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On the K–theory of subgroups of virtually connected Lie groups

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We prove that for every finitely generated subgroup G of a virtually connected Lie group which admits a finite-dimensional model for $\underline{E}G$, the assembly map in algebraic K–theory is split injective. We also prove a similar statement for algebraic L–theory which, in particular, implies the generalized integral Novikov conjecture for such groups.

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1 Introduction

For every group G and every ring R there is a functor $\mathbb{K}_R\colon \operatorname{Or} G \to \mathfrak{Spectra}$ from the orbit category of G to the category of spectra, sending G/H to (a spectrum weakly equivalent to) the K–theory spectrum $\mathbb{K}(R[H])$ for every subgroup $H \le G$. By K–theory we will always mean nonconnective K–theory as constructed by Pedersen and Weibel [26]. For any such functor $F\colon \operatorname{Or} G \to \mathfrak{Spectra}$ a G–homology theory $\mathbb{F}$ can be constructed via

$$\mathbb{F}(X) := \operatorname{Map}_G(\,\cdot\,, X_+) \wedge_{\operatorname{Or} G} F;$$

see Davis and Lück [14]. We will denote its homotopy groups by $H_n^G(\,\cdot\,, F) := \pi_n \mathbb{F}(X)$. Let $\mathcal{F}$ be a family of subgroups of G. The K–theoretic assembly map for $\mathcal{F}$ is the map

$$\alpha_{\mathcal{F}}\colon H_n^G(E_{\mathcal{F}}G; \mathbb{K}_R) \to H_n^G(\mathrm{pt}; \mathbb{K}_R) \cong K_n(R[G]),$$

induced by the map $E_{\mathcal{F}}G \to \mathrm{pt}$. Here $E_{\mathcal{F}}G$ denotes the classifying space for the family $\mathcal{F}$; see Lück [22]. The assembly map is a helpful tool to relate the K–theory of the group ring $R[G]$ to the K–theory of the group rings over $H \in \mathcal{F}$. The assembly map can be defined more generally for any small additive G–category instead of R; see Bartels and Reich [11]. In this article all additive categories will be small.

Analogously, for every additive G–category $\mathcal{A}$ with involution and every family of subgroups $\mathcal{F}$ we can define the L–theoretic assembly map

$$\alpha_{\mathcal{F}}\colon H_n^G(E_{\mathcal{F}}G; \mathbb{L}_{\mathcal{A}}^{\langle -\infty\rangle}) \to H_n^G(\mathrm{pt}; \mathbb{L}_{\mathcal{A}}^{\langle -\infty\rangle}).$$

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The Farrell–Jones conjecture [15] states that the assembly maps $\alpha_{\mathcal{V}cyc}$ for the family of virtually cyclic subgroups in K– and L–theory are isomorphisms for all additive G–categories $\mathcal{A}$ (with involution) and all $n \in \mathbb{Z}$. The Farrell–Jones conjecture has been proven for a large class of groups, for example hyperbolic and CAT(0)–groups (Bartels and Lück [7; 8], Bartels, Lück and Reich [9; 10] and Wegner [29]), virtually solvable groups (Wegner [30]), and lattices in virtually connected Lie groups (Bartels, Farrell and Lück [4] and Kammeyer, Lück and Rüping [19]). The Farrell–Jones conjecture implies that the assembly maps $\alpha_{\mathcal{F}in}$ for the family of finite subgroups are split injective; see Bartels [2, Theorem 1.3]. The rational split injectivity of the map $\alpha_{\mathcal{F}in}$ in L–theory implies the Novikov conjecture. The integral split injectivity of $\alpha_{\mathcal{F}in}$ is called the *generalized integral Novikov conjecture*; for more details see Section 6. Kasparov proved the Novikov conjecture for all discrete subgroups of virtually connected Lie groups in [20, Theorem 6.9]. More generally, the Novikov conjecture is true for groups which uniformly embed into a Hilbert space; see Skandalis, Tu and Yu [27]. This includes all amenable groups and all groups with finite asymptotic dimension. By Carlsson and Goldfarb [12, Section 3] and Ji [17, Corollary 3.4], discrete subgroups of virtually connected Lie groups have finite asymptotic dimension, giving a second proof that the Novikov conjecture holds for these groups. Here we will show that, in particular, discrete subgroups of virtually connected Lie groups also satisfy the generalized integral Novikov conjecture.

In [21] the author proved the split injectivity of the assembly map for finitely generated subgroups G of $\mathrm{GL}_n(\mathbb{C})$ which have an upper bound on the Hirsch length of the unipotent subgroups. For a definition of the Hirsch length see Section 3. The bound on the Hirsch length exists if and only if G has finite virtual cohomological dimension by Alperin and Shalen [1]. Since G is virtually torsion-free, this is the case if and only if there is a finite-dimensional model for $\underline{E}G$ where we consider G with the discrete topology; see Lück [22, Theorem 3.1]. In this article we want to extend this theorem to subgroups of all virtually connected Lie groups. Note that in the theorem we again consider G with the discrete topology.

Theorem 1.1 *Let G be a finitely generated subgroup of a virtually connected Lie group, and assume there exists a finite-dimensional model for $\underline{E}G$. Then the K–theoretic assembly map*

$$H_n^G(\underline{E}G;\mathbb{K}_{\mathcal{A}}) \to K_n(\mathcal{A}[G])$$

is split injective for every additive G–category $\mathcal{A}$.

A similar version holds for L–theory as well, which implies, in particular, the generalized integral Novikov conjecture for these groups; see Section 6.

If G is a discrete subgroup of a virtually connected Lie group H, and K the maximal compact subgroup of H, then H/K is a finite-dimensional model for $\underline{E}G$; see Lück [23, Theorem 4.4]. In particular, we get the following corollary.

Corollary 1.2 *Let G be a finitely generated discrete subgroup of a virtually connected Lie group. Then the K–theoretic assembly map*

$$H_n^G(\underline{E}G;\mathbb{K}_{\mathcal{A}}) \to K_n(\mathcal{A}[G])$$

is split injective for every additive G–category $\mathcal{A}$.

