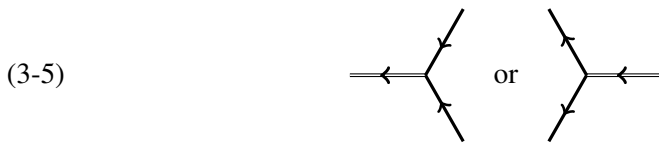
  1– (respectively 2–) labeled edges are directed out from points labeled by 1 (respectively 2) in the domain and directed into such labeled points in the codomain. No edges are attached to points labeled by 0.

- 2–morphisms are formal matrices of $\Bbbk$–linear combinations of degree-zero $\mathfrak{sl}_2$ foams — surfaces with oriented singular seams which locally look like the product of the letter Y with an interval — considered up to isotopy (relative to the boundary) and local relations.
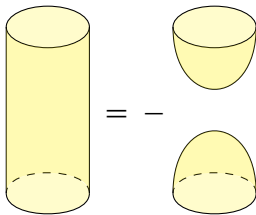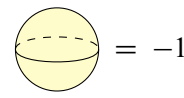
There are two types of facets of an $\mathfrak{sl}_2$ foam, 1–labeled and 2–labeled, depending on which type of edge they are incident upon when the foam is expressed as a composition of elementary foams. The degree of a foam $F: q^{t_1} W_1 \to q^{t_2} W_2$ is given by the degree of the cobordism resulting from deleting all the 2–labeled facets and edges and forgetting the orientation of the 1–labeled edges.

---

[5]Actually, they work over the ring $\mathbb{Z}[\frac{1}{2}, i]$.
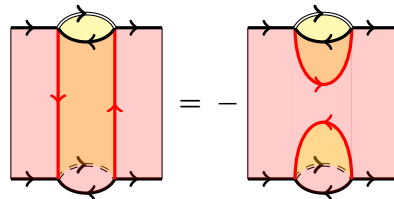
As in $\mathcal{U}_Q(\mathfrak{sl}_m)$, we shall read diagrammatic depictions of webs and foams from right to left and from bottom to top. The orientation of a singular seam gives a cyclic ordering of the facets incident upon the seam via the right hand rule. By convention, a seam travels down through the first vertex in (3-5) and up through the second; this corresponds to the cyclic orientation of web vertices from [4].
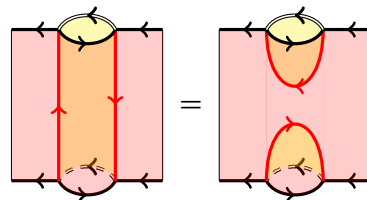
The relations for $\mathfrak{sl}_2$ foams come from a non-local, universal construction detailed in [4]; however, we can exhibit a complete set of local relations giving an equivalent description of Blanchet's work. In what follows, the 2–labeled facets are depicted in yellow and 1–labeled facets are drawn in red.
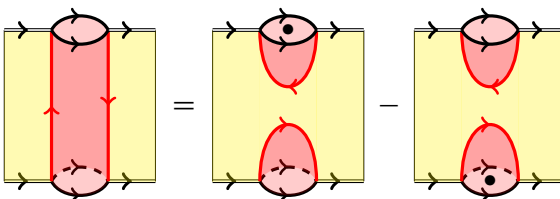
We impose the relations (3-2) and (3-3) for 1–labeled facets, as well as the following relations involving 2–labeled facets.

(3-6) $\qquad$ = −

(3-7) $\qquad$ = −1
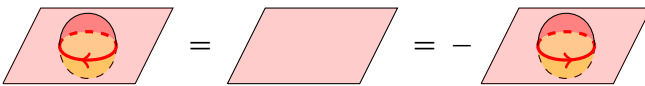
(3-8) $\qquad$ $= \begin{cases} 1 & \text{if } (\alpha, \beta) = (1, 0) \\ -1 & \text{if } (\alpha, \beta) = (0, 1) \\ 0 & \text{if } (\alpha, \beta) = (0, 0) \text{ or } (1, 1) \end{cases}$

(3-9) $\qquad$ = −

(3-10) $\qquad$ =
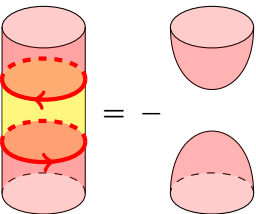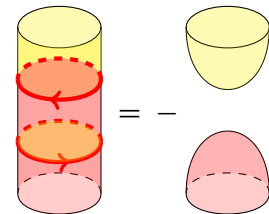
(3-11) $\qquad$ = $\qquad$ −
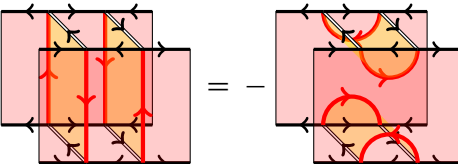
Relations (3-2), (3-3) and (3-6)–(3-8) allow for the evaluation of any closed $\mathfrak{sl}_2$ foam. Additionally, note that these relations imply that we can reverse the direction of any closed, singular seam at the cost of multiplying by -1. Equations (3-9), (3-10), and (3-11) guarantee that if a linear combination of foams evaluates to zero whenever it is closed off to give a closed foam, then that linear combination is zero; the latter is a non-local relation from [4]. The equivalence of this relation to the collection of local relations given above can be proved in a manner similar to the proof of [59, Lemma 3.5] using the above local relations. The proof utilizes several relations that follow from the above local relations, allowing a web with a digon or square face to be expressed in terms of webs with fewer faces.
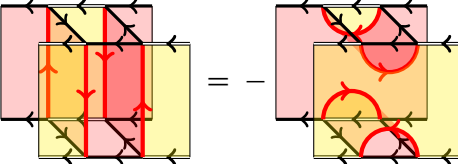
(3-12)
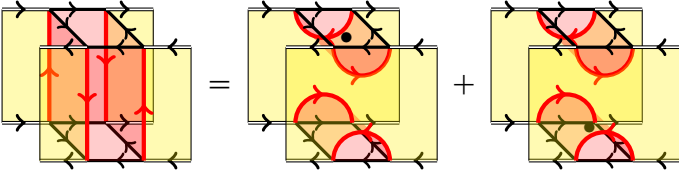
(3-13)
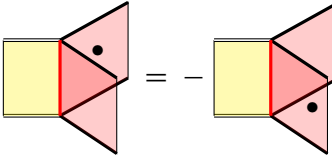
(3-14)

(3-15)

(3-16)

(3-17)

(3-18)

(3-19)

(3-20)

The neck-cutting relation (3-6) implies that the topology of the 2–labeled facets plays a limited role. One may hence ask if there is a way to coherently delete such facets and obtain a forgetful 2–functor from $2\mathbf{BFoam}_m(N)$ to (an appropriate version of) the Bar-Natan 2–category.

Such a 2–functor would act via
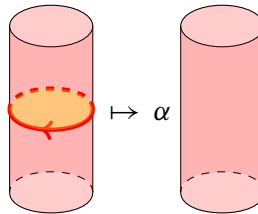
for some scalar α; (3-12) shows that composing the left-hand foam with a cap produces a cap, while pre-composing with a cup gives -1 multiplied by a cup. It is therefore impossible to define such a 2–functor which acts as the identity on foams containing no 2–labeled facets (unless working over a field of characteristic 2).

3A3 Foamation We now define $\mathfrak{sl}_2$ foamation 2–functors $\mathcal{U}_Q(\mathfrak{sl}_m) \to 2\mathbf{BFoam}_m(N)$ categorifying the skew Howe map to webs discussed in Section 1. As in the de-categorified case, we define the 2–functor on objects by sending an $\mathfrak{sl}_m$ weight $\lambda = (\lambda_1, \ldots, \lambda_{m-1})$ to the sequence $(a_1, \ldots, a_m)$ with $a_i \in \{0, 1, 2\}$, $\lambda_i = a_{i+1} - a_i$, and $\sum_{i=1}^m a_i = N$ provided it exists and to the zero object otherwise.

The map is given on 1–morphisms by

$$\mathbf{1}_\lambda\{t\} \mapsto q^t \quad \begin{array}{c} \underleftarrow{\hspace{3cm}} \; a_m \\ \vdots \\ \underleftarrow{\hspace{3cm}} \; a_1 \end{array}$$

when the boundary values lie in $\{0, 1, 2\}$ and to the zero 1–morphism otherwise. The labelings of the edges incident upon the boundary are given by the boundary labels; edges incident upon boundary points labeled by zero should be deleted. Note that we have not depicted $m - 2$ horizontal strands in each of the latter two formulae.

We will make use of a lemma to deduce the existence of the foamation functors. Let the images of $\mathbf{1}_\lambda$, $\mathcal{E}_i\mathbf{1}_\lambda$, and $\mathcal{F}_i\mathbf{1}_\lambda$ given above be denoted $\mathbb{1}_\lambda$, $\mathsf{E}_i\mathbb{1}_\lambda$ and $\mathsf{F}_i\mathbb{1}_\lambda$.

Lemma 3.2 *There are isomorphisms*

$$\mathsf{F}_i\mathsf{E}_i\mathbb{1}_\lambda \cong \mathsf{E}_i\mathsf{F}_i\mathbb{1}_\lambda \oplus \bigoplus_{[-\langle i,\lambda\rangle]} \mathbb{1}_\lambda \quad \text{if } \langle i,\lambda\rangle \leq 0,$$

$$\mathsf{E}_i\mathsf{F}_i\mathbb{1}_\lambda \cong \mathsf{F}_i\mathsf{E}_i\mathbb{1}_\lambda \oplus \bigoplus_{[\langle i,\lambda\rangle]} \mathbb{1}_\lambda \quad \text{if } \langle i,\lambda\rangle \geq 0,$$

and $\mathsf{F}_j\mathsf{E}_i\mathbb{1}_\lambda \cong \mathsf{E}_i\mathsf{F}_j\mathbb{1}_\lambda$ *for* $i \neq j \in I$ *in* $2\mathbf{Foam}_m(N)$.

Proof We'll prove only the first relation since the proof of the second is analogous and the third is straightforward. The condition on weights implies that λ maps to a sequence with $a_{i+1} \leq a_i$.

If $a_i = 0$, $a_{i+1} = 0$ and both sides of the equation map to the zero foam. If $a_i = 1$ and $a_{i+1} = 0$, then the web
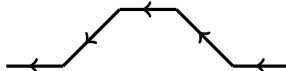
is isomorphic to $\mathbb{1}_\lambda$ via the foam realizing the web isotopy.

If $a_i = a_{i+1} = 1$, then the relevant webs are isotopic:
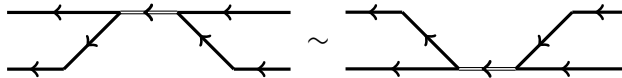
hence isomorphic.

If $a_i = 2$ and $a_{i+1} = 0$, then the web
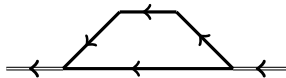
is isomorphic to $q^{-1}\mathbb{1}_\lambda \oplus q\mathbb{1}_\lambda$ using equations (3-11), (3-13), and (3-14). This confirms the relation since $\mathsf{E}_i\mathsf{F}_i\mathbb{1}_\lambda \mapsto 0$ for such a weight.

If $a_i = 2$ and $a_{i+1} = 1$, then the web
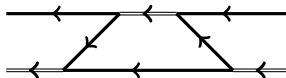
is isomorphic to $\mathbb{1}_\lambda$ using equations (3-16) and (3-18). Finally, if $a_i = a_{i+1} = 2$, both sides of the equation map to zero. $\qquad\square$

Proposition 3.3 *There is a 2–representation*

$$\Phi_2\colon \mathcal{U}_Q(\mathfrak{sl}_m) \to 2\mathbf{BFoam}_m(N)$$

for each $N > 0$ specified on single strand 2–morphisms by

on crossings by

where $j - i > 1$, and on caps and cups by

where in the above diagrams the ith sheet is always in the front. To obtain the specific image foam from these (generic[6]) pictures, use the labelings of the domain and codomain webs (which are determined by the $\mathfrak{sl}_m$ weight λ) to induce a labeling on all foam facets, then remove all 0–labeled facets[7] and re-color the remaining facets appropriately. In particular, the blue-colored facets in the above are used only to make the pictures more readable.

Note that the singular seams may degenerate in such examples, eg

$$\Phi_2\left(\underset{i}{\overset{}{\times}}_i , \lambda \right) = $$

when λ maps to a sequence with $a_i = 2$ and $a_{i+1} = 0$.

Moreover, the generic depiction of the foam in this example contained an intersection of 2 seams (before removing facets). It is not difficult to check that all pictures containing such a singularity yield well-defined foams, where some singular seams degenerate and the intersection between seams disappears.

Additionally, to make the pictures above more readable, we haven't depicted the orientation of the seams in Proposition 3.3. In the case where they do not degenerate, the orientation is induced by the corresponding trivalent vertices in the boundary webs. Explicitly, seams are directed into trivalent vertices which split a 2–labeled edge into 1–labeled edges in the codomain of a foam, and out from vertices which merge 1–labeled edges into 2–labeled ones (and vice-versa in the domain web).

Proof While it is not difficult to verify all relations by hand, we apply Theorem 2.5 to $2\mathbf{BFoam}_m(N)$ to reduce the number of relations that need to be verified. For each m and N, the non-zero objects of this 2–category are indexed by the non-zero $\mathfrak{sl}_m$ weight spaces of the finite-dimensional $U_q(\mathfrak{sl}_m)$–module $\bigwedge_q^N(\mathbb{C}^2 \otimes \mathbb{C}^m)$, so condition (1) is satisfied. Furthermore, it is clear from the definitions that $\mathcal{E}_i \mathbb{1}_\lambda$ has $\mathcal{F}_i \mathbb{1}_{\lambda+\alpha_i}$ as a left and right adjoint, up to a grading shift. Lemma 3.2 establishes conditions (4) and (6), thus, it suffices to show that conditions (2) and (5) are satisfied.

We first check condition (2). Given a foam in $\mathrm{Hom}(\mathbb{1}_\lambda, q^t \mathbb{1}_\lambda)$, we can apply the neck-cutting relations (3-3) and (3-6) in the neighborhood of each boundary component to express the foam as a linear combination of foams which are the disjoint unions of

---

[6]These pictures more generally depict $\mathfrak{sl}_n$ foams; see the follow-up paper [62] for complete details.

[7]One can check that any foam which would contain facets not labeled by 0, 1 or 2 would necessarily have either domain or codomain equal to the zero 1–morphism, hence is zero.

closed foams, 2–labeled sheets and 1–labeled sheets, which may carry dots. Relations (3-2), (3-3) and (3-6)–(3-8) give that any closed foam is equal to an element of $\Bbbk[\,\overset{3\,\bullet}{\bigcirc}\,]$. (3-1) then shows that $\mathrm{Hom}(\mathbb{1}_\lambda, q^t\mathbb{1}_\lambda)$ is zero for $t < 0$ and 1–dimensional for $t = 0$.

Using the neck-cutting relations, we can express any foam mapping between fixed webs W_1 and W_2 as a linear combination of foams in which every 2–labeled facet is a disk incident upon the boundary; such facets are determined by the collection of singular seams incident upon the web vertices. The union of the 1–labeled facets gives a (dotted) cobordism between the 1–labeled edges of W_1 and W_2. Using the neck-cutting relations and (3-11) we can assume that this cobordism consists of (dotted) disks. Since there are only finitely many ways to connect the vertices of the boundary webs with singular seams lying on the cobordism (up to isotopy), it follows from (3-1) that $\mathrm{Hom}(q^{t_1}W_1, q^{t_2}W_2)$ is finite-dimensional for all values of t_1 and t_2.

Finally, we check condition (5), ie that the KLR relations are satisfied.

Relation (2-16) The 2–morphism

automatically maps to the zero foam unless λ maps to a sequence with $a_i = 2$ and $a_{i+1} = 0$. In this case, we compute the image:
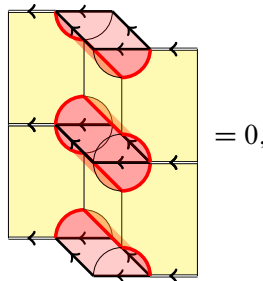
which follows from (3-13). The images of the 2–morphisms
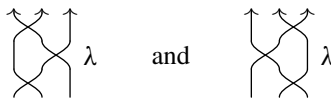
are both zero since either λ or $\lambda + 3\alpha_i$ maps to the zero object, confirming the relation.

Relation (2-17) As before, the only non-trivial case is when λ maps to a sequence with $a_i = 2$ and $a_{i+1} = 0$. In this case, we must have the equalities
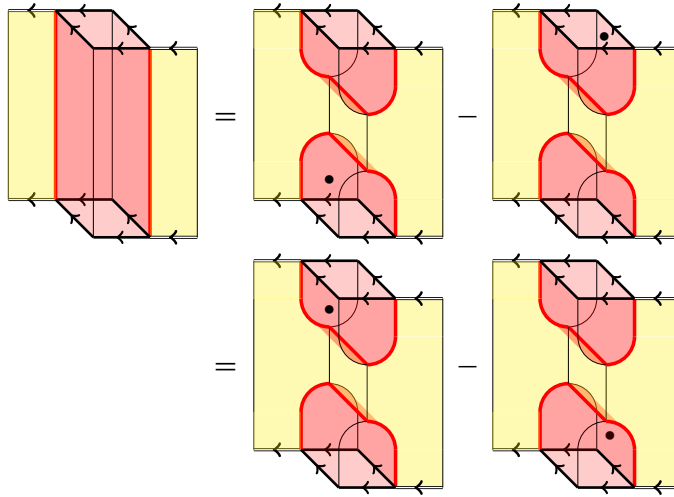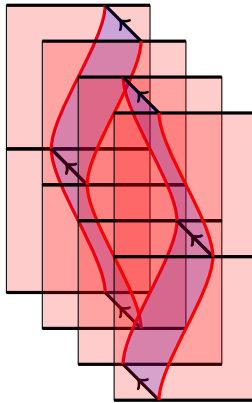
both of which follow from (3-11).

Relation (2-18) The equality

$$(\alpha_i, \alpha_j) = 0$$

corresponds to $|i - j| \geq 2$ in which case the image of the relation is realized via an isotopy. For example, when $i < j$ we see that
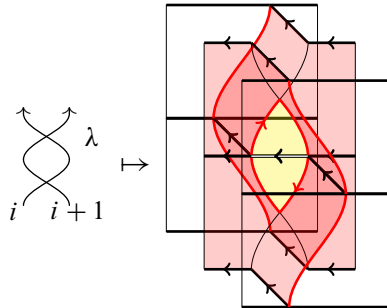

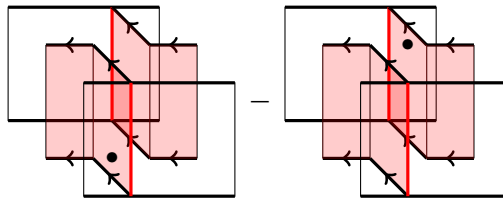is isotopic to the identity foam for any values of the a_l's.

