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Exactly fourteen intrinsically knotted graphs have 21 edges

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Johnson, Kidwell, and Michael showed that intrinsically knotted graphs have at least 21 edges. Also it is known that K_7 and the thirteen graphs obtained from K_7 by ∇Y moves are intrinsically knotted graphs with 21 edges. We prove that these 14 graphs are the only intrinsically knotted graphs with 21 edges.

57M25, 57M27

1 Introduction

Throughout the article we will take an embedded graph to mean a graph embedded in R^3. We call a graph G *intrinsically knotted* if every embedding of the graph contains a knotted cycle. Conway and Gordon [2] showed that K_7, the complete graph with seven vertices, is an intrinsically knotted graph. A graph H is *minor* of another graph G if it can be obtained from G by contracting or deleting some edges. An intrinsically knotted graph is *minor minimal intrinsically knotted* provided no proper minor is intrinsically knotted. Robertson and Seymour [9] proved that there are only finite minor minimal intrinsically knotted graphs, but finding the complete set of them is still an open problem. However, it is well known that K_7 and the thirteen graphs obtained from this graph by ∇Y moves are minor minimal intrinsically knotted; see Conway and Gordon [2], and Kohara and Suzuki [6].

A ∇Y *move* is an exchanging operation that removes all edges of a triangle abc and inserts a new vertex v and three edges va, vb and vc as in Figure 1. Its reverse operation is called a $Y\nabla$ *move*. Since ∇Y moves preserve intrinsic knottedness (see Motwani, Raghunathan, and Saran [7]), we will only consider triangle-free graphs in the article.

From the work of Johnson, Kidwell, and Michael [5], it follows that any intrinsically knotted graph consists at least 21 edges. Here is the main theorem.

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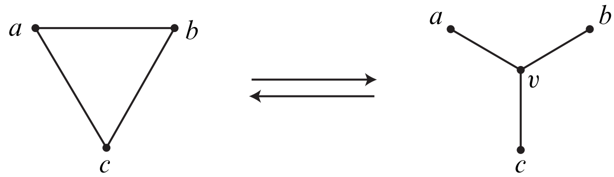

Figure 1: ∇Y and $Y\nabla$ moves

Theorem 1 *The only triangle-free intrinsically knotted graphs with exactly* 21 *edges are* H_{12} *and* C_{14}. (H_{12} *and* C_{14} *were described by Kohara and Suzuki in [6].*)

Kohara and Suzuki [6] found fourteen intrinsically knotted graphs. Goldberg, Mattman, and Naimi [3] constructed twenty graphs derived from H_{12} and C_{14} by $Y\nabla$ moves as in Figure 2, and they showed that these six graphs, N_9, N_{10}, N_{11}, N'_{10}, N'_{11}, and N'_{12}, are not intrinsically knotted. This fact was proved by Hanaki, Nikkuni, Taniyama, and Yamazaki [4] independently. Theorem 1 guarantees that all intrinsically knotted graphs with 21 edges can be obtained from H_{12} and C_{14} by $Y\nabla$ moves. Thus, we have the following theorem.

Theorem 2 *The only intrinsically knotted graphs with exactly* 21 *edges are* K_7 *and the thirteen graphs obtained from* K_7 *by* ∇Y *moves.*

This theorem gives us the complete set of fourteen minor minimal intrinsically knotted graphs with 21 edges.

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2 Terminology

From now on let $G = (V, E)$ denote a triangle-free graph with 21 edges. Here V and E denote the sets of all vertices and edges of G, respectively. For any two distinct vertices a and b, let $\widehat{G}_{a,b} = (\widehat{V}_{a,b}, \widehat{E}_{a,b})$ denote the graph obtained from G by deleting two vertices a and b, and then contracting an edge incident to a vertex of degree 1 or

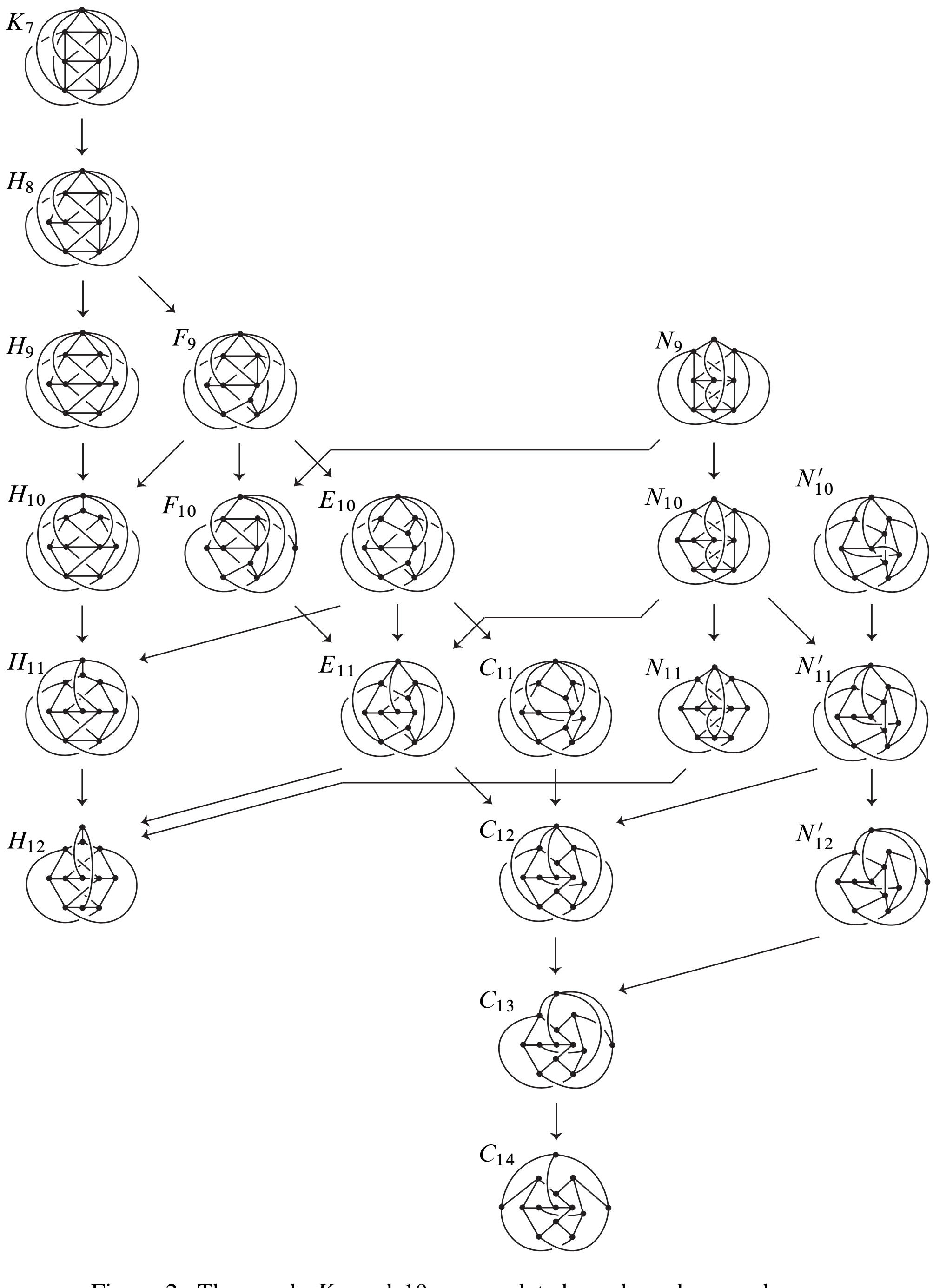


Figure 2: The graph K_7 and 19 more related graphs, where each arrow represents a ∇Y move

2 repeatedly until no vertices of degree 1 or 2 exist. Removing vertices means deleting interiors of all edges incident to these vertices as well as the resulting isolated vertices.