The condition on the existence of a finite-dimensional model for $\underline{E}G$ can be reformulated in the following way.

Proposition 1.3 *A finitely generated subgroup G of a virtually connected Lie group admits a finite-dimensional model for $\underline{E}G$ if and only if there exists $N \in \mathbb{N}$ such that every finitely generated abelian subgroup of G has rank at most N.*

The rank of an abelian group A is defined as $\mathrm{rk}(A) := \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q})$ or, equivalently, as the cardinality of a maximal linearly independent subset of A. The statement that every finitely generated abelian subgroup of G has rank at most N is equivalent to the statement that every abelian subgroup of G has rank at most N. For a proof of the proposition, see Section 3.

In Section 7, we prove that Theorem 1.1 and its L–theoretic analog also hold without the assumption that G is finitely generated.

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2 Lie groups

A Lie group is virtually connected if it has only finitely many connected components. For the rest of this section let H be a virtually connected Lie group with Lie algebra $\mathfrak{h}$ (which we identify with T_eH). The Lie group H acts on itself by conjugation;

$$c\colon H \to \mathrm{Aut}(H),\ g \mapsto (h \mapsto ghg^{-1}).$$

Taking the derivative yields a map

$$\mathrm{Ad}\colon H \to \mathrm{Aut}(\mathfrak{h}),\ g \mapsto D_e(c(g)).$$

Since $\mathrm{Aut}(\mathfrak{h})$ is a Lie subgroup of $\mathrm{GL}(\mathfrak{h})$, Ad gives a representation of H. The kernel of the representation Ad is the centralizer $C_H(H_0)$ of the unit component H_0 of H.

By definition of the centralizer, the group $C_H(H_0) \cap H_0$ is abelian, and since H is virtually connected the centralizer $C_H(H_0)$ is, therefore, virtually abelian. For every subgroup G of H we obtain a short exact sequence

$$1 \to C_H(H_0) \cap G \to G \to \mathrm{Ad}(G) \to 1,$$

with virtually abelian kernel and linear quotient. We will use this sequence to extend the results of [21] to general virtually connected Lie groups. Before we can do so, we first need to prove Proposition 1.3, which will be done in the next chapter.

3 A bound on the rank of abelian subgroups

In the proof of Proposition 1.3, a bound on the Hirsch length of the finitely generated nilpotent subgroups is needed. First we review some facts about nilpotent groups to see that this is the same as a bound on the ranks of the finitely generated abelian subgroups.

Let G be a group. Define $G_1 := G$ and, recursively, $G_{n+1} := [G_n, G]$. The series $G = G_1 \geq G_2 \geq \cdots$ is called the *lower central series* of G. A group G is *nilpotent* if there exists $c \in \mathbb{N}$ with $G_{c+1} = 1$. The smallest such c is called the *nilpotency class* of G; we denote it by $c(G)$. The *upper central series* $1 = Z_0(G) \leq Z_1(G) \leq \cdots$ of G is recursively defined by

$$Z_{i+1}(G) := \{g \in G \mid \forall h \in G : [g,h] \in Z_i(G)\}.$$

If G is nilpotent, then $Z_{c(G)}(G) = G$ and the length of the upper and lower central series agree. For any normal subgroup $H \leq G$ the quotient G/H is again nilpotent.

The *Hirsch length* $h(G)$ of G is

$$h(G) := \mathrm{rk}(G_1/G_2) + \cdots + \mathrm{rk}(G_{c-1}/G_c) + \mathrm{rk}(G_c),$$

where $\mathrm{rk}(H)$ denotes the rank of an abelian group H; ie $\mathrm{rk}(H) := \dim_{\mathbb{Q}}(H \otimes_{\mathbb{Z}} \mathbb{Q})$.

Let $n(G)$ denote

$$\max\{\mathrm{rk}(A) \mid A \trianglelefteq G \text{ an abelian normal subgroup}\}.$$

Let H be and G be a group acting on H. G *acts nilpotently* if there is a series

$$1 = H_0 \leq H_1 \leq \cdots \leq H_n = H$$

of G–invariant normal subgroups of H such that the induced action on H_i/H_{i-1} is trivial. In the special case where $H = G$ and the action is by conjugation, G acts nilpotently on itself if and only if G is nilpotent.

Proposition 3.1 *Let G be finitely generated nilpotent. Then $h(G) \leq \frac{n(G)(n(G)+1)}{2}$.*

The proposition is proved in Möhres [25, Theorem 2] for torsion-free nilpotent groups instead of finitely generated nilpotent groups. For the convenience of the reader we give a proof. For this we need the following well-known statements about nilpotent groups.

Lemma 3.2 *A subgroup of a finitely generated nilpotent group is finitely generated.*

Proof The statement follows by induction on the nilpotency class. □

Lemma 3.3 [28, Theorem 1.3] *Let G be nilpotent and $N \trianglelefteq G$ a nontrivial normal subgroup. Then $N \cap Z(G)$ is nontrivial, where $Z(G)$ denotes the center of G.*

Lemma 3.4 *Let G be nilpotent and A a maximal abelian normal subgroup. Then $C_G(A) = A$, where $C_G(A)$ is the centralizer of A in G.*

Proof Since $A \trianglelefteq G$ is normal, so is $C_G(A)$. Suppose $A \neq C_G(A)$. Then $C_G(A)/A$ is a nontrivial normal subgroup of G/A, and $H := C_G(A)/A \cap Z(G/A)$ is nontrivial by the previous lemma. Let $C = \langle c \rangle$ be a cyclic subgroup of H. Then $C \trianglelefteq Z(G/A) \trianglelefteq G/A$ and, since C lies in the center, it is a normal subgroup of G/A. Let $c' \in C_G(A)$ be a preimage of c; then the preimage of C is $\langle A, c' \rangle$. This is abelian and normal in G; hence, A was not maximal with this property. □