If $i \neq j$ and

$$(\alpha_i, \alpha_j) \neq 0$$

we must have $j = i \pm 1$. We begin with the case $j = i + 1$. The image of the left-hand side is zero unless $a_{i+1} = 1$ since the intermediate objects in the relation map to the zero object in the image; similarly, the right-hand image is zero unless $a_{i+1} = 1, 2$.

When $a_{i+1} = 1$ we have

where we have omitted the shading on the front and back sheets for clarity. Applying (3-3) on both sides of the singular seam and evaluating the resulting theta-foams using (3-8), this gives

which is the image of

$$\overset{\uparrow\,\uparrow}{\underset{i}{\Big|}\overset{\bullet}{\Big|}}\underset{i+1}{\Big|} - \underset{i}{\Big|}\overset{\uparrow\,\uparrow}{\Big|}\overset{\bullet}{\underset{i+1}{\Big|}}.$$

When $a_{i+1} = 2$, we must confirm that
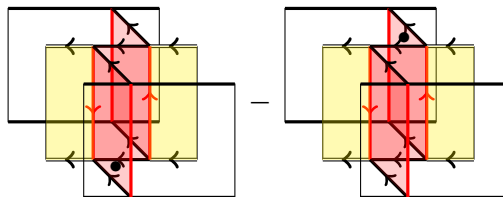
is the zero foam. This follows from the dot-sliding relation (3-20).

When $j = i-1$, it similarly suffices to confirm the relation when $a_i = 0, 1$. For $a_i = 0$, the images of

$$\overset{\uparrow\,\uparrow}{\underset{i}{\Big|}\overset{\bullet}{\Big|}}\underset{i-1}{\Big|} \quad \text{and} \quad \underset{i}{\Big|}\overset{\uparrow\,\uparrow}{\Big|}\overset{\bullet}{\underset{i-1}{\Big|}}$$

are isotopic, so both sides of the relation map to zero.

For $a_i = 1$, we compute as follows.
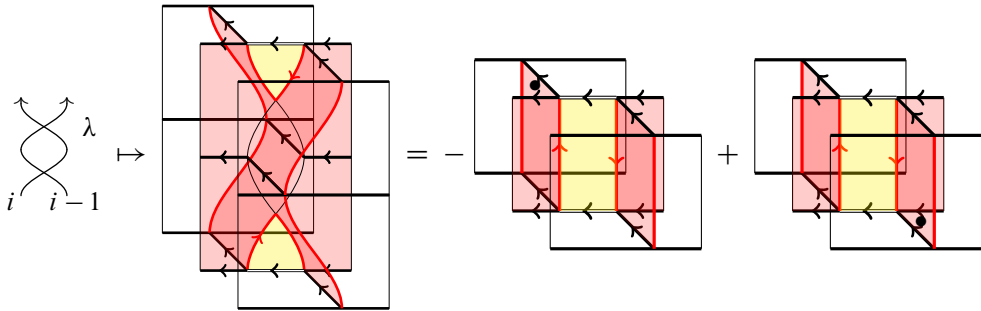
To see this, note the middle of the image foam takes the form of the left-hand side of Equation (3-11) turned sideways, ie there is a sideways 1–labeled cylinder meeting a pair of 2–labeled facets along oppositely oriented singular seams. Applying this relation and using Equation (3-20) shows that it equals the specified linear combination of foams.

Relation (2-19) This relation follows by sliding a dot along a facet, ie via isotopy.

Relation (2-20) For all choices of i, j and k this relation holds via isotopy (or since both sides map to zero). This is obvious in the case that two of the three values (α_i, α_j), (α_i, α_k) and (α_j, α_k) are zero. In the other cases a computation is necessary; note that we can assume $i \neq j \neq k$ since otherwise both sides of the equation automatically map to zero (an intermediate weight must map to the zero object).

If $j = i + 1$ and $k = i + 2$, then we compute both sides of the relation to be

(3-21) and
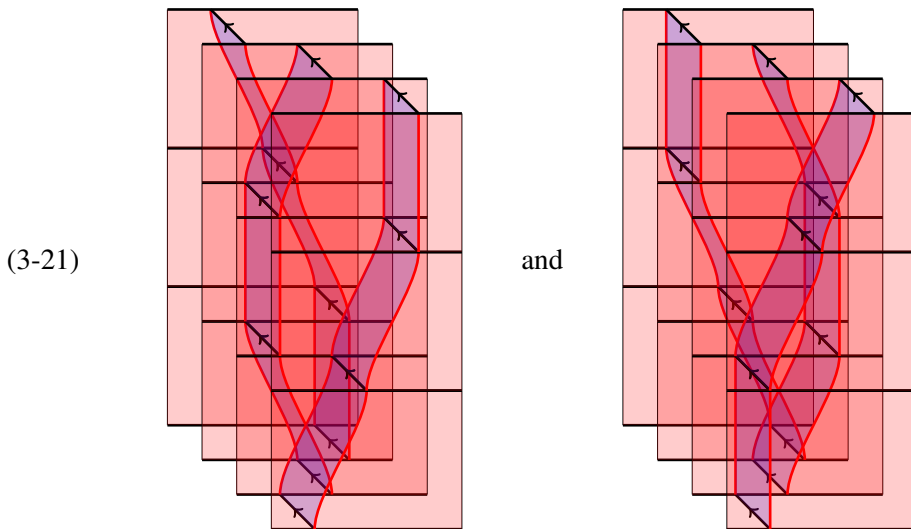
which are equal up to isotopy for any value of the a_l's. The other cases follow similarly.

Relation (2-21) We must have $j = i \pm 1$ and we'll only compute for $j = i + 1$ since the other case is analogous. Note that all 2–morphisms involved automatically map to zero if λ is sent to a sequence with $a_{i+1} = 2$ or with $a_i = 0, 1$, so we'll compute for the remaining values.
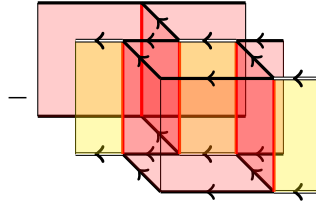
When $a_i = 2$ and $a_{i+1} = 0$ we have
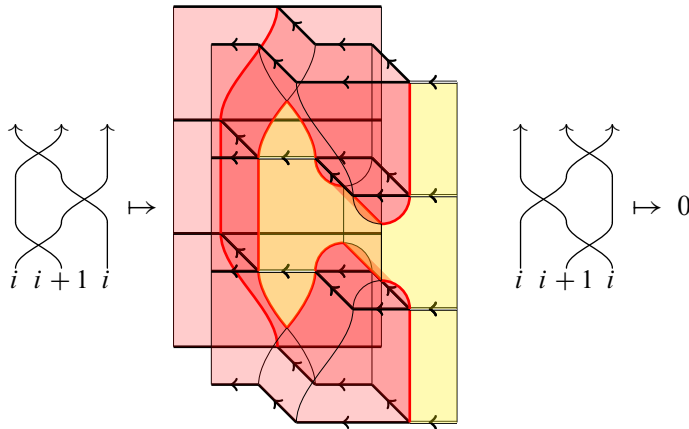
which gives the relation since the former is isotopic to the identity.

Finally, when $a_i = 2$ and $a_{i+1} = 1$ we compute that
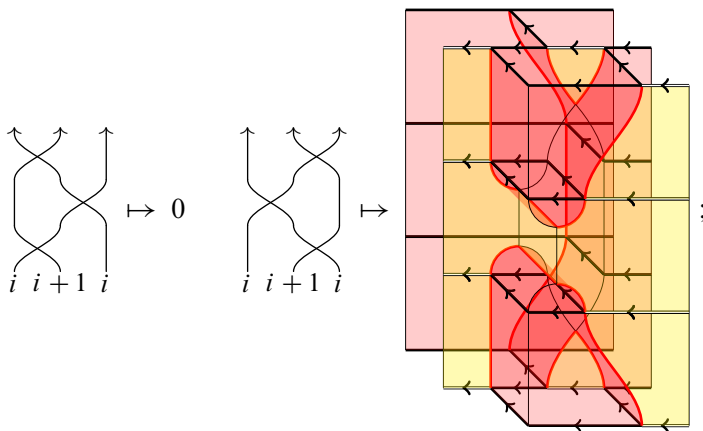
applying (3-18) to the above (which, after forgetting the back-most facets and applying an isotopy, is precisely equal to the negative of the right-hand side of (3-18)) gives the

foam

which confirms the relation. □

Note that the scalings of the images of the cap and cup 2–morphisms play no role in the proof of the proposition. They are determined by the proof of Theorem 2.5.

3A4 Clark–Morrison–Walker foams In the original construction of functorial Khovanov homology [22], Clark, Morrison and Walker use a variation of Bar-Natan's 2–category involving disoriented surfaces defined over the Gaussian integers. We can define foamation 2–functors to a family of 2–categories related to their construction. We will assume some familiarity with their work.

We fix once and for all ω to be a primitive fourth root of unity.

Definition 3.4 **CMWFoam$_m(N)$** is the 2–category defined as follows:

- Objects are sequences $(a_1, \ldots, a_m)$ labeling points in the interval $[0, 1]$ with $a_i \in \{0, 1, 2\}$ and $N = \sum_{i=1}^{m} a_i$, together with a zero object.

- 1–morphisms are formal direct sums of $\mathbb{Z}$–graded disoriented planar tangles directed out from 1–labeled points in the domain and into such points in the codomain.

- 2–morphisms are formal matrices of $\mathbb{k}[\omega]$–linear combinations of degree-zero dotted disoriented cobordisms between such disoriented planar tangles, modulo isotopy and local relations.

The disorientations are represented by fringed seams; the local relations are given by (3-2) and (3-3) in regions where no seams are present and the following local seam relations.
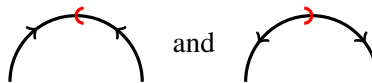
(3-22)

By adjusting some coefficients in the formulation of Proposition 3.3 and appropriately interpreting the image foams as 2–morphisms in **CMWFoam**$_m(N)$, we obtain the desired 2–functor. The interpretation is as follows.
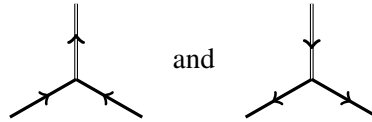
should be read as the disoriented tangles

and (in addition to removing 0–labeled sheets from the generic pictures below) the 2–labeled sheets should be deleted, retaining the seams where they meet 1–labeled facets and adding fringes aligned with the disorientation "tags" on the tangles.

Proposition 3.5 *For each $N > 0$ there is a 2–representation* $\Phi_{\mathrm{CMW}} \colon \mathcal{U}_Q(\mathfrak{sl}_m) \to$ **CMWFoam**$_m(N)$ *defined on single strand 2–morphisms by*

on crossings by

$$\Phi_{\mathrm{CMW}}\left(\,{}_{j}\!\!\diagdown\!\!\!\times\!\!\!\diagup{}_{i}\;\lambda\right) =$$

$$\Phi_{\mathrm{CMW}}\left(\,{}_{i}\!\!\diagdown\!\!\!\times\!\!\!\diagup{}_{j}\;\lambda\right) =$$

where $j - i > 1$, and on caps and cups by

$$\Phi_{\mathrm{CMW}}\left(\,{}^{i}\!\!\frown\!\!{}^{\lambda}\right) =$$

$$\Phi_{\mathrm{CMW}}\left(\,{}^{i}\!\!\frown\!\!{}^{\lambda}\right) = (-1)^{a_i}(-\omega)^{\delta}$$

$$\Phi_{\mathrm{CMW}}\left(\,{}_{i}\!\!\smile\!\!{}_{\lambda}\right) = (-1)^{a_i+1}(-\omega)^{\delta}$$

$$\Phi_{\mathrm{CMW}}\left(\,{}_{i}\!\!\smile\!\!{}_{\lambda}\right) =$$
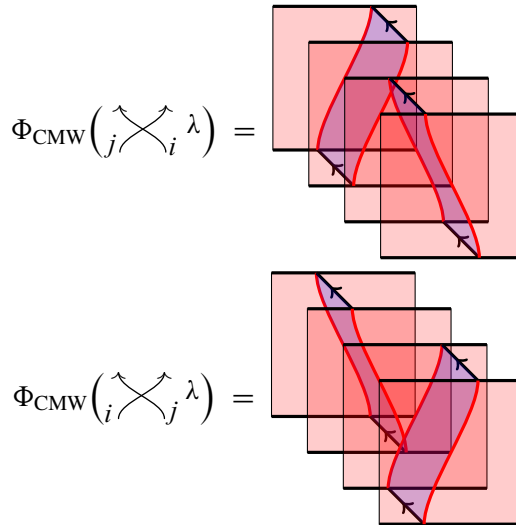
where in the above diagrams the ith sheet is always in the front, and $\delta = 1$ if λ_i is even and $\delta = 0$ otherwise. As before, we have drawn generic versions of the image foams, and one should apply the procedure outlined before this statement to obtain the actual CMW foams.

The proof is the same as for Proposition 3.3: we apply Theorem 2.5. Most of the work involves checking the KLR relations and is straightforward, so we omit almost all of the details. The following calculation confirms the nilHecke relation (2-17), which we include to show the importance of the disorientation seams:

In the above pictures we have applied isotopies to the disoriented tangles and cobordisms for clarity (so that all fringes and dots lie on the front side of the cobordisms). This relation determines the scaling of the (i, i) crossing in the definition above. The KLR relations also fix the scaling on the composition of an $(i, i+1)$ and an $(i+1, i)$ crossing; we choose to rescale the $(i+1, i)$ crossing. The scalings of the other 2–morphisms again follow from the proof of Theorem 2.5.

At present time, we don't have a good explanation for the rescalings in the above 2–functor. For this reason, we believe that $2\mathbf{BFoam}_m(N)$ is a more natural setting for the foamation 2–functors; in particular, we'll see in Proposition 3.10 that the definition of the foamation functor from Proposition 3.3 generalizes mutatis mutandis to give a foamation 2–functor to an enhanced version of $\mathfrak{sl}_3$ foams.

3B $\mathfrak{sl}_3$ foam 2–categories

In this section, we recall the definition of the $\mathfrak{sl}_3$ foam 2–category and prove the existence of the $\mathfrak{sl}_3$ foamation 2–functor. We then define an enhanced $\mathfrak{sl}_3$ foam 2–category similar to Blanchet's $\mathfrak{sl}_2$ foam category which appears naturally in the categorical skew Howe context. Finally, we give a functor from enhanced foams to standard foams, contrasting the $\mathfrak{sl}_2$ case.

3B1   Standard $\mathfrak{sl}_3$ foams   In [34], Khovanov gives a foam based categorification of the $\mathfrak{sl}_3$ link invariant. This construction was generalized by Mackaay and Vaz in [54] and Morrison and Nieh in [59] in the spirit of Bar-Natan's $\mathfrak{sl}_2$ cobordism

2–category [2], giving a categorification of Kuperberg's $\mathfrak{sl}_3$ spider [45]. Mackaay and Vaz showed [55] that these foam based constructions of $\mathfrak{sl}_3$ link homologies coincide with the $n = 3$ case of Khovanov and Rozansky's $\mathfrak{sl}_n$ link homologies defined via matrix factorizations [42].

We now recall the definition of this 2–category, which we denote 3**Foam**, using a hybrid of the above approaches:

- Objects are sequences of points in the interval $[0, 1]$ labeled by 1 or 2, together with a zero object.

- 1–morphisms are formal direct sums of $\mathbb{Z}$–graded shifts of $\mathfrak{sl}_3$ webs — directed, trivalent planar graphs with boundary in which each (interior) vertex is a sink or a source — where an edge is directed out from a point labeled by 1 and into a point labeled by 2 in the domain and vice-versa in the codomain.

- 2–morphisms are formal matrices of $\Bbbk$–linear combinations of degree-zero $\mathfrak{sl}_3$ foams — dotted surfaces with oriented singular seams which locally look like the product of the letter Y with an interval — considered up to isotopy (relative to the boundary) and local relations.

Denoting the $\mathbb{Z}$–grading of a web by the monomial q^t for $t \in \mathbb{Z}$, the degree of a foam $F : q^{t_1} W_1 \to q^{t_2} W_2$ is given by the formula

$$\deg(F) = 2\chi(F) - \#\partial + \tfrac{1}{2}\#V + t_2 - t_1,$$

where $\chi(F)$ is the Euler characteristic of the foam F, $\#\partial$ is the number of boundary points in either W_1 or W_2 (they agree!), and $\#V$ is the total number of trivalent vertices in $W_1 \coprod W_2$. A dot should be viewed as a puncture for the sake of computing an $\mathfrak{sl}_3$ foam's Euler characteristic. We shall depict $\mathfrak{sl}_3$ foams using the colors red and blue, for clarity; unlike the $\mathfrak{sl}_2$ case, these colors have no meaning as all facets are treated equally.

The local $\mathfrak{sl}_3$ foam relations are as follows (where a number next to a dot denotes the number of dots present).

(3-23) $= 0 =$ $= -1$

(3-24)

(3-25)

$$\text{(theta foam)} = \begin{cases} 1 & \text{if } (\alpha, \beta, \gamma) = (0, 1, 2) \text{ or a cyclic permutation} \\ -1 & \text{if } (\alpha, \beta, \gamma) = (0, 2, 1) \text{ or a cyclic permutation} \\ 0 & \text{all other triples with } \alpha, \beta, \gamma \leq 2 \end{cases}$$

(3-26)

(3-27)

Note that the local foam relations are all degree-homogeneous. The direction of the singular seams keeps track of a cyclic ordering of the incident facets; by convention, we take this ordering to be given by the right-hand rule. This convention is opposite to that used in [34] and [54], hence one would expect to see opposite foam relations above; however, we also reverse the relation between singular seams and trivalent vertices (seams are directed up through source vertices and down through sink vertices) which corresponds to taking different generating morphisms for the 2–category. It is easy to see that the above 2–category is isomorphic to the universal $\mathfrak{sl}_3$ foam 2–category from [54] and [59, Section A]. Our conventions are chosen to align with those in the definition of the 2–category $2\mathbf{BFoam}_m(N)$.