In a graph, the distance between two vertices a and b is the number of edges in the shortest path connecting them and is denoted by $\mathrm{dist}(a,b)$. The degree of a vertex a is denoted by $\deg(a)$. To count the number of edges of $\widehat{G}_{a,b}$, we introduce some notation.

- $E(a)$ is the set of edges which are incident to a.
- $V(a) = \{c \in V \mid \mathrm{dist}(a,c) = 1\}$.
- $V_n(a) = \{c \in V \mid \mathrm{dist}(a,c) = 1,\ \deg(c) = n\}$.
- $V_n(a,b) = V_n(a) \cap V_n(b)$.
- $V_Y(a,b) = \{c \in V \mid \exists\ d \in V_3(a,b) \text{ such that } c \in V_3(d) \setminus \{a,b\}\}$.

First consider the graph $G \setminus \{a,b\}$ for some distinct vertices a and b. In this graph each vertex of $V_3(a,b)$ has degree 1, and each vertex of $V_3(a)$, $V_3(b)$ (not in $V_3(a,b)$), and $V_4(a,b)$ has degree 2. To derive $\widehat{G}_{a,b}$, we first delete all edges incident to a and b from G, and then we also delete the remaining edges incident to $V_3(a,b)$, and finally we contract one edge of the remaining pair of edges incident to each vertex of $V_3(a)$, $V_3(b)$ (not in $V_3(a,b)$), $V_4(a,b)$, and $V_Y(a,b)$ as dotted lines in Figure 3(a). Thus, we have the following equation counting the number of edges of $\widehat{G}_{a,b}$ which is called a *count equation*:

$$|\widehat{E}_{a,b}| = 21 - |E(a) \cup E(b)| - (|V_3(a)| + |V_3(b)| - |V_3(a,b)| + |V_4(a,b)| + |V_Y(a,b)|).$$

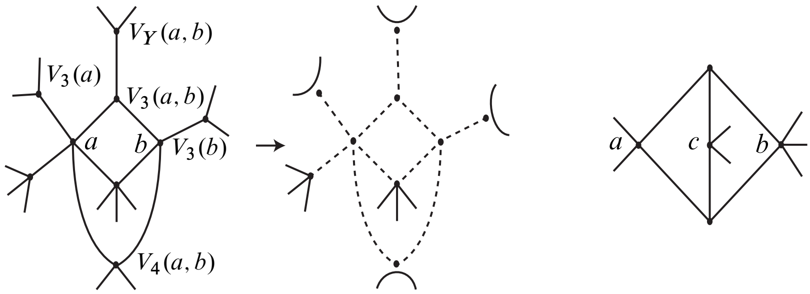
For short, $NE(a,b) = |E(a) \cup E(b)|$ and $NV_3(a,b) = |V_3(a)| + |V_3(b)| - |V_3(a,b)|$. If a and b are adjacent vertices (ie $\mathrm{dist}(a,b) = 1$), then all of $V_3(a,b)$, $V_4(a,b)$, and $V_Y(a,b)$ are empty because G is triangle-free. Note that this manner of deriving $\widehat{G}_{a,b}$ must be handled in a slightly different way when there is a vertex c in V such that more than one vertex of $V(c)$ are contained in $V_3(a,b)$ as in Figure 3(b). In this case, we usually delete or contract more edges incident to c, even though c is not in $V_Y(a,b)$.

A graph is *n–apex* if one can remove n vertices from the graph to obtain a planar graph. The following lemma gives an important condition for a graph to be not intrinsically knotted.

Lemma 3 [1; 8] *If G is 2–apex, then G is not intrinsically knotted.*

The following two lemmas play an important role for G to be 2–apex.

Lemma 4 *If $|\widehat{E}_{a,b}| \leq 8$, then $\widehat{G}_{a,b}$ is a planar graph. Thus, G is not intrinsically knotted.*



(a) (b)

Figure 3: Deriving $\widehat{G}_{a,b}$

Lemma 5 *If $|\widehat{E}_{a,b}| = 9$, then $\widehat{G}_{a,b}$ is either a planar graph or homeomorphic to $K(3,3)$. Furthermore, if $\widehat{G}_{a,b}$ is not homeomorphic to $K(3,3)$, then G is not intrinsically knotted.*

The graph $K(3,3)$ is a bipartite graph where each part has three vertices and each vertex is adjacent to every vertex in the opposite part, and so it is a triangle-free graph and every vertex has degree 3.

To prove Theorem 1, we will show that any triangle-free graph with 21 edges is eventually either a 2–apex or homeomorphic to one of H_{12} or C_{14}. Since intrinsically knotted graphs have at least 21 edges [5], it is sufficient to consider simple and connected graphs having no vertex of degree 1 or 2. Our process is constructing all possible such triangle-free graph G with 21 edges, deleting two suitable vertices a and b of G, and then counting the number of edges of $\widehat{G}_{a,b}$. If $\widehat{G}_{a,b}$ has 9 edges or less, we can use Lemma 4 or Lemma 5 in order to show that G is not intrinsically knotted. In the event that $\widehat{G}_{a,b}$ is not planar, we will show that G is homeomorphic to H_{12} or C_{14}.

Before describing the proof of Theorem 1, we introduce more notation. Since G is triangle-free, for any vertex a of G, no two vertices in $V(a)$ are adjacent. This means that $E(b)$ and $E(c)$ do not contain an edge in common for any two distinct vertices b and c in $V(a)$. We set:

- $E^2(a) = \bigcup_{b \in V(a)} E(b)$.
- $E \setminus E^2(a) = \{e_1(a), \dots, e_{21-n}(a)\}$ if $|E^2(a)| = n < 21$.

$e_i(a)$ is called an *extra edge*, and the two endpoints of the edge are denoted as $x_i(a)$ and $y_i(a)$, where $\deg(x_i(a)) \geq \deg(y_i(a))$.

In order to visualize G, we perform the following steps. First choose a vertex a with the maximal degree among all vertices and draw $E^2(a)$. If $|E^2(a)| < 21$, draw $E \setminus E^2(a)$ apart from $E^2(a)$ as in Figure 4(a). Then all vertices of degree 1 of $E^2(a)$ and $E \setminus E^2(a)$ are merged into some vertices of degree at least 3 without adding new edges as in Figure 4(b). Let $\overline{V}(a)$ denote the set of all such vertices, and let $[\overline{V}(a)]$ denote a sequence of the degrees of vertices in $\overline{V}(a)$ as follows:

- $\overline{V}(a) = V \setminus (V(a) \cup \{a\}) = \{\overline{v}_1(a), \dots, \overline{v}_m(a)\}$ with $\deg(\overline{v}_i(a)) \geq \deg(\overline{v}_{i+1}(a))$.
- $[\overline{V}(a)] = [\deg(\overline{v}_1(a)), \dots, \deg(\overline{v}_m(a))]$.
- $\|[\overline{V}(a)]\| = \deg(\overline{v}_1(a)) + \cdots + \deg(\overline{v}_m(a))$.

The graph in Figure 4(b) is an example satisfying $\deg(a) = 5$, $|V_3(a)| = 1$, $|E^2(a)| = 19$, and $[\overline{V}(a)] = [4, 4, 4, 3, 3]$.

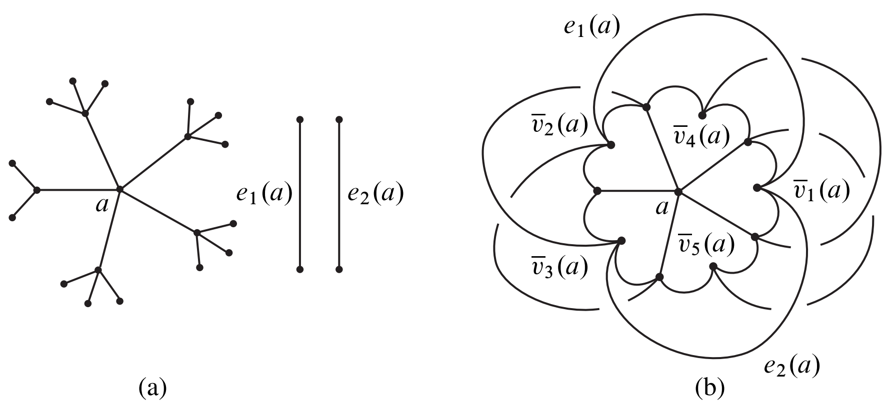


Figure 4: Visualization of G

The remaining three sections of the article are devoted to the proof of Theorem 1. From now on, a denotes one of vertices with maximal degree in G. The proof is divided into three parts according to the degree of a. In Section 3 we show that any graph G with $\deg(a) \geq 5$ cannot be intrinsically knotted. In Section 4 we show that an intrinsically knotted graph with $\deg(a) = 4$ is exactly H_{12}. Finally, in Section 5 we show that any intrinsically knotted graph, all of whose vertices have degree 3, is always C_{14}.

