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Categorified $\mathfrak{sl}_N$ invariants of colored rational tangles

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We use categorical skew Howe duality to find recursion rules that compute categorified $\mathfrak{sl}_N$ invariants of rational tangles colored by exterior powers of the standard representation. Further, we offer a geometric interpretation of these rules which suggests a connection to Floer theory. Along the way we make progress towards two conjectures about the colored HOMFLY homology of rational links and discuss consequences for the corresponding decategorified invariants.

57M25, 81R50; 57R58

1 Introduction and statement of results

Reshetikhin and Turaev [43] define invariants of framed oriented tangles (and thus also knots and links) with components labeled ("colored") by irreducible representations of a semi-simple Lie algebra. Starting from the work of Khovanov [21] and Khovanov and Rozansky [27] on the case of $\mathfrak{sl}_N$ with the standard representation, much of the $\mathfrak{sl}_N$ package of Reshetikhin–Turaev invariants has been categorified using a variety of different methods; for a recent survey see eg Turner [48]. The best-studied case is the one of fundamental (minuscule) $\mathfrak{sl}_N$ representations, ie the exterior powers Λ^k of the standard representation.

On the decategorified level it is well-known that $\mathfrak{sl}_N$ Reshetikhin–Turaev invariants stabilize for $N \to \infty$ and hence can be interpreted as specializations of two-variable HOMFLY-type invariants via setting $a = q^N$. Khovanov–Rozansky HOMFLY homology [28; 22] is a categorification of the HOMFLY polynomial [12] and an extension to *colored HOMFLY homology* with respect to labelings Λ^k was developed by Mackaay, Stošić and Vaz [36] and proved to be a link invariant by Webster and Williamson [49].

Colored HOMFLY homology is poorly understood even for the simplest non-trivial knots and links. A good starting point for understanding link homologies are rational knots and links, which have been proven to have particularly simple uncolored $\mathfrak{sl}_N$ and HOMFLY homologies; they are essentially determined by decategorified invariants. This paper is guided by two conjectures about the colored HOMFLY homology of rational knots and links.

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Conjecture 1.1 (Changing color on unknot components only shifts q–grading) *Let L be a rational two-component link with components colored by a fixed color Λ^j and a variable color Λ^i and let $\mathrm{HP}_j(L,\Lambda^i) \in \mathbb{N}[a^{\pm 1}, t^{\pm 1}, q^{-1}][\![q]\!]$ be the Hilbert–Poincaré series of the Λ^i–reduced (Λ^i,Λ^j)–colored HOMFLY homology of L, where powers of t indicate homological degree.*

Then there exist rational functions $F_j(L) \in \mathbb{N}[a^{\pm 1}, s^{\pm 1}, t^{\pm 1}](q)$ such that

$$\mathrm{HP}_j(L,\Lambda^i) = F_j(L)(a, s = q^{i-j}, t, q)$$

for all $i \geq j$, after expanding the right-hand side into a power series in q.

Conjecture 1.1 is due to Stošić (private communication).

Conjecture 1.2 (Homologies of higher colors are like powers of uncolored homology) *Let L be a rational knot or link. There exists a corrected Q–grading on reduced triply graded Λ^j–colored HOMFLY homology with Poincaré polynomials $\widetilde{P}_j(L)(a,Q,t)$, such that*

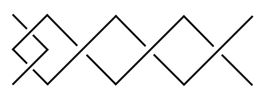
$$\widetilde{P}_j(L)(a,Q,t) = (\widetilde{P}_1(L)(a,Q,t))^j.$$

In particular, the ordinary Poincaré polynomials $P_j(L)(a,q,t)$ of Λ^j–colored HOMFLY homology satisfy

$$P_j(L)(a,1,t) = (P_1(L)(a,1,t))^j.$$

This is a special case of the "refined exponential growth" conjecture of Gorsky, Gukov and Stošić [19; 18], where a similar behavior is conjectured for rational knots and torus knots in the more general setting of labelings by rectangular Young diagrams. Moreover, there the correction Q of the standard q–grading results from comparing two alternative homological gradings.