Lemma 3.5 *Let $\mathrm{Tr}(n, \mathbb{Z}) \leq \mathrm{GL}_n(\mathbb{Z})$ denote the subgroup of unitriangular matrices; ie every element of $\mathrm{Tr}(n, \mathbb{Z})$ has 1's on the diagonal and 0's below the diagonal. If $G \leq \mathrm{GL}_n(\mathbb{Z})$ acts nilpotently on $\mathbb{Z}^n$, then it is unipotent and conjugate to a subgroup of $\mathrm{Tr}(n, \mathbb{Z})$.*

Proof Since $\mathrm{Tr}(n, \mathbb{Z})$ is unipotent, it suffices to prove that G is conjugate to a subgroup of it. Let

$$0 = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \ldots \trianglelefteq H_k = \mathbb{Z}^n$$

be a sequence of G–invariant subspaces and let G act trivially on H_i/H_{i-1} for all $i = 1, \ldots, k$. The lemma is obvious for $k = 1$, and we will prove it by induction on k. Let $H' := \{z \in \mathbb{Z}^n \mid \exists\ l \in \mathbb{Z} : lz \in H_1\}$. Let $z \in H'$ and $l \in \mathbb{Z}$ with $lz \in H_1$. For every

$g \in G$ we have $lg(z) = g(lz) = lz$ and thus also $g(z) = z$; ie G acts trivially on H'. By construction, $\mathbb{Z}^n/H'$ is torsion-free, and we obtain a splitting $\mathbb{Z}^n \cong H' \oplus \mathbb{Z}^n/H'$. The sequence

$$0 = H'/H' \trianglelefteq H_2 + H'/H' \trianglelefteq \cdots \trianglelefteq H_k + H'/H' = \mathbb{Z}^n/H'$$

consists of G–invariant subspaces, and G acts trivially on the quotients. By induction there is a basis of $\mathbb{Z}^n/H$ such that $G \leq \mathrm{GL}(\mathbb{Z}^n/H)$ is unitriangular. Using this basis together with a basis of H' yields a basis of $\mathbb{Z}^n$ for which G lies in $\mathrm{Tr}(n, \mathbb{Z})$. □

Proof of Proposition 3.1 Let $n := n(G)$ and A be a maximal abelian normal subgroup. Then A again is finitely generated by Lemma 3.2, and $A \cong \mathbb{Z}^n \oplus F$ with F a finite group. The group G acts by conjugation on A and, since $C_G(A) = A$, the induced map $G/A \to \mathrm{Aut}(A)$ is injective. Since F is finite, the projection to $\mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$ has finite kernel. The group G is nilpotent, and thus it acts nilpotently on $\mathbb{Z}^n$ (by conjugation). This implies that the image G/A in $\mathrm{GL}_n(\mathbb{Z})$ is conjugate to a subgroup of the unitriangular matrices $\mathrm{Tr}(n, \mathbb{Z})$. Since $h(\mathrm{Tr}(n, \mathbb{Z})) = n(n-1)/2$, we have

$$\begin{aligned} h(G) &\leq h(A) + h(\ker(\mathrm{Aut}(A) \to \mathrm{GL}_n(\mathbb{Z}))) + h(\mathrm{Tr}(n, \mathbb{Z})) \\ &= n + 0 + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}. \end{aligned}$$ □

A direct corollary of Proposition 3.1 is the following.

Corollary 3.6 *Let G be a group. Then G has a bound on the Hirsch length of its finitely generated nilpotent subgroups if and only if it has a bound on the rank of its finitely generated abelian subgroups.*

Before we can prove Proposition 1.3 we need the following lemma.

Lemma 3.7 *Let A be a (countable) abelian group with finite rank, then there is a finite-dimensional model for $\underline{E}A$.*

Proof Let $\operatorname{rk} A = n$. Then there exists a subgroup $B \leq A$ isomorphic to $\mathbb{Z}^n$. The quotient $Q := A/B$ has rank 0 and thus is a torsion group. For $n \in \mathbb{N}$ let $F_n \leq Q$ be finite subgroups with $F_n \leq F_{n+1}$ and $Q = \bigcup_{n \in \mathbb{N}} F_n$. Define a Q–CW-complex X by taking $\coprod_{n \in \mathbb{N}} Q/F_n$ as the zero skeleton and for every $n \in \mathbb{N}$ adding a 1–cell with stabilizer F_n between the 0–cells Q/F_n and Q/F_{n+1}. This defines a 1–dimensional model X for $\underline{E}Q$. Let $p\colon A \to Q$ be the quotient map. For every finite subgroup $F \leq Q$, the preimage $p^{-1}(F)$ is finitely generated abelian of rank n and thus has $\mathbb{R}^n$ as an n–dimensional model for $\underline{E}p^{-1}(Q)$. Therefore, the proof of Lück [22, Theorem 3.1] shows that A has a model for $\underline{E}A$ of dimension $n+1$. □

Let G be a subgroup of $\mathrm{GL}_n(\mathbb{C})$ and assume there exists $N \in \mathbb{N}$ such that the rank of every finitely generated unipotent subgroup of G is at most N. Then, by Alperin and Shalen [1], the virtual cohomological dimension of G is bounded and therefore admits a finite-dimensional model for $\underline{E}G$ by [22, Theorem 6.4]. Using this, we now can prove Proposition 1.3.

Proof of Proposition 1.3 Let G be a subgroup of a virtually connected Lie group H such that there exists a finite dimensional model X for $\underline{E}G$. Then, in particular, X is a model for $\underline{E}A$ for every abelian subgroup $A \leq G$ and $\operatorname{rk} A \leq \dim X$.