3 deg(a) ≥ 5

In this section we will show that for some $a', b' \in V$ either $|\widehat{E}_{a',b'}| \leq 8$ or $|\widehat{E}_{a',b'}| = 9$, but that $\widehat{G}_{a',b'}$ is not homeomorphic to $K(3,3)$ by showing that it contains a vertex of degree more than 3 or a triangle (or sometimes a bigon). Then, as a conclusion, G is not intrinsically knotted by Lemmas 4 and 5. Recall that G has 21 edges, every vertex has degree at least 3, and a has the maximal degree among them.

3.1 Case $\deg(a) \geq 6$ or $\deg(a) = 5$ with $|V_3(a)| \geq 4$

If $\deg(a) \geq 6$, then $|V_3(a)| \geq 3$. Let c be any vertex in $V_3(a)$. Choose a vertex b which has the maximal degree among $V(c) \setminus \{a\}$. Then $|E(b)| + |V_Y(a,b)| \geq 4$, since $|V_Y(a,b)| \geq 1$ when $\deg(b) = 3$. Note that $|V_3(b)| \geq |V_3(a,b)|$. By the count equation, $|\widehat{E}_{a,b}| \leq 8$ in $\widehat{G}_{a,b}$.

Suppose that $\deg(a) = 5$ and $|V_3(a)| \geq 4$. The proof is similar to the previous paragraph.

3.2 Case $\deg(a) = 5$ and $|V_3(a)| = 3$

Let b and c be two vertices of $V(a) \setminus V_3(a)$. First, suppose that both of them have degree 5. Then $NE(a,b) = 9$ and $|V_3(a)| = 3$, so $|\widehat{E}_{a,b}| \leq 9$. Furthermore, the vertex c has degree 4 in $\widehat{G}_{a,b}$, so it follows that $\widehat{G}_{a,b}$ is not homeomorphic to $K(3,3)$. Thus, G is not intrinsically knotted by Lemma 5.

Now assume that one of them, say b, has degree 4. If $V(b) \setminus \{a\}$ consists of three vertices, all of which are of degree 3, then $NE(a,b) = 8$ and $NV_3(a,b) = 6$, so $|\widehat{E}_{a,b}| \leq 7$. If not, let d be a vertex of $V(b)$ which has degree at least 4. Then $NE(a,d) \geq 9$, $|V_3(a)| = 3$, and $|V_4(a,d)| \geq 1$, because $V_4(a,d) \ni b$. This implies that $|\widehat{E}_{a,d}| \leq 8$.

3.3 Case $\deg(a) = 5$ and $|V_3(a)| = 0$

First, suppose that $V(a)$ contains a vertex of degree 5, say c. Since G has 21 edges, the other four vertices of $V(a)$ have degree 4. By the previous cases, it is sufficient to suppose that $|V_3(c)| \leq 2$. So $V(c) \setminus \{a\}$ has at least two vertices, say b and d, of degree 4 or 5. Since $|E^2(a)| = 21$ and G is triangle-free, all edges of $E(b)$ must be incident to different vertices of $V(a)$, so $|V_4(a,b)| \geq 3$. This implies that $|\widehat{E}_{a,b}| \leq 9$. Since $\widehat{G}_{a,b}$ has the vertex d of degree at least 4, it follows that $\widehat{G}_{a,b}$ is not homeomorphic to $K(3,3)$.

Now, assume that all vertices of $V(a)$ have degree 4, giving $|E^2(a)| = 20$. Let $e_1(a)$ be the extra edge and recall that two endpoints of $e_1(a)$ are $x_1(a)$ and $y_1(a)$ with $\deg(x_1(a)) \geq \deg(y_1(a))$. Since G is triangle-free, all edges of $E(x_1(a)) \cup E(y_1(a))$ except $e_1(a)$ must be incident to different vertices of $V(a)$. Thus the degrees of $x_1(a)$ and $y_1(a)$ must be either 4 and 3, or 3 and 3, respectively. If $\deg(x_1(a)) = 4$, then $|V_4(a, x_1(a))| = 3$ and $|V_3(x_1(a))| = 1$, so $|\widehat{E}_{a,x_1(a)}| = 8$. If not, $[\overline{V}(a)]$ is either $[5,3,3,3,3]$ or $[4,4,3,3,3]$, because $|[\overline{V}(a)]| = 17$. Thus $\overline{v}_1(a)$ has degree 5 or 4 and differs from $x_1(a)$ and $y_1(a)$, so $|V_4(a, \overline{v}_1(a))| \geq 4$. Therefore, $|\widehat{E}_{a,\overline{v}_1(a)}| \leq 8$.

3.4 Case deg(a) = 5 and $|V_3(a)| = 1$

In this case, $V(a)$ contains four vertices of degree 4 or 5. Let n be the number of such vertices of degree 4, and so we have $4-n$ vertices of degree 5, where $n = 2, 3, 4$. This implies that $|E^2(a)| = 21+(2-n)$, and $n-2$ extra edges exist. If $\overline{V}(a)$ contains a vertex $\overline{v}_1(a)$ of degree 5, then five edges of $E(\overline{v}_1(a))$ are extra edges or incident to different vertices in $V(a)$. For any of the above n, at least two among these edges are incident to vertices of degree 4 in $V(a)$. Then $NE(a, \overline{v}_1(a)) = 10$, $|V_3(a)| = 1$, and $|V_4(a, \overline{v}_1(a))| \geq 2$, implying $|\widehat{E}_{a,\overline{v}_1(a)}| \leq 8$.

Now, suppose that $\overline{V}(a)$ contains vertices of degree 3 or 4 only. If $n=2$, $\|[\overline{V}(a)]\| = 16$, and so $[\overline{V}(a)]$ is either $[4,4,4,4]$ or $[4,3,3,3,3]$. For any vertex b in $V_5(a)$, four edges of $E(b)$ must be incident to different vertices of $\overline{V}(a)$. Indeed, these four edges are incident to four vertices of degree 4, or at least three edges among them are incident to vertices of degree 3 in $\overline{V}(a)$. This means that the vertex b has degree 5 with either $V_3(b) = 0$ or $V_3(b) \geq 3$. Both cases are dealt with in previous cases 3.3, 3.1, and 3.2.

If $n = 3$, $\|[\overline{V}(a)]\| = 17$, and so $[\overline{V}(a)] = [4,4,3,3,3]$. Let $V_5(a) = \{b\}$. To avoid the case 3.2, four edges of $E(b)$ must be incident to two vertices of degree 4 and two vertices of degree 3 in $\overline{V}(a)$, which are $\overline{v}_1(a)$, $\overline{v}_2(a)$, $\overline{v}_3(a)$, and $\overline{v}_4(a)$. Then there is a vertex c of $V_4(a)$ such that at most one edge of $E(c)$ is incident to $\overline{v}_3(a)$ and $\overline{v}_4(a)$, ie two edges of $E(c)$ are incident to $\overline{v}_1(a)$, $\overline{v}_2(a)$, or $\overline{v}_5(a)$. This implies that $NE(b,c) = 9$ and $NV_3(b,c) + |V_4(b,c)| \geq 4$, implying $|\widehat{E}_{b,c}| \leq 8$.