While we cannot prove these conjectures for rational knots and links, we make progress towards them by establishing related results for categorified $\mathfrak{sl}_N$ invariants of colored rational tangles. Rational tangles $T(p,q)$ are built recursively, starting from a trivial two-strand tangle, by planar composition with crossings on the top and on the right, according to the continued fraction expansion of p/q. All rational links are closures of rational tangles. We give an example:

The tangle $T(7,2)$ associated to $\frac{7}{2} = [3,2]$ is: 

Using the categorical skew Howe duality framework developed by Cautis, Kamnitzer and Licata [6; 7] we study categorified $\mathfrak{sl}_N$ invariants of rational tangles with strands labeled by exterior powers of the standard representation. These invariants take values in a homotopy 2–category of chain complexes over categories that depend on the colors on the strands of the tangle; see Section 2.5. In Section 3 we demonstrate that for fixed colors Λ^i and Λ^j these invariants can be written as complexes with chain spaces being direct sums of grading shifts of a finite number of basic objects represented by webs (MOY graphs). We use results of Cautis [6] to compute explicit twist rules that describe how the invariant of a rational tangle changes under composing with a crossing on the top or on the right. The following example illustrates twist rules in the special case of Bar-Natan's geometric version of Khovanov homology for tangles [2].

Example 1.3 The Khovanov homology of a positive rational tangle can be computed recursively by using the following twist rules, which describe what happens to a generator of a chain space under tensoring with a crossing complex. We write top and right twists as operators T and R:

$$
\begin{aligned}
T(\,)\,(\,) &= \times &&= 0 \to qt^{-1} \asymp \to)\,(\to 0,\\
T(\asymp) &= \overset{\times}{\underset{\frown}{\bigcirc}} &&= 0 \to qt^{-1} \overset{\smile}{\underset{\frown}{\bigcirc}} \to \asymp \to 0 \quad \sim \quad 0 \to q^2t^{-1} \asymp \to 0 \to 0,\\
R(\,)\,(\,) &=)\bigcirc\!\!\times &&= 0 \to)\,(\to q^{-1}t\,)\bigcirc(\to 0 \quad \sim \quad 0 \to 0 \to q^{-2}t\,)\,(\to 0,\\
R(\asymp) &= \times &&= 0 \to \asymp \to q^{-1}t\,)\,(\to 0.
\end{aligned}
$$

Here all possible non-trivial differentials are given by saddle cobordisms, and compared to [2] we use slightly different grading conventions that are more compatible with the colored $\mathfrak{sl}_N$ case treated in this paper.

To be more precise, a priori the rules in this example only compute the chain spaces and the component of the differential coming from the last added crossing. To get the full information about the complex that is the tangle invariant, one would need to compute the induced differentials coming from previous crossings.

Definition 1.4 The *colored HOMFLY complex* of a colored rational tangle T is the chain complex representing the categorified Reshetikhin–Turaev $\mathfrak{sl}_N$ invariant of T that is obtained recursively by applying the twist rules from Section 3.3 term-wise and computing the induced differentials.

In the general case of an $\mathfrak{sl}_N$ colored HOMFLY complex we find it difficult to compute induced differentials and thus restrict our attention to counting the number and grading shifts of basic objects in its chain spaces. The correct data structure is, therefore, the following modified notion of Poincaré polynomial.

Definition 1.5 The *Poincaré polynomial* $\mathcal{P}^N_{i,j}(T)$ of the (Λ^i, Λ^j)–colored HOMFLY complex of a rational tangle T with $i \geq j$ is an element of $\mathbb{N}[q^{\pm 1}, t^{\pm 1}]\langle X_0, \dots, X_j \rangle$, the free $\mathbb{N}[q^{\pm 1}, t^{\pm 1}]$–module (over a semiring) spanned by basic objects X_k. It is the formal sum of the basic objects X_k appearing in the chain spaces of the complex, weighted by powers of t and q, indicating shifts in homological and q–grading, respectively. The basic objects X_k of weight k are introduced in Definition 3.1.

Example 1.6 Bar-Natan's complex for the $(2,k)$ torus tangle $T(k,1)$ is (with the grading convention introduced above)

$$q^{2k-1}t^{-k} \asymp \to \cdots \to q^3 t^{-2} \asymp \to q t^{-1} \asymp \to)\,(.$$

The Poincaré polynomial of this complex is $)\,(+ \sum_{i=1}^{k} q^{2i-1} t^{-i} \asymp$.

The twist rules for the case of colored $\mathfrak{sl}_N$ invariants of rational tangles, which we compute in Section 3.3, stabilize in an obvious way for large N and exhibit a very simple dependence on the higher color i, thus immediately implying the following theorem.