Using the neck-cutting relation (3-24) and the theta-foam relation (3-25), we can derive the following local relations.

(3-28) (3-29)

(3-30)

The values of the 3–, 4– and 5–dotted spheres should be viewed as (graded) parameters which are typically set equal to zero in the literature, eg in [34] and [59], although this is not required for our considerations. In the case that the 3–dotted sphere is zero, Morrison–Nieh show the relation

which allows for a completely topological description of this 2–category when 3 is invertible in $\Bbbk$.

Note that the set of relations above does not explicitly correspond with that from either [54] or [59]. The relations (3-23)–(3-25), together with a non-local relation constitute the relations from [54] (although in that work the authors introduce parameters a, b and c in place of the dotted-sphere parameters above). In [59], Morrison and Nieh show that the relations (3-26) and (3-27) imply the non-local relation. Our chosen set of relations above almost agree with those of Morrison–Nieh (when the dotted surface parameters equal zero): they impose the relation that reversing the orientation of a singular seam negates the foam instead of specifying the values of the theta-foams; this seam reversal relation follows from (3-24) and (3-25).

As in the $\mathfrak{sl}_2$ case, we are interested in a related family of 2–categories which is natural to consider from the perspective of categorical skew Howe duality.

Definition 3.6 $3\textbf{Foam}_m(N)$ is the 2–category defined as follows:

- Objects are sequences $(a_1, \ldots, a_m)$ labeling points in the interval $[0, 1]$ with $a_i \in \{0, 1, 2, 3\}$ and $N = \sum_{i=1}^{m} a_i$ together with a zero object.

- 1–morphisms are formal direct sums of $\mathbb{Z}$–graded shifts of $\mathfrak{sl}_3$ webs mapping between the points labeled by 1 and 2 as in $3\textbf{Foam}$.

- 2–morphisms are formal matrices of $\mathbb{k}$–linear combinations of degree-zero $\mathfrak{sl}_3$ foams mapping between such webs.

Note that the objects in $3\textbf{Foam}_m(N)$ correspond with the direct summands appearing in the decomposition of $\bigwedge_q^N(\mathbb{C}^3 \otimes \mathbb{C}^m)$ into $\mathfrak{sl}_m$ weight spaces and 1–morphisms correspond to $\mathfrak{sl}_3$ intertwiners between such summands. For each m and N, there is an obvious 2–functor $3\textbf{Foam}_m(N) \to 3\textbf{Foam}$ which forgets the 0's and 3's.

3B2 Foamation We now define $\mathfrak{sl}_3$ foamation 2–functors $\mathcal{U}_Q(\mathfrak{sl}_m) \to 3\textbf{Foam}_m(N)$. As in the $\mathfrak{sl}_2$ case, the 2–functor is defined on objects by sending an $\mathfrak{sl}_m$ weight $\lambda = (\lambda_1, \ldots, \lambda_{m-1})$ to the sequence $(a_1, \ldots, a_m)$ with $a_i \in \{0, 1, 2, 3\}$, $\lambda_i = a_{i+1} - a_i$ and $\sum_{i=1}^{m} a_i = N$ provided such a sequence exists and to the zero object otherwise.

The map on 1–morphisms is again given by

when the boundary values lie in $\{0, 1, 2, 3\}$ and to the zero 1–morphism otherwise. Note that the orientation of the undirected strands (and whether they become deleted) is determined by these boundary values and that we have not depicted $m - 2$ horizontal strands in each of the latter two formulae.

We will make use of a lemma to deduce the existence of the foamation 2–functors. Let the images of $\mathbf{1}_\lambda$, $\mathcal{E}_i \mathbf{1}_\lambda$ and $\mathcal{F}_i \mathbf{1}_\lambda$ above be denoted $\mathbb{1}_\lambda$, $\mathsf{E}_i \mathbb{1}_\lambda$ and $\mathsf{F}_i \mathbb{1}_\lambda$.

Lemma 3.7 *There are isomorphisms*

$$\mathsf{F}_i \mathsf{E}_i \mathbb{1}_\lambda \cong \mathsf{E}_i \mathsf{F}_i \mathbb{1}_\lambda \oplus_{[-\langle i, \lambda \rangle]} \mathbb{1}_\lambda \quad \textit{if } \langle i, \lambda \rangle \leq 0, \quad \mathsf{E}_i \mathsf{F}_i \mathbb{1}_\lambda \cong \mathsf{F}_i \mathsf{E}_i \mathbb{1}_\lambda \oplus_{[\langle i, \lambda \rangle]} \mathbb{1}_\lambda \quad \textit{if } \langle i, \lambda \rangle \geq 0,$$

and $\mathsf{F}_j \mathsf{E}_i \mathbb{1}_\lambda \cong \mathsf{E}_i \mathsf{F}_j \mathbb{1}_\lambda$ *for* $i \neq j \in I$ *in* $3\textbf{Foam}_m(N)$.

Proof The proof is similar to the proof of Lemma 3.2. $\square$

Proposition 3.8 *There is a 2–representation* $\Phi_3\colon \mathcal{U}_Q(\mathfrak{sl}_m) \to 3\mathbf{Foam}_m(N)$ *for each* $N > 0$ *defined on single strand morphisms by*

$$\Phi_3\left(\begin{array}{c}\uparrow \; \lambda \\ i\end{array}\right) = \quad\qquad\qquad \Phi_3\left(\begin{array}{c}\bullet \; \lambda \\ i\end{array}\right) = \qquad$$

on crossings by

$$\Phi_3\left(\begin{array}{c}\lambda \\ i \quad i\end{array}\right) = \qquad\qquad\qquad \Phi_3\left(\begin{array}{c}\lambda \\ i \quad i+1\end{array}\right) = \qquad$$

$$\Phi_3\left(\begin{array}{c}\lambda \\ i+1 \quad i\end{array}\right) = (-1)^{a_{i+1}+1}$$

$$\Phi_3\left(\begin{array}{c}\lambda \\ j \quad i\end{array}\right) = \qquad\qquad\qquad \Phi_3\left(\begin{array}{c}\lambda \\ i \quad j\end{array}\right) = \qquad$$

where $j - i > 1$, *and on caps and cups by*

$$\Phi_3\left(\begin{array}{c}i \quad \lambda\end{array}\right) = \pm \qquad\qquad\qquad \Phi_3\left(\begin{array}{c}i \quad \lambda\end{array}\right) = \pm$$

$$\Phi_3\left(\begin{array}{c}i \quad \lambda\end{array}\right) = \pm \qquad\qquad\qquad \Phi_3\left(\begin{array}{c}i \quad \lambda\end{array}\right) = \pm$$

where the $\pm$ *above depend on (the image of) the weight* λ *and are given by*[8] *Table 1.*

------

[8]In fact, we will see that the $\pm$ signs involved in the definition of the 2–functor on caps and cups play no role in the proof of this proposition; they are determined by the proof of Theorem 2.5. In the next section, we will give a topological interpretation of this system of signs.

Counterclockwise cap		Sign	Clockwise cap		Sign
$N_i = 3$	$\lambda_i = 1$	$-$	$N_i = 3$	$\lambda_i = -1$	$-$
$N_i = 2$	$\lambda_i = 0$	$+$	$N_i = 2$	$\lambda_i = 0$	$-$
$N_i = 4$	$\lambda_i = 0$	$-$	$N_i = 4$	$\lambda_i = 0$	$-$
$N_i = 1$	$\lambda_i = -1$	$+$	$N_i = 1$	$\lambda_i = 1$	$+$
$N_i = 3$	$\lambda_i = -1$	$+$	$N_i = 3$	$\lambda_i = 1$	$-$
$N_i = 5$	$\lambda_i = -1$	$-$	$N_i = 5$	$\lambda_i = 1$	$-$
$N_i = 2$	$\lambda_i = -2$	$+$	$N_i = 2$	$\lambda_i = 2$	$+$
$N_i = 4$	$\lambda_i = -2$	$+$	$N_i = 4$	$\lambda_i = 2$	$-$
$N_i = 3$	$\lambda_i = -3$	$+$	$N_i = 3$	$\lambda_i = 3$	$+$

Counterclockwise cup		Sign	Clockwise cup		Sign
$N_i = 3$	$\lambda_i = 1$	$+$	$N_i = 3$	$\lambda_i = -1$	$+$
$N_i = 2$	$\lambda_i = 0$	$+$	$N_i = 2$	$\lambda_i = 0$	$+$
$N_i = 4$	$\lambda_i = 0$	$-$	$N_i = 4$	$\lambda_i = 0$	$+$
$N_i = 1$	$\lambda_i = -1$	$+$	$N_i = 1$	$\lambda_i = 1$	$+$
$N_i = 3$	$\lambda_i = -1$	$-$	$N_i = 3$	$\lambda_i = 1$	$+$
$N_i = 5$	$\lambda_i = -1$	$-$	$N_i = 5$	$\lambda_i = 1$	$-$
$N_i = 2$	$\lambda_i = -2$	$-$	$N_i = 2$	$\lambda_i = 2$	$+$
$N_i = 4$	$\lambda_i = -2$	$-$	$N_i = 4$	$\lambda_i = 2$	$-$
$N_i = 3$	$\lambda_i = -3$	$-$	$N_i = 3$	$\lambda_i = 3$	$-$

Table 1: Signs for relations in Proposition 3.8, where $N_i = a_i + a_{i+1}$

Again, the foams drawn above are generic depictions of the image. As such, up to signs, they are exactly the same as those appearing in Proposition 3.3; however, the process we apply to obtain the actual $\mathfrak{sl}_3$ foams differs from the one applied in Proposition 3.3. We now kill any web or foam containing a label strictly smaller than 0 or strictly larger than 3, and delete any web strand (or foam facet) labeled 0 or 3. The singular seams in such pictures may again degenerate in such situations (and in particular, all intersections of seams will disappear), eg in the case that λ maps to a sequence with $a_i = 1, a_{i+1} = 2$ we have the following, which is a saddle cobordism.

Proof As with the proof of Proposition 3.3, we apply Theorem 2.5. Conditions (1) and (3) follow as before and Lemma 3.7 gives conditions (4) and (6).

Working in the setting where the 3–, 4– and 5–dotted spheres are set equal to zero, it is shown in [59] that the vector space $\mathrm{Hom}(\mathbb{1}_\lambda, q^t \mathbb{1}_\lambda)$ is zero for $t < 0$ and one-dimensional for $t = 0$ (provided $\mathbb{1}_\lambda$ is non-zero) and that for any webs W_1 and W_2, the vector space $\mathrm{Hom}(q^{t_1} W_1, q^{t_2} W_2)$ is finite-dimensional. The same arguments show that this holds when these dotted spheres are not set equal to zero (since they have negative Euler characteristic). This confirms condition (2).

We thus conclude by checking condition (5), the KLR relations:

Relation (2-16) The 2–morphism

maps to a foam which can only possibly be non-zero for λ mapping to sequences with

$$a_i = 2, 3 \quad \text{and} \quad a_{i+1} = 0, 1.$$

When $(a_i, a_{i+1}) = (2, 0)$ we compute the image:
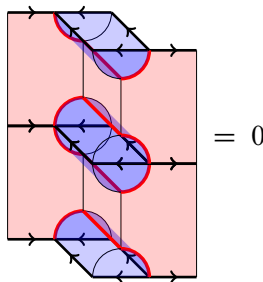
which follows by applying the neck-cutting relation (3-24) on the red sheet in a neighborhood of the center singular seam, and noting that all of the resulting theta-foams evaluate to zero. The computation for the remainder of the values of (a_i, a_{i+1}) follows similarly.

We next compute the images of the 2–morphisms
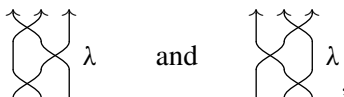
noting that the images can only be non-zero for λ mapping to sequences with $a_i = 3$ and $a_{i+1} = 0$. The above 2–morphisms map to the following foams, respectively,
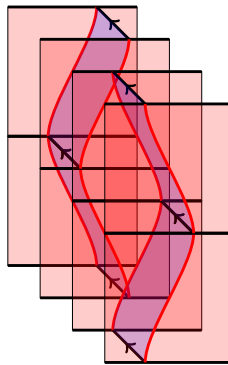
which are equal by (3-30).

and
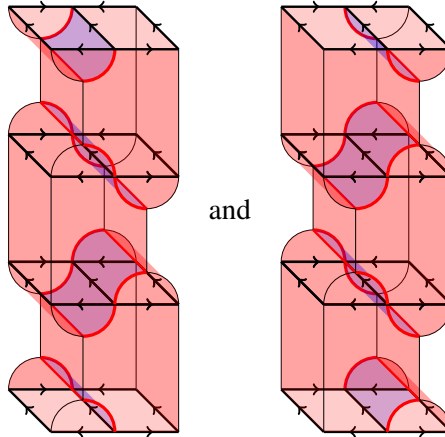
Relation (2-17) The images of this relation are simply a restatement of (3-26).

Relation (2-18) The equality

$$(\alpha_i, \alpha_j) = 0$$

corresponds to $|i - j| \geq 2$ in which case the image of the relation is realized via isotopy. For example, when $i < j$


is isotopic to the identity foam over the bottom (or equivalently top) boundary.

When $i \neq j$ and

$$(\alpha_i, \alpha_j) \neq 0$$

we must have $j = i \pm 1$. We'll compute the image of this relationship in the case $j = i + 1$ (the other case is similar). The image of the left-hand side is zero when $a_{i+1} = 0, 3$ since the intermediate objects in the relation map to the zero object in the image; the same is true on the right-hand side when $a_{i+1} = 0$. When $a_{i+1} = 3$, both

2–morphisms involved in the expression on the right-hand side map to the same foam, so the relation is satisfied since $t_{i,i+1} = -t_{i+1,i}$. When $a_{i+1} = 1$ we have
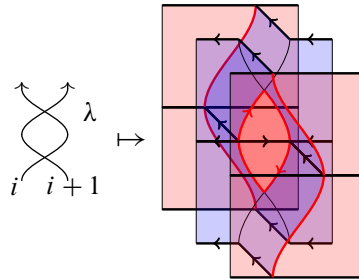
and noticing that the center singular seam is a circle bounding a disk, applying (3-28) to this seam gives the following foam: the image of

and noticing that the center singular seam is a circle bounding a disk, applying (3-28) to this seam gives the following foam: the image of
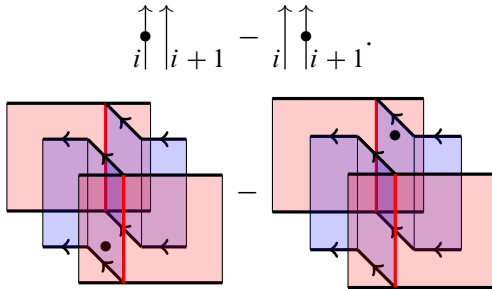
Finally, when $a_{i+1} = 2$
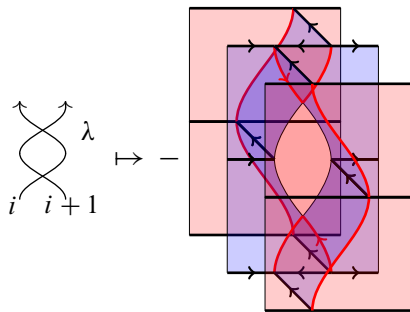
since the central facet is 3–labeled, hence erased. Applying (3-26) (turned sideways!) to the region between the semi-circular seams gives the foam
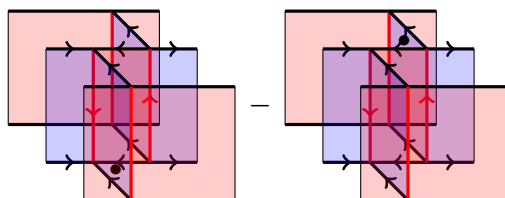
which again is the image of

$$\overset{\uparrow}{\underset{i}{\bullet}}\overset{\uparrow}{\underset{i+1}{|}} - \overset{\uparrow}{\underset{i}{|}}\overset{\uparrow}{\underset{i+1}{\bullet}}.$$

**Relation (2-19)** These hold by sliding a dot along a facet, ie via isotopy.

Relation (2-20) For all choices of i, j and k this relation holds via isotopy. This is obvious whenever one of the strands carries a label which is at least 2 bigger or smaller than both other labels. In the remaining cases a computation is necessary; we'll exhibit this only for two cases, since the remaining cases follow similarly.

First, suppose that $j = i$ and $k = i + 1$; then both sides of the relation automatically map to zero unless λ maps to a sequence with $a_{i+1} = 1$. We hence compute the image (of both sides of the relation) in this case, finding them to be

 and
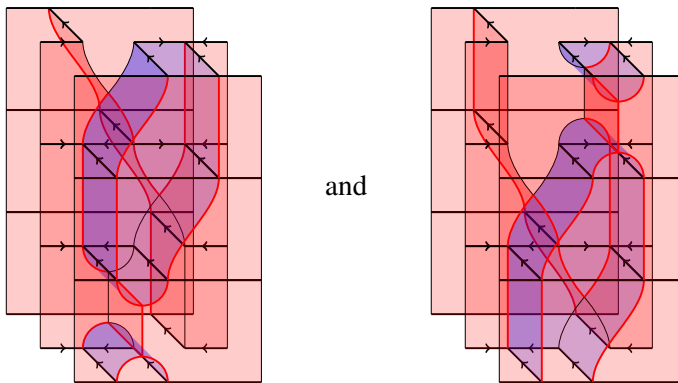
which are related via isotopy (no matter which values of a_i and a_{i+2} we choose).

Next, suppose $j = i + 1$ and $k = i + 2$; then we compute both sides of the relation to be

 and
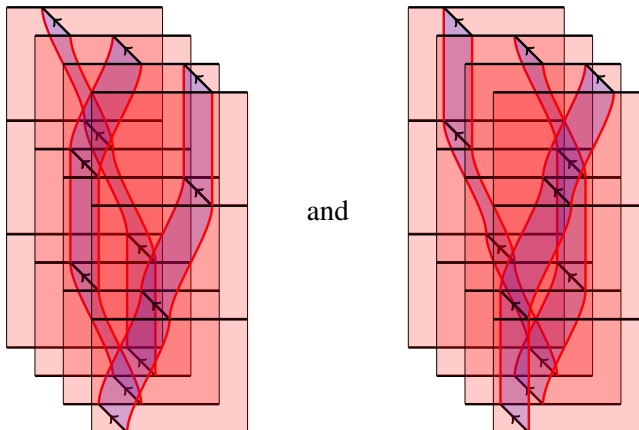
which are equal up to isotopy.