Finally, if $n = 4$, $\|[\overline{V}(a)]\| = 18$, and so $[\overline{V}(a)]$ is either $[4,4,4,3,3]$ or $[3,3,3,3,3,3]$. Recall that two extra edges exist. In the former case let $\{\overline{v}_1(a), \overline{v}_2(a), \overline{v}_3(a)\}$ be the three vertices of degree 4 in $\overline{V}(a)$. For each $i = 1,2,3$, if more than two edges of $E(\overline{v}_i(a))$ are incident to $V_4(a)$, then $NE(a, \overline{v}_i(a)) = 9$, $|V_3(a)| = 1$, and $|V_4(a, \overline{v}_i(a))| \geq 3$, implying $|\widehat{E}_{a,\overline{v}_i(a)}| \leq 8$. So, each of at least two edges of $E(\overline{v}_i(a))$ must be either incident to the unique vertex of $V_3(a)$ or an extra edge. Since G is triangle-free, one of three vertices, say $\overline{v}_1(a)$, has the property that $E(\overline{v}_1(a))$ contains both extra edges, and $V(\overline{v}_1(a))$ and $V(\overline{v}_i(a))$ for each $i = 2,3$ cannot share a vertex in $V(a)$. This implies that $V(\overline{v}_2(a))$ and $V(\overline{v}_3(a))$ coincide as in Figure 5(a). Then $NE(\overline{v}_2(a), \overline{v}_3(a)) = 8$, and either $|V_4(\overline{v}_2(a), \overline{v}_3(a))| = 4$ or $|V_4(\overline{v}_2(a), \overline{v}_3(a))| = 3$ and $|V_3(\overline{v}_2(a))| = 1$. Thus, $|\widehat{E}_{\overline{v}_2(a),\overline{v}_3(a)}| \leq 9$. In $\widehat{G}_{\overline{v}_2(a),\overline{v}_3(a)}$ the vertex a still has degree 4 or 5 so that $\widehat{G}_{\overline{v}_2(a),\overline{v}_3(a)}$ is not homeomorphic to $K(3,3)$.

In the latter case, let $V_4(a) = \{b_1, b_2, b_3, b_4\}$. We claim that for some $i, j = 1,2,3,4$, $|V_3(b_i, b_j)| \leq 1$. Suppose not; that is, $|V_3(b_i, b_j)| \geq 2$ for all combinations of i and j. By some combinatorics we can derive that all 12 edges of $E(b_1) \cup E(b_2) \cup E(b_3) \cup E(b_4) \setminus E(a)$ are incident to only four vertices of $\overline{V}(a)$ as in Figure 5(b).

This means that two extra edges must be incident to the remaining two vertices of $\overline{V}(a)$ at both endpoints. But a bigon is not allowed. Therefore, without loss of generality, $|V_3(b_1, b_2)| \leq 1$. Then $NE(b_1, b_2) = 8$ and $NV_3(b_1, b_2) \geq 5$, implying $|\widehat{E}_{b_1,b_2}| \leq 8$.

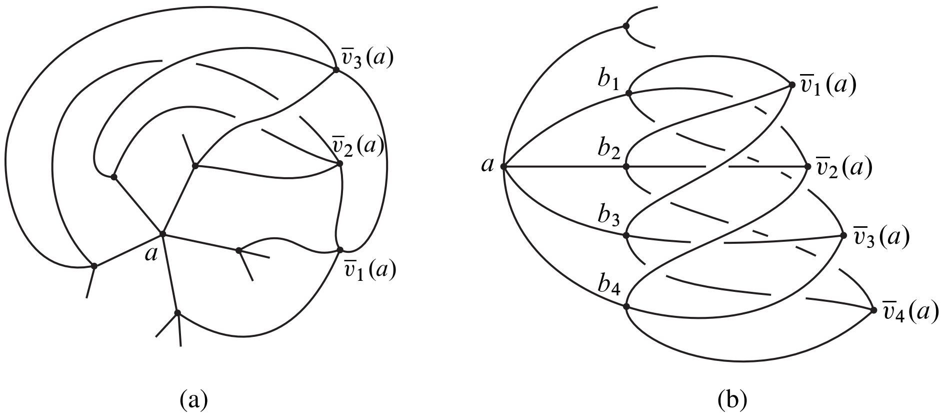


Figure 5: [4, 4, 4, 3, 3] and [3, 3, 3, 3, 3, 3] cases

3.5 Case deg$(a) = 5$ and $|V_3(a)| = 2$

If $V(a)$ contains a vertex of degree 5, say b, then the previous four cases guarantee that we only consider that $|V_3(b)| = 2$, so $NV_3(a, b) = 4$, which implies $|\widehat{E}_{a,b}| = 8$. Therefore we assume that $V(a)$ contains three vertices of degree 4. In this case three extra edges exist. Since $\|[\overline{V}(a)]\| = 19$, $[\overline{V}(a)]$ is one of [5, 5, 5, 4], [5, 5, 3, 3, 3], [5, 4, 4, 3, 3], [4, 4, 4, 4, 3], or [4, 3, 3, 3, 3, 3].

If, for some vertex $\overline{v}_i(a)$ with degree 5, one edge of $E(\overline{v}_i(a))$ is incident to $V_4(a)$, then $NE(a, \overline{v}_i(a)) = 10$, $|V_3(a)| = 2$, and $|V_4(a, \overline{v}_i(a))| \geq 1$, implying $|\widehat{E}_{a,\overline{v}_i(a)}| \leq 8$. Thus, three edges of $E(\overline{v}_i(a))$ are extra edges and the remaining two edges are incident to $V_3(a)$. In the first two cases, [5, 5, 5, 4] and [5, 5, 3, 3, 3], both $E(\overline{v}_1(a))$ and $E(\overline{v}_2(a))$ share three extra edges, but G does not have a bigon. In the third case, [5, 4, 4, 3, 3], $E(\overline{v}_1(a))$ contains three extra edges and one of these extra edges must be incident to $\overline{v}_4(a)$ or $\overline{v}_5(a)$, both of which have degree 3. Then $NE(a, \overline{v}_1(a)) = 10$ and $NV_3(a, \overline{v}_1(a)) \geq 3$, implying $|\widehat{E}_{a,\overline{v}_1(a)}| \leq 8$.

If, for some vertex $\overline{v}_i(a)$ with degree 4, two edges of $E(\overline{v}_i(a))$ are incident to $V_4(a)$, then $NE(a, \overline{v}_i(a)) = 9$, $|V_3(a)| = 2$, and $|V_4(a, \overline{v}_i(a))| \geq 2$, implying $|\widehat{E}_{a,\overline{v}_i(a)}| \leq 8$. Thus, at most one edge of $E(\overline{v}_i(a))$ is incident to $V_4(a)$. In the fourth case, [4, 4, 4, 4, 3], at least twelve among sixteen edges incident to four vertices of degree 4 in $\overline{V}(a)$ are not incident to $V_4(a)$. This is impossible because there are only two vertices in $V_3(a)$ and three extra edges. In the last case, [4, 3, 3, 3, 3, 3], since only one edge of $E(\overline{v}_1(a))$ is possibly incident to $V_4(a)$, there is a vertex b in $V_4(a)$ such that three edges of $E(b)$ are incident to vertices of degree 3 in $\overline{V}(a)$. Then $NE(a,b)=8$ and $NV_3(a,b)\geq 5$, implying $|\widehat{E}_{a,b}|\leq 8$.

4 deg(a) = 4

Since $|V|=|V_4|+|V_3|$ and $4|V_4|+3|V_3|=2|E|$, the pair $(|V_4|,|V_3|)$ has three choices: $(3,10)$, $(6,6)$, and $(9,2)$. Here, V_n denotes the set of vertices of degree n. As in the preceding section, we will show that for some $a',b'\in V$ either $|\widehat{E}_{a',b'}|\leq 8$ or $|\widehat{E}_{a',b'}|=9$, but $\widehat{G}_{a',b'}$ is not homeomorphic to $K(3,3)$, implying that G is not intrinsically knotted. But one exception occurs so that G can possibly be H_{12} when $(|V_4|,|V_3|)=(6,6)$.