Theorem 1.7 *The $\mathfrak{sl}_N$ colored HOMFLY complexes of a positive rational tangle T colored by representations Λ^i and Λ^j, with $N \gg i \geq j$, depend essentially only on j up to shifts in q–grading. More precisely, there exist elements*

$$\mathcal{P}_j(T) \in \mathbb{N}[a^{\pm 1}, q^{\pm 1}, s^{\pm 1}, t^{\pm 1}]\langle X_0, \dots X_j \rangle$$

such that

$$\mathcal{P}^N_{i,j}(T) = \mathcal{P}_j(T)(a = q^N, q, s = q^{i-j}, t).$$

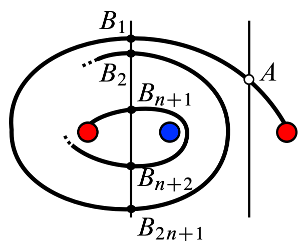
We thus sometimes consider a and s as two additional gradings of basic objects in the colored HOMFLY complex.

In Section 5 we show that in the decategorified setting a statement similar to Theorem 1.7 holds for any link with an unknot component:

Proposition 1.8 *Let $L' = L \cup U$ be a link with an unknot component U and some fixed coloring on the components of L. Then there exists $P^{\mathrm{st}}(L', U) \in \mathbb{Z}[a^{\pm 1}, r^{\pm 1}](q)$ such that, for any $i \in \mathbb{N}$, $P^{\mathrm{st}}(L', U)(a, r = q^i, q)$ is the colored HOMFLY polynomial of L with color Λ^i on U, reduced with respect to Λ^i. We call $P^{\mathrm{st}}(L', U)$ the* color-stable HOMFLY polynomial *of L' with respect to the unknot component U.*

In Section 4 we propose a geometric algorithm for computing the colored HOMFLY complex of a colored rational tangle, which is very similar to Bigelow's geometric model for the Jones polynomial [3]. The basis for this geometric algorithm is a special planar diagram P of the tangle in the plane $\mathbb{C}$, displaying only one strand α. In the case of colors Λ^i and Λ^j with $i \geq j$ we establish a bijection between the generators of the colored HOMFLY complex and intersections of $\mathrm{Sym}^j(\alpha)$ with certain half-dimensional submanifolds V_h in $\mathrm{Sym}^j(\mathbb{C})$. We further expect that in this picture differentials are realized as discs connecting intersection points with boundary on V_h and $\mathrm{Sym}^j(\alpha)$. Their gradings are determined by graded intersection numbers with certain divisors in $\mathrm{Sym}^j(\mathbb{C})$. In particular, using the language from Conjecture 1.2, we identify the difference between the original q–grading and the corrected Q–grading as coming from intersections of such discs with the big diagonal in $\mathrm{Sym}^j(\mathbb{C})$.

The diagram P for the tangle $T(2n+1, 1)$ is given by:



In the case of colors (Λ^1, Λ^1) the generators of the colored HOMFLY complex are in bijection with the set $\{A, B_1, \dots, B_{2n+1}\}$. In the case of Bar-Natan's version of Khovanov homology for tangles, A and B_j are generators of type $)\,($ and $\asymp$ respectively.

In Theorem 4.5 we prove that this geometric model observes the twist rules from Section 3.3 and, hence, correctly computes the Poincaré polynomial of the colored rational tangle. As a corollary we get the following theorem.

Theorem 1.9 *The generators of the (Λ^i, Λ^j)–colored HOMFLY complex of a rational tangle with $i \geq j$ are in bijection with certain j–tuples of generators of the (Λ^i, Λ^1)–colored HOMFLY complex. The a–, s– and t–gradings of these j–tuples are additive, ie they are the sums of the a–, s– and t–gradings of the components. The q–grading, on the other hand, splits into an additive component Q and a non-additive component $q-Q$. With respect to these gradings, colored HOMFLY complexes satisfy an analogue of Conjecture 1.2.*

There are obvious structural similarities between our geometric set-up and various Floer-theoretic constructions. In particular we want to mention Manolescu's interpretation [37] of Seidel–Smith symplectic Khovanov homology [46] in terms of Bigelow's model for the Jones polynomial [3]; see also Section 4.5. It is an interesting question whether it is possible to phrase our geometric algorithm in completely symplectic language and hence give an A–model realization of $\mathfrak{sl}_N$ link homologies for colored rational tangles. In the meantime we present a connection to link Floer homology that follows from the geometric model for the colored HOMFLY complex:

Corollary 1.10 *Let T be a rational tangle whose denominator closure is a two-component link L. Then the (Λ^i, Λ^1)–colored HOMFLY complex of T computes the link Floer homology of L.*

In the final section we compare the invariants described in this paper with (multivariable) link invariants arising from Lie superalgebras $\mathfrak{sl}_{m|n}$, as defined by Geer and Patureau-Mirand [14; 15], which satisfy very interesting color-stability properties.