For the other direction, let G be a finitely generated subgroup of a virtually connected Lie group H such that there exists a bound on the rank of the finitely generated abelian subgroups of G. Then, by Corollary 3.6, G has also a bound on the Hirsch length of its finitely generated nilpotent subgroups. Let $G_0 := G \cap H_0$, and consider the extension

$$1 \to C_H(H_0) \cap G_0 \to G_0 \to \mathrm{Ad}(G_0) \to 1$$

from Section 2. Since $C_H(H_0) \cap G_0$ is contained in the center of G_0, $\mathrm{Ad}(G_0)$ also has a bound on the Hirsch length of its finitely generated nilpotent subgroups and, thus, on the finitely generated unipotent subgroups. By the above it admits a finite dimensional model for $\underline{E}\,\mathrm{Ad}(G_0)$. And since also $K := C_H(H_0) \cap G_0$ has finite rank, there is a finite dimensional model for $\underline{E}K$ by Lemma 3.7. Consider the extensions

$$1 \to K \to G_0 \to \mathrm{Ad}(G_0) \to 1,$$
$$1 \to G_0 \to G \to F \to 1,$$

with F finite. The group G_0 is finitely generated since finite index subgroups of finitely generated groups are again finitely generated. Thus $\mathrm{Ad}(G_0)$ is virtually torsion-free by Selberg's lemma, and we can use [22, Theorem 3.1] to obtain a finite dimensional model for $\underline{E}G$ from these sequences. □

Remark 3.8 Using the results of the author from [21], the short exact sequence

$$1 \to C_H(H_0) \cap G \to G \to \mathrm{Ad}(G) \to 1$$

implies that G has fqFDC, which also is defined in [21]. In particular, if G has a bound on the order of the finite subgroups, then the main result of [21] directly implies the split injectivity of the K–theoretic assembly map and a similar result in L–theory. Since we do not know if this always holds, we use a different approach using inheritance properties; see Sections 4 and 5.

4 Inheritance properties

To use the short exact sequence from Section 2 we want to show the following inheritance property.

Proposition 4.1 *Assume there is a short exact sequence of groups*

$$1 \to J \to G \xrightarrow{\phi} Q \to 1$$

such that for every virtually cyclic subgroup $V \le Q$ the preimage $\phi^{-1}(V)$ satisfies the Farrell–Jones conjecture. Furthermore, assume that the assembly map

$$H_n^G(\underline{E}\,Q;\mathbb{K}_{\mathcal{B}}) \to K_n(\mathcal{B}[Q])$$

is split injective for every $n \in \mathbb{Z}$ and every additive Q–category $\mathcal{B}$. Then the K–theoretic assembly map

$$H_n^G(\underline{E}G;\mathbb{K}_{\mathcal{A}}) \to K_n(\mathcal{A}[G])$$

is split injective for every $n \in \mathbb{Z}$ and every additive G–category $\mathcal{A}$.

Proof Let $\mathcal{A}$ be an additive G–category. The fact that $\phi^{-1}(V)$ satisfies the Farrell–Jones conjecture for every virtually cyclic subgroup $V \le Q$ implies that the natural map $H_n^G(E_{\mathcal{V}cyc}G;\mathbb{K}_{\mathcal{A}}) \to H_n^G(E_{\phi^*\mathcal{V}cyc}G;\mathbb{K}_{\mathcal{A}})$ is an isomorphism, by Bartels and Lück [6, Lemma 2.2], where $\phi^*\mathcal{V}cyc := \{K \le G \mid \phi(K) \in \mathcal{V}cyc\}$. Here we used that the projection $E_{\mathcal{V}cyc}G \times E_{\phi^*\mathcal{V}cyc}G \to E_{\phi^*\mathcal{V}cyc}G$ is a model for the natural map $E_{\mathcal{V}cyc}G \to E_{\phi^*\mathcal{V}cyc}G$. Furthermore, the natural map $H_n^G(\underline{E}G;\mathbb{K}_{\mathcal{A}}) \to H_n^G(E_{\mathcal{V}cyc}G;\mathbb{K}_{\mathcal{A}})$ is split injective by Bartels [2]. Now the commutative diagram

$$\begin{tikzcd}
H_n^G(\underline{E}G;\mathbb{K}_{\mathcal{A}}) \arrow[r] \arrow[d, hook] & H_n^G(E_{\phi^*\mathcal{F}in}G;\mathbb{K}_{\mathcal{A}}) \arrow[d] \\
H_n^G(E_{\mathcal{V}cyc}G;\mathbb{K}_{\mathcal{A}}) \arrow[r, "\cong"] & H_n^G(E_{\phi^*\mathcal{V}cyc}G;\mathbb{K}_{\mathcal{A}})
\end{tikzcd}$$

implies that the map $H_n^G(\underline{E}G;\mathbb{K}_{\mathcal{A}}) \to H_n^G(E_{\phi^*\mathcal{F}in}G;\mathbb{K}_{\mathcal{A}})$ is split injective, where $\phi^*\mathcal{F}in := \{K \le G \mid \phi(K) \in \mathcal{F}in\}$. By Bartels and Reich [11, Corollary 4.3] the split injectivity for Q implies that the assembly map $H_n^G(E_{\phi^*\mathcal{F}in}G;\mathbb{K}_{\mathcal{A}}) \to K_n(\mathcal{A}[G])$ is split injective. Combining these results yields the proposition. □

To apply the above proposition for the short exact sequence from the previous section, we need the following.

Lemma 4.2 *The class of virtually solvable groups is closed under group extensions.*

The idea of the proof is taken from math.stackexchange.com; see [13].

Proof Let
$$1 \to N \to G \xrightarrow{p} Q \to 1$$
be a short exact sequence, and let N and Q be virtually solvable. Let $Q' \leq Q$ be a solvable subgroup with $[Q : Q'] < \infty$; then $[G : p^{-1}(Q')] < \infty$. Thus we can assume that Q is solvable. We will first consider the case that N is finite. Since N is normal in G, G acts on N by conjugation, which induces a map $c\colon G \to \mathrm{Aut}(N)$. The centralizer $C_G(N)$ of N in G is the kernel of c. Since the class of solvable groups is closed under extension, and $C_G(N) \cap N$ is abelian, the exact sequence
$$1 \to C_G(N) \cap N \to C_G(N) \to p(C_G(N)) \to 1$$
shows that $C_G(N)$ is solvable. The group N is finite; thus $C_G(N)$ has finite index in G.

