The L^2–Alexander torsion is symmetric

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We show that the L^2–Alexander torsion of a 3–manifold is a symmetric function. This can be viewed as a generalization of the symmetry of the Alexander polynomial of a knot.

1 Introduction

An *admissible triple* (N, ϕ, γ) consists of an irreducible, orientable, compact 3–manifold $N \neq S^1 \times D^2$ with empty or toroidal boundary, a class $\phi \neq 0 \in H^1(N; \mathbb{Z}) = \text{Hom}(\pi_1(N), \mathbb{Z})$ and a homomorphism $\gamma \colon \pi_1(N) \to G$ such that ϕ factors through γ. In [4; 5] we used the L^2–torsion (see for example Lück [14]) to associate to an admissible triple (N, ϕ, γ) the L^2–Alexander torsion $\tau^{(2)}(N, \phi, \gamma)$ which is a function

$$\tau^{(2)}(N, \phi, \gamma) \colon \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$$

that is well defined up to multiplication by a function of the type $t \mapsto t^m$ for some $m \in \mathbb{Z}$. We recall the definition in Section 6.

The goal of this paper is to show that the L^2–Alexander torsion is symmetric. In order to state the precise symmetry result we need to recall that given a 3–manifold N the *Thurston norm* [16] of some $\phi \in H^1(N; \mathbb{Z}) = \text{Hom}(\pi_1(N), \mathbb{Z})$ is defined as

$$x_N(\phi) := \min\{\chi_-(S) \mid S \subset N \text{ properly embedded surface dual to } \phi\}.$$

Here, given a surface S we define its complexity as $\chi_-(S) := -\chi(S')$, where S' is the result of deleting all components from S that are disks or spheres. Thurston [16] showed that x_N is a (possibly degenerate) norm on $H^1(N; \mathbb{Z})$. Now we can formulate the main result of this paper.

Theorem 1.1 *Let (N, ϕ, γ) be an admissible triple. Then for any representative τ of $\tau^{(2)}(N, \phi, \gamma)$ there exists an $n \in \mathbb{Z}$ with $n \equiv x_N(\phi)$ mod 2 such that*

$$\tau(t^{-1}) = t^n \cdot \tau(t) \text{ for any } t \in \mathbb{R}_{>0}.$$

It is worth looking at the case that $N = S^3 \setminus \nu K$ is the complement of a tubular neighborhood νK of an oriented knot $K \subset S^3$. We denote by $\phi_K \colon \pi_1(N) \to \mathbb{Z}$ the epimorphism sending the oriented meridian to 1. Let $\gamma \colon \pi_1(N) \to G$ be a homomorphism such that ϕ_K factors through γ. We define

$$\tau^{(2)}(K, \gamma) := \tau^{(2)}(S^3 \setminus \nu K, \phi_K, \gamma).$$

If we take $\gamma = \text{id}$ to be the identity, then we showed in [4] that

$$\tau^{(2)}(K, \text{id}) = \Delta_K^{(2)}(t) \cdot \max\{1, t\},$$

where $\Delta_K^{(2)}(t) \colon \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ denotes the L^2–Alexander invariant of Li and Zhang [12; 13], which was also studied by Dubois and Wegner [6; 7] and Aribi [1; 2].

If we take $\gamma = \phi_K$, then we showed in [4] that the L^2–Alexander torsion $\tau^{(2)}(K, \phi_K)$ is fully determined by the Alexander polynomial $\Delta_K(t)$ of K and that in turn $\tau^{(2)}(K, \phi_K)$ almost determines the Alexander polynomial $\Delta_K(t)$. In this sense the L^2–Alexander torsion can be viewed as a "twisted" version of the Alexander polynomial, and at least morally it is related to the twisted Alexander polynomial of Wada [20] and to the higher-order Alexander polynomials of Cochran [3] and Harvey [10]. We refer to [5] for more on the relationship and similarities between the various twisted invariants.

If K is a knot, then any Seifert surface is dual to ϕ_K and it immediately follows that $x(\phi_K) \leq \max\{2 \cdot \text{genus(K)} - 1, 0\}$. In fact an elementary argument shows that for any *non-trivial* knot we have the equality $x(\phi_K) = 2 \cdot \text{genus(K)} - 1$. In particular the Thurston norm of ϕ_K is odd. We thus obtain the following corollary to Theorem 1.1.

Theorem 1.2 *Let $K \subset S^3$ be an oriented non-trivial knot and let $\gamma \colon \pi_1(N) \to G$ be a homomorphism such that ϕ_K factors through γ. Then there exists an odd n with*

$$\tau^{(2)}(K, \gamma)(t^{-1}) = t^n \cdot \tau^{(2)}(K, \gamma)(t) \quad \text{for any } t \in \mathbb{R}_{>0}.$$

The proof of Theorem 1.1 has many similarities with the proof of the main theorem in Friedl, Kim and Kitayama [9], which in turn builds on the ideas of Turaev [17; 18; 19]. In an attempt to keep the proof as short as possible we will on several occasions refer to [9] and [17] for definitions and results.