Structure of this paper Section 2 is a review of technology developed by Cautis, Kamnitzer, Licata and Morrison in [6; 9] and related papers. In Section 3 we compute the twist rules that recursively determine the Poincaré polynomials of colored rational tangles and, thus, prove Theorem 1.7. Section 4 introduces a graphical method of computing the colored HOMFLY complexes of rational tangles, which proves Theorem 1.9 and suggests a Floer-theoretic interpretation. Section 5 contains proofs of Proposition 1.8 and Corollary 1.10 and makes contact with multivariable link invariants from Lie superalgebras.

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2 Categorified quantum link invariants via categorical skew Howe duality

In this section we set up the framework in which our calculations take place.

2.1 Reshetikhin–Turaev tangle invariants

Definition 2.1 A *tangle* is an embedding of pairs $(T, \partial T) \to (D^2 \times I, D^2 \times \partial I)$, where T is the disjoint union of a finite number of oriented arcs and circles, I is the interval $[-1, 1]$ and D^2 is the unit disc in $\mathbb{C}$. The ends of the arcs lying on $D^2 \times \{-1\}$ and $D^2 \times \{1\}$ are called *bottom and top ends*, respectively. A *tangle diagram* is a generic projection of the tangle onto $I \times I$, sending $D^2 \times \{\pm 1\}$ to $I \times \{\pm 1\}$.

We consider tangles up to regular isotopies that fix the boundary. This means we identify tangles that are given by tangle diagrams which differ only by planar isotopy and Reidemeister moves of type 2 and 3. To emphasize that we do not allow Reidemeister moves of type 1, we sometimes refer to tangles as framed tangles. In the following, the arcs and circles of tangles (and hence tangle diagrams) are allowed to carry orientations and labels (colors).

In [51] Witten gave a physical interpretation of the Jones polynomial and the $a = q^N$ specializations of the HOMFLY polynomial via Chern–Simons theory with gauge group $G = \mathrm{SU}(2)$ and $G = \mathrm{SU}(N)$ respectively. Later, Reshetikhin and Turaev [43] provided a rigorous mathematical framework for such quantum link invariants using the representation theory of quantum groups $U_q(\mathfrak{g})$, which are deformations of the enveloping algebras of the Lie algebras of the gauge groups G. More generally, they defined invariants of framed tangles T with strands labeled by irreducible representations of $U_q(\mathfrak{g})$, or equivalently by dominant weights of the semi-simple Lie algebra $\mathfrak{g}$. If the bottom end strands are labeled by $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ and the top end strands by $\underline{\mu} = (\mu_1, \dots, \mu_{n'})$, they associate to T a map of $U_q(\mathfrak{g})$–representations

$$\psi(T)\colon V_{\underline{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \to V_{\mu_1} \otimes \cdots \otimes V_{\mu_{n'}} = V_{\underline{\mu}},$$

where V_λ denotes the irreducible $U_q(\mathfrak{g})$ representation of highest weight λ. This map is an invariant of the framed labeled tangle. The invariant is computed by isotoping T into generic position, then scanning it from bottom to top and assembling the map from the maps associated to cups, caps and crossings.

In the case of links L and complex semi-simple Lie algebras $\mathfrak{g}$, where the bottom and top representations are both the trivial $U_q(\mathfrak{g})$ representation $\mathbb{C}(q)$, the map $\psi(L)$ is given by multiplying by $\psi(L)(1) \in \mathbb{C}(q)$. We list some prominent instances:

- If $\mathfrak{g} = \mathfrak{sl}_2$ and L is labeled by (symmetric powers of) the standard representation, then $\psi(L)(1)$ is the (colored) Jones polynomial of L.
- If $\mathfrak{g} = \mathfrak{sl}_N$ and L is labeled by the standard representation, then $\psi(L)(1)$ is the $a = q^N$ specialization of the HOMFLY polynomial of L.