4.1 Case $(|V_4|,|V_3|)=(3,10)$

First suppose that V_4 has a vertex a such that all four vertices of $V(a)$ have degree 3. Let b_1 and b_2 be the other vertices of V_4. For each $i=1,2$, $NE(a,b_i)=8$. If there is a vertex of $V_3(b_i)$ which is not contained in $V(a)$, then $NV_3(a,b_i)\geq 5$, implying $|\widehat{E}_{a,b_i}|\leq 8$. Thus each vertex of $V(b_1)$ is the vertex b_2 or contained in $V(a)$, and similarly for b_2. This implies that the number of vertices of V_3 which have distance 1 or 2 from the vertex a is at most 6. Take a vertex c of V_3 with distance at least 3 from a. Since each vertex of $V(c)$ is neither b_1 nor b_2, it has degree 3. Thus $NE(a,c)=7$ and $NV_3(a,c)\geq 7$, implying $|\widehat{E}_{a,c}|\leq 7$.

Now, we only need to consider the case that each vertex of V_4 is adjacent to at least one vertex of degree 4. Then, without loss of generality, we have vertices a, b and c of V_4 such that $V(b)$ contains a and c. If $V_3(a)$ and $V_3(c)$ do not coincide, then $|V_4(a,c)|=1$ and $NV_3(a,c)\geq 4$, implying $|\widehat{E}_{a,c}|\leq 8$. If $V_3(a)$ and $V_3(c)$ coincide and $|V_Y(a,c)|\geq 2$, then $|V_4(a,c)|=1$ and $NV_3(a,c)=3$, implying $|\widehat{E}_{a,c}|\leq 7$. If not, for the unique vertex d of $V_Y(a,c)$, $V_3(a)=V_3(c)=V(d)$. Then, for a vertex b' of $V_3(b)$, $V_3(b')$ is disjoint from $V_3(a)$. Thus $NE(a,b')=7$, $NV_3(a,b')=5$, and $|V_4(a,b')|=1$, implying $|\widehat{E}_{a,b'}|\leq 8$.

4.2 Case $(|V_4|,|V_3|)=(6,6)$

Consider the subgraph H of G consisting of all edges whose both end vertices have degree 4. Since G has six vertices of degree 3 and the same number of vertices of degree 4, H is not empty set.

Claim 1 *If H has a vertex of degree 1, then G is not intrinsically knotted.*

Proof Suppose that H has a vertex a of degree 1. Let b be the unique vertex of degree 4 in $V(a)$. If $|V_3(b)|=3$, then $NE(a,b)=7$ and $NV_3(a,b)=6$, implying $|\widehat{E}_{a,b}|\leq 8$. Thus, there is another vertex c of $V_4(b)$, and so we let $V(c)=\{b,d_1,d_2,d_3\}$.

First, assume that $|V_3(c)|=0$. So the two vertices of $V(b)\setminus\{a,c\}$ must have degree 3, because the six vertices a,b,c,d_1,d_2, and d_3 in V_4 are all different. Thus $NE(a,b)=7$ and $NV_3(a,b)=5$, so $|\widehat{E}_{a,b}|\leq 9$. Since $\widehat{G}_{a,b}$ has another vertex d_1 of degree 4, it follows that $\widehat{G}_{a,b}$ is not homeomorphic to $K(3,3)$.

Second, assume that $|V_3(c)|=1$, say $d_1\in V_3(c)$. If d_1 is not one of the vertices in $V(a)$, then $NE(a,c)=8$ and $NV_3(a,c)+|V_4(a,c)|=5$, implying $|\widehat{E}_{a,c}|\leq 8$. So we may assume that d_1 is in $V(a)$ and let $V(d_1)=\{a,c,v_1\}$. If v_1 has degree 3, then $NV_3(a,c)+|V_4(a,c)|=4$ and $V_Y(a,c)=\{v_1\}$, implying $|\widehat{E}_{a,c}|\leq 8$. Otherwise v_1 has degree 4 and it is different from d_2 and d_3. For any $i=2,3$, each vertex of $V(d_i)\setminus\{c\}$ either has degree 3 or is v_1. Thus $NE(d_2,d_3)=8$ and $NV_3(d_2,d_3)+|V_4(d_2,d_3)|\geq 4$, implying $|\widehat{E}_{d_2,d_3}|\leq 9$. But $\widehat{G}_{d_2,d_3}$ has a triangle containing vertices a, b and d_1. See Figure 6(a).

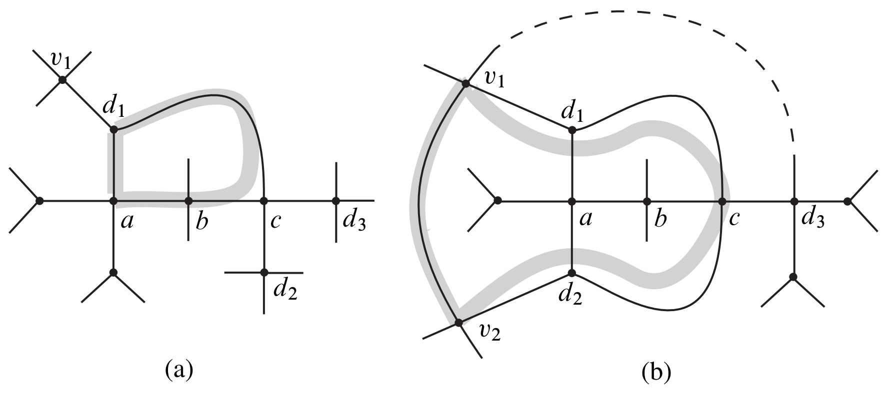


Figure 6: Some nonintrinsically knotted cases

Last, assume that $|V_3(c)|\geq 2$ and let d_1 and d_2 be two such vertices. As in the previous case, we may say that d_1 and d_2 are in $V(a)$, and $V(d_i)=\{a,c,v_i\}$ for $i=1,2$ where v_i has degree 4. When $v_1=v_2$, $|V_3(a)|=3$, $|V_4(a,c)|=1$, and v_1 has degree 2 when we construct $\widehat{G}_{a,c}$, implying $|\widehat{E}_{a,c}|\leq 8$. When $\mathrm{dist}(v_1,v_2)\geq 2$, three cases occur as follows: $|V_3(v_1)|\geq 3$, $|V_3(v_2)|\geq 3$, or for both $i=1,2$ $|V_3(v_i)|=2$ and $V_4(v_i)=V_4\setminus\{a,c,v_1,v_2\}$. All three cases satisfy that $NV_3(v_1,v_2)+|V_4(v_1,v_2)|\geq 4$, implying $|\widehat{E}_{v_1,v_2}|\leq 9$. But $\widehat{G}_{v_1,v_2}$ has a bigon containing vertices a and c. Finally, when $\mathrm{dist}(v_1,v_2)=1$, two cases occur as follows. If d_3 has degree 3, then by the same reason as before we may say that d_3 is also in $V(a)$, and $V(d_3)=\{a,c,v_3\}$ where v_3 has degree 4. By the previous argument any pair of v_1, v_2 and v_3 has distance 1. This implies that G contains a triangle. If d_3 has degree 4, then $|V_3(d_3)| \geq 2$, because at most one vertex of $V(d_3)$ can be v_1 or v_2. Thus, $NV_3(a, d_3) \geq 4$, implying $|\widehat{E}_{a,d_3}| \leq 9$. But $\widehat{G}_{a,d_3}$ has a triangle containing vertices c, v_1 and v_2. See Figure 6(b). □

Claim 2 *If H is not a cycle with 6 edges, then G is not intrinsically knotted.*

Proof By Claim 1, if H is not a cycle with 6 edges, then H contains a cycle with 4 or 5 edges. First assume that H contains a cycle with 5 edges. Let $\{a_1, \dots, a_5\}$ be the set of five vertices of the cycle appearing in clockwise order. If the remaining vertex b of V_4 is contained in some $V(a_i)$, say $i = 1$, then b must have distance 1 from one of a_3 and a_4, say a_3, by Claim 1. See Figure 7. If $V_3(a_2) \neq V_3(b)$, $NV_3(a_2, b) + |V_4(a_2, b)| \geq 5$, implying $|\widehat{E}_{a_2,b}| \leq 8$. Otherwise, $V_3(a_2) = V_3(b)$. Let c_1 and c_3 be the vertices of $V_3(a_1)$ and $V_3(a_3)$, respectively. If $c_1 = c_3$, we still have $|\widehat{E}_{a_2,b}| \leq 9$ and $\widehat{G}_{a_2,b}$ has a triangle containing vertices a_5, a_4 and $c_1 = c_3$. If $c_1 \neq c_3$, then $|\widehat{E}_{a_1,a_3}| \leq 9$ and $\widehat{G}_{a_1,a_3}$ has a bigon as in the figure.