Now let N be any virtually solvable group. And let $\mathcal{S}$ be the set of all normal, solvable, finite-index subgroups of N, ordered by inclusion. This is not empty, and we can choose K to be a maximal element of S. For every $g \in G$ also $gKg^{-1}K$ is a solvable, normal, finite-index subgroup of N. Since K was maximal, it therefore has to be normal in G. From the short exact sequence
$$1 \to N/K \to G/K \to Q \to 1,$$
it follows from the first case that G/K is virtually solvable. Since K is solvable, the sequence
$$1 \to K \to G \to G/K \to 1$$
implies that G is virtually solvable. □

5 Proof of Theorem 1.1

For this section let H be a virtually connected Lie group and $G \leq H$ a finitely generated subgroup such that there exists a finite dimensional model for $\underline{E}G$. The proof of Theorem 1.1 follows easily from the statements of the previous section.

Proof of Theorem 1.1 Let $\Gamma := \mathrm{Ad}(G)$ be the image of G under $\mathrm{Ad}\colon H \to \mathrm{GL}(\mathfrak{h})$. Since $C_H(H_0) \cap G \cap H_0$ is contained in the center of G, the preimage of any unipotent subgroup U of $\mathrm{Ad}(G \cap H_0)$ is a nilpotent subgroup of $G \cap H_0$. By Corollary 3.6 and

Proposition 1.3 there is a bound on the Hirsch length of the nilpotent subgroups of $G \cap H_0$ and, in particular, there is a bound on the Hirsch length of U. Since $G \cap H_0$ has finite index in G, this implies that there also is a bound on the Hirsch length of the unipotent subgroups of Γ. Now we can apply the following:

[21, Corollary 3] *Let F be a field of characteristic zero, and let Γ be a finitely generated subgroup of $\mathrm{GL}_n(F)$ with a global upper bound on the Hirsch rank of its unipotent subgroups. Then the K–theoretic assembly map*

$$H_*^{\Gamma}(\underline{E}G; \mathbb{K}_{\mathcal{A}}) \to H_*^{\Gamma}(pt; \mathbb{K}_{\mathcal{A}}) \cong K_*(\mathcal{A}[\Gamma])$$

is split injective for every additive Γ–category $\mathcal{A}$.

Note that [21, Corollary 3] is stated only for rings instead of additive Γ–categories, but by [21, Theorem 8.1] it is true for any additive Γ–category.

Furthermore, by Wegner [30], every virtually solvable group satisfies the Farrell–Jones conjecture. Using this and Lemma 4.2, we see that the sequence

$$1 \to C_H(H_0) \cap G \to G \to \mathrm{Ad}(G) \to 1$$

satisfies the conditions of Proposition 4.1. Therefore, the assembly map

$$H_*^{G}(\underline{E}G; \mathbb{K}_{\mathcal{A}}) \to K_*(\mathcal{A}[G])$$

is split injective for every additive G–category $\mathcal{A}$. □

6 L–theory

Most of the statements from the previous sections also hold for L–theory. For the rest of the section let G be a finitely generated subgroup of a virtually connected Lie group H with a finite dimensional model for $\underline{E}G$, and let Q be the image of G under $\mathrm{Ad}\colon H \to \mathrm{GL}(\mathfrak{h})$. Furthermore, let ϕ denote $\mathrm{Ad}|_G$, and let $\mathcal{A}$ be an additive G–category with involution. As above we obtain the commutative diagram

$$\begin{tikzcd}
H_n^G(\underline{E}G; \mathbb{L}_{\mathcal{A}}^{\langle -\infty\rangle}) \arrow[r] \arrow[d] & H_n^G(E_{\phi^*\mathcal{F}in}G; \mathbb{L}_{\mathcal{A}}^{\langle -\infty\rangle}) \arrow[d] \\
H_n^G(E_{\mathcal{V}cyc}G; \mathbb{L}_{\mathcal{A}}^{\langle -\infty\rangle}) \arrow[r, "\cong"] & H_n^G(E_{\phi^*\mathcal{V}cyc}G; \mathbb{L}_{\mathcal{A}}^{\langle -\infty\rangle}),
\end{tikzcd}$$

and the lower horizontal map is still an isomorphism by Bartels and Lück [6, Lemma 2.2] and Wegner [30]. But for the vertical map on the left to be injective we need that for every virtually cyclic subgroup $V \subseteq G$ there is an $i_0 \in \mathbb{N}$ such that for every $i \geq i_0$ we have $K_{-i}(\mathcal{A}[V]) = 0$; see Bartels [2]. Then it remains to show that

$$H_n^G(E_{\phi^*\mathcal{F}in}G; \mathbb{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(\mathcal{A}[G])$$

is split injective. By Bartels and Reich [11, Proposition 4.2 and Corollary 4.3], this follows if

$$H_n^Q(\underline{E}G; \mathbb{L}_{\mathrm{ind}_\phi \mathcal{A}}^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}((\mathrm{ind}_\phi \mathcal{A})[Q])$$

is split injective. See [11] for the definition of $\mathrm{ind}_\phi \mathcal{A}$. To apply [21, Theorem 9.1] as above, we need the further assumption that for every finite subgroup $H \leq Q$ there is an $i_0 \in \mathbb{N}$ such that for every $i \geq i_0$ we have

$$0 = K_{-i}((\mathrm{ind}_\phi \mathcal{A})[H]) \cong K_{-i}(\mathcal{A}[\phi^{-1}(H)]).$$

Since $\phi^{-1}(H)$ is virtually abelian, we obtain the following version of the main theorem for L–theory.

Theorem 6.1 *Let G be a finitely generated subgroup of a virtually connected Lie group, and assume there exists an $N \in \mathbb{N}$ such that every finitely generated abelian subgroup of G has rank at most N. Let $\mathcal{A}$ be an additive G–category with involution. Assume further that for every virtually abelian subgroup H of G there is an $i_0 \in \mathbb{N}$ such that for every $i \geq i_0$ we have $K_{-i}(\mathcal{A}[H]) = 0$; then the L–theoretic assembly map*

$$H_n^G(\underline{E}G; \mathbb{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(\mathcal{A}([G])$$

is split injective.