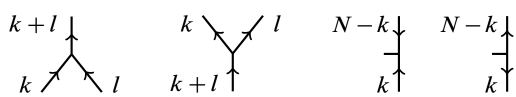
- If $\mathfrak{g} = \mathfrak{sl}_N$ and L is labeled by some other irreducible representations, then $\psi(L)(1)$ is the $a = q^N$ specialization of the corresponding colored HOMFLY polynomial of L.

2.2 The $U_q(\mathfrak{sl}_N)$ representation category and web relations

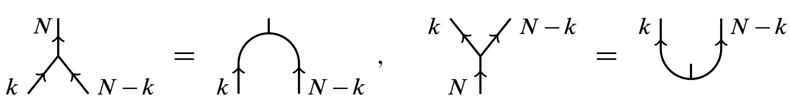
In this paper we are only interested in the tangle invariants for $\mathfrak{g} = \mathfrak{sl}_N = \mathfrak{sl}_N(\mathbb{C})$ and fundamental representations $\Lambda^i V$, where V is the standard representation. The representation category of the quantum group $U_q(\mathfrak{sl}_N)$ is well-understood and has a nice graphical description which ties in nicely with the Reshetikhin–Turaev picture. We follow the exposition of Cautis, Kamnitzer and Morrison in [9].

Definition 2.2 The *free spider category* $\mathrm{FSp}(\mathfrak{sl}_N)$ has as objects finite sequences $\underline{k}$ in $\{1^\pm, \dots, (N-1)^\pm\}$ and morphisms in $\mathrm{Mor}(\underline{k}, \underline{l})$ are $\mathbb{C}(q)$–linear combinations of oriented planar graphs, called *webs*, in the unit square $[-1,1] \times [-1,1] \subset \mathbb{R}^2$, up to planar isotopy, such that:

- The edges of the web are labeled by elements of the set $\{1, \dots, N-1\}$.
- The boundary of the web splits into a bottom boundary on $[-1,1] \times \{-1\}$ and a top boundary on $[-1,1] \times \{1\}$.
- The sequence of labels and orientations on the bottom and top boundary agree with the sequences $\underline{k}$ and $\underline{l}$, where j^+ and j^- stand for upward and downward oriented strands labeled by j.
- The internal vertices are of only these four types:



The latter two bivalent vertices are called *tags*. In general it does matter on which side the tag is placed and for now we distinguish these local sub-graphs from their mirror images. In some formulas we further allow labelings of edges by integers outside the range of $\{1, \dots, N-1\}$. Edges labeled by 0 are to be deleted, N–labeled edges incident to a trivalent vertex get reduced to tags as shown below:



Any diagram containing a labeling outside the range $\{0, \dots, N\}$ is defined to be 0.

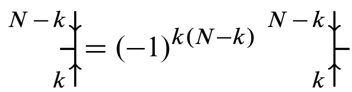
Morphisms are composed by vertically stacking graphs and gluing boundary points.

We further define, for $n \in \mathbb{Z}$, $k \in \mathbb{N}$, $k \leq |n|$:

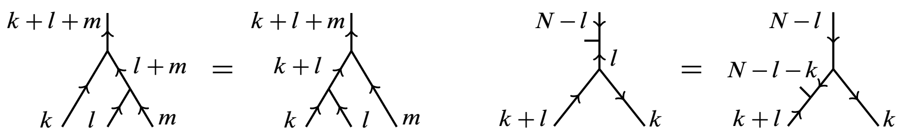
- *Quantum integers* $[n] := \frac{q^n - q^{-n}}{q^1 - q^{-1}}$.
- *Quantum factorials* $[n]! := [1][2]\cdots[n]$ if $n \geq 0$ and $[n]! := [-1][-2]\cdots[n]$ if $n \leq 0$.
- *Quantum binomial coefficients* $\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n][n-1]\cdots[n-k+1]}{[k]!}$.

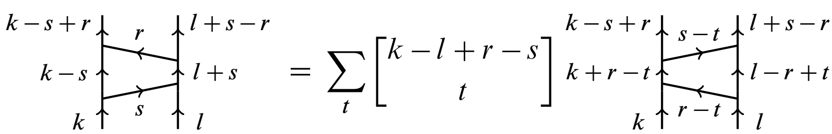
Note that $[-n] = -[n]$ and $[-n]! = (-1)^n[n]$. Negative quantum integers are only used in the last relation in the following definition. For the remainder of this paper we only deal with positive quantum integers.