Conventions All manifolds are assumed to be connected, orientable and compact. All CW–complexes are assumed to be finite and connected. If G is a group then we equip $\mathbb{C}[G]$ with the involution given by complex conjugation and by $\bar{g} := g^{-1}$ for $g \in G$. We extend this involution to matrices over $\mathbb{C}[G]$ by applying the involution to each entry. Given a ring R we will view all modules as left R–modules, unless we say explicitly otherwise. Furthermore, given a matrix $A \in M_{m,n}(R)$ we denote by $A: R^m \to R^n$ the R–homomorphism of left R–modules obtained by right multiplication with A and thinking of elements in R^m as the only row in a $(1, m)$–matrix.

2 Euler structures

In this section we recall the notion of an Euler structure of a pair of CW–complexes and manifolds which is due to Turaev. We refer to [17; 18; 9] for full details. Throughout this paper, given a space X, we denote by $\mathcal{H}_1(X)$ the first integral homology group viewed as a multiplicative group.

2A Euler structures on CW–complexes

Let X be a CW–complex of dimension m and let Y be a proper subcomplex. We denote by $p: \widetilde{X} \to X$ the universal covering of X and we write $\widetilde{Y} := p^{-1}(Y)$. An *Euler lift* is a set of cells in $\widetilde{X}$ such that each cell of $X \setminus Y$ is covered by precisely one of the cells in the Euler lift.

Using the canonical left action of $\pi = \pi_1(X)$ on $\widetilde{X}$ we obtain a free and transitive action of π on the set of cells of $\widetilde{X} \setminus \widetilde{Y}$ lying over a fixed cell in $X \setminus Y$. If c and c' are two Euler lifts, then we can order the cells such that $c = \{c_{ij}\}$ and $c' = \{c'_{ij}\}$ and such that for each i and j the cells c_{ij} and c'_{ij} lie over the same i–cell in $X \setminus Y$. In particular there exist unique $g_{ij} \in \pi$ such that $c'_{ij} = g_{ij} \cdot c_{ij}$. We denote the projection map $\pi \to \mathcal{H}_1(X)$ by Ψ. We define

$$c'/c := \prod_{i=0}^{m} \prod_{j} \Psi(g_{ij})^{(-1)^i} \in \mathcal{H}_1(X).$$

We say that c and c' are *equivalent* if $c'/c \in \mathcal{H}_1(X)$ is trivial. An equivalence class of Euler lifts will be referred to as an *Euler structure*. We denote by $\mathrm{Eul}(X, Y)$ the set of Euler structures. If $Y = \varnothing$ then we will also write $\mathrm{Eul}(X) = \mathrm{Eul}(X, Y)$.

Given $g \in \mathcal{H}_1(X)$ and $e \in \mathrm{Eul}(X, Y)$ we define $g \cdot e \in \mathrm{Eul}(X, Y)$ as follows: pick representatives c for e and $\widetilde{g} \in \pi_1(X)$ for g, then act on one i–cell of c by $g^{(-1)^i}$. The resulting Euler lift represents an element in $\mathrm{Eul}(X, Y)$ which is independent of the choice of the cell. We denote by $g \cdot e$ the Euler structure represented by this new Euler lift. This defines a free and transitive $\mathcal{H}_1(X)$–action on $\mathrm{Eul}(X, Y)$, with $(g \cdot e)/e = g$.

If (X', Y') is a cellular subdivision of (X, Y), then there exists a canonical $\mathcal{H}_1(X)$–equivariant bijection $\sigma \colon \mathrm{Eul}(X, Y) \to \mathrm{Eul}(X', Y')$ which is defined as follows. Let $e \in \mathrm{Eul}(X, Y)$ and pick an Euler lift for (X, Y) which represents e. There exists a unique Euler lift for (X', Y') such that the cells in the Euler lift of (X', Y') are contained in the cells of the Euler lift of (X, Y). We denote by $\sigma(e)$ the Euler structure represented by this Euler lift. This map equals the map of Turaev [17, Section 1.2].

2B Euler structures of smooth manifolds

Let N be a manifold and let $\partial_0 N \subset \partial N$ be a union of components of ∂N such that $\chi(N, \partial_0 N) = 0$. A *triangulation* of N is a pair (X, t) where X is a simplicial complex and $t \colon |X| \to N$ is a homeomorphism. Throughout this section we write $Y := t^{-1}(\partial_0 N)$. For the most part we will suppress t from the notation. Following [17, Section I.4.1] we consider the projective system of sets $\{\mathrm{Eul}(X, Y)\}_{(X,t)}$, where (X, t) runs over all C^1–triangulations of N and where the maps are the $\mathcal{H}_1(N)$–equivariant bijections between these sets induced either by C^1–subdivisions or by smooth isotopies in N. We define $\mathrm{Eul}(N, \partial_0 N)$ by identifying the sets $\{\mathrm{Eul}(X, Y)\}_{(X,t)}$ via these bijections. We refer to $\mathrm{Eul}(N, \partial_0 N)$ as the set of *Euler structures on* $(N, \partial_0 N)$. For a C^1–triangulation X of N we get a canonical $\mathcal{H}_1(N)$–equivariant bijection $\mathrm{Eul}(X, Y) \to \mathrm{Eul}(N, \partial_0 N)$.