Relation (2-21) We must have $j = i \pm 1$ and we'll only compute for $j = i + 1$ since the other case is analogous. Note that all 2–morphisms involved automatically map to zero if λ is sent to a sequence with

$$a_{i+1} = 3 \quad \text{or with} \quad a_i = 0, 1,$$

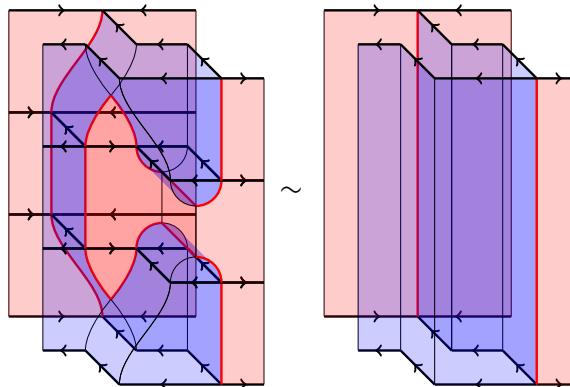so we'll compute for the remaining values.
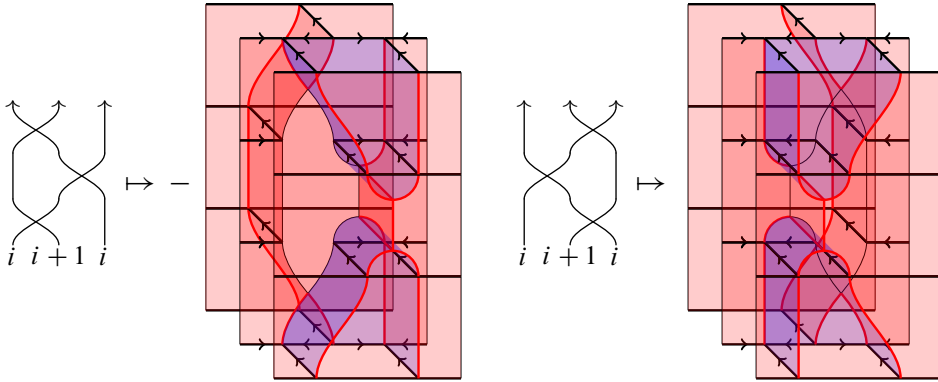
First, let $a_{i+1} = 0$; then
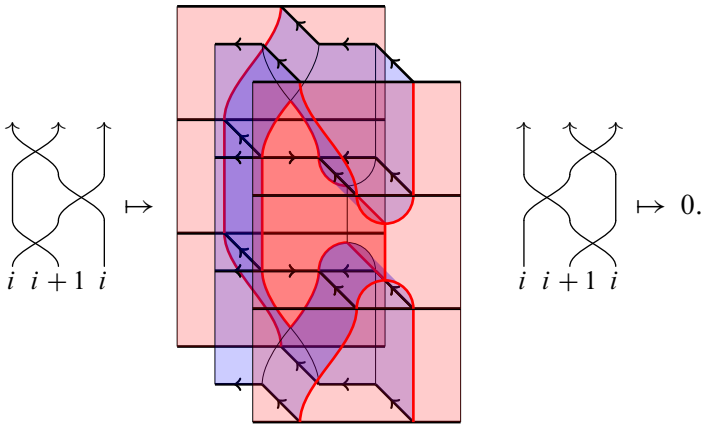
This confirms the relation since the former is isotopic to the identity foam when
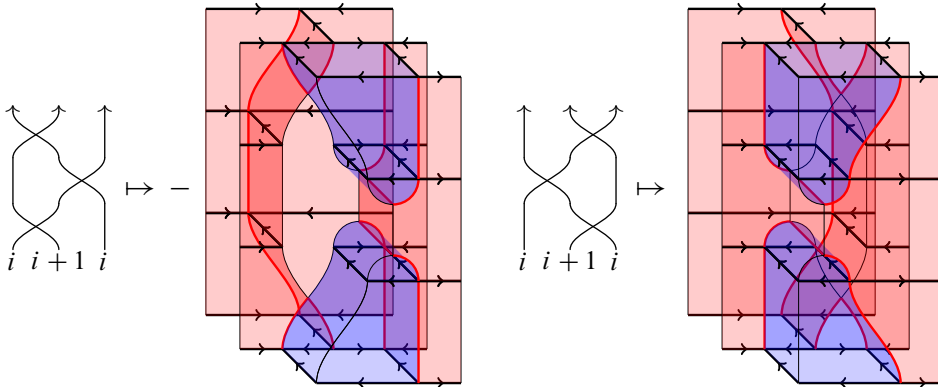
$$a_i = 2, 3,$$

noting that in these cases either the leftmost or rightmost front facet is deleted. For example, the $a_i = 2$, $a_{i+2} = 1$ case is the following isotopy.

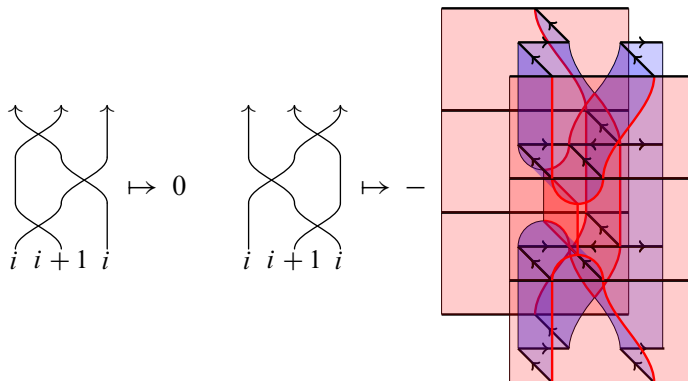Next, let $a_{i+1} = 1$; then we compute as follows.

The relation then follows from Equation (3-27) when $a_i = 2, 3$, again noting that either the leftmost or rightmost front facet is deleted for both foams. For example, taking $a_i = 2$ and $a_{i+2} = 1$ gives

and forgetting the back facets then exactly allows us to apply Equation (3-27).

Finally, if $a_{i+1} = 2$, then

and this confirms the relation since the latter is the identity foam over its boundary webs when $a_i = 2, 3$. □

3B3 Enhanced $\mathfrak{sl}_3$ foams We now aim to better explain the signs in Table 1 which give the scalings of the cap and cup 2–morphisms. We take inspiration from Blanchet's $\mathfrak{sl}_2$ foam construction in which the edges of webs labeled by 2 and the corresponding facets of foams play a special role (and in particular are not deleted).

We hence define an $\mathfrak{sl}_3$ foam 2–category in which we retain 3–labeled edges and the corresponding 3–labeled facets. Although such a construction is not suggested at the decategorified level (as it was in the $\mathfrak{sl}_2$ case) we will see that the foamation functor is much more natural to define in this context and that an appropriately defined functor which forgets the 3–labeled data gives a topological interpretation of the scalings. We believe that such n–labeled facets will play a role in extending the work in this paper to the $n \geq 4$ case; this is explored in detail in the follow-up paper [62].

Definition 3.9 $3\mathbf{BFoam}_m(N)$ is the 2–category defined as follows:

- Objects are sequences $(a_1, \ldots, a_m)$ labeling points in the interval $[0, 1]$ with $a_i \in \{0, 1, 2, 3\}$ and $N = \sum_{i=1}^{m} a_i$ together with a zero object.

- 1–morphisms are formal direct sums of $\mathbb{Z}$–graded enhanced $\mathfrak{sl}_3$ webs, that is, trivalent planar graphs with boundary with edges of two types: directed edges ⟶ and undirected 3–*labeled* edges ═══ where vertices involving only the directed edges are as in 3**Foam** and vertices involving the 3–labeled edges take the form below.
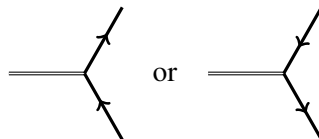

Oriented edges are directed out from points labeled by 1 and into points labeled by 2 in the domain and vice-versa in the codomain and 3–labeled edges are attached to points labeled by 3 in both the domain and codomain. As in $3\mathbf{Foam}_m(N)$, no edges are attached to points labeled by 0.

- 2–morphisms are $\mathfrak{sl}_3$ foams between such webs where we allow additional 3–labeled facets incident upon the 3–labeled strands of the webs and attached to the remainder of the foam along singular seams which are allowed to intersect the traditional singular seams; the 3–labeled facets are not allowed to carry dots. We impose local relations on these foams.

We shall refer to the 2–morphisms in this category as Blanchet $\mathfrak{sl}_3$ foams and depict the 3–labeled facets in yellow. The traditional facets of these foams will continue to be depicted in both red and blue. As in the $\mathfrak{sl}_2$ case, the degree of a Blanchet $\mathfrak{sl}_3$ foam is defined to be the same as the degree of the standard $\mathfrak{sl}_3$ foam which results when the 3–labeled web edges and foam facets are deleted.

The local relations are given by the relations in 3**Foam** in regions where 3–labeled facets are not present with additional relations for the 3–labeled facets. The seams where a 3–labeled facet meet the traditional facets are allowed to move freely on the foam (relative to the points where such seams meet the web vertices depicted above). This additional foam relation is typified by the following local relation.
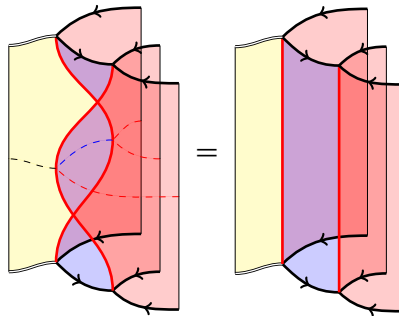
We also impose a strong neck-cutting relation for these facets,
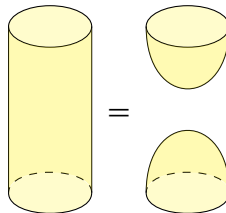
and the condition that we may delete any 3–labeled facet F not incident upon the boundary (and delete its boundary seam) at the cost of multiplying by $(-1)^{\chi(F)}$. Finally, we have the relation

(3-31)

which we impose for all coherent orientations of the singular seams. An Euler characteristic argument shows that these relations are consistent. Using such foams, we have the following result.

Proposition 3.10 *The definition in Proposition 3.3 describes a family of 2–functors*

$$\mathcal{U}_Q(\mathfrak{sl}_m) \to 3\mathbf{BFoam}_m(N).$$

As before, we view the definition as showing the general image of each generating 2–morphism; edges connected to points labeled by 0 and facets incident upon them are understood to be deleted. The proof of this proposition follows as in the proof of Propositions 3.3 and 3.8.

The relations for the 3–labeled facets allow us to delete any facet which does not meet the boundary; however, this is not enough to define a forgetful functor $3\mathbf{BFoam}_m(N) \to 3\mathbf{Foam}_m(N)$. We can give such a rule by taking into account the boundary data.

Given a Blanchet $\mathfrak{sl}_3$ foam F, define $\chi_3(F)$ to be the Euler characteristic of its 3–labeled facets and let $n_u(F)$ denote the number of 3–labeled edges in the codomain of F. Let $\Psi(F) = (-1)^{\chi_3(F)-n_u(F)}\overline{F}$ where $\overline{F}$ is the $\mathfrak{sl}_3$ foam obtained from F by deleting the 3–labeled facets (and the 3–labeled edges from the boundary webs). Similarly, define $n_b(F)$ to be the number of 3–labeled edges in the domain of F and $n_l(F)$ and $n_r(F)$ to be the number of points labeled by 3 in the codomain and domain (respectively) of the webs between which F maps; of course, these later two denote the number of 3–labeled vertical intervals on the left and right boundary of F.

Proposition 3.11 *The assignment $F \mapsto \Psi(F)$ defines a 2–functor $3\mathbf{BFoam}_m(N) \to 3\mathbf{Foam}_m(N)$ where objects are sent to themselves and enhanced webs are sent to the webs obtained by deleting the 3–labeled edges.*

Proof It suffices to show that Ψ is compatible with horizontal and vertical composition of foams. To this end, consider foams F_1, F_2, and F_3 which can be composed as indicated by the following schematic.
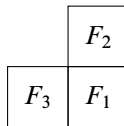
We have

$$\chi_3(F_1 \cup F_2) - n_u(F_1 \cup F_2) = \chi_3(F_1 \cup F_2) - n_u(F_2)$$
$$= \chi_3(F_1) - n_u(F_1) + \chi_3(F_2) - n_u(F_2),$$
$$\chi_3(F_1 \cup F_3) - n_u(F_1 \cup F_3) = \chi_3(F_1) - n_l(F_1) + \chi_3(F_3) - n_u(F_1) + n_l(F_1) - n_u(F_3)$$
$$= \chi_3(F_1) - n_u(F_1) + \chi_3(F_3) - n_u(F_3),$$

which gives the result. $\square$

One can now consider the composition of the 2–functors defined in Propositions 3.10 and 3.11.

Proposition 3.12 *The composition* $\mathcal{U}_Q(\mathfrak{sl}_m) \to 3\mathbf{BFoam}_m(N) \to 3\mathbf{Foam}_m(N)$ *gives the 2–functor from Proposition 3.8.*

Proof   The proof follows from a routine, yet tedious, calculation. We'll exhibit a few of the more interesting cases:

• Let λ map to a sequence where $a_{i+1} = 2$; then
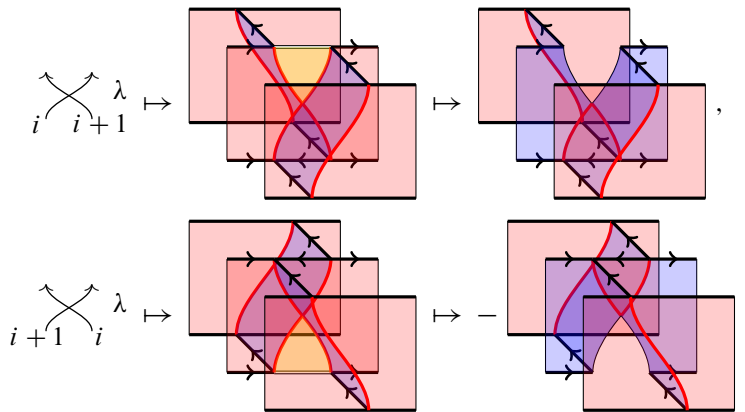
• Let λ map to a sequence with $a_i = 0$ and $a_{i+1} = 3$ (ie $N_i = 3$ and $\lambda_i = 3$), then
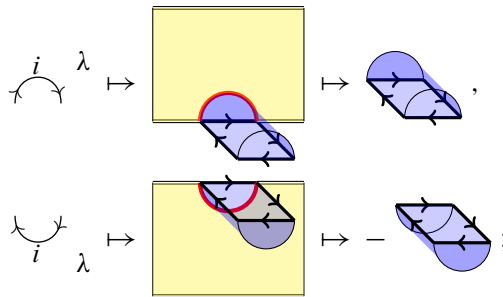
note that this guarantees that the relevant degree-zero bubble ⟨figure⟩ is sent to the identity 2–morphism in $3\mathbf{Foam}_m(N)$.   □

3B4   Clark–Morrison–Walker $\mathfrak{sl}_3$ foams   One may notice that the topology of a 3–labeled facet is relatively unimportant; the signs obtained by removing any 3–labeled facet (not incident upon the boundary webs) depend only on the facet's boundary seams. One may then ask why one needs these facets at all: couldn't we instead introduce a Clark–Morrison–Walker (CMW) version of $\mathfrak{sl}_3$ foams?

Indeed, we can define such a theory by removing all 3–labeled facets and edges of webs from the definitions in the previous subsection, keeping only the new CMW seams and imposing the relation that one may remove a closed seam at the cost of multiplying by -1. It is easy to see that we obtain a family of 2–functors from $\mathcal{U}_Q(\mathfrak{sl}_m)$ to the 2–category of CMW $\mathfrak{sl}_3$ foams. Note that the CMW seams in such a theory do not need fringes, unlike the $\mathfrak{sl}_2$ case.

However, when one tries to define a forgetful 2–functor to the category of (traditional) $\mathfrak{sl}_3$ foams as before, it surprisingly appears that the rigidity obtained from the interaction of the 3–labeled facets with the 3–labeled edges plays a non-trivial role. Indeed, there is no hope to define such a 2–functor as we now demonstrate.
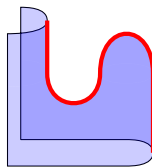
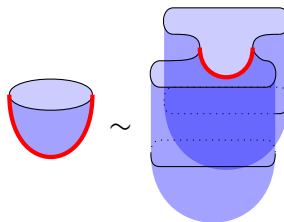Assume a forgetful 2–functor exists. We need maps
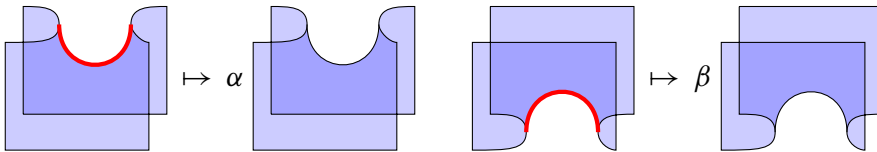
with $\alpha\beta = -1$ and
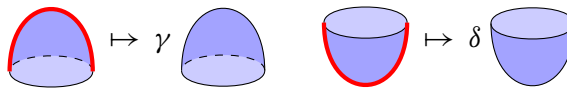
with $\gamma\delta = -1$. Since the foam

is isotopic to the identity, we must have $\alpha\gamma = 1$ and since

we must have $\delta = \alpha$. This then gives

$$1 = \alpha\gamma = \delta\gamma = -1,$$

a contradiction.

Note that this argument remains valid even if we enhance the CMW seams with fringes (repeat the above argument with all seam directed into the page). The above argument no longer remains valid if the seams are given an orientation; however, similar arguments prevent a definition of CMW $\mathfrak{sl}_3$ foams possessing a forgetful functor to $\mathfrak{sl}_3$ foams.

3C   Extended foamation 2–functors

In order to construct categorified link invariants, we will need to consider the images of the Rickard complexes under the foamation 2–functors, and hence the images of divided powers. Recall that the later are 1–morphisms which lie in $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$ — the Karoubi envelope of $\mathcal{U}_Q(\mathfrak{sl}_m)$ — but not in $\mathcal{U}_Q(\mathfrak{sl}_m)$ itself.