Definition 2.3 The $\mathfrak{sl}_N$ *spider category* $\mathrm{Sp}(\mathfrak{sl}_N)$ is the quotient of the free spider category $\mathrm{FSp}(\mathfrak{sl}_N)$ by the following local relations on morphisms

(2-1) 

(2-2) 

(2-3) 

(2-4) 

together with the mirror images and arrow reversals of these.

We briefly recall some facts about the representation category of $U_q(\mathfrak{sl}_N)$ and collect them in the following lemma. Details can be found in Cautis, Kamnitzer and Morrison [9].

Lemma 2.4 *The representation category of $U_q(\mathfrak{sl}_N)$ is a pivotal category and it is the idempotent completion (Karoubi envelope) of the full subcategory generated by objects isomorphic to tensor products of fundamental representations, which we denote by $\mathrm{Rep}(U_q(\mathfrak{sl}_N))$.*

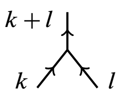
As a pivotal category $\mathrm{Rep}(U_q(\mathfrak{sl}_N))$ is generated by up-to-scaling unique intertwiners

$$M_{k,l}\colon \Lambda^k V \otimes \Lambda^l V \to \Lambda^{k+l} V \quad \text{and} \quad M'_{k,l}\colon \Lambda^{k+l} V \to \Lambda^k V \otimes \Lambda^l V.$$

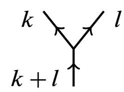
Theorem 2.5 [9, Theorems 3.2.1 and 3.3.1] *There is an equivalence of pivotal categories* $\phi\colon \mathrm{Sp}(\mathfrak{sl}_N) \to \mathrm{Rep}(U_q(\mathfrak{sl}_N))$*, defined on objects by*

$$\underline{k} = (k_1^{\epsilon_1}, \dots, k_s^{\epsilon_s}) \to (\Lambda^{k_1} V)^{\epsilon_1} \otimes \cdots \otimes (\Lambda^{k_s} V)^{\epsilon_s},$$

where $\epsilon_k \in \{-1, 1\}$ *and* W^{-1} *stands for the representation dual to* $W^1 := W$*, and on generating morphisms by*



$\to M_{k,l}$ *and*

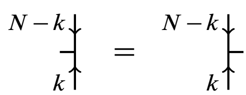


$\to M'_{k,l}.$

As a by-product, this theorem says that all computations of Reshetikhin–Turaev invariants of tangles labeled with fundamental representations can be done in the spider category of $\mathfrak{sl}_N$. However, this idea is much older and goes back at least to Murakami, Ohtsuki and Yamada [42]. Thus the morphisms in the $\mathfrak{sl}_N$ spider category are also known as *MOY graphs*.

Remark 2.6 In [42] tags do not appear and edges labeled by N can be erased completely. Not distinguishing between the two ways of inserting a tag causes an error of a factor of ± 1 on the level of tangle invariants. We adopt this convention and hence work with a smaller category in which the relation

(2-5)



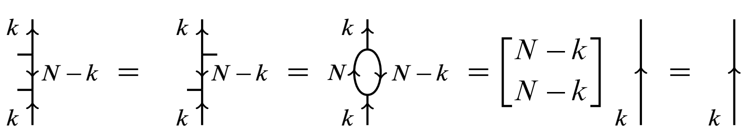
holds. This new spider category is equivalent to the projective representation category $\mathrm{Rep}(U_q(\mathfrak{sl}_N))/\pm 1$ and it will suffice for our purposes.

2.3 Constructing Reshetikhin–Turaev tangle invariants

We now describe the maps used to define Reshetikhin–Turaev tangle invariants.[1]

Lemma 2.7 *Every oriented strand in a web can be inverted at the expense of changing its label* k *into the complementary label* $N - k$ *and introducing two tags.*

Proof To see this, we use a relation from (2-2) in the projective $\mathfrak{sl}_N$ spider category:

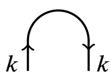
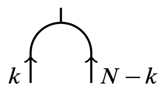
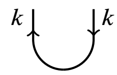
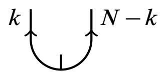


□

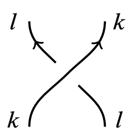
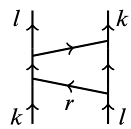
[1] The maps we present agree with the original Reshetikhin–Turaev maps up to multiplication by non-zero scalars and powers of q.