For torsion-free groups G the *integral Novikov conjecture* states that the assembly map

$$H_n^G(EG; \mathbb{L}_{\mathbb{Z}}^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(\mathbb{Z}[G])$$

is injective. It is known that the integral Novikov conjecture is false for groups containing torsion. Following Ji [18], we say that G satisfies the *generalized integral Novikov conjecture* if the assembly maps

$$H_n^G(\underline{E}G; \mathbb{L}_{\mathbb{Z}}^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(\mathbb{Z}[G]), \quad H_n^G(\underline{E}G; \mathbb{K}_{\mathbb{Z}}) \to K_n(\mathbb{Z}[G])$$

are injective. By Lück and Reich [24, Propostion 2.20], the relative rational assembly map

$$H_n^G(EG; \mathbb{L}_{\mathbb{Z}}^{\langle -\infty \rangle}) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_n^G(\underline{E}G; \mathbb{L}_{\mathbb{Z}}^{\langle -\infty \rangle}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective. Observe that, since the $\mathbb{Z}/2$–Tate cohomology groups vanish rationally, there is no difference between the various decorations in L–theory as can be seen using the Rothenberg sequence. Therefore, by [24, Proposition 1.53], the injectivity of the rational assembly map

$$H_n^G(EG;\mathbb{L}_{\mathbb{Z}}^{\langle -\infty\rangle})\otimes_{\mathbb{Z}}\mathbb{Q}\to L_n^{\langle -\infty\rangle}(\mathbb{Z}[G])\otimes_{\mathbb{Z}}\mathbb{Q}$$

implies the Novikov conjecture about the homotopy invariance of higher signatures. In particular, the generalized integral Novikov conjecture implies the (classical) Novikov conjecture.

We will show that $K_{-n}(\mathbb{Z}[G])=0$ for $n>1$ and any virtually abelian group A. Therefore, Theorem 6.1 implies the generalized integral Novikov conjecture for the groups G appearing in the theorem; ie we get the following corollary.

Corollary 6.2 *Let G be a finitely generated subgroup of a virtually connected Lie group, and assume there exists an $N\in\mathbb{N}$ such that every finitely generated abelian subgroup of G has rank at most N. Then G satisfies the generalized integral Novikov conjecture.*

By Farrell and Jones [16, Theorem 2.1], for every virtually cyclic group V and $n>1$,

$$K_{-n}(\mathbb{Z}[V])=0.$$

Let G be a group and let X be a finite G–CW-complex with virtually cyclic stabilizers. By induction on the dimension of X we prove that

$$H_{-n}^G(X;\mathbb{K}_{\mathbb{Z}})=0$$

for every $n>1$. For $\dim X=0$, we have

$$H_{-n}^G(X;\mathbb{K}_{\mathbb{Z}})\cong\bigoplus_{x\in X}\mathbb{K}_{-n}(\mathbb{Z}[G_x])=0,$$

where the stabilizers G_x are virtually cyclic by assumption. Assume the above holds for m and let $\dim X=m+1$. Then we have the exact sequence

$$0=H_{-n}^G(X^{(m)};\mathbb{K}_{\mathbb{Z}})\to H_{-n}^G(X;\mathbb{K}_{\mathbb{Z}})\to H_{-n}^G(X,X^{(m)};\mathbb{K}_{\mathbb{Z}}),$$

and

$$H_{-n}^G(X,X^{(m)};\mathbb{K}_{\mathbb{Z}})\cong\bigoplus_{c\in C_m}\mathbb{K}_{-n-m-1}(\mathbb{Z}[G_c])=0,$$

where C_m denotes the set of m–cells of X and G_c the (virtually cyclic) stabilizer of the cell c. Since every virtually abelian group A satisfies the Farrell–Jones conjecture, we have

$$K_{-n}(\mathbb{Z}[A]) \cong H^A_{-n}(X;\mathbb{K}_\mathbb{Z}) \cong \operatorname*{colim}_K H^A_{-n}(AK;\mathbb{K}_\mathbb{Z}) = 0,$$

where X is an A–CW-complex model for $E_{\mathcal{V}cyc}A$, and the colimit is taken over all finite subcomplexes $K \subseteq X$.

7 Inheritance under colimits

In this section we want to show that Theorem 1.1 and Theorem 6.1 hold without the assumption that G is finitely generated.

By Bartels, Echterhoff and Lück [3, Lemma 2.4 and Lemma 6.2] for every system G_α of finitely generated subgroups of G such that $\operatorname{colim}_\alpha G_\alpha \cong G$, the assembly map

$$H^G_n(\underline{E}G;\mathbb{K}_\mathcal{A}) \to K_n(\mathcal{A}[G])$$

is the colimit of the assembly maps

$$H^{G_\alpha}_n(\underline{E}G_\alpha;\mathbb{K}_\mathcal{A}) \to K_n(\mathcal{A}[G_\alpha]),$$

for any additive G–category $\mathcal{A}$. The same statement holds in L–theory for any additive G–category with involution. Note that the statement in [3] is formulated for rings with G–action instead of additive G–categories, but the statement for G–categories holds in the same way. Furthermore, a finite-dimensional model for $\underline{E}G$ gives a finite-dimensional model for $\underline{E}G_\alpha$ by restricting the action to G_α. So taking the colimit over all finitely generated subgroups proves that injectivity holds without the assumption that G is finitely generated. For the construction of a splitting we need to see that the splittings for the finitely generated subgroups are natural with respect to the structure maps of the colimit. In the proof of Theorem 1.1 and Theorem 6.1 the assumption that G is finitely generated is only needed to apply [21, Corollary 3] and its L–theoretic analog, respectively. So it suffices to see that the splittings constructed in [21] are natural with respect to the structure maps of the colimit.