3 The L^2–torsion of a manifold

3A The L^2–torsion of a chain complex

First we recall some key properties of the Fuglede–Kadison determinant and the definition of the L^2–torsion of a chain complex of free based left $\mathbb{C}[G]$–modules. Throughout the section we refer to [14] and to [4] for details and proofs.

We fix a group G. Let A be a $k \times l$–matrix over $\mathbb{C}[G]$. There exists the notion of A being of *determinant class*. (To be more precise, we view the $k \times l$–matrix A as a map

$\mathcal{N}(G)^l \to \mathcal{N}(G)^k$, where $\mathcal{N}(G)$ is the von Neumann algebra of G, and then there is the notion of being of *determinant class*.) We treat this entirely as a black box, but we note that if G is residually amenable, eg a 3–manifold group [11] or solvable, then by [8] any matrix over $\mathbb{Q}[G]$ is of determinant class. If the matrix A is not of determinant class then for the purpose of this paper we define $\det_{\mathcal{N}(G)}(A) = 0$. On the other hand, if A is of determinant class, then we define

$$\det_{\mathcal{N}(G)}(A) := \text{Fuglede–Kadison determinant of } A \in \mathbb{R}_{>0}.$$

Here we do not assume that A is a square matrix. In an attempt to keep the paper as short as possible we will not provide the (somewhat lengthy) definition of the Fuglede–Kadison determinant. Instead we summarize a few key properties in the following theorem which is a consequence of [14, Example 3.12] and [14, Theorem 3.14].

Theorem 3.1   (1)   *If A is a square matrix with complex entries such that the usual determinant $\det(A) \in \mathbb{C}$ is non-zero, then $\det_{\mathcal{N}(G)}(A) = |\det(A)|$.*

  (2)   *The determinant does not change if we swap two rows or two columns.*

  (3)   *Right multiplication of a column by $\pm g$, $g \in G$ does not change the determinant.*

  (4)   *For any matrix A over $\mathbb{C}[G]$ we have $\det_{\mathcal{N}(G)}(A) = \det_{\mathcal{N}(G)}(\overline{A}^t)$.*

Note that (2) implies that when we study Fuglede–Kadison determinants of homomorphisms we can work with unordered bases. Now let

$$C_* = (0 \to C_l \xrightarrow{\partial_l} C_{l-1} \xrightarrow{\partial_{l-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0)$$

be a chain complex of free left $\mathbb{C}[G]$–modules. We refer to [14] for the definition of the L^2–Betti numbers $b_i^{(2)}(C_*) \in \mathbb{R}_{\geq 0}$. Now suppose that the chain complex is equipped with bases $B_i \subset C_i$, $i = 0, \ldots, l$. If one of the L^2–Betti numbers $b_i^{(2)}(C_*)$ is non-zero or if one the boundary maps is not of determinant class, then we define the L^2–torsion $\tau^{(2)}(C_*, B_*) := 0$. Otherwise we define the L^2–*torsion* to be

$$\tau^{(2)}(C_*, B_*) := \prod_{i=1}^{l} \det_{\mathcal{N}(G)}(A_i)^{(-1)^i} \in \mathbb{R}_{>0},$$

where the A_i denote the boundary matrices corresponding to the given bases. This definition is the multiplicative inverse of the exponential of the L^2–torsion as defined in [14, Definition 3.29].

3B   The twisted L^2–torsion of CW–complexes and manifolds

Let (X, Y) be a pair of CW–complexes and let $e \in \mathrm{Eul}(X, Y)$. We denote by $p\colon \widetilde{X} \to X$ the universal covering of X and we write $\widetilde{Y} := p^{-1}(Y)$. The deck transformation turns $C_*(\widetilde{X}, \widetilde{Y})$ naturally into a chain complex of left $\mathbb{Z}[\pi_1(X)]$–modules.

Now let G be a group and let $\varphi\colon \pi(X) \to \mathrm{GL}(d, \mathbb{C}[G])$ be a representation. We view elements of $\mathbb{C}[G]^d$ as row vectors. Right multiplication via $\varphi(g)$ thus turns $\mathbb{C}[G]^d$ into a right $\mathbb{Z}[\pi_1(X)]$–module. We consider the chain complex

$$C_*^{\varphi}(X, Y; \mathbb{C}[G]^d) := \mathbb{C}[G]^d \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X}, \widetilde{Y})$$

of left $\mathbb{C}[G]$–modules. Let $e \in \mathrm{Eul}(X, Y)$. We pick an Euler lift $\{c_{ij}\}$ that represents e. Throughout this paper we denote by $v_1, \ldots, v_d$ the standard basis for $\mathbb{C}[G]^d$. We equip the chain complex $C_*^{\varphi}(X, Y; \mathbb{C}[G]^d)$ with the basis provided by the $v_k \otimes c_{ij}$. Therefore we can define

$$\tau^{(2)}(X, Y, \varphi, e) := \tau^{(2)}\big(C_*^{\varphi}(X, Y; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}\}\big) \in \mathbb{R}_{\geq 0}.$$