The universal property of the Karoubi envelope implies that the foamation 2–functors $\mathcal{U}_Q(\mathfrak{sl}_m) \to n(\mathbf{B})\mathbf{Foam}_m(N)$ extend to 2–functors

$$\dot{\mathcal{U}}_Q(\mathfrak{sl}_m) \to \mathrm{Kar}(n(\mathbf{B})\mathbf{Foam}_m(N)),$$

where $n(\mathbf{B})\mathbf{Foam}_m(N)$ is shorthand for all $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ categories used as targets of foamation 2–functors so far in this paper. We would like to consider these extended 2–functors; however, the Karoubi envelope of the foam 2–categories is not easy to work with. Indeed, the indecomposable 1–morphisms in these 2–categories are closely related to (dual) canonical bases, and hence are similarly complicated to study.

It turns out, however, that the foam 2–categories are "closer" to their Karoubi envelopes than the 2–categories $\mathcal{U}_Q(\mathfrak{sl}_m)$ are to $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$. We shall see that the images of the divided powers already exist in the foam 2–categories. To this end, recall from Definition 2.2 that $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$ is the sub-2–category of $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$ whose 1–morphisms are generated by (shifts of) divided powers.

Proposition 3.13   *The 2–functors defined in Propositions 3.3, 3.8, and 3.10 extend to 2–functors from $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$ to the relevant foam 2–categories.*

Proof   Recall that any category $\mathcal{C}$ embeds fully faithfully into $\mathrm{Kar}(\mathcal{C})$ by sending an object c to the pair (c, id_c) and a morphism $f : c \to d$ to the triple $(\mathrm{id}_c, f, \mathrm{id}_d)$; see [5, Chapter 6.5]. It thus suffices to show that when we restrict the 2–functor
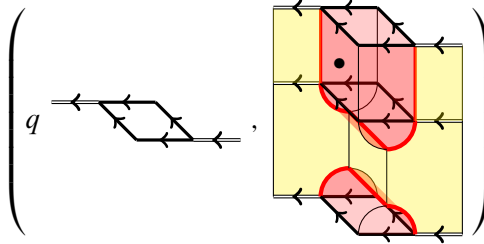
$$\dot{\mathcal{U}}_Q(\mathfrak{sl}_m) \to \mathrm{Kar}(n(\mathbf{B})\mathbf{Foam}_m(N))$$

to $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$, the images of all 1–morphisms are isomorphic to 1–morphisms lying in the sub-2–category $n(\mathbf{B})\mathbf{Foam}_m(N) \subset \mathrm{Kar}(n(\mathbf{B})\mathbf{Foam}_m(N))$.
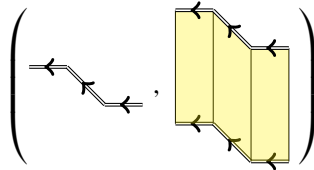
We begin with the $\mathfrak{sl}_2$ case. Since $\mathcal{E}_i^k \mathbf{1}_\lambda \mapsto 0$ and $\mathcal{F}_i^k \mathbf{1}_\lambda \mapsto 0$ for all $k \geq 3$, we need only consider the 1–morphisms $\mathcal{E}_i^{(2)} \mathbf{1}_\lambda$ and $\mathcal{F}_i^{(2)} \mathbf{1}_\lambda$. Note that the 1–morphism

$$\mathcal{E}_i^{(2)} \mathbf{1}_\lambda = \left(\mathcal{E}_i^2 \mathbf{1}_\lambda\{1\}, \;\;\begin{array}{c}\includegraphics\end{array} \; \lambda \right)$$

is mapped to zero unless $a_i = 2$ and $a_{i+1} = 0$. In this case, the above is mapped to

$$\left(q \;\; \begin{array}{c}\includegraphics\end{array} \;\;,\;\; \begin{array}{c}\includegraphics\end{array} \right)$$

which is isomorphic to

$$\left(\begin{array}{c}\includegraphics\end{array} \;,\; \begin{array}{c}\includegraphics\end{array} \right)$$
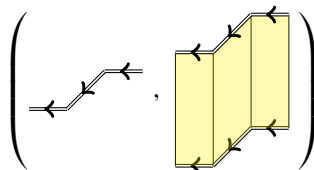
in $\mathrm{Kar}(2\mathbf{BFoam}_m(N))$. Explicitly, we use Equation (3-11) to show that the following give a pair of inverse isomorphisms.

$$\left(\begin{array}{c}\includegraphics\end{array} \;,\; \begin{array}{c}\includegraphics\end{array} \;,\; \begin{array}{c}\includegraphics\end{array} \right)$$

$$\left(\begin{array}{c}\includegraphics\end{array} \;,\; \begin{array}{c}\includegraphics\end{array} \;,\; \begin{array}{c}\includegraphics\end{array} \right)$$
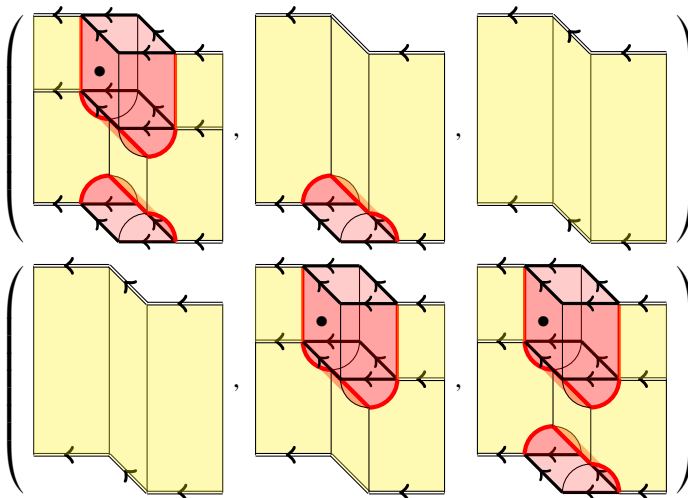
Similarly, we find that the image of $\mathcal{F}_i^{(2)} \mathbf{1}_\lambda$ is isomorphic to

$$\left(\begin{array}{c}\includegraphics\end{array} \;,\; \begin{array}{c}\includegraphics\end{array} \right)$$

when $a_i = 0$ and $a_{i+1} = 2$ (the only case when $\mathcal{F}_i^2 \mathbf{1}_\lambda$ is not mapped to zero).

In the $\mathfrak{sl}_3$ case, it suffices to consider the 2–functor

$$\mathcal{U}_Q(\mathfrak{sl}_m) \to 3\mathbf{BFoam}_m(N),$$

since the 2–functor

$$\mathcal{U}_Q(\mathfrak{sl}_m) \to 3\mathbf{Foam}_m(N)$$

is obtained via composition with the forgetful functor. We find that $\mathcal{E}_i^k \mathbf{1}_\lambda$ and $\mathcal{F}_i^k \mathbf{1}_\lambda$ are both sent to zero for $k \geq 4$, so it suffices to consider the 1–morphisms

$$\mathcal{E}_i^{(2)} \mathbf{1}_\lambda, \quad \mathcal{E}_i^{(3)} \mathbf{1}_\lambda, \quad \mathcal{F}_i^{(2)} \mathbf{1}_\lambda \quad \text{and} \quad \mathcal{F}_i^{(3)} \mathbf{1}_\lambda.$$

We see that $\mathcal{E}_i^{(2)} \mathbf{1}_\lambda$ is (again) mapped to the 1–morphism

which is isomorphic in $\mathrm{Kar}(3\mathbf{BFoam}_m(N))$ to

for any value of the a_l's using the $\mathfrak{sl}_3$ foam relations. Similarly, the image of $\mathcal{F}_i^{(2)} \mathbf{1}_\lambda$ is isomorphic to

The directed lines in the latter two 1–morphisms above can be viewed as 2–labeled edges directed in the opposite direction; from this perspective, the boundary labels appear more appropriate.
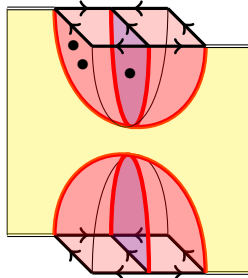
The only case where $\mathcal{E}_i^{(3)}\mathbf{1}_\lambda$ is not sent to zero is for $a_i = 3$ and $a_{i+1} = 0$. and in this case $\mathcal{E}_i^{(3)}\mathbf{1}_\lambda$ is mapped to

(3-32)

which is isomorphic to

in $\mathrm{Kar}(3\mathbf{BFoam}_m(N))$. The isomorphism above is evident after noticing that the foam in (3-32) is equal to

using (3-30) and (3-31). Finally, the image of $\mathcal{F}_i^{(3)}\mathbf{1}_\lambda$ is isomorphic to

when $a_i = 0$ and $a_{i+1} = 3$ (ie the only non-zero case). $\qquad\square$

We can summarize the 2–functors $\check{\mathcal{U}}_Q(\mathfrak{sl}_m) \to n(\mathbf{B})\mathbf{Foam}_m(N)$ using ladders with labelings on the diagonal edges:

$$\mathcal{E}_i^{(k)}\mathbf{1}_\lambda \mapsto \quad \begin{array}{c} a_{i+1}+k \longleftarrow \quad a_{i+1} \\ \raisebox{2pt}{$\scriptstyle k$} \\ a_i-k \longleftarrow \quad a_i \end{array} \qquad \mathcal{F}_i^{(k)}\mathbf{1}_\lambda \mapsto \quad \begin{array}{c} a_{i+1}-k \longleftarrow \quad a_{i+1} \\ \raisebox{2pt}{$\scriptstyle k$} \\ a_i+k \longleftarrow \quad a_i \end{array}$$

where 2–labeled edges in $3\mathbf{Foam}_m(N)$ and $3\mathbf{BFoam}_m(N)$ should be viewed as 1–labeled edges oriented in the opposite direction and 3–labeled edges should be deleted in $3\mathbf{Foam}_m(N)$ and un-oriented in $3\mathbf{BFoam}_m(N)$. For example, the image of $\mathcal{E}^{(3)}\mathbf{1}_{-3}$ in $3\mathbf{Foam}_2(N)$ is the empty web between the sequences $(3,0)$ and $(0,3)$.

4 Applications

Many known constructions in link homology follow from the 2–functors defined in the previous section. Indeed, we will re-construct Khovanov homology, $\mathfrak{sl}_3$ link homology and categorified highest weight projectors in these theories using the categorified quantum Weyl group action. The skew Howe perspective also provides a framework for showing that Cautis and Kamnitzer's algebro-geometric formulation of $\mathfrak{sl}_3$ link homology is isomorphic to Khovanov's $\mathfrak{sl}_3$ link homology (the latter is known to be the same as $\mathfrak{sl}_3$ Khovanov–Rozansky homology and the category $\mathcal{O}$ $\mathfrak{sl}_3$ link invariant). We also explain how foam relations follow as consequences of the relations in the categorified quantum group.

4A   Link homology via skew Howe duality

In this section, we show that all of the ingredients needed to define $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ link homology theories can be recovered from the foamation functors. We also show how the invariant of any link can be given as the image of a complex in $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$. This suggests that the graphical calculus in the categorified quantum group can be used to explore properties of categorified link invariants.

4A1   Categorified braidings   In [15, Theorem 4.3], Cautis, Kamnitzer and Licata show that the action of the quantum Weyl group elements $T_i 1_\lambda$ on the skew Howe representation $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ gives the braiding on the category of finite-dimensional $\dot{\mathbf{U}}(\mathfrak{sl}_n)$ representations, up to a factor of $\pm q^r$. In the language of webs, this says that the value of a crossing is given by the image of the corresponding quantum Weyl group element.

The same holds true at the categorified level (after extending the foamation 2–functors to 2–categories of complexes), ie the complexes assigned to crossings in $\mathfrak{sl}_2$ and

$\mathfrak{sl}_3$ link homology can be recovered, up to shifts in quantum and homological degree, as the images of the Rickard complexes. Recall these are given by

$$\mathcal{T}_i \mathbf{1}_\lambda = \mathcal{E}_i^{(-\lambda_i)} \mathbf{1}_\lambda \xrightarrow{d_1} \mathcal{E}_i^{(-\lambda_i+1)} \mathcal{F}_i \mathbf{1}_\lambda \{1\} \xrightarrow{d_2} \cdots \xrightarrow{d_s} \mathcal{E}_i^{(-\lambda_i+s)} \mathcal{F}_i^{(s)} \mathbf{1}_\lambda \{s\} \xrightarrow{d_{s+1}} \cdots$$

when $\lambda_i \leq 0$ and

$$\mathcal{T}_i \mathbf{1}_\lambda = \mathcal{F}_i^{(\lambda_i)} \mathbf{1}_\lambda \xrightarrow{d_1} \mathcal{F}_i^{(\lambda_i+1)} \mathcal{E}_i \mathbf{1}_\lambda \{1\} \xrightarrow{d_2} \cdots \xrightarrow{d_s} \mathcal{F}_i^{(\lambda_i+s)} \mathcal{E}_i^{(s)} \mathbf{1}_\lambda \{s\} \xrightarrow{d_{s+1}} \cdots$$

when $\lambda_i \geq 0$, where in the above formulae the leftmost term is in homological degree zero. The above complexes are isomorphic when $\lambda_i = 0$.

The complexes $\mathcal{T}_i \mathbf{1}_\lambda$ are invertible, up to homotopy, with inverses given by

$$\mathbf{1}_\lambda \mathcal{T}_i^{-1} = \cdots \xrightarrow{d_{s+1}^*} \mathbf{1}_\lambda \mathcal{E}_i^{(s)} \mathcal{F}_i^{(-\lambda_i+s)} \{-s\} \xrightarrow{d_s^*} \cdots \xrightarrow{d_2^*} \mathbf{1}_\lambda \mathcal{E}_i \mathcal{F}_i^{(-\lambda_i+1)} \{-1\} \xrightarrow{d_1^*} \mathbf{1}_\lambda \mathcal{F}_i^{(-\lambda_i)}$$

when $\lambda_i \leq 0$ and

$$\mathbf{1}_\lambda \mathcal{T}_i^{-1} = \cdots \xrightarrow{d_{s+1}^*} \mathbf{1}_\lambda \mathcal{F}_i^{(s)} \mathcal{E}_i^{(\lambda_i+s)} \{-s\} \xrightarrow{d_s^*} \cdots \xrightarrow{d_2^*} \mathbf{1}_\lambda \mathcal{F}_i \mathcal{E}_i^{(\lambda_i+1)} \{-1\} \xrightarrow{d_1^*} \mathbf{1}_\lambda \mathcal{E}_i^{(\lambda_i)}$$

when $\lambda_i \geq 0$, where in these formulae the rightmost term is in homological degree zero. Note that these complexes are obtained by taking the adjoints of the above (in the category of complexes).

We begin with the $\mathfrak{sl}_2$ case. When λ maps to a sequence with $a_i = 1 = a_{i+1}$,

(4-1) $\qquad \Phi_2(\mathcal{T}_i \mathbf{1}_\lambda) = \left(\begin{array}{c} \end{array} \right)$

which gives the value of the positive $(1,1)$ crossing . This complex is the Blanchet foam analog of the formula for the crossing given in [32] and [2]. The negative crossing is given by

(4-2) $\qquad \Phi_2(\mathbf{1}_\lambda \mathcal{T}_i^{-1}) = \left(\begin{array}{c} \end{array} \right).$

In both (4-1) and (4-2), the identity web appears in homological degree zero.

In order to give a construction of the link invariant via the foamation 2–functors, we will also need the formulae for the braidings involving 0's and 2's. Defining the

positive crossings to be the images of the $\mathcal{T}_i \mathbf{1}_\lambda$ in the appropriate weights and the negative crossings to be the images of the $\mathbf{1}_\lambda \mathcal{T}_i^{-1}$, this gives the formulae

$$\mathllap{} = \mathllap{} = \Phi_2(\mathcal{F}_i \mathbf{1}_\lambda) = \mathllap{}$$

$$\mathllap{} = \mathllap{} = \Phi_2(\mathcal{E}_i \mathbf{1}_\lambda) = \mathllap{}$$

$$\mathllap{} = \mathllap{} = \Phi_2(\mathcal{F}_i^{(2)} \mathbf{1}_\lambda) = \mathllap{}$$

(4-3)
$$\mathllap{} = \mathllap{} = \Phi_2(\mathcal{E}_i^{(2)} \mathbf{1}_\lambda) = \mathllap{}$$

$$\mathllap{} = \mathllap{} = \Phi_2(\mathcal{F}_i \mathbf{1}_\lambda) = \mathllap{}$$

$$\mathllap{} = \mathllap{} = \Phi_2(\mathcal{E}_i \mathbf{1}_\lambda) = \mathllap{}$$

$$\mathllap{} = \mathllap{} = \Phi_2(\mathbf{1}_\lambda) = \mathllap{}$$

where the dotted strands are meant to indicate a 0–labeled edge, ie an edge that is not actually present. The braiding on two such 0–labeled edges is simply the empty web mapping between the appropriate labels.

In the $\mathfrak{sl}_3$ case, we give the formulae for the braidings in $3\mathbf{BFoam}_m(N)$, since those in $3\mathbf{Foam}_m(N)$ can be recovered from these via the forgetful 2–functor. We'll first compute the braidings for the traditional $\mathfrak{sl}_3$ edges. The $(1,1)$ crossings are again given as

(4-4)
$$\mathllap{} := \Phi_3(\mathcal{T}_i \mathbf{1}_\lambda)$$
$$= \left(\xrightarrow{\qquad\qquad} q \right)$$

(4-5)
$$\mathllap{} := \Phi_3(\mathbf{1}_\lambda \mathcal{T}_i^{-1})$$
$$= \left(q^{-1} \xrightarrow{\qquad\qquad} \right)$$

where the identity webs are in homological degree zero in both of the above formulae.