We will use the definitions of controlled categories and bounded mapping spaces from [21, Sections 5 and 7]. In the following let X denote a finite dimensional simplicial model for $\underline{E}G$. By Bartels, Farrell, Jones and Reich [5, Section 6] the assembly map

$$H^G_n(\underline{E}G;\mathbb{K}_\mathcal{A}) \to K_n(\mathcal{A}[G])$$

can be identified with the map

$$\operatorname*{colim}_{K\subseteq X \text{ fin.}} \pi_{n+1}(\mathbb{K}\mathcal{A}_G(GK)^\infty)^G \to \operatorname*{colim}_{K\subseteq X \text{ fin.}} \pi_n(\mathbb{K}\mathcal{A}_G(GK)_0)^G.$$

Now consider the diagram

$$\begin{tikzcd}
\operatorname*{colim}_{K\subseteq X \text{ fin.}} \pi_{n+1}(\mathbb{K}\mathcal{A}_G(GK)^\infty)^G \arrow[r] \arrow[d, "f"] & \operatorname*{colim}_{K\subseteq X \text{ fin.}} \pi_n(\mathbb{K}\mathcal{A}_G(GK)_0)^G \arrow[d, "j"] \\
\operatorname*{colim}_{K\subseteq X \text{ fin.}} \pi_{n+1} \operatorname{Map}_G^{bd}(X, \mathbb{K}\mathcal{A}_G(GK)^\infty) \arrow[r, "h"'] & \operatorname*{colim}_{K\subseteq X \text{ fin.}} \pi_n \operatorname{Map}_G^{bd}(X, \mathbb{K}\mathcal{A}_G(GK)_0).
\end{tikzcd}$$

By [21, Remark 7.7] the map f is an isomorphism and the map h is an isomorphism in the situation of [21, Corollary 3].

Let $\Gamma \to \Lambda$ be an injective group homomorphism. For every Λ–set J and every subcomplex $K \subseteq X$ we can define a map

$$\left(\prod_J^{bd} \mathcal{A}_\Gamma(\Gamma K)^\infty\right)^\Gamma \to \left(\prod_J^{bd} \mathcal{A}_\Lambda(\Lambda K)^\infty\right)^\Lambda$$

as follows. A controlled module $(M_j) \in (\prod_J^{bd} \mathcal{A}_\Gamma(\Gamma K))^\Gamma$ is sent to $(M'_j)_j$ with $(M'_j)_{h',x,t} := \bigoplus_{[h]\in\Lambda/\Gamma}(M_{h^{-1}j})_{h^{-1}h',h^{-1}x,t}$ and analogously on morphisms. This map is well defined since (M_j) is Γ–invariant. The above maps induce a map

$$\operatorname{Map}_\Gamma^{bd}(X, \mathbb{K}\mathcal{A}_\Gamma(\Gamma K)) \to \operatorname{Map}_\Lambda^{bd}(X, \mathbb{K}\mathcal{A}_\Lambda(\Lambda K))$$

for every finite subcomplex $K \subseteq X$, and in the special case where $J = \{\mathrm{pt}\}$ we obtain a map

$$(\mathbb{K}\mathcal{A}_\Gamma(\Gamma K)^\infty)^\Gamma \to (\mathbb{K}\mathcal{A}_\Lambda(\Lambda K)^\infty)^\Lambda.$$

The same maps can be constructed with $\mathcal{A}_\Gamma(\Gamma K)^\infty$ and $\mathcal{A}_\Lambda(\Lambda K)^\infty$ replaced by $\mathcal{A}_\Gamma(\Gamma K)_0$ and $\mathcal{A}_\Lambda(\Lambda K)_0$, respectively. So they induce maps from the above diagram for Γ to the same diagram for Λ. We will omit the technical proofs that the maps of the diagram are natural with respect to these maps and that under the identification with the assembly map they correspond to the structure maps of the colimit from [5]. This shows that the splitting $f^{-1} \circ h^{-1} \circ j$ is natural with respect to the structure maps of the colimit.

Now let us consider the L–theoretic version. For [21, Remark 7.7] it was used that the category

$$\left(\prod_{j\in J} \mathbb{K}\mathcal{A}_G(GK)^\infty\right)^G \simeq \prod_{[j]\in G\backslash J} \mathbb{K}\mathcal{A}_G^{G_j}(GK)^\infty$$

is weakly equivalent to

$$\left(\mathbb{K}\prod_{j\in J}\mathcal{A}_G(GK)^\infty\right)^G \simeq \mathbb{K}\prod_{[j]\in G\backslash J}\mathcal{A}_G^{G_j}(GK)^\infty$$

for every G–set J with finite stabilizers and every finite subcomplex $K\subseteq X$, where G_j is the stabilizer of $j\in J$. Let $H\leq G$ be finite; then

$$K_n(\mathcal{A}_G^{G_j}(G/H)^\infty)\cong\prod_{G_j\backslash G/H}K_n(\mathcal{A}_G^{G_j}(G_j/(G_j\cap H))^\infty)\cong\prod_{G_j\backslash G/H}K_{n-1}(\mathcal{A}[G_j\cap H]).$$

If for each finite subgroup $H\leq G$ there exists $N\in\mathbb{N}$ such that for each $n\geq N$ the groups $K_{-n}\mathcal{A}[H]$ vanish, then by induction on the cells this implies that for every finite subcomplex $K\subseteq X$ there exists $N\in\mathbb{N}$ such that for $n\geq N$ the groups $K_{-n}(\mathcal{A}_G^{G_j}(GK)^\infty)$ vanish. Therefore, under this assumption, L–theory commutes with the above product, and we get that the map

$$\phi\colon \mathrm{Map}_G^{bd}(X,\mathbb{L}\mathcal{A}_G(GK)^\infty)\to\mathrm{Map}_G(X,\mathbb{L}\mathcal{A}_G(GK)^\infty)$$

is an isomorphism. Also, under the above assumption,

$$\psi\colon (\mathbb{L}\mathcal{A}_G(GK)^\infty)^G\to\mathrm{Map}_G(X,\mathbb{L}\mathcal{A}_G(GK)^\infty)$$

is an isomorphism; see [21, Section 9]. Since ψ factors over ϕ, the map

$$(\mathbb{L}\mathcal{A}_G(GK)^\infty)^G\to\mathrm{Map}_G^{bd}(X,\mathbb{L}\mathcal{A}_G(GK)^\infty)$$

is an isomorphism as well. Therefore, we obtain the naturality of the splitting as in the case for K–theory.