Lemma 3.2   (1)   *The number $\tau^{(2)}(X, Y, \varphi, e)$ is well defined.*

(2)   *If $g \in \mathcal{H}_1(X)$, then*

$$\tau^{(2)}(X, Y, \varphi, ge) = \det_{\mathcal{N}(G)}(\varphi(g^{-1})) \cdot \tau^{(2)}(X, Y, \varphi, e).$$

(3)   *If (X', Y') is a cellular subdivision of (X, Y) and if $e' \in \mathrm{Eul}(X', Y')$ is the Euler structure corresponding to e, then*

$$\tau^{(2)}(X', Y', \varphi, e') = \tau^{(2)}(X, Y, \varphi, e).$$

The proofs are completely analogous to the proofs for ordinary Reidemeister torsion as given in [18; 9]. In the interest of space we will not provide the proofs.

Finally let N be a manifold and let $\partial_0 N \subset \partial N$ be a union of components of ∂N with $\chi(N, \partial_0 N) = 0$. Let G be a group and let $\varphi\colon \pi(N) \to \mathrm{GL}(d, \mathbb{C}[G])$ be a representation. Let $e \in \mathrm{Eul}(N, \partial_0 N)$. Recall that for any C^1–triangulation $f\colon X \to N$ we get a bijection $\mathrm{Eul}(X, Y) \xrightarrow{f_*} \mathrm{Eul}(N, \partial_0 N)$. We define

$$\tau^{(2)}(N, \partial_0 N, \varphi, e) := \tau^{(2)}(X, Y, \varphi \circ f_*, f_*^{-1}(e)).$$

By Lemma 3.2(3) and the discussion in [17] the invariant $\tau^{(2)}(N, \partial_0 N, \varphi, e) \in \mathbb{R}_{\geq 0}$ is well defined, ie independent of the choice of the triangulation.

4 Duality for torsion of manifolds equipped with Euler structures

4A The algebraic duality theorem for L^2–torsion

Let G be a group and let V be a right $\mathbb{C}[G]$–module. We denote by $\overline{V}$ the left $\mathbb{C}[G]$–module with the same underlying abelian group but with the module structure given by $p \cdot_{\overline{V}} v := v \cdot_V \overline{p}$ for $p \in \mathbb{C}[G]$ and $v \in V$. If V is a left $\mathbb{C}[G]$–module then we can consider $\mathrm{Hom}_{\mathbb{C}[G]}(V, \mathbb{C}[G])$, the set of all left $\mathbb{C}[G]$–module homomorphisms. Since the range $\mathbb{C}[G]$ is a $\mathbb{C}[G]$–bimodule we can naturally view $\mathrm{Hom}_{\mathbb{C}[G]}(V, \mathbb{C}[G])$ as a right $\mathbb{C}[G]$–module.

In the following let C_* be a chain complex of length m of left $\mathbb{C}[G]$–modules with boundary operators ∂_i. Suppose that C_* is equipped with a basis B_i for each C_i. We denote by $C^{\#}$ the *dual chain complex* whose chain groups are the $\mathbb{C}[G]$–left modules $C_i^{\#} := \overline{\mathrm{Hom}_{\mathbb{C}[G]}(C_{m-i}, \mathbb{C}[G])}$ and where the boundary map $\partial_i^{\#} \colon C_{i+1}^{\#} \to C_i^{\#}$ is given by $(-1)^{m-i} \partial_{m-i-1}^*$. This means that for any $c \in C_{m-i}$ and $d \in C_{i+1}^{\#}$ we have $\partial_i^{\#}(d)(c) = (-1)^{m-i} d(\partial_{m-i}(c))$. We denote by $B_*^{\#}$ the bases of $C^{\#}$ dual to the bases B_*.

Lemma 4.1 *If $\tau^{(2)}(C_*, B_*) = 0$, then $\tau^{(2)}(C_*^{\#}, B_*^{\#}) = 0$, otherwise we have*

$$\tau^{(2)}(C_*, B_*) = \tau^{(2)}(C_*^{\#}, B_*^{\#})^{(-1)^{m+1}}.$$

Proof By the proof of [14, Theorem 1.35(3)] the L^2–Betti numbers of C_* vanish if and only if the L^2–Betti numbers of $C_*^{\#}$ vanish. In particular, if either L^2–Betti number does not vanish, then both torsions are zero.

Now we suppose that the L^2–Betti numbers of C_* vanish. We denote by A_i the corresponding matrices of the boundary maps of C_*. The boundary matrices of the chain complex $C_*^{\#}$ with respect to the basis $B_*^{\#}$ are given by $(-1)^{m-i} \overline{A_i^t}$. Now the lemma is an immediate consequence of the definitions and of Theorem 3.1(4). $\square$

4B The duality theorem for manifolds

Before we state our main technical duality theorem we need to introduce two more definitions.