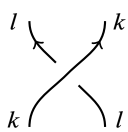
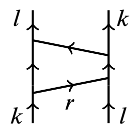
Let T be a tangle in general position labeled by fundamental representations of $U_q(\mathfrak{sl}_N)$. In order to define the Reshetikhin–Turaev invariant of T, it suffices to define the maps that correspond to cups, caps and crossings. Via Theorem 2.5 we can present these maps in terms of linear combinations of webs.

Definition 2.8 (Building blocks for RT invariants) To define the maps associated to caps and cups, replace each downward-oriented k–strand by an upward-oriented strand labeled by $N-k$ and place tags on caps and cups, eg

 $\to$  $= M_{k,N-k}$,  $\to$  $= M'_{k,N-k}$.

The maps associated to positive crossings are

 $\mapsto \displaystyle\sum_{r=0}^{l}(-1)^{r+(k+1)l}q^r$  if $l \le k$,

 $\mapsto \displaystyle\sum_{r=0}^{k}(-1)^{r+(l+1)k}q^r$  if $l > k$.

The formulas for negative crossings are obtained by replacing q by q^{-1}. These formulas are taken from Murakami, Ohtsuki and Yamada [42] with the q–grading adjusted to our purposes.

Remark 2.9 For Reshetikhin–Turaev invariants of links colored by fundamental representations of $\mathfrak{sl}_N$ there is an algorithm that does not use any tags at all: without replacing any cups or caps, immediately use the description of the crossing maps to locally replace all crossings by linear combination of webs. The result is a linear combination of closed webs which can then be evaluated to elements in the ground ring. An explicit algorithm for evaluating closed webs that does not use tags is described by Wu in [52, Corollary 14.8 and proof of Theorem 14.7].

Remark 2.10 Similarly as in the uncolored case, where all involved representations on the strands are standard, there is a stability in the sequence of colored $\mathfrak{sl}_N$ invariants for varying N. Using this stability, one can define two-variable colored HOMFLY invariants. However, we have to explain what it means to fix representations on the strands when dealing with $\mathfrak{sl}_N$ invariants of different N. Formally, colored HOMFLY link invariants can be defined as invariants of oriented links labeled by

Young diagrams, or equivalently by partitions of integers. These partitions specify an irreducible representation as highest weight representation in a tensor product of fundamental representations, independent of $N \gg 0$. In particular, a one-part partition (k) stands for the k^{th} exterior power of the standard representation. To compute the colored HOMFLY invariant of a link, pick an $N \gg 0$, interpret labels (k) as $\Lambda^k V$ and proceed in the same way as when computing the corresponding $\mathfrak{sl}_N$ invariant. However, in doing this treat $a = q^N$ as an independent variable and replace every occurrence of

$$\begin{bmatrix} N+b \\ c \end{bmatrix} = \prod_{k=1}^{c} \frac{[N+b-k+1]}{[k]} = \prod_{k=1}^{c} \frac{q^{N+b-k+1} - q^{-N-b+k-1}}{q^k - q^{-k}}$$

by

$$\begin{bmatrix} b \\ c \end{bmatrix}_a := \prod_{k=1}^{c} \frac{aq^{b-k+1} - a^{-1}q^{-b+k-1}}{q^k - q^{-k}}.$$

This recipe still produces invariants of framed colored links and it takes values in $\mathbb{C}(q)[a^{\pm 1}]$, but usually not in $\mathbb{C}[a^{\pm 1}, q^{\pm 1}]$. Nevertheless, these invariants are sometimes called *colored HOMFLY polynomials*. Similarly, one can define colored HOMFLY invariants of tangles; see Section 5.2.

2.4 Web relations via skew Howe duality

Quantum skew Howe duality and its categorical analogues are powerful tools for studying the (2–)representation category of $U_q(\mathfrak{sl}_N)$.

Let X be the weight lattice of $\mathfrak{sl}_N$ with simple roots $\alpha_1, \dots, \alpha_{N-1}$ and fundamental weights $\Lambda_1, \dots, \Lambda_{N-1}$ and the non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ given by

$$\langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1},$$

which furthermore satisfies $\langle \alpha_i, \Lambda_j \rangle = \delta_{ij}$. Denote by Y the root lattice of $\mathfrak{sl}_N$ and $Y_{\mathbb{C}} := Y \otimes_{\mathbb{Z}} \mathbb{C}$.