If b is not contained in $V(a_i)$ for any $i = 1, \dots, 5$, then $|V_3(a_i)| = 2$. If there is a pair of vertices a_i and a_{i+2} (or a_{i-3} if $i = 4, 5$) such that $V_3(a_i)$ and $V_3(a_{i+2})$ are disjoint, then $NV_3(a_i, a_{i+2}) + |V_4(a_i, a_{i+2})| = 5$, implying $|\widehat{E}_{a_i,a_{i+2}}| \leq 8$. Otherwise, for any pair of vertices a_i and a_{i+2} (or a_{i-3} if $i = 4, 5$), $V_3(a_i)$ and $V_3(a_{i+2})$ share vertices. Then they must share only one vertex as in Figure 7(b). Since there is only one extra vertex b of degree 4, for some pair of vertices a_i and a_{i+2}, $NV_3(a_i, a_{i+2}) + |V_4(a_i, a_{i+2})| = 4$ and $V_Y(a_i, a_{i+2}) \geq 1$, implying $|\widehat{E}_{a_i,a_{i+2}}| \leq 8$.

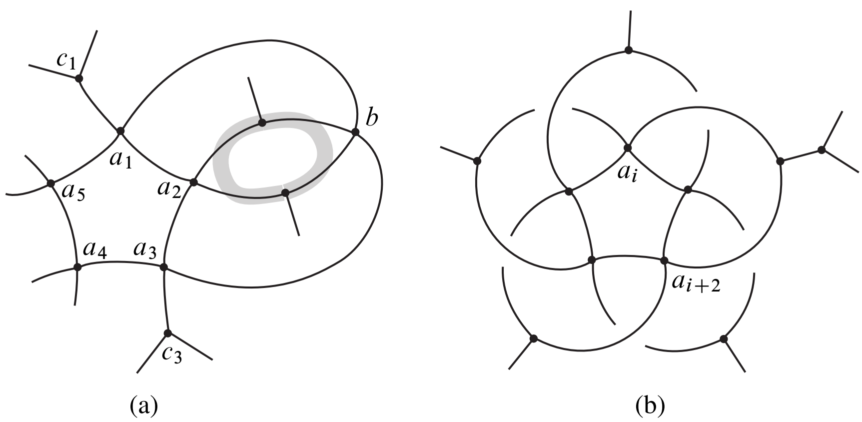


Figure 7: Cycle with 5 edges

Now, assume that H contains a cycle with 4 edges. Let $\{a_1, \dots, a_4\}$ be the set of four vertices of the cycle appearing in clockwise order. If $V(a_1)$ and $V(a_3)$ (or similarly for $V(a_2)$ and $V(a_4)$) share only two vertices, a_2 and a_4, then the remaining two vertices of V_4 must be contained in $V(a_1) \cup V(a_3)$. Otherwise, since $V(a_1) \cup V(a_3)$ has four more vertices other than a_2 and a_4, $NV_3(a_1, a_3) \geq 3$ and $|V_4(a_1, a_3)| = 2$, implying $|\widehat{E}_{a_1,a_3}| \leq 8$. By Claim 1, the two vertices have distance 1, so H contains a cycle with 5 edges which was dealt in the previous case. If $V(a_1)$ and $V(a_3)$ (or similarly for $V(a_2)$ and $V(a_4)$) share exactly three vertices, a_2, a_4 and b, then let c_1 and c_3 be the remaining vertices of $V(a_1)$ and $V(a_3)$, respectively. If both c_1 and c_3 have degree 3, then $NV_3(a_1, a_3) + |V_4(a_1, a_3)| \geq 5$. If both have degree 4, then H contains a cycle with 5 edges as in the previous case. Finally, if only c_1 (or similarly c_3) has degree 4, then, by Claim 1, $V(c_1)$ contains another vertex, say d, of V_4, and also d must have distance 1 from one of a_2 and a_4, say a_4, as in Figure 8(a). So $NV_3(a_4, c_1) + |V_4(a_4, c_1)| \geq 4$, implying $|\widehat{E}_{a_4,c_1}| \leq 9$, and $\widehat{G}_{a_4,c_1}$ has a triangle containing vertices a_2, a_3, and b. Now we may assume that $V(a_1) = V(a_3)$ and $V(a_2) = V(a_4)$. Then $NV_3(a_1, a_3) + |V_4(a_1, a_3)| = 4$, implying $|\widehat{E}_{a_1,a_3}| \leq 9$, and so $\widehat{G}_{a_1,a_3}$ has a bigon as in Figure 8(b). □

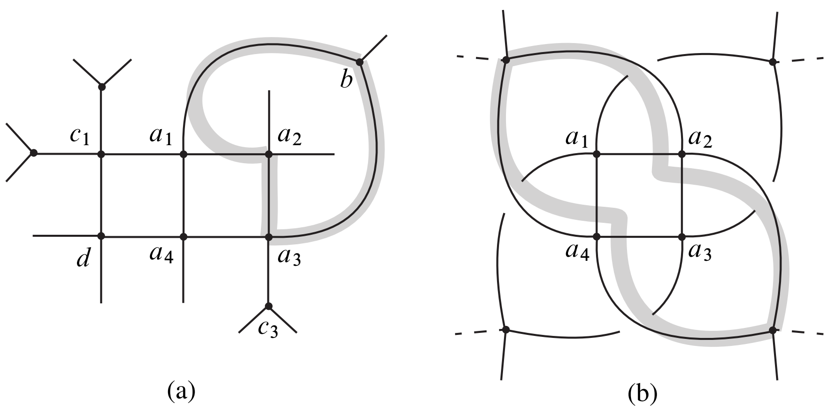


Figure 8: Cycle with 4 edges

By Claim 2, H is exactly a cycle with 6 edges. Let $\{a_1, \dots, a_6\}$ be the set of six vertices of the cycle with a_i adjacent to a_{i+1} for $i = 1, \dots, 5$, and a_6 adjacent to a_1. First, suppose that there is not a vertex b in V_3 such that $V(b) = \{a_1, a_3, a_5\}$. If $V_3(a_1)$ and $V_3(a_3)$ are disjoint, then $NV_3(a_1, a_3) + |V_4(a_1, a_3)| = 5$. If $V_3(a_1)$ and $V_3(a_3)$ share exactly one vertex c, then the vertex of $V(c) \setminus \{a_1, a_3\}$ is not a_5, so it should be one of $V_Y(a_1, a_3)$. Thus $NV_3(a_1, a_3) + |V_4(a_1, a_3)| + |V_Y(a_1, a_3)| = 5$. If $V_3(a_1)$ and $V_3(a_3)$ are same, then $NV_3(a_1, a_5) + |V_4(a_1, a_5)| = 5$, because $V_3(a_1)$ and $V_3(a_5)$ are disjoint. All three cases guarantee that G is not intrinsically knotted. Therefore we may assume that there are two vertices b_1 and b_2 so that $V(b_1) = \{a_1, a_3, a_5\}$ and $V(b_2) = \{a_2, a_4, a_6\}$. See Figure 9(a).