(1) Let G be a group and let $\varphi \colon \pi \to \mathrm{GL}(d, \mathbb{C}[G])$ be a representation. We denote by $\varphi^{\dagger}$ the representation which is given by $g \mapsto \overline{\varphi(g^{-1})^t}$.

(2) Let N be an m–manifold and let $e \in \mathrm{Eul}(N, \partial N)$. Pick a triangulation X for N. We denote by Y the subcomplex corresponding to ∂N. Let $X^{\dagger}$ be the CW–complex that is given by the cellular decomposition of N dual to X. Pick

an Euler lift $\{c_{ij}\}$ that represents $e \in \mathrm{Eul}(X, Y) = \mathrm{Eul}(N, \partial N)$. For any i–cell c in $\widetilde{X} \setminus \widetilde{Y}$ we denote by $c^{\dagger}$ the unique oriented $(m - i)$–cell in $\widetilde{X}^{\dagger}$ which has intersection number $+1$ with c. The Euler lift $\{c_{ij}^{\dagger}\}$ defines an element in $\mathrm{Eul}(X^{\dagger}) = \mathrm{Eul}(N)$ that we denote by $e^{\dagger}$. This map is an $\mathcal{H}_1(N)$–equivariant bijection and we denote the inverse map $\mathrm{Eul}(N, \partial N) \to \mathrm{Eul}(N)$ again by $e \mapsto e^{\dagger}$. We refer to [15, Chapter 70], [18, Section 14] and [9, Section 4] for details.

Theorem 4.2 *Let N be an m–manifold. Let G be a group and let $\varphi \colon \pi(N) \to \mathrm{GL}(d, \mathbb{C}[G])$ be a representation. Let $e \in \mathrm{Eul}(N, \partial N)$. Then either $\tau^{(2)}(N, \partial N, \varphi, e)$ and $\tau^{(2)}(N, \varphi^{\dagger}, e^{\dagger})$ are both zero, or the following equality holds:*

$$\tau^{(2)}(N, \partial N, \varphi, e) = \tau^{(2)}(N, \varphi^{\dagger}, e^{\dagger})^{(-1)^{m+1}}.$$

Proof Pick a triangulation X for N and denote by Y the subcomplex corresponding to ∂N. Let $X^{\dagger}$ be the CW–complex which is given by the cellular decomposition of N dual to X. We identify $\pi = \pi_1(X) = \pi_1(N) = \pi_1(X^{\dagger})$. We pick an Euler lift $\{c_{ij}\}$ which represents $e \in \mathrm{Eul}(N, \partial N) = \mathrm{Eul}(X, Y)$. We denote by $c_{ij}^{\dagger}$ the corresponding dual cells. The theorem follows from the definitions and the following claim.

Claim *Either both $\tau^{(2)}(C_*^{\varphi}(X, Y; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}\})$ and $\tau^{(2)}(C_*^{\varphi^{\dagger}}(X^{\dagger}; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}^{\dagger}\})$ are zero, or the following equality holds:*

$$\tau^{(2)}\big(C_*^{\varphi}(X, Y; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}\}\big) = \tau^{(2)}\big(C_*^{\varphi^{\dagger}}(X^{\dagger}; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}^{\dagger}\}\big)^{(-1)^{m+1}}.$$

In order to prove the claim we first note that there is a unique, sesquilinear paring

$$C_{m-i}(\widetilde{X}, \widetilde{Y}) \times C_i(\widetilde{X}^{\dagger}) \to \mathbb{Z}[\pi],$$

$$(a, b) \mapsto \langle a, b \rangle := \sum_{g \in \pi} (a \cdot gb) g^{-1}$$

such that $a \cdot b^{\dagger} = \delta_{ab}$ for any two cells a and b of $\widetilde{X} \setminus \widetilde{Y}$. Here *sesquilinear* means that for any $a \in C_{m-i}(\widetilde{X}, \widetilde{Y})$, $b \in C_i(\widetilde{X}^{\dagger})$ and $p, q \in \mathbb{Z}[\pi]$ we have $\langle pa, qb \rangle = q \langle a, b \rangle \bar{p}$. It is straightforward to see that the pairing is non-singular. It follows immediately from [18, Claim 14.4]) that these maps give rise to well-defined maps

$$C_i(\widetilde{X}, \widetilde{Y}) \to \overline{\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_{m-i}(\widetilde{X}^{\dagger}), \mathbb{Z}[\pi])},$$

$$a \mapsto (b \mapsto \langle a, b \rangle)$$

that define an isomorphism of based chain complexes of right $\mathbb{Z}[\pi]$–modules. In fact it follows easily from the definitions that the maps define an isomorphism

$$(C_*(\widetilde{X}, \widetilde{Y}), \{c_{ij}\}) \to \big(\overline{\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_{m-*}(\widetilde{X}^{\dagger}), \mathbb{Z}[\pi])}, \{(c_{ij}^{\dagger})^*\}\big)$$

of based chain complexes of left $\mathbb{Z}[\pi]$–modules. Tensoring these chain complexes with $\mathbb{C}[G]^d$ we obtain an isomorphism