Similarly, the $(1, 2)$ braidings are given by

(4-6) $\quad \succ\!\!\prec \; := \Phi_3(\mathcal{T}_i \mathbf{1}_\lambda)$

$$= \left(\underset{}{\underline{\hspace{1.5cm}}} \xrightarrow{\hspace{2cm}} q \; \right),$$

(4-7) $\quad \succ\!\!\prec \; := \Phi_3(\mathbf{1}_\lambda \mathcal{T}_i^{-1})$

$$= \left(q^{-1} \underset{}{\underline{\hspace{1.5cm}}} \xrightarrow{\hspace{2cm}} \underset{}{\underline{\hspace{1.5cm}}} \right);$$

note that (4-6) is a positive $(1, 2)$ braiding and (4-7) is a negative $(1, 2)$ braiding, although topologically the former is a left-handed crossing and the latter is right-handed. The $(2, 1)$ braidings are given by

(4-8) $\quad \succ\!\!\prec \; := \Phi_3(\mathcal{T}_i \mathbf{1}_\lambda)$

$$= \left(\underset{}{\underline{\hspace{1.5cm}}} \xrightarrow{\hspace{2cm}} q \; \right)$$

(4-9) $\quad \succ\!\!\prec \; := \Phi_3(\mathbf{1}_\lambda \mathcal{T}_i^{-1})$

$$= \left(q^{-1} \underset{}{\underline{\hspace{1.5cm}}} \xrightarrow{\hspace{2cm}} \underset{}{\underline{\hspace{1.5cm}}} \right)$$

and the $(2, 2)$ braidings are given by

(4-10) $\quad \succ\!\!\prec \; := \Phi_3(\mathcal{T}_i \mathbf{1}_\lambda)$

$$= \left(\underset{}{\underline{\hspace{1.5cm}}} \xrightarrow{\hspace{2cm}} q \; \right)$$

(4-11) $\quad \succ\!\!\prec \; := \Phi_3(\mathbf{1}_\lambda \mathcal{T}_i^{-1})$

$$= \left(q^{-1} \quad \rightarrow \quad \right).$$

As in the $\mathfrak{sl}_2$ case, we will also need the formulae for the braidings between sequences involving 0's and 3's in order to construct the link invariant. Defining the positive crossings to be the images of the $\mathcal{T}_i \mathbf{1}_\lambda$ (for appropriate λ) and the negative crossings to be the images of the $\mathbf{1}_\lambda \mathcal{T}_i^{-1}$, this gives the formulae

$$\qquad = \qquad = \Phi_3(\mathcal{F}_i \mathbf{1}_\lambda) = \qquad$$

$$\qquad = \qquad = \Phi_3(\mathcal{E}_i \mathbf{1}_\lambda) = \qquad$$

$$\qquad = \qquad = \Phi_3(\mathcal{F}_i^{(2)} \mathbf{1}_\lambda) = \qquad$$

$$\qquad = \qquad = \Phi_3(\mathcal{E}_i^{(2)} \mathbf{1}_\lambda) = \qquad$$

$$\qquad = \qquad = \Phi_3(\mathcal{F}_i^{(3)} \mathbf{1}_\lambda) = \qquad$$

(4-12)
$$\qquad = \qquad = \Phi_3(\mathcal{E}_i^{(3)} \mathbf{1}_\lambda) = \qquad$$

$$\qquad = \qquad = \Phi_3(\mathcal{F}_i^{(2)} \mathbf{1}_\lambda) = \qquad$$

$$\qquad = \qquad = \Phi_3(\mathcal{E}_i^{(2)} \mathbf{1}_\lambda) = \qquad$$

$$\qquad = \qquad = \Phi_3(\mathcal{F}_i \mathbf{1}_\lambda) = \qquad$$

$$\qquad = \qquad = \Phi_3(\mathcal{E}_i \mathbf{1}_\lambda) = \qquad$$

$$\qquad = \qquad = \Phi_3(\mathbf{1}_\lambda) = \qquad$$

where again the dotted strands are meant to indicate 0–labeled (non-)edges. The braiding on two such edges is the empty web.

Let $n = 2, 3$; it will be useful to note that any object $\mathbf{a} = (a_1, \cdots, a_m) \in n(\mathbf{B})\mathbf{Foam}_m(N)$ can be identified with a canonical object which corresponds to the same $\mathfrak{sl}_n$ representation as $\mathbf{a}$ (up to isomorphism). Given an object $\mathbf{a}$ in $n(\mathbf{B})\mathbf{Foam}_m(N)$, denote by $\bar{\mathbf{a}}$ the associated *reduced sequence* defined to be the same sequence as $\mathbf{a}$ with all values 0 and n deleted. For example, if $\mathbf{a} = (1, 3, 0, 2, 0, 1)$ and $n = 3$, then $\bar{\mathbf{a}} = (1, 2, 1)$.

Definition 4.1 Given an object $\mathbf{a}$ of $n(\mathbf{B})\mathbf{Foam}_m(N)$, the associated *canonical sequence* is the unique object $\mathbf{a}'$ in $n(\mathbf{B})\mathbf{Foam}_m(N)$ such that $\bar{\mathbf{a}}' = \bar{\mathbf{a}}$ and

$$\mathbf{a}' = (0, \ldots, 0, a'_k, a'_{k+1}, \ldots, a'_{k+r}, n, \ldots, n)$$

with $0 < a'_{k+s} < n$ for $0 \le s \le r$.

The trivial braidings (4-3) and (4-12) can be used to give an equivalence between an object $\mathbf{a}$ in $n(\mathbf{B})\mathbf{Foam}_m(N)$ and its canonical sequence $\mathbf{a}'$; this is the analog of [10, Corollaries 7.3 and 7.8] in the web and foam setting. Let the web $\mathbf{a} \xrightarrow{\beta_\mathbf{a}} \mathbf{a}'$ be given by the (composition of) braidings involving $0-$ and $n-$labeled edges and let the web $\mathbf{a}' \xrightarrow{\beta_\mathbf{a}^{-1}} \mathbf{a}$ be given using the inverses of the above braidings. Since the images of the Rickard complexes braid in any (integrable) $2-$representation [14], the above maps are uniquely defined up to coherent isomorphism. Fix once and for all choices of $\beta_\mathbf{a}$ and $\beta_\mathbf{a}^{-1}$ for each object $\mathbf{a}$ in each of the foam $2-$categories.

4A2 The $\mathfrak{sl}_2$ tangle invariant The webs that appear in the image of the foamation $2-$functors are all in ladder form; hence, we require a method for assigning a complex of ladders in the foam $2-$categories to each tangle. A process which transforms any web to a ladder is detailed in [19]; however, an adaptation of a construction from [10] is more useful for our purposes.

Let τ be an oriented $(r, t)-$tangle diagram, ie a tangle diagram with r endpoints on the right and t endpoints on the left, which we assume to be in Morse position with respect to the horizontal axis. We now describe a method for assigning to this diagram a complex $[\![\tau]\!]_2$ in $2\mathbf{BFoam}_{r+2l}(r + 2l)$, for l sufficiently large.

We assign to each basic tangle a complex of $1-$morphisms mapping between canonical sequences; the complex assigned to a tangle will then be the horizontal composition (in the $2-$category of complexes) of the basic complexes.
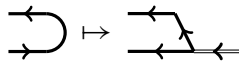
A tangle involving no crossings, cups or caps is mapped to the identity web of the sequence

$$(0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2),$$

where the number of 1's is equal to the number of strands in the tangle. For example observe the following, where the bottom dotted web edges are zero-labeled, ie not actually present.
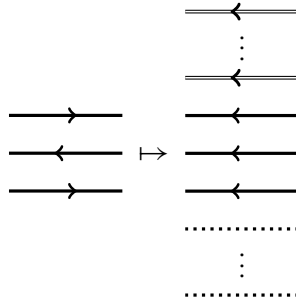
We'd like to map the cup as follows,

however the domain on this web will not be a canonical sequence (especially when other strands of the tangle are present). We will hence pre-compose with the relevant web $\beta_{\mathbf{a}}^{-1}$. For example,
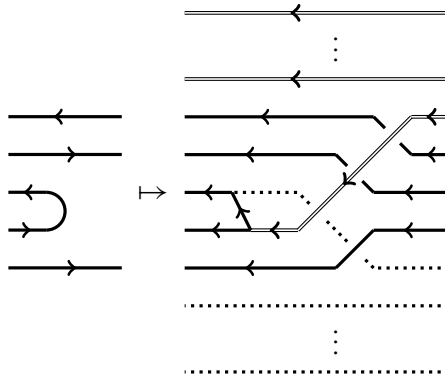
where again we have depicted the 0–labeled edges.

We similarly define the map on the remaining cup and caps to be given by
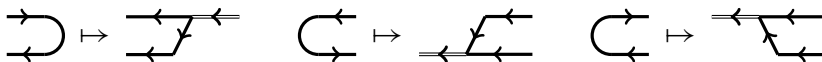
where we pre- or post-compose with the appropriate braiding maps β^{-1} and β as necessary to ensure that the webs map between canonical sequences.

We assign the complexes (4-1) and (4-2) to the positive and negative left-oriented crossings. This assignment determines the value of the invariant on the remainder of the crossings (up to isomorphism) since they can be obtained from the left-oriented

crossings by composing with caps and cups; eg we have
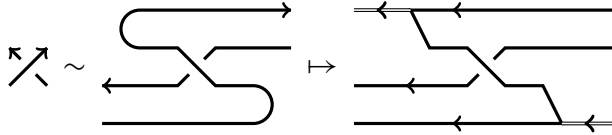
where the latter is understood to represent the complex assigned to a left-oriented crossing horizontally composed in the category of complexes with the indicated webs (and the necessary braiding maps β^{-1} and β so that the webs in the complex map between canonical sequences). Formulae for the other crossings can be obtained similarly.

Proposition 4.2 *The complex $[\![\tau]\!]_2$ assigned to a tangle diagram τ, viewed in the homotopy category of complexes of $2\mathbf{BFoam}_m(N)$, gives an invariant of framed tangles.*

Proof It suffices to check the tangle Reidemeister moves (see [31] or [12]); this is a standard computation following the argument detailed in [2] adapted to the Blanchet foam setting. Alternatively, one can simplify the computation using the proof of [10, Proposition 7.9], where it is shown that (most of) the desired relations hold already in the categorified quantum group. $\square$

One can check that (locally)

$$\left[\!\!\left[\,\vcenter{\hbox{}}\,\right]\!\!\right]_2 \simeq q^{-1}\left[\!\!\left[\,\vcenter{\hbox{}}\,\right]\!\!\right]_2 \quad \text{and} \quad \left[\!\!\left[\,\vcenter{\hbox{}}\,\right]\!\!\right]_2 \simeq q\left[\!\!\left[\,\vcenter{\hbox{}}\,\right]\!\!\right]_2$$

so renormalizing the invariant using the writhe $w(\tau)$ of the tangle

$$[\![\tau]\!]_2^r := q^{w(\tau)}[\![\tau]\!]_2$$

gives an invariant independent of framing.

Given a link L, the invariant $[\![L]\!]_2^r$ is a complex of webs mapping between the sequence $(\underline{0}, \underline{2}) := (0, \ldots, 0, 2, \ldots, 2)$ and itself. Applying the functor

$$\mathrm{HOM}(\mathrm{id}_{(\underline{0},\underline{2})}, -) := \bigoplus_{t \in \mathbb{Z}} \mathrm{Hom}(q^{-t}\mathrm{id}_{(\underline{0},\underline{2})}, -)$$

to this complex (where $\mathrm{id}_{(\underline{0},\underline{2})}$ is the identity web) and setting the parameter $\bigcirc = 0$ gives a complex of finite-dimensional graded vector spaces, which we denote $\mathrm{Kh}_2(L)$. As the notation indicates, we have the following result.

**Proposition 4.3** *The homology of the complex* $\mathrm{Kh}_2(L)$ *is the Khovanov homology of the link* L.

Proof Let D be a diagram of the link L. The complex $[\![D]\!]_2^r$ consists of $\mathfrak{sl}_2$ webs with no 1–labeled boundary, ie these webs consist of 1–labeled circles joined to each other (and to the boundary) by 2–labeled edges. Such a web in $[\![D]\!]_2^r$ contributes a direct summand of dimension $2^{\#\,\text{of circles}}$ to the complex $\mathrm{Kh}_2(D)$. Indeed, if W is such a web then $\mathrm{HOM}(\mathbb{1}_{(\underline{0},2)}, W)$ is a free $\Bbbk[\,\circled{3^\bullet}\,]$–module with basis given by 1–labeled cups with one or no dots, intersecting 2–labeled sheets transversely.

The complex $\mathrm{Kh}_2(D)$ is hence obtained from a cube of resolutions in which the nodes of the cube are exactly those appearing in the construction of Khovanov homology. One can check that the maps labeling the edges of this cube are, up to a $\pm$ sign, the maps m and Δ from [32]. Since the squares in this cube of resolutions anti-commute (by construction), an argument[9] from [61] shows that this complex is isomorphic to the complex assigned to D in [32]. $\qquad\square$

**4A3 An explicit example** The invariant of the Hopf link

$$L \;=\;$$
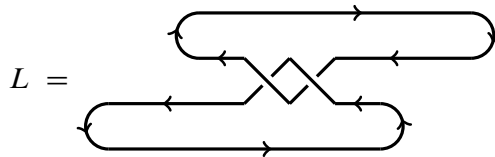
can be constructed in $2\mathbf{BFoam}_{2l}(2l)$ for any $l \geq 2$; we'll take the minimal case $l = 2$. By the procedure detailed above, we have that $[\![L]\!]_2$ is given by the complex

(4-13)
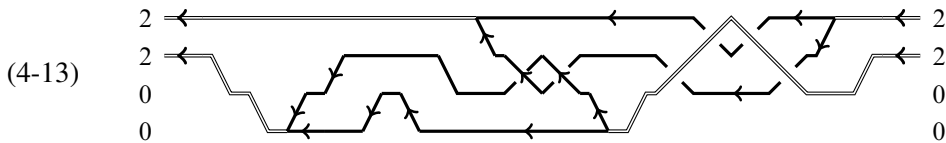
which is shorthand for the complex obtained from the following cube of resolutions (after applying some web isomorphisms), where the foams α in the complex are those depicted in (4-1) horizontally composed with the relevant identity foams.

---

[9]Although [61] deals with odd Khovanov homology, Lemma 2.2 of that paper, which proves that the homology doesn't depend on a choice of sign assignment, can be adapted to the (even) Khovanov homology case as well. The proof of [61, Lemma 2.2] shows that given a commutative cube and two sign assignments ϵ, ϵ' making all squares anti-commute, the product $\epsilon \cdot \epsilon'$ is a cocycle. This is the fact used to show that the complexes corresponding to ϵ and ϵ' are isomorphic.

(4-14)
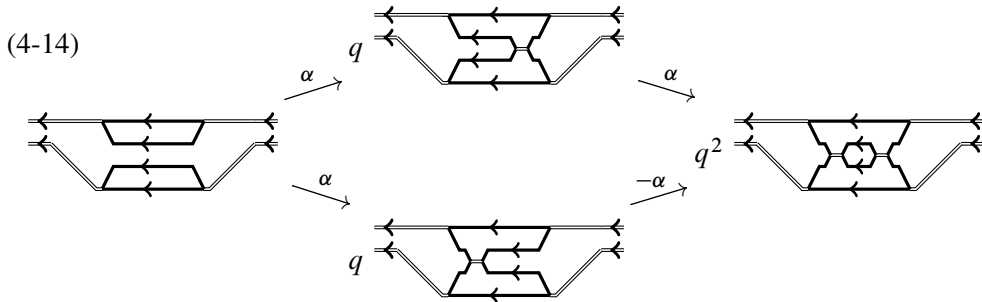
Note that the complex (4-13) is the image under Φ_2 of the complex

$$\mathcal{E}_2^{(2)}\mathcal{E}_1^{(2)}\mathcal{F}_1\mathcal{F}_2\mathcal{F}_1\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3\mathcal{T}_2\mathcal{T}_2\mathcal{E}_1\mathcal{F}_1^{(2)}\mathcal{T}_2^{-1}\mathcal{T}_3^{-1}\mathcal{T}_3\mathcal{T}_2\mathcal{F}_3\mathcal{F}_2^{(2)}\mathbf{1}_{(0,2,0)}$$

in $\mathrm{Kom}(\check{\mathcal{U}}_Q(\mathfrak{sl}_4))$. Indeed, for any tangle τ, we can realize the complex $[\![\tau]\!]_2$ as the image of a complex in the categorified quantum group by pulling back the various pieces assigned to elementary tangles to $\mathrm{Kom}(\check{\mathcal{U}}_Q(\mathfrak{sl}_m))$. One may then use the graphical calculus of the categorified quantum group to perform calculations in link homology; see eg [10, Section 10].

4A4 The $\mathfrak{sl}_3$ tangle invariant We define the $\mathfrak{sl}_3$ tangle invariant $[\![\tau]\!]_3 \in 3\mathbf{Foam}_m(N)$ in a similar manner as above.[10] An oriented tangle (diagram) with no caps, cups or crossings determines a sequence **s** of 1's and 2's (corresponding to the strands directed to the left and right respectively) and we map such a tangle to the identity web of the sequence $(\underline{0}, \mathbf{s}, \underline{3})$, eg
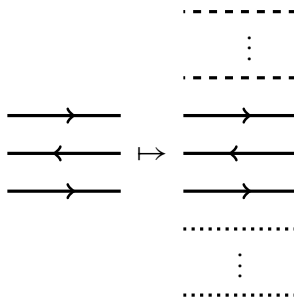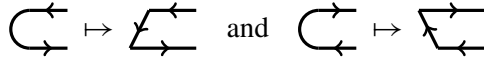

where the dotted and dashed lines denote web edges which are not actually present, ie $0-$ and $3-$labeled edges.

The invariant is defined on cups by

---
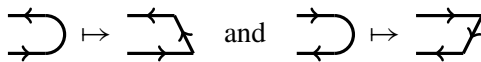
[10]We could define this invariant in $3\mathbf{BFoam}_m(N)$ as well; however, the invariant in $3\mathbf{Foam}_m(N)$ is (essentially) the $\mathfrak{sl}_3$ invariant found in the literature.

and on caps by

$$\overset{\rightarrow}{\subset} \mapsto \overset{\leftarrow}{\diagdown} \quad \text{and} \quad \overset{\rightarrow}{\subset} \mapsto \overset{\rightarrow}{\diagdown}$$

where, as in the $\mathfrak{sl}_2$ case, we pre- and post-compose with the relevant braidings so that the webs map between canonical sequences. These braidings are given by deleting the 3–labeled edges from those given in (4-12).

We define the invariant on left-oriented crossings by equations (4-4) and (4-5). We'd like to define the image of the remainder of the crossings using the images of the braidings (4-6)–(4-11) under the forgetful functor $3\mathbf{BFoam}_m(N) \to 3\mathbf{Foam}_m(N)$; however, this assignment would not be invariant under planar isotopy as the complexes differ by factors of $q^{\pm 1}$. It is possible to rescale the Rickard complexes $\mathcal{T}_i \mathbf{1}_\lambda$ depending on the weight λ to fix this issue, but this introduces unwanted scalings on the trivial braidings (4-12). We instead follow our $\mathfrak{sl}_2$ approach and define the remainder of the crossings in terms of the left-oriented crossings and caps and cups.