Suppose that there is a vertex c, with $c \neq b_1$, so that $V(c)$ contains a_1 and a_3. Let d_2 and d_5 be the vertices of $V_3(a_2)$ and $V_3(a_5)$, other than b_1 and b_2, respectively. If $d_2 \neq d_5$, then $NV_3(a_2, a_5) = 4$. If $d_2 = d_5$, then $NV_3(a_2, a_5) = 3$ and $V_Y(a_2, a_5)$ is not empty. Both cases provide $|\widehat{E}_{a_2,a_5}| \leq 9$, and $\widehat{G}_{a_2,a_5}$ has a triangle containing vertices a_1, a_3, and c. Therefore we may assume in general that for any vertex c, except b_1 and b_2, $V(c)$ does not contain both a_i and a_{i+2} for any $i = 1, 2, 3, 4$, and both a_i and a_{i-4} for any $i = 5, 6$.

Now we conclude $E \setminus \{E^2(b_1) \cup E^2(b_2)\}$ consists of three extra edges. Note that each vertex of these edges has degree 3, and there are four more vertices of degree 3 besides b_1 and b_2. These two facts guarantee that these extra edges must be connected as a tree. This tree can be of two types; either all three edges are incident to one vertex d, or two edges are incident to different endpoints of the other edge e, respectively. In both cases, any two edges adjoined to the tree at the same vertex at the end must be also incident to a_i and a_{i+3}, respectively, for some $i = 1, 2, 3$. Therefore, G is one of three graphs as in Figure 9(b)–(c), depending on the type of the tree. The graph G in Figure 9(b) is H_{12}, which is intrinsically knotted. But the two graphs in Figure 9(c) are not intrinsically knotted because, for some i, $|\widehat{E}_{a_i,a_{i+2}}| \leq 9$, and $\widehat{G}_{a_i,a_{i+2}}$ has a triangle.

4.3 Case $(|V_4|, |V_3|) = (9, 2)$

Let b_1 and b_2 be the vertices of V_3. Since $|V_3| = 2$, there are at least three vertices, a_1, a_2, and a_3, in V_4 such that all vertices of each $V(a_i)$ have degree 4. If $\text{dist}(a_1, a_2) = 1$, then $V(a_1) \cup V(a_2)$ consists of 8 vertices of V_4, and so let c be the ninth vertex. Let d be any vertex among $V(a_1) \cup V(a_2) \setminus \{a_1, a_2\}$ which is not contained in $V(c)$. We assume that d is in $V(a_1)$. Then $V(d)$ should be contained in $V(a_2) \cup \{b_1, b_2\}$. This implies that $NE(a_2, d) = 8$ and $|V_3(d)| + |V_4(a_2, d)| \geq 4$, implying $|\widehat{E}_{a_2,d}| \leq 9$. Since c has degree 4 in $\widehat{G}_{a_2,d}$, it follows that $\widehat{G}_{a_2,d}$ is not homeomorphic to $K(3,3)$. We have the same result for any choices of pairs among a_1, a_2, and a_3.

Now assume that the distance between any pair among a_1, a_2, and a_3 is at least 2. We separate into several cases according to the number $|V_4(a_1, a_2)|$. If $V_4(a_1, a_2) = \varnothing$ (ie $\text{dist}(a_1, a_2) > 2$), then $|V_4| \geq 10$, a contradiction. If $V_4(a_1, a_2) = \{d\}$, then $V_4 = V(a_1) \cup V(a_2) \cup \{a_1, a_2\}$. This implies that $a_3 \in V(a_1) \cup V(a_2)$, so $\text{dist}(a_1, a_3) = 1$ or $\text{dist}(a_2, a_3) = 1$, both of which were dealt with in the previous case. If $V_4(a_1, a_2) = \{d_1, d_2\}$, then $V(d_1) \cup V(d_2) \setminus \{a_1, a_2\}$ is contained in $\{a_3, b_1, b_2\}$. This implies that each $V(d_i) \setminus \{a_1, a_2\}$ is a set of two vertices among $\{a_3, b_1, b_2\}$, so that $|V_3(d_1, d_2)| + |V_4(d_1, d_2)| \geq 4$, implying $|\widehat{E}_{d_1,d_2}| \leq 9$. Since at least two of four vertices in $V(a_1) \cup V(a_2) \setminus \{d_1, d_2\}$ still have degree 4 in $\widehat{G}_{d_1,d_2}$, it follows