$$\left(\mathbb{C}[G]^d \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}, \widetilde{Y}), \{v_k \otimes c_{ij}\}\right)$$
$$\to \left(\mathbb{C}[G]^d \otimes_{\mathbb{Z}[\pi]} \overline{\operatorname{Hom}_{\mathbb{Z}[\pi]}(C_{m-*}(\widetilde{X}^\dagger), \mathbb{Z}[\pi])}, \{v_k \otimes (c_{ij}^\dagger)^*\}\right)$$

of based chain complexes of $\mathbb{C}[G]$–modules. Furthermore the maps

$$\mathbb{C}[G]^d \otimes_{\mathbb{Z}[\pi]} \overline{\operatorname{Hom}_{\mathbb{Z}[\pi]}(C_i(\widetilde{X}^\dagger), \mathbb{Z}[\pi])} \to \overline{\operatorname{Hom}_{\mathbb{C}[G]}\left(C_i^{\varphi^\dagger}(X^\dagger; \mathbb{C}[G]^d), \mathbb{C}[G]\right)},$$

$$v \otimes f \mapsto \left(\begin{array}{c} C_i^{\varphi^\dagger}(X^\dagger; \mathbb{C}[G]^d) \to \mathbb{C}[G], \\ w \otimes \sigma \mapsto v\varphi(\overline{f(\sigma)})\overline{w}^t \end{array}\right)$$

induce an isomorphism

$$\left(C_*^\varphi(X, Y; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}\}\right) \to \left(C_*^{\varphi^\dagger}(X^\dagger; \mathbb{C}[G]^d)^\#, \{(v_k \otimes c_{ij}^\dagger)^\#\}\right)$$

of based chain complexes of $\mathbb{C}[G]$–modules. The claim follows from Lemma 4.1. $\square$

5   Twisted L^2–torsion of 3–manifolds

5A   Canonical structures on tori

Let T be a torus. We equip T with a CW–structure with one 0–cell p, two 1–cells x and y and one 2–cell s. We write $\pi = \pi_1(T, p)$ and by a slight abuse of notation we denote by x and y the elements in π represented by x and y. We denote by $\widetilde{T}$ the universal cover of T. There exist lifts of the cells such that the chain complex $C_*(\widetilde{T})$ of left $\mathbb{Z}[\pi]$–modules with respect to the bases given by these lifts is of the form

$$(1) \qquad 0 \to \mathbb{Z}[\pi] \xrightarrow{(y-1 \;\; 1-x)} \mathbb{Z}[\pi]^2 \xrightarrow{\begin{pmatrix} 1-x \\ 1-y \end{pmatrix}} \mathbb{Z}[\pi] \to 0.$$

We refer to the corresponding Euler structure of T as the *canonical Euler structure on T*. Given a group G we say that a representation $\varphi\colon \pi \to \operatorname{GL}(1, \mathbb{C}[G])$ is *monomial* if for any $x \in \pi$ we have $\varphi(x) = zg$ for some $z \in \mathbb{C}$ and $g \in G$. The following is [4, Lemma 5.6].

Lemma 5.1 Let $\varphi\colon \pi_1(T) \to \operatorname{GL}(1, \mathbb{C}[G])$ be a monomial representation such that $b_*^{(2)}(T; \mathbb{C}[G]) = 0$ and e be the canonical Euler structure on T. Then $\tau^{(2)}(T, \varphi, e) = 1$.

5B Chern classes on 3–manifolds with toroidal boundary

Let N be a 3–manifold with toroidal incompressible boundary and let $e \in \mathrm{Eul}(N, \partial N)$. Let X be a triangulation for N. We denote the subcomplexes corresponding to the boundary components of N by $S_1 \cup \cdots \cup S_b$. We denote by $p\colon \widetilde{X} \to X$ and $p_i\colon \widetilde{S}_i \to S_i, i = 1, \ldots, b$ the universal covering maps of X and $S_i, i = 1, \ldots, b$. For each i we identify a component of $p^{-1}(S_i)$ with $\widetilde{S}_i$

Pick an Euler lift c that represents e. For each boundary torus S_i pick an Euler lift $\widetilde{s}_i$ to $\widetilde{S}_i \subset p^{-1}(S_i) \subset \widetilde{X}$ that represents the canonical Euler structure. The set of cells $\{\widetilde{s}_1, \ldots, \widetilde{s}_b, c\}$ defines an Euler structure $K(e)$ for N, which only depends on e. Put differently, we defined a map $K\colon \mathrm{Eul}(N, \partial N) \to \mathrm{Eul}(N)$ which is easily seen to be $\mathcal{H}_1(N)$–equivariant. Given $e \in \mathrm{Eul}(N)$ there exists a unique element $g \in \mathcal{H}_1(N)$ such that $e = g \cdot K(e^\dagger)$. Following Turaev [19, page 11] and [9, Section 6.3] we define $c_1(e) := g \in H_1(N; \mathbb{Z})$ and we refer to $c_1(e)$ as the *Chern class of e*.