Proposition 4.4 *The complex $[\![\tau]\!]_3$ assigned to a tangle diagram τ, viewed in the homotopy category of complexes of $3\mathbf{Foam}_m(N)$, is an invariant of framed tangles.*

Renormalizing this invariant via $[\![\tau]\!]_3^r = q^{2w(\tau)}[\![\tau]\!]_3$ gives an invariant independent of framing which is (essentially) the same as Morrison–Nieh's [59] extension of Khovanov's $\mathfrak{sl}_3$ link homology [34] to tangles, after setting the 3–, 4– and 5–dotted spheres equal to zero.

4A5 Categorified clasps In [10], Cautis showed that given any categorification of the skew Howe representations $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$, one obtains a categorification of $\mathfrak{sl}_n$ clasps, the $\mathfrak{sl}_n$ analogs of the Jones–Wenzl projectors, using the higher representation theory of the categorified quantum group. He conjectured that these representations could be categorified in the foam setting and that this construction would give the categorified Jones–Wenzl projectors from [23] and [65] and the categorified $\mathfrak{sl}_3$ projectors from [63]. Although the foam categories only categorify the intertwiners between such representations (and not the representations themselves), Cautis' methods indeed give a uniform construction of categorified clasps in the $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ foam 2–categories. We now recall the details of this construction.

Fix a reduced expression $w = s_{i_1} \cdots s_{i_k}$ for the longest word w in the Weyl group for $\mathfrak{sl}_m$ and consider the complex $\mathcal{T}_w \mathbf{1}_\lambda := \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_k} \mathbf{1}_\lambda$ in $\mathrm{Kom}(\check{\mathcal{U}}_Q(\mathfrak{sl}_m))$; this complex gives the invariant assigned to a half-twist tangle. Cautis shows that the images of the complexes $\mathcal{T}_w^{2k} \mathbf{1}_\lambda$ in any integrable 2–representation stabilize as $k \to \infty$. Denote the image of $\mathcal{T}_w \mathbf{1}_\lambda$ in such a 2–representation by $\mathsf{T}_w \mathbb{1}_\lambda$ and let $\mathsf{T}_w^\infty \mathbb{1}_\lambda :=$

$\lim_{k \to \infty} \mathsf{T}_w^{2k} \mathbb{1}_\lambda$. The complexes $\mathsf{T}_w^\infty \mathbb{1}_\lambda$ are idempotents (with respect to horizontal composition of complexes) and give categorified clasps in any 2–representation categorifying $\bigwedge_q^N (\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$.

We first consider the $\mathfrak{sl}_2$ case. Let $\mathsf{P}_m^+ := \mathsf{T}_w^\infty \mathbb{1}_{(0,\dots,0)}$ in $2\mathbf{BFoam}_m(m)$ (which is a complex of webs mapping from the sequence $(1, \dots, 1)$ to itself); then we have the following result, which should be viewed as the Blanchet foam analog of [23, Theorem 3.2].

Proposition 4.5 *The complex P_m^+ satisfies the following properties:*

(1) P_m^+ *is supported only in positive homological degree.*

(2) *The identity web* $\mathrm{id}_{(1,\dots,1)}$ *appears only in homological degree zero.*

(3) P_m^+ *annihilates the webs*
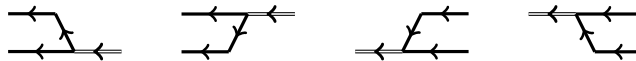
in $2\mathbf{BFoam}_m(m)$, *up to homotopy.*

Proof Properties (1) and (2) follow via inspection. Property (3) follows from arguments in [10, Section 5] or by adapting arguments from [63] to the $\mathfrak{sl}_2$ foam setting. □

It follows that P_m^+ categorifies the analog of the Jones–Wenzl projector p_m in the category of Blanchet webs. Using the foamation 2–functor Φ_{CMW}, the above procedure also gives a construction of the categorified Jones–Wenzl projectors from [23] and [65] in the CMW foam setting.

In the $\mathfrak{sl}_3$ case, let $\mathbf{s}$ denote a sequence of 1's and 2's of length m; let $\#\mathbf{s}_1$ denote the number of 1's and $\#\mathbf{s}_2$ the number of 2's in $\mathbf{s}$. Define $\mathsf{P}_\mathbf{s}^+ := \mathsf{T}_w^\infty \mathbb{1}_\lambda$ in $3\mathbf{Foam}_m(m + \#\mathbf{s}_2)$ where λ maps to $\mathbf{s}$ under Φ_3.

Proposition 4.6 *The complex $\mathsf{P}_\mathbf{s}^+$ is the categorified clasp $\tilde{P}_\mathbf{s}$ constructed in [63].*

There is nothing to prove here; the categorified $\mathfrak{sl}_3$ clasps in [63] are constructed precisely as the limit of the complexes $\mathsf{T}_w^{2k} \mathbb{1}_\lambda$ as $k \to \infty$. Note that the $+$'s and $-$'s in the sequences in that work correspond to our 1's and 2's, respectively.

Having constructed categorified clasps, we can extend our $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ tangle invariants to give categorified invariants of framed tangles in which each component is labeled by an irreducible representation. This construction is detailed in many places in the literature, in particular in [23; 63; 10], so we will be brief. Given a framed tangle τ with components labeled by irreducible representations, choose for each component a

tensor product of fundamental representations having the corresponding irreducible as a highest weight subrepresentation. Assign to the tangle the complex assigned to a cabling of the tangle (we use here the fact that τ is framed) with the categorified projector inserted along the cabling. The number of strands in the cabling of each component (and the direction of such strands in the $\mathfrak{sl}_3$ case) as well as which projector P^+ is inserted is given by the relevant tensor product of fundamental representations; this corresponds to a sequence of 1's in the $\mathfrak{sl}_2$ case and a sequence of 1's and 2's in the $\mathfrak{sl}_3$ case.

One can show (see [23; 63; 10]) that the above invariant doesn't depend on the choice of where the projector is inserted or which tensor product of fundamentals is used (up to equivalence in the case that the tangle is not a link) and gives a categorification of the Reshetikhin–Turaev invariant of framed tangles.

4B   Comparing knot homologies

Let $\Phi\colon \dot{\mathcal{U}}_Q(\mathfrak{sl}_m) \to \mathcal{K}$ be any 2–representation giving a categorification of $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$. Cautis shows that for N and m sufficiently large, this 2–representation assigns to any framed, oriented link K a complex of 1–morphisms $\Psi(K) \in \mathrm{End}(\mathrm{Kom}(\Phi(\lambda)))$, where λ is the highest weight in $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ [10, Section 7.5]. His framework does not require the full structure of a 2–representation of $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$, but rather the weaker data encoded in what he calls a categorical 2–representation. This weaker action is more like the data described in Theorem 2.5 without requiring the KLR action.

The KLR relations greatly simplify the resulting complexes; in particular, they imply analogues of the higher Serre relation [67] and commutativity relations for divided powers [40]. Using these relations, the complex $\Psi(K)$ associated to a link K can be reduced to a complex that only involves direct sums of the identity 1–morphism $\mathbb{1}_\lambda$ of $\Phi(\lambda)$, with various grading shifts. In fact, one does not actually need to know that the 2–representation $\mathcal{K}$ categorifies $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ to apply this reduction procedure; one only needs that the nonzero weight spaces of $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ parametrize the nonzero objects in $\mathcal{K}$. In this case, all simplifications can be performed in a quotient of the 2–category $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$ obtained by killing the weights which do not appear in $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ (this idea is used extensively in the sequel paper [62]).

Applying the functor $\mathrm{HOM}(\mathbb{1}_\lambda, -)$ to the (reduced) complex $\Psi(K)$ maps it to a complex of graded vector spaces. The number of $\mathbb{1}_\lambda$ summands and their grading shifts are formally determined by the categorified quantum group, hence so are the vector spaces appearing in the complex. The differentials depend only on the map $\mathrm{HOM}(\mathbf{1}_\lambda, \mathbf{1}_\lambda) \to \mathrm{HOM}(\mathbb{1}_\lambda, \mathbb{1}_\lambda)$, so it follows that this map completely determines the link homology theory.

When the graded algebra $A := \mathrm{HOM}(\mathbb{1}_\lambda, \mathbb{1}_\lambda)$ is 1–dimensional in degree zero and 0 in all other degrees, only one such map exists; hence all constructions of $\mathfrak{sl}_n$ link homology satisfying this condition are equivalent. After quotienting by the 3–dotted sphere in the $\mathfrak{sl}_2$ case and the 3–, 4– and 5–dotted spheres in the $\mathfrak{sl}_3$ case, the foam 2–categories satisfy this condition.

This observation gives a method for showing that Cautis–Kamnitzer link homology is equivalent to Khovanov–Rozansky homology. Using constructions from previous work [17; 18], Cautis describes (weak) categorical 2–representations on derived categories $\mathcal{K}_{Gr,m}$ of coherent sheaves on varieties arising as orbits in the affine Grassmannian, as well as on coherent sheaves on Nakajima quiver varieties $\mathcal{K}_{Q,m}$. Both of these categorical 2–representations are conjectured by Cautis, Kamnitzer and Licata to extend to 2–representations of $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$. By the results of [20] it suffices to prove that the KLR algebras act; this was done in the $m = 2$ case in [16] and (while this paper was under review) was generalized to symmetric Kac–Moody algebras (in particular for arbitrary m) in [11]. Moreover, in this setting the algebra A satisfies the 1–dimensionality condition, so this will show that the link homology theory from [13] fits into the framework described above.

The results from this paper will hence show that the foam based constructions of $\mathfrak{sl}_n$ link homology agree with the Cautis–Kamnitzer construction for $n = 2, 3$. This re-proves [12, Theorem 8.2] and pairs with the results from [55] to give the $n = 3$ case of [13, Conjecture 6.4] equating $\mathfrak{sl}_3$ Cautis–Kamnitzer and Khovanov–Rozansky link homology. In the sequel to this paper, we will establish the analogous results for general n.

## 4C Deriving foam relations from categorified quantum groups

In [19], Cautis, Kamnitzer and Morrison showed that the relations on $\mathfrak{sl}_n$ webs could be derived via skew Howe duality from the relations in $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$. Here we categorify this result in the $n = 2, 3$ case to show that many foam relations can be deduced from the assignments defining the foamation 2–functors Φ_2 and Φ_3. The main result of this section is that all $\mathfrak{sl}_3$ foam relations, all CMW $\mathfrak{sl}_2$ foam relations (assuming a strong form of locality), and many Blanchet $\mathfrak{sl}_2$ foam relations follow from relations in the categorified quantum group $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$. In a follow-up paper, we study foam categories for arbitrary n using this framework [62].

4C1 Blanchet $\mathfrak{sl}_2$ foam relations Since the Blanchet foams arising as images under our 2–functors must contain both 1– and 2–labeled facets (unless they are identity foams) and always bound webs whose edges are oriented leftward, we cannot expect

to recover all defining relations from the relations in $\mathcal{U}_Q(\mathfrak{sl}_m)$. For example, we have no hope of recovering the 1– and 2–labeled neck-cutting relations (3-3) and (3-6) or closed foam relations.

There are nevertheless numerous foam relations arising from the quantum group relations, which we list below. Note that some of the relations we obtain actually slightly generalize Blanchet's original relations, using 2– and 3–dotted enhanced spheres as graded parameters. The category one would obtain by only considering images of the quantum group relations would be a left-directed foam category providing a universal version of Blanchet's construction. In particular, setting the 3–dotted enhanced sphere to 1 and the 2–dotted one to zero gives a framework for Lee degenerations of Khovanov homology.

- The nilHecke relation (2-17) implies the enhanced neck-cutting relation below.
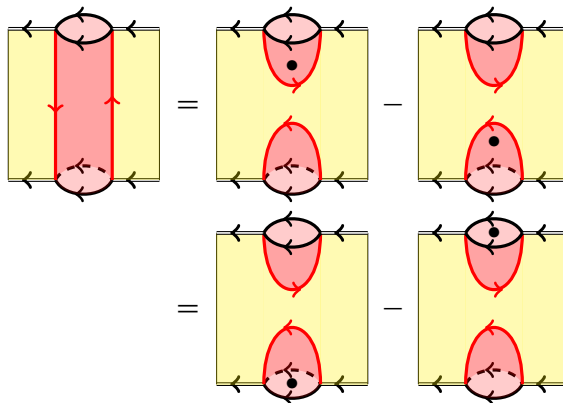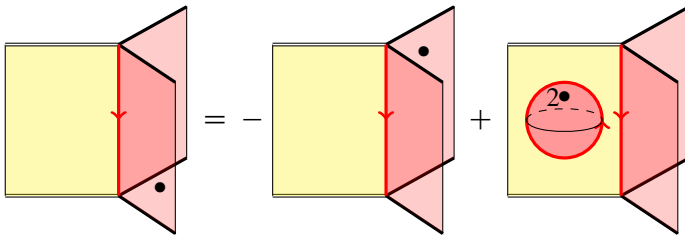
  The first is relation (3-11) (note the orientation of the seams) and the second is equivalent to this using an isotopy of the 1–labeled tube.

- Degree-zero bubbles in weight ± 2 imply the blister relation (3-14). The LHS of the blister relation (3-13) follows from the non-dotted bubble in weight ± 2.

- Composing the second enhanced neck-cutting relation with
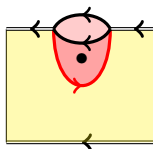
  and using the previous blister relations, we obtain the following generalization of the dot-migration relation (3-20).

(4-15)
$$\text{[diagram]} = - \text{[diagram]} + \text{[diagram]}$$

Relation (2-18) with $j = i + 1$ and $a_{i+1} = 2$ implies that twice-dotted blisters can migrate between 2–labeled facets; this allows us to view them as graded parameters.

- Composing the enhanced neck-cutting relation with

$$\text{[diagram]} \quad \text{and} \quad \text{[diagram]}$$
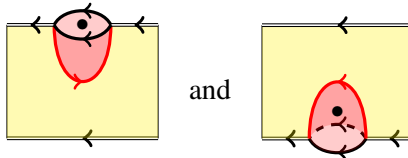
  gives the RHS of (3-13), using (3-14). We do not obtain the analog of relation (3-13) with two dots on the same facet, since the dot-sliding relation has an additional term.

- Relation (2-18) with $j = i + 1$ and $a_{i+1} = 1$ implies the relation

$$\text{[diagram]} = \text{[diagram]} - \text{[diagram]}$$
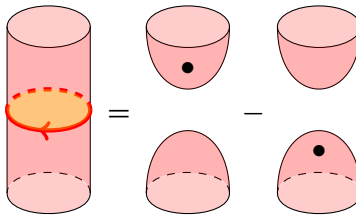
  which can be viewed as another enhanced version of neck-cutting.

- Degree zero bubbles in weight ± 1 with $m = 2$ and $N = 3$ imply relation (3-16).

- Relation (2-21) implies the foam relation (3-18).

Finally, we comment on the behavior of a twice-dotted foam facet. When $\lambda = -2$, we compute the following.

$$2 \, \bullet \, \Big\uparrow \, \lambda \;=\; \text{[diagram]} \;=\; - \text{[diagram]} \lambda \;+\; \sum_{g_1 + g_2 + g_3 = 1} g_1 \, \text{[diagram]} \lambda$$

The term on the left in the last part is sent to zero under the foamation functor. We obtain the following, providing a way to decompose twice-dotted foam facets using the image of 2– and 3–dotted bubbles.

$$\Phi_2 \left(2\!\!\bullet \uparrow \, \lambda \right) = \Phi_2 \left(\uparrow \bigcirc\!\!\bullet\!\!{}_{3}\, \lambda \,\uparrow + \uparrow \begin{matrix} \bigcirc\!\!\bullet\!\!{}_{2} \\ \bigcirc\!\!\bullet\!\!{}_{2} \end{matrix} \lambda \,\uparrow + \bullet \bigcirc\!\!\bullet\!\!{}_{2}\, \lambda \right)$$

4C2 CMW $\mathfrak{sl}_2$ foam relations We would like to derive the relations in the Clark–Morrison–Walker $\mathfrak{sl}_2$ foam category from the relations in $\mathcal{U}_Q(\mathfrak{sl}_m)$. Recall that this category[11] is a less rigid version of the one presented in Definition 3.4. Its objects are formal $\mathbb{Z}$–graded direct sums of disoriented planar curves (as depicted in Section 3A4) and morphisms are matrices of linear combinations of disoriented cobordisms, modulo isotopy, Relations (3-2), (3-3) and (3-22).

However, since the CMW seam relations (3-22) involve complex coefficients, we cannot expect to derive them from the categorical skew Howe action of $\mathcal{U}_Q(\mathfrak{sl}_m)$. We hence must impose the additional requirement that some relations can be performed completely locally (which in practice says that some relations have a "square root"). We will show that by imposing relations derived from this additional assumption, we can derive a slightly more general CMW foam 2–category in which both the 2– and 3–dotted spheres are (graded) parameters. Specializing these dotted spheres to zero then recovers the usual CMW category (which also shows that this more general category we obtain does not collapse).

- **Seam relations** Considering (2-21) with $a_i = 2$ and $a_{i+1} = 1$, we find that the first term of the relation is mapped to zero and the remaining foams give

  (up to isotopy). Assuming this relation can be expressed locally, this requires

  with ω' a primitive fourth root of unity (a priori, this is not required to equal the fourth root ω from Section 3A4).

---

[11]Clark, Morrison and Walker work in the setting of a canopolis, which is a version of a 2–category. In this section we'll only be concerned with the details of the Hom categories in this 2–category, and hence work in terms of categories.

Having fixed a value for ω', the values of degree-zero bubbles in weight ± 1 with $m = 2$ and $N = 3$ give that a circular seam squares to give -1 (in both cases). Again assuming complete locality, this gives that a circle can be removed from a foam at the cost of multiplying by a primitive fourth root of unity. Using the above, we determine

$$(4\text{-}16) \qquad \boxed{\bigcirc} = \omega' \, \boxed{} \,, \qquad \boxed{\bigcirc} = -\omega' \, \boxed{} \,.$$

- **Closed foam relations** Since negative degree bubbles are zero, we deduce that a non-dotted sphere is zero by considering the image of (non-dotted) bubbles in weight ± 2 with $N = 2$ and $m = 2$. The values of once-dotted bubbles in weight ± 2 give the value of a once-dotted sphere, depending on the value of ω. Choosing $\omega' = \omega$ (which we do for the duration), we obtain that a once-dotted sphere has value 1. After we deduce a neck-cutting relation, we will be able to evaluate n–dotted spheres with $n \geq 4$ in terms of spheres with fewer dots.