References

[1] **R C Alperin**, **P B Shalen**, *Linear groups of finite cohomological dimension*, Bull. Amer. Math. Soc. 4 (1981) 339–341 MR609046

[2] **A C Bartels**, *On the domain of the assembly map in algebraic K–theory*, Algebr. Geom. Topol. 3 (2003) 1037–1050 MR2012963

[3] **A Bartels**, **S Echterhoff**, **W Lück**, *Inheritance of isomorphism conjectures under colimits*, from: "K–theory and noncommutative geometry", (G Cortiñas, J Cuntz, M Karoubi, R Nest, C A Weibel, editors), Eur. Math. Soc., Zürich (2008) 41–70 MR2513332

[4] **A Bartels**, **F T Farrell**, **W Lück**, *The Farrell–Jones conjecture for cocompact lattices in virtually connected Lie groups*, J. Amer. Math. Soc. 27 (2014) 339–388 MR3164984

[5] **A Bartels**, **T Farrell**, **L Jones**, **H Reich**, *On the isomorphism conjecture in algebraic K–theory*, Topology 43 (2004) 157–213 MR2030590

[6] **A Bartels**, **W Lück**, *Isomorphism conjecture for homotopy K–theory and groups acting on trees*, J. Pure Appl. Algebra 205 (2006) 660–696 MR2210223

[7] **A Bartels**, **W Lück**, *The Borel conjecture for hyperbolic and* CAT(0)*–groups*, Ann. of Math. 175 (2012) 631–689 MR2993750

[8] **A Bartels**, **W Lück**, *Geodesic flow for* CAT(0)*–groups*, Geom. Topol. 16 (2012) 1345–1391 MR2967054

[9] **A Bartels**, **W Lück**, **H Reich**, *Equivariant covers for hyperbolic groups*, Geom. Topol. 12 (2008) 1799–1882 MR2421141

[10] **A Bartels**, **W Lück**, **H Reich**, *The K–theoretic Farrell–Jones conjecture for hyperbolic groups*, Invent. Math. 172 (2008) 29–70 MR2385666

[11] **A Bartels**, **H Reich**, *Coefficients for the Farrell–Jones conjecture*, Adv. Math. 209 (2007) 337–362 MR2294225

[12] **G Carlsson**, **B Goldfarb**, *On homological coherence of discrete groups*, J. Algebra 276 (2004) 502–514 MR2058455

[13] **Y de Cornulier**, *Answer to "If N and G/N are virtually solvable, then G is virtually solvable?"* (2013) Available at `http://math.stackexchange.com/q/437063`

[14] **J F Davis**, **W Lück**, *Spaces over a category and assembly maps in isomorphism conjectures in K– and L–theory*, K–Theory 15 (1998) 201–252 MR1659969

[15] **F T Farrell**, **L E Jones**, *Isomorphism conjectures in algebraic K–theory*, J. Amer. Math. Soc. 6 (1993) 249–297 MR1179537

[16] **F T Farrell**, **L E Jones**, *The lower algebraic K–theory of virtually infinite cyclic groups*, K–Theory 9 (1995) 13–30 MR1340838

[17] **L Ji**, *Asymptotic dimension and the integral K–theoretic Novikov conjecture for arithmetic groups*, J. Differential Geom. 68 (2004) 535–544 MR2144540

[18] **L Ji**, *Integral Novikov conjectures and arithmetic groups containing torsion elements*, Comm. Anal. Geom. 15 (2007) 509–533 MR2379803

[19] **H Kammeyer**, **W Lück**, **H Rüping**, *The Farrell–Jones conjecture for arbitrary lattices in virtually connected Lie groups*, preprint (2014) `arXiv:1401.0876v1` To appear in Geom. Topol.

[20] **G G Kasparov**, *Equivariant KK–theory and the Novikov conjecture*, Invent. Math. 91 (1988) 147–201 MR918241

[21] **D Kasprowski**, *On the K–theory of groups with finite decomposition complexity*, Proc. Lond. Math. Soc. 110 (2015) 565–592 MR3342098

[22] **W Lück**, *The type of the classifying space for a family of subgroups*, J. Pure Appl. Algebra 149 (2000) 177–203 MR1757730

[23] **W Lück**, *Survey on classifying spaces for families of subgroups*, from: "Infinite groups: Geometric, combinatorial and dynamical aspects", (L Bartholdi, T Ceccherini-Silberstein, T Smirnova-Nagnibeda, A Zuk, editors), Progr. Math. 248, Birkhäuser, Basel (2005) 269–322 MR2195456

[24] **W Lück**, **H Reich**, *The Baum–Connes and the Farrell–Jones conjectures in K– and L–theory*, from: "Handbook of K–theory, Vol. 1, 2", (E M Friedlander, D R Grayson, editors), Springer, Berlin (2005) 703–842 MR2181833

[25] **W Möhres**, *Torsion-free nilpotent groups with bounded ranks of the abelian subgroups*, from: "Group theory", (O H Kegel, F Menegazzo, G Zacher, editors), Lecture Notes in Math. 1281, Springer, Berlin (1987) 115–117 MR921696

[26] **E K Pedersen**, **C A Weibel**, *A nonconnective delooping of algebraic K–theory*, from: "Algebraic and geometric topology", (A Ranicki, N Levitt, F Quinn, editors), Lecture Notes in Math. 1126, Springer, Berlin (1985) 166–181 MR802790

[27] **G Skandalis**, **J L Tu**, **G Yu**, *The coarse Baum–Connes conjecture and groupoids*, Topology 41 (2002) 807–834 MR1905840

[28] **R B Warfield, Jr**, *Nilpotent groups*, Lecture Notes in Mathematics 513, Springer, Berlin (1976) MR0409661

[29] **C Wegner**, *The K–theoretic Farrell–Jones conjecture for CAT(0)–groups*, Proc. Amer. Math. Soc. 140 (2012) 779–793

[30] **C Wegner**, *The Farrell–Jones conjecture for virtually solvable groups*, preprint (2013) `arXiv:1308.2432v2` To appear in J. Topol. Available at `http://dx.doi.org/10.1112/jtopol/jtv026`

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