5C Torsions of 3–manifolds

Let π and G be groups and let $\varphi\colon \pi \to \mathrm{GL}(1, \mathbb{C}[G])$ be a monomial representation. By the multiplicativity of the Fuglede–Kadison determinant, see [14, Theorem 3.14], given $g \in \pi$ the invariant $\det_{\mathcal{N}(G)}(\varphi(g))$ only depends on the homology class of g. Put differently, $\det_{\mathcal{N}(G)} \circ \varphi\colon \pi \to \mathbb{R}_{\geq 0}$ descends to a map $\det_{\mathcal{N}(G)} \circ \varphi\colon H_1(\pi; \mathbb{Z}) \to \mathbb{R}_{\geq 0}$.

Theorem 5.2 *Let N be a 3–manifold which is either closed or which has toroidal, incompressible boundary. Let G be a group and let $\varphi\colon \pi(N) \to \mathrm{GL}(1, \mathbb{C}[G])$ be a monomial representation such that $b_*^{(2)}(\partial N; \mathbb{C}[G]) = 0$. For any $e \in \mathrm{Eul}(N)$ we have*

$$\tau^{(2)}(N, \partial N, \varphi, e^\dagger) = \det_{\mathcal{N}(G)}(\varphi(c_1(e))) \cdot \tau^{(2)}(N, \varphi, e).$$

Proof The assumption that $b_*^{(2)}(\partial N; \mathbb{C}[G]) = 0$ together with the proof of [14, Theorem 1.35(2)] implies that $b_*^{(2)}(N; \mathbb{C}[G]) = 0$ if and only if $b_*^{(2)}(N, \partial N; \mathbb{C}[G]) = 0$. If both are non-zero, then both torsions $\tau^{(2)}(N, \partial N, \varphi, e^\dagger)$ and $\tau^{(2)}(N, \varphi, e)$ are zero. For the remainder of this proof we assume that $b_*^{(2)}(N; \mathbb{C}[G]) = 0$.

Pick a triangulation X for N. As usual denote by Y the subcomplex corresponding to ∂N. Let $e \in \mathrm{Eul}(N)$. Pick an Euler lift c_* which represents $e^\dagger \in \mathrm{Eul}(N, \partial N) = \mathrm{Eul}(X, Y)$. Denote the components of Y by $Y_1 \cup \cdots \cup Y_b$ and pick $\widetilde{s}_*^1, \ldots, \widetilde{s}_*^b$ as in the previous section. We write $\widetilde{s}_* = \widetilde{s}_*^1 \cup \cdots \cup \widetilde{s}_*^b$. Denote by $\{\widetilde{s}_* \cup c_*\}$ the resulting Euler lift for X. Recall that this Euler lift represents $K(e^\dagger)$.

Claim $\qquad \tau^{(2)}\big(C_*^\varphi(X, Y; \mathbb{C}[G]), \{c_*\}\big) = \tau^{(2)}\big(C_*^\varphi(X; \mathbb{C}[G]), \{\widetilde{s}_* \cup c_*\}\big).$

We consider the following short exact sequence of chain complexes

$$0 \to \bigoplus_{i=1}^{b} C_*^\varphi(Y_i; \mathbb{C}[G]) \to C_*^\varphi(X; \mathbb{C}[G]) \to C_*^\varphi(X, Y; \mathbb{C}[G]) \to 0,$$

with the bases $\{s_*^i\}_{i=1,\ldots,b}$, $\{\tilde{s}_* \cup c_*\}$ and $\{c_*\}$. These bases are in fact compatible, in the sense that the middle basis is the image of the left basis together with a lift of the right basis. By Lemma 5.1 we have $\tau^{(2)}(C_*^\varphi(Y_i; \mathbb{C}[G]), \{\tilde{s}_*^i\}) = 1$ for $i = 1, \ldots, b$. Now it follows from the multiplicativity of torsion, see [14, Theorem 3.35], that

$$\tau^{(2)}\big(C_*^\varphi(X, Y; \mathbb{C}[G]), \{c_*\}\big) = \tau^{(2)}\big(C_*^\varphi(X; \mathbb{C}[G]), \{c_* \cup \tilde{s}_*\}\big).$$

Here we used that the complexes are acyclic. This concludes the proof of the claim.

Finally it follows from this claim, the definitions and Lemma 3.2 that

$$\tau^{(2)}(N, \partial N, \varphi, e^\dagger) = \tau^{(2)}\big(C_*^\varphi(X, Y; \mathbb{C}[G]), \{c_*\}\big) = \tau^{(2)}\big(C_*^\varphi(X; \mathbb{C}[G]), \{\tilde{s}_* \cup c_*\}\big)$$
$$= \tau^{(2)}(N, \varphi, K(e^\dagger)) = \tau^{(2)}(N, \varphi, c_1(e)^{-1}e)$$
$$= \det_{\mathcal{N}(G)}(\varphi(c_1(e))) \cdot \tau^{(2)}(N, \varphi, e). \qquad \square$$