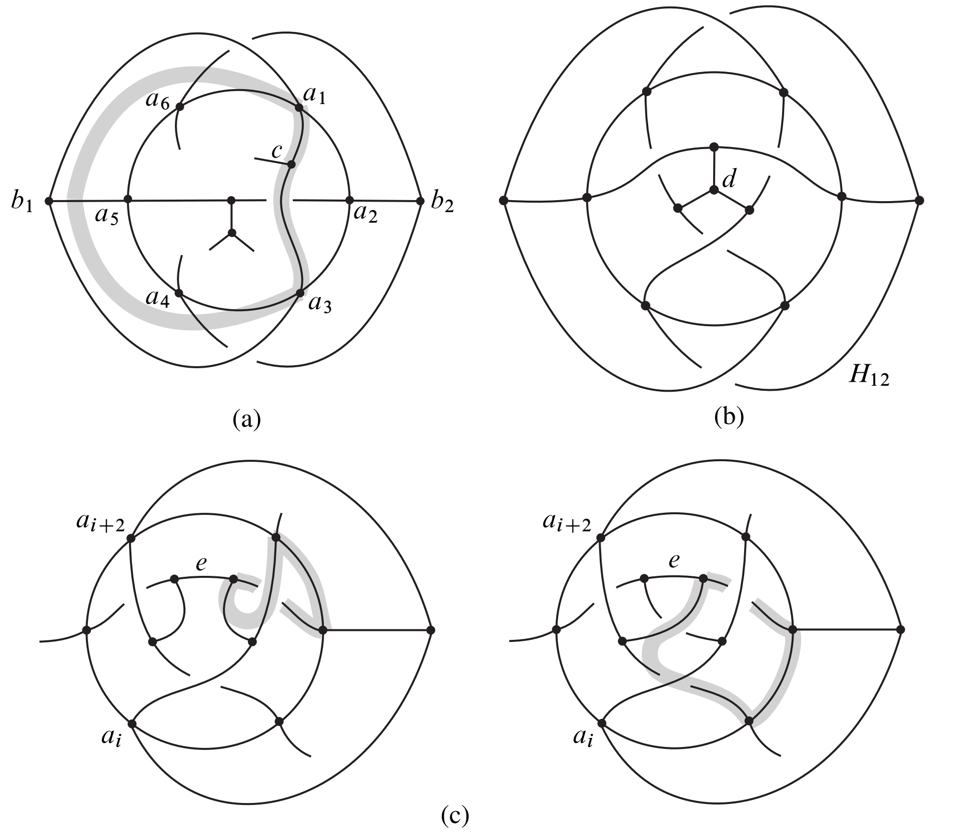


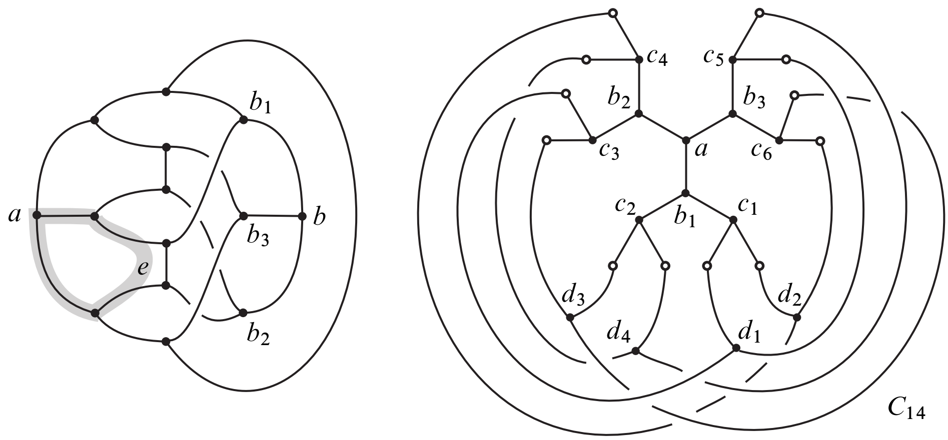
Figure 9: Constructing H_{12}

that $\widehat{G}_{d_1,d_2}$ is not homeomorphic to $K(3,3)$. If $V_4(a_1,a_2) = \{d_1,d_2,d_3\}$, then $V(d_1) \cup V(d_2) \cup V(d_3) \setminus \{a_1,a_2\}$ is contained in $\{a_3,a_4,b_1,b_2\}$, where a_3 and a_4 are the remaining two vertices of degree 4 other than $V(a_1) \cup V(a_2) \cup \{a_1,a_2\}$. Thus each $V(d_i) \setminus \{a_1,a_2\}$ is the set of two vertices among $\{a_3,a_4,b_1,b_2\}$. This implies that $|V_3(d_i,d_j)| + |V_4(d_i,d_j)| \geq 4$ for some $i, j = 1,2,3$, implying $|\widehat{E}_{d_i,d_j}| \leq 9$. Since at least one of three vertices $V(a_1) \cup V(a_2) \setminus \{d_i,d_j\}$ still has degree 4 in $\widehat{G}_{d_i,d_j}$, it follows that $\widehat{G}_{d_i,d_j}$ is not homeomorphic to $K(3,3)$. Finally, if $|V_4(a_1,a_2)| = 4$, then $|\widehat{E}_{a_1,a_2}| \leq 9$. Since $\widehat{G}_{a_1,a_2}$ still has the remaining three vertices of degree 4, it follows that $\widehat{G}_{a_1,a_2}$ is not homeomorphic to $K(3,3)$.

5 deg(a) = 3

Since we are working on the graph with 21 edges and every vertex has degree 3, there are exactly 14 vertices. First, suppose that there exists a pair of vertices a and b with

$\operatorname{dist}(a,b) \geq 4$. Then $E^2(a)$ and $E^2(b)$ can share vertices, but they do not share edges in common. Since $|E^2(a) \cup E^2(b)| = 18$ and $|V(a) \cup V(b) \cup \{a,b\}| = 8$, the 18 endpoints of $E^2(a)$, $E^2(b)$, and three extra edges which are $E \setminus \{E^2(a) \cup E^2(b)\}$, meet at six vertices. If any two edges of $E^2(a) \setminus E(a)$ (and similarly for b) are incident to one vertex c of these six vertices, take the unique vertex d of $V(a)$ which is not an endpoint of these two edges. Then $NE(b,d) = 6$ and $NV_3(b,d) = 6$, implying $|\widehat{E}_{b,d}| = 9$. But $\widehat{G}_{b,d}$ has a triangle containing c and the two vertices of $V(a) \setminus \{d\}$, so it follows that $\widehat{G}_{b,d}$ is not homeomorphic to $K(3,3)$. If not, each of these six vertices is a common endpoint of one edge of $E^2(a)$, one edge of $E^2(b)$, and one extra edge. Now, take an extra edge e and let b_1 and b_2 be the two vertices of $V(b)$ which have distance 1 from the endpoints of e. Let b_3 be the remaining vertex of $V(b)$. Then $NE(b_1,b_2) = 6$, $NV_3(b_1,b_2) = 5$, and $V_Y(b_1,b_2) = \{b_3\}$, implying $|\widehat{E}_{b_1,b_2}| = 9$. But $\widehat{G}_{b_1,b_2}$ has a triangle containing a and two vertices of $V(a)$, so it follows that $\widehat{G}_{b_1,b_2}$ is not homeomorphic to $K(3,3)$. See Figure 10(a).



(a) (b)

Figure 10: Constructing C_{14}

Therefore, we assume that the distance between any pair of vertices cannot exceed 3. Now we construct the intrinsically knotted graph G satisfying these conditions. Take a vertex a and let $V(a) = \{b_1, b_2, b_3\}$ and $V(b_i) = \{a, c_{2i-1}, c_{2i}\}$ for $i = 1,2,3$. As in Figure 10(b), the graph $E(a) \cup E(c_1) \cup \cdots \cup E(c_6)$ consists of 21 edges and 22 vertices. We show this is the only way to draw the graph with 21 edges such that all vertices have distance at most 3 from a and 10 vertices $a, b_1, b_2, b_3, c_1, \ldots, c_5$, and c_6 have degree 3. Now we join 12 white dots in Figure 10(b) into 4 groups indicating the remaining 4 vertices by d_1, d_2, d_3 and d_4. Thus each $V(d_j)$, $j = 1,2,3,4$, has three vertices among $c_1, \ldots, c_6$. Since the distance between any c_i and $c_{i'}$ cannot exceed 3, the following two properties must be satisfied. The first property is that $V(d_j)$ contains exactly one vertex from each group $\{c_{2i-1}, c_{2i}\}$ for $i = 1, 2, 3$. For example, if $V(d_1) = \{c_1, c_2, c_3\}$ (ie two vertices from the group $\{c_1, c_2\}$), then we can connect c_1 to at most two vertices among $\{c_4, c_5, c_6\}$ through some $E(d_j)$. This means that the distance between c_1 and one among $\{c_4, c_5, c_6\}$ exceeds 3. The second property is that different $V(d_j)$ and $V(d_{j'})$ share at most one vertex. For example, if they share two vertices c_1 and c_3, then $\mathrm{dist}(c_1, c_4) = 4$. From these two properties, without loss of generality, we may say that

$$V(d_1) = \{c_1, c_3, c_5\}, \quad V(d_2) = \{c_1, c_4, c_6\},$$
$$V(d_3) = \{c_2, c_3, c_6\}, \quad V(d_4) = \{c_2, c_4, c_5\}$$

as drawn in Figure 10(b). This graph is exactly C_{14}.

References

[1] **P Blain**, **G Bowlin**, **T Fleming**, **J Foisy**, **J Hendricks**, **J Lacombe**, *Some results on intrinsically knotted graphs*, J. Knot Theory Ramifications 16 (2007) 749–760 MR2341314

[2] **J H Conway**, **C M Gordon**, *Knots and links in spatial graphs*, J. Graph Theory 7 (1983) 445–453 MR722061

[3] **N Goldberg**, **T W Mattman**, **R Naimi**, *Many, many more intrinsically knotted graphs*, Algebr. Geom. Topol. 14 (2014) 1801–1823 MR3212585

[4] **R Hanaki**, **R Nikkuni**, **K Taniyama**, **A Yamazaki**, *On intrinsically knotted or completely 3–linked graphs*, Pacific J. Math. 252 (2011) 407–425 MR2860431

[5] **B Johnson**, **M E Kidwell**, **T S Michael**, *Intrinsically knotted graphs have at least 21 edges*, J. Knot Theory Ramifications 19 (2010) 1423–1429 MR2746195

[6] **T Kohara**, **S Suzuki**, *Some remarks on knots and links in spatial graphs*, from: "Knots 90", (A Kawauchi, editor), de Gruyter, Berlin (1992) 435–445 MR1177440

[7] **R Motwani**, **A Raghunathan**, **H Saran**, *Constructive results from graph minors: Linkless embeddings*, from: "Proc. 29^{th} annual symposium on foundations of computer science", IEEE, Piscataway, NJ (1988) 398–409

[8] **M Ozawa**, **Y Tsutsumi**, *Primitive spatial graphs and graph minors*, Rev. Mat. Complut. 20 (2007) 391–406 MR2351115

[9] **N Robertson**, **P D Seymour**, *Graph minors, XX: Wagner's conjecture*, J. Combin. Theory Ser. B 92 (2004) 325–357 MR2099147

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