- **Neck-cutting** The nilHecke relation (2-17) gives us two neck-cutting relations:

Using the seam and closed foam relations, we can recover a deformed version of the neck-cutting relation from Equation (3-3). Caping with a dotted disk containing a disorientation seam, we have

which gives a relation for sliding a dot through a seam.

We then compute

which gives

ie the following deformation of the neck-cutting relation.

Specializing $\left(\overset{2\bullet}{\bigcirc}\right) = 0$, we recover the foam 2–category from [22].
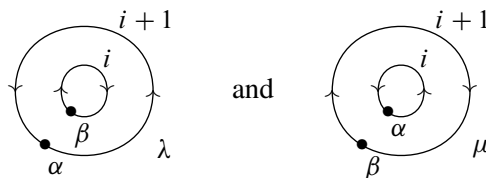
4C3 $\mathfrak{sl}_3$ **foam relations** In the $\mathfrak{sl}_3$ setting, all foam relations are consequences of the relations in $\mathcal{U}_Q(\mathfrak{sl}_m)$:

- **Dotted spheres** The values in relation (3-23) are recovered by the value of

in weight 3 with $m = 2$ and $N = 3$ for $\alpha = 0, 1, 2$.

- **Neck-cutting** The image of Equation (2-28) in weight 3 and with $m = 2$ and $N = 3$ gives the neck-cutting relation (3-24) (note that the cup gives a -1 coefficient and the cap gives $+1$). One can obtain the simpler neck-cutting relations found in the literature by quotienting the categorified quantum group by the relevant bubbles (or equivalently passing to the quotient of the foam category where we set the 3– and 4–dotted spheres equal to zero).

- Equation (3-26) is a consequence of the nilHecke relation (2-17).

- Equation (3-27) is a consequence of Equation (2-21).

- **Theta-foams** For $\alpha + \beta \leq 3$, the values of

(4-17)

and
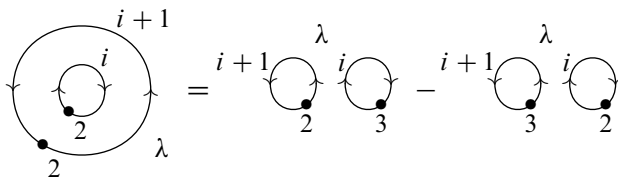
when λ maps to a sequence with $a_i = 0$, $a_{i+1} = 3$ and $a_{i+2} = 0$ and μ maps to a sequence with $a_i = 3$, $a_{i+1} = 0$ and $a_{i+2} = 3$ give the values in relation (3-25) when $\alpha + \beta \leq 3$ and $\gamma = 0$. In fact, these values, together with the remainder of the foam relations, determine the values of all theta-foams.

Using the values of theta-foams we have already determined, we can deduce the blister relations

from the neck-cutting and dotted sphere relations. The equality

implies that $\begin{pmatrix} 2^\bullet \\ 2^\bullet \end{pmatrix} = 0$, and then the neck-cutting relation gives the following additional blister relation.
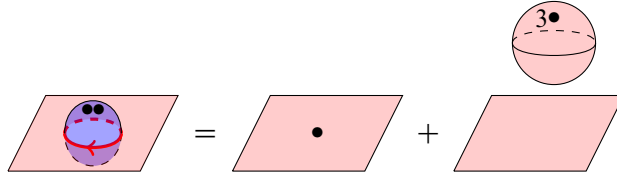
Composing (3-26) with the foam
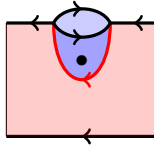
then gives the dot migration relation (compare to [34, Figure 17]):

(4-18)

Using this relation, in conjunction with (3-29), we can evaluate the remaining theta-foams from Equation (3-25).

Note that we may also recover many of the relations which follow as consequences of the defining relations:

- Equation (3-28) is a consequence of Equation (2-18).

- Using (2-28) with $\lambda_i = 1$ and $N_i = 3$, we compute

which gives (3-29).

- Equation (3-30) follows from the degree-zero bubble

$$\overset{i}{\bigcirc}{}^{\lambda} = \mathrm{id}$$

when $a_i = 1$ and $a_{i+1} = 2$.

References

[1] **D Bar-Natan**, *On Khovanov's categorification of the Jones polynomial*, Algebr. Geom. Topol. 2 (2002) 337–370   MR1917056

[2] **D Bar-Natan**, *Khovanov's homology for tangles and cobordisms*, Geom. Topol. 9 (2005) 1443–1499   MR2174270

[3] **J Bernstein**, **I Frenkel**, **M Khovanov**, *A categorification of the Temperley–Lieb algebra and Schur quotients of $U(\mathfrak{sl}_2)$ via projective and Zuckerman functors*, Selecta Math. 5 (1999) 199–241   MR1714141

[4] **C Blanchet**, *An oriented model for Khovanov homology*, J. Knot Theory Ramifications 19 (2010) 291–312   MR2647055

[5] **F Borceux**, *Handbook of categorical algebra, 1*, Encyclopedia of Mathematics and its Applications 50, Cambridge Univ. Press (1994)   MR1291599

[6] **J Brundan**, **C Stroppel**, *Highest weight categories arising from Khovanov's diagram algebra, III: Category $\mathcal{O}$*, Represent. Theory 15 (2011) 170–243   MR2781018

[7] **C L Caprau**, *An sl(2) tangle homology and seamed cobordisms*, PhD thesis, University of Iowa (2007)   MR2710797   Available at http://search.proquest.com/docview/304848972

[8] **C L Caprau**, sl(2) *tangle homology with a parameter and singular cobordisms*, Algebr. Geom. Topol. 8 (2008) 729–756   MR2443094

[9] **C Caprau**, *The universal $\mathfrak{sl}(2)$ cohomology via webs and foams*, Topology Appl. 156 (2009) 1684–1702   MR2521705

[10] **S Cautis**, *Clasp technology to knot homology via the affine Grassmannian*   arXiv:1207.2074

[11] **S Cautis**, *Rigidity in higher representation theory*   arXiv:1409.0827

[12] **S Cautis**, **J Kamnitzer**, *Knot homology via derived categories of coherent sheaves, I: The $\mathfrak{sl}(2)$–case*, Duke Math. J. 142 (2008) 511–588   MR2411561

[13] **S Cautis**, **J Kamnitzer**, *Knot homology via derived categories of coherent sheaves, II: $\mathfrak{sl}_m$ case*, Invent. Math. 174 (2008) 165–232   MR2430980

[14] **S Cautis**, **J Kamnitzer**, *Braiding via geometric Lie algebra actions*, Compos. Math. 148 (2012) 464–506   MR2904194

[15] **S Cautis**, **J Kamnitzer**, **A Licata**, *Categorical geometric skew Howe duality*, Invent. Math. 180 (2010) 111–159   MR2593278

[16] **S Cautis**, **J Kamnitzer**, **A Licata**, *Coherent sheaves and categorical $\mathfrak{sl}_2$ actions*, Duke Math. J. 154 (2010) 135–179   MR2668555

[17] **S Cautis**, **J Kamnitzer**, **A Licata**, *Coherent sheaves on quiver varieties and categorification*, Math. Ann. 357 (2013) 805–854   MR3118615

[18] **S Cautis**, **J Kamnitzer**, **A Licata**, *Derived equivalences for cotangent bundles of Grassmannians via categorical $\mathfrak{sl}_2$ actions*, J. Reine Angew. Math. 675 (2013) 53–99   MR3021447

[19] **S Cautis**, **J Kamnitzer**, **S Morrison**, *Webs and quantum skew Howe duality*, Math. Ann. 360 (2014) 351–390   MR3263166

[20] **S Cautis**, **A D Lauda**, *Implicit structure in* 2*–representations of quantum groups*, Selecta Math. 21 (2015) 201–244   MR3300416

[21] **J Chuang**, **R Rouquier**, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$–categorification*, Ann. of Math. 167 (2008) 245–298   MR2373155

[22] **D Clark**, **S Morrison**, **K Walker**, *Fixing the functoriality of Khovanov homology*, Geom. Topol. 13 (2009) 1499–1582   MR2496052

[23] **B Cooper**, **V Krushkal**, *Categorification of the Jones–Wenzl projectors*, Quantum Topol. 3 (2012) 139–180   MR2901969

[24] **T Dyckerhoff**, **D Murfet**, *The Kapustin–Li formula revisited*, Adv. Math. 231 (2012) 1858–1885   MR2964627

[25] **I Frenkel**, **M Khovanov**, **C Stroppel**, *A categorification of finite-dimensional irreducible representations of quantum $\mathfrak{sl}_2$ and their tensor products*, Selecta Math. 12 (2006) 379–431   MR2305608

[26] **I Frenkel**, **C Stroppel**, **J Sussan**, *Categorifying fractional Euler characteristics, Jones–Wenzl projectors and* $3j$*–symbols*, Quantum Topol. 3 (2012) 181–253   MR2901970

[27] **D Hill**, **J Sussan**, *The Khovanov–Lauda* 2*–category and categorifications of a level two quantum $\mathfrak{sl}_n$ representation*, Int. J. Math. Math. Sci. (2010)   MR2669068

[28] **M Jacobsson**, *An invariant of link cobordisms from Khovanov homology*, Algebr. Geom. Topol. 4 (2004) 1211–1251   MR2113903

[29] **J Kamnitzer**, **P Tingley**, *The crystal commutor and Drinfeld's unitarized R–matrix*, J. Algebraic Combin. 29 (2009) 315–335   MR2496310

[30] **A Kapustin**, **Y Li**, *Topological correlators in Landau–Ginzburg models with boundaries*, Adv. Theor. Math. Phys. 7 (2003) 727–749   MR2039036

[31] **C Kassel**, *Quantum groups*, Graduate Texts in Mathematics 155, Springer, New York (1995)   MR1321145

[32] **M Khovanov**, *A categorification of the Jones polynomial*, Duke Math. J. 101 (2000) 359–426   MR1740682

[33] **M Khovanov**, *A functor-valued invariant of tangles*, Algebr. Geom. Topol. 2 (2002) 665–741   MR1928174

[34] **M Khovanov**, *sl(3) link homology*, Algebr. Geom. Topol. 4 (2004) 1045–1081 MR2100691

[35] **M Khovanov**, *An invariant of tangle cobordisms*, Trans. Amer. Math. Soc. 358 (2006) 315–327   MR2171235

[36] **M Khovanov**, **A D Lauda**, *A diagrammatic approach to categorification of quantum groups, I*, Represent. Theory 13 (2009) 309–347   MR2525917

[37] **M Khovanov**, **A D Lauda**, *A categorification of quantum* sl(n), Quantum Topol. 1 (2010) 1–92   MR2628852

[38] **M Khovanov**, **A D Lauda**, *A diagrammatic approach to categorification of quantum groups, II*, Trans. Amer. Math. Soc. 363 (2011) 2685–2700   MR2763732

[39] **M Khovanov**, **A D Lauda**, *Erratum to: "A categorification of quantum* sl(n) *"*, Quantum Topol. 2 (2011) 97–99   MR2763088

[40] **M Khovanov**, **A D Lauda**, **M Mackaay**, **M Stošić**, *Extended graphical calculus for categorified quantum* sl(2), Mem. Amer. Math. Soc. 1029 (2012)   MR2963085

[41] **M Khovanov**, **L Rozansky**, *Topological Landau–Ginzburg models on the world-sheet foam*, Adv. Theor. Math. Phys. 11 (2007) 233–259   MR2322554

[42] **M Khovanov**, **L Rozansky**, *Matrix factorizations and link homology*, Fund. Math. 199 (2008) 1–91   MR2391017

[43] **M Khovanov**, **L Rozansky**, *Matrix factorizations and link homology, II*, Geom. Topol. 12 (2008) 1387–1425   MR2421131

[44] **D Kim**, *Graphical calculus on representations of quantum Lie algebras*, PhD thesis, University of California, Davis (2003)   MR2704398   Available at `http://search.proquest.com/docview/304595724`

[45] **G Kuperberg**, *Spiders for rank 2 Lie algebras*, Comm. Math. Phys. 180 (1996) 109–151   MR1403861

[46] **A D Lauda**, *A categorification of quantum* sl(2), Adv. Math. 225 (2010) 3327–3424 MR2729010

[47] **A D Lauda**, *An introduction to diagrammatic algebra and categorified quantum* $\mathfrak{sl}_2$, Bull. Inst. Math. Acad. Sin. 7 (2012) 165–270   MR3024893

[48] **G Lusztig**, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. 3 (1990) 447–498   MR1035415

[49] **G Lusztig**, *Introduction to quantum groups*, Progress in Mathematics 110, Birkhäuser, Boston (1993)   MR1227098

[50] **M Mackaay**, $\mathfrak{sl}(3)$–*foams and the Khovanov–Lauda categorification of quantum sl(k)* `arXiv:0905.2059`

[51] **M Mackaay**, **W Pan**, **D Tubbenhauer**, *The $\mathfrak{sl}_3$–web algebra*, Math. Z. 277 (2014) 401–479 MR3205780

[52] **M Mackaay**, **M Stošić**, **P Vaz**, $\mathfrak{sl}(N)$–*link homology* ($N \geq 4$) *using foams and the Kapustin–Li formula*, Geom. Topol. 13 (2009) 1075–1128 MR2491657

[53] **M Mackaay**, **M Stošić**, **P Vaz**, *A diagrammatic categorification of the q–Schur algebra*, Quantum Topol. 4 (2013) 1–75 MR2998837

[54] **M Mackaay**, **P Vaz**, *The universal sl₃–link homology*, Algebr. Geom. Topol. 7 (2007) 1135–1169 MR2336253

[55] **M Mackaay**, **P Vaz**, *The foam and the matrix factorization sl₃ link homologies are equivalent*, Algebr. Geom. Topol. 8 (2008) 309–342 MR2443231

[56] **C Manolescu**, *Link homology theories from symplectic geometry*, Adv. Math. 211 (2007) 363–416 MR2313538

[57] **V Mazorchuk**, **C Stroppel**, *A combinatorial approach to functorial quantum $\mathfrak{sl}_k$ knot invariants*, Amer. J. Math. 131 (2009) 1679–1713 MR2567504

[58] **S E Morrison**, *A diagrammatic category for the representation theory of $U_q(\mathfrak{sl}_n)$*, PhD thesis, University of California, Berkeley (2007) MR2710589 Available at `http://search.proquest.com/docview/304900559`

[59] **S Morrison**, **A Nieh**, *On Khovanov's cobordism theory for $\mathfrak{su}_3$ knot homology*, J. Knot Theory Ramifications 17 (2008) 1121–1173 MR2457839

[60] **H Murakami**, **T Ohtsuki**, **S Yamada**, *Homfly polynomial via an invariant of colored plane graphs*, Enseign. Math. 44 (1998) 325–360 MR1659228

[61] **P S Ozsváth**, **J Rasmussen**, **Z Szabó**, *Odd Khovanov homology*, Algebr. Geom. Topol. 13 (2013) 1465–1488

[62] **H Queffelec**, **D E V Rose**, *The $\mathfrak{sl}_n$ foam 2–category: A combinatorial formulation of Khovanov–Rozansky homology via categorical skew Howe duality* `arXiv:1405.5920`

[63] **D E V Rose**, *A categorification of quantum $\mathfrak{sl}_3$ projectors and the $\mathfrak{sl}_3$ Reshetikhin–Turaev invariant of tangles*, Quantum Topol. 5 (2014) 1–59 MR3176309

[64] **R Rouquier**, *q–Schur algebras and complex reflection groups*, Mosc. Math. J. 8 (2008) 119–158 MR2422270

[65] **L Rozansky**, *An infinite torus braid yields a categorified Jones–Wenzl projector*, Fund. Math. 225 (2014) 305–326 MR3205575

[66] **P Seidel**, **I Smith**, *A link invariant from the symplectic geometry of nilpotent slices*, Duke Math. J. 134 (2006) 453–514 MR2254624

[67] **M Stošić**, *Indecomposable* 1*–morphisms of* $\dot{U}_3^+$ *and the canonical basis of* $U_q^+(sl_3)$ arXiv:1105.4458

[68] **C Stroppel**, *Categorification of the Temperley–Lieb category, tangles, and cobordisms via projective functors*, Duke Math. J. 126 (2005) 547–596 MR2120117

[69] **C Stroppel**, *Parabolic category* $\mathcal{O}$*, perverse sheaves on Grassmannians, Springer fibres and Khovanov homology*, Compos. Math. 145 (2009) 954–992 MR2521250

[70] **C Stroppel**, **J Sussan**, *Categorified Jones–Wenzl projectors: A comparison*, from: "Perspectives in representation theory", Contemp. Math. 610, Amer. Math. Soc. (2014) 333–351 MR3220633

[71] **J Sussan**, *Category O and sl(k) link invariants*, PhD thesis, Yale University (2007) MR2710319 Available at http://search.proquest.com/docview/304773953

[72] **C Vafa**, *Topological Landau–Ginzburg models*, Modern Phys. Lett. A 6 (1991) 337–346 MR1093562

[73] **P Vaz**, *The diagrammatic Soergel category and* sl(2) *and* sl(3) *foams*, Int. J. Math. Math. Sci. (2010) MR2652384

[74] **B Webster**, *Knot invariants and higher representation theory, I: Diagrammatic and geometric categorification of tensor products* arXiv:1001.2020

[75] **B Webster**, *Knot invariants and higher representation theory, II: The categorification of quantum knot invariants* arXiv:1005.4559

[76] **B Webster**, *Canonical bases and higher representation theory*, Compos. Math. 151 (2015) 121–166 MR3305310

[77] **H Wu**, *A colored* $\mathfrak{sl}(N)$ *homology for links in* S^3, Dissertationes Math. (Rozprawy Mat.) 499 (2014) 1–217 MR3234803

[78] **Y Yonezawa**, *Matrix factorizations and intertwiners of the fundamental representations of quantum group* $U_q(sl_n)$ arXiv:0806.4939

[79] **Y Yonezawa**, *Quantum* $(\mathfrak{sl}_n, \wedge V_n)$ *link invariant and matrix factorizations*, Nagoya Math. J. 204 (2011) 69–123 MR2863366

Department of Mathematics, University of Southern California
Los Angeles, CA 90089, USA

CNRS, Institut Montpelliérain Alexander Grothendieck, Université de Montpellier
34095 Montpellier Cedex 5, France

Department of Mathematics, University of Southern California
Los Angeles, CA 90089, USA

lauda@usc.edu, hoel.queffelec@umontpellier.fr, davidero@usc.edu