6 The symmetry of the L^2–Alexander torsion

Let $(N, \phi, \gamma \colon \pi_1(N) \to G)$ be an admissible triple and let $e \in \mathrm{Eul}(N)$. Given $t \in \mathbb{R}_{>0}$ we consider the representation $\gamma_t \colon \pi_1(N) \to \mathrm{GL}(1, \mathbb{C}[G])$ that is given by $\gamma_t(g) := (t^{\phi(g)}\gamma(g))$. We denote by $\tau^{(2)}(N, \phi, \gamma, e)$ the function

$$\tau^{(2)}(N, \phi, \gamma, e) \colon \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0},$$
$$t \mapsto \tau^{(2)}(N, \gamma_t, e).$$

For another $e' \in \mathrm{Eul}(N)$ we have $e' = ge$ for some $g \in \mathcal{H}_1(N)$. By Lemma 3.2

$$\tau^{(2)}(N, \phi, \gamma, ge)(t) = t^{-\phi(g)} \tau^{(2)}(N, \phi, g, e)(t) \text{ for all } t \in \mathbb{R}_{>0}.$$

Put differently, the functions $\tau^{(2)}(N, \phi, \gamma, e)$ and $\tau^{(2)}(N, \phi, \gamma, ge)$ are equivalent. We denote by $\tau^{(2)}(N, \phi, \gamma)$ the equivalence class of the functions $\tau^{(2)}(N, \phi, \gamma, e)$ and we refer to $\tau^{(2)}(N, \phi, \gamma)$ as the L^2–*Alexander torsion* of (N, ϕ, γ).

Proof of Theorem 1.1 Let $e \in \mathrm{Eul}(N)$ and $t \in \mathbb{R}_{>0}$. We write $\tau = \tau^{(2)}(N, \gamma, \phi, e)$. Note that $(\gamma_t)^\dagger = \gamma_{t^{-1}}$. It follows from Theorems 4.2 and 5.2 that

$$
\begin{aligned}
\tau(t) = \tau^{(2)}(N, \gamma, \phi, e) &= \tau^{(2)}(N, \gamma_t, e) \\
&= \tau^{(2)}(N, \partial N, (\gamma_t)^\dagger, e^\dagger) = \tau^{(2)}(N, \partial N, \gamma_{t^{-1}}, e^\dagger) \\
&= \det_{\mathcal{N}(G)}(\gamma_{t^{-1}}(c_1(e))) \cdot \tau^{(2)}(N, \gamma_{t^{-1}}, e) \\
&= \det_{\mathcal{N}(G)}(t^{-\phi(c_1(e))} c_1(e)) \cdot \tau^{(2)}(N, \gamma_{t^{-1}}, e) \\
&= t^{-\phi(c_1(e))} \cdot \tau^{(2)}(N, \gamma_{t^{-1}}, e) = t^{-\phi(c_1(e))} \cdot \tau(t^{-1}).
\end{aligned}
$$

Now it suffices to show that for any $\phi \in H^1(N; \mathbb{Z})$ we have $\phi(c_1(e)) = x_N(\phi)$ mod 2.

So let S be a Thurston norm minimizing surface which is dual to some $\phi \in H^1(N; \mathbb{Z})$. Since N is irreducible and since $N \neq S^1 \times D^2$ we can arrange that S has no disk components. Therefore we have

$$
x_N(\phi) \equiv \chi_-(S) \equiv b_0(\partial S) \mod 2\mathbb{Z}.
$$

On the other hand, by [19, Lemma VI.1.2] and [19, Section XI.1] we have that $b_0(\partial S) \equiv c_1(e) \cdot S \mod 2\mathbb{Z}$ where $c_1(e) \cdot S$ is the intersection number of $c_1(e) \in H_1(N) = \mathcal{H}_1(N)$ with S. Since S is dual to ϕ, we obtain the desired equality

$$
\phi(c_1(e)) \equiv c_1(e) \cdot S \equiv b_0(\partial S) \equiv \chi_-(S) \equiv x_N(\phi) \mod 2\mathbb{Z}. \qquad \square
$$

Finally, a *real admissible triple* (N, ϕ, γ) is defined like an admissible triple, except that now we also allow ϕ to lie in $H^1(N; \mathbb{R}) = \mathrm{Hom}(\pi_1(N), \mathbb{R})$. The same definition as in Section 6 associates to (N, ϕ, e) a function $\tau^{(2)}(N, \phi, e): \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ that is well defined up to multiplication by a function of the form $t \mapsto t^r$ for some $r \in \mathbb{R}$. The same argument as in the proof of Theorem 1.1 gives us the following result.

Theorem 6.1 Let (N, ϕ, γ) be a real admissible triple. Then for any representative τ of $\tau^{(2)}(N, \phi, \gamma)$ there exists an $r \in \mathbb{R}$ such that $\tau(t^{-1}) = t^r \cdot \tau(t)$ for any $t \in \mathbb{R}_{>0}$.

The only difference to Theorem 1.1 is that for real cohomology classes $\phi \in H^1(N; \mathbb{R})$ we cannot relate the exponent r to the Thurston norm of ϕ.

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