Definition 2.11 The quantum group $U_q(\mathfrak{sl}_N)$ of $\mathfrak{sl}_N$ is the $\mathbb{C}(q)$–algebra with generators E_i, F_i, K_i for $i = 1, \dots, N-1$ and the relations

$$K_i K_j = K_j K_i, \quad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$K_j E_i K_j^{-1} = q^{\langle \alpha_i, \alpha_j \rangle}, \quad K_j F_i K_j^{-1} = q^{-\langle \alpha_i, \alpha_j \rangle},$$

$$[2] E_i E_j E_i = E_i^2 E_j + E_j E_i^2 \quad \text{if } |i-j| = 1, \quad [E_i, E_j] = 0 \quad \text{if } |i-j| > 1,$$

and similarly for Fs. If $\mu = \sum l_i \Lambda_i \in \mathbb{Z}\langle \Lambda_1, \dots, \Lambda_{N-1}\rangle$ is an integral weight, then we write K_μ for the product $K_1^{l_1} \cdots K_{N-1}^{l_{N-1}}$.

$U_q(\mathfrak{sl}_N)$ is a deformation of the universal enveloping algebra of $\mathfrak{sl}_N$. We now recall the Beilinson–Lusztig–MacPherson idempotent modification $\dot{U}(\mathfrak{sl}_N)$ of $U_q(\mathfrak{sl}_N)$.

Definition 2.12 The idempotent version $\dot{U}(\mathfrak{sl}_N)$ of the quantum group of $\mathfrak{sl}_N$ is defined by adjoining orthogonal idempotents 1_λ for integral $\mathfrak{sl}_N$–weights $\lambda \in X$ to $U_q(\mathfrak{sl}_N)$, subject to the relations

$$K_\mu 1_\lambda = 1_\lambda K_\mu = q^{\langle \mu, \lambda\rangle} 1_\lambda, \quad E_i 1_\lambda = 1_{\lambda+\alpha_i} E_i, \quad F_i 1_\lambda = 1_{\lambda-\alpha_i} F_i.$$

Alternatively, $\dot{U}(\mathfrak{sl}_N)$ can be considered as a $\mathbb{C}(q)$–linear category with

- objects given by integral weights $\lambda \in X$, and
- morphisms generated by $E_i 1_\lambda \in \mathrm{Hom}(\lambda, \lambda+\alpha_i)$ and $F_i 1_\lambda \in \mathrm{Hom}(\lambda, \lambda-\alpha_i)$, where 1_λ denotes the identity morphism of λ.

Of special importance to categorification is the $\mathbb{Z}[q^{\pm 1}]$–subalgebra ${}_{\mathcal{A}}\dot{U}(\mathfrak{sl}_N)$ of $\dot{U}(\mathfrak{sl}_N)$, which is generated by divided powers $E_i^{(k)} := E_i^k/[k]!$ and $F_i^{(k)} := F_i^k/[k]!$.

There exist several variants of the following theorem in the literature. For a more thorough treatment we refer to Cautis, Kamnitzer and Morrison [6; 9]. Let $\mathbb{C}_q^s$ and $\mathbb{C}_q^N$ be the standard representations of $U_q(\mathfrak{sl}_s)$ and $U_q(\mathfrak{sl}_N)$, respectively, and denote by Λ the appropriate q–deformed exterior algebra functor. Then we have:

Theorem 2.13 (Quantum skew Howe duality) *The algebra $\Lambda(\mathbb{C}_q^s \otimes \mathbb{C}_q^N)$ carries commuting actions of $U_q(\mathfrak{sl}_s)$ and $U_q(\mathfrak{sl}_N)$ that constitute a Howe pair. That is, for any $M \in \mathbb{N}$,*

$$\mathrm{End}_{U_q(\mathfrak{sl}_N)} \Lambda^M(\mathbb{C}_q^s \otimes \mathbb{C}_q^N) \text{ is generated by } U_q(\mathfrak{sl}_s),$$
$$\mathrm{End}_{U_q(\mathfrak{sl}_s)} \Lambda^M(\mathbb{C}_q^s \otimes \mathbb{C}_q^N) \text{ is generated by } U_q(\mathfrak{sl}_N).$$

Moreover, $\Lambda(\mathbb{C}_q^s \otimes \mathbb{C}_q^N)$ splits into $U_q(\mathfrak{sl}_s)$ weight spaces as

$$\Lambda^M(\mathbb{C}_q^s \otimes \mathbb{C}_q^N) \cong \bigoplus_{i_1+\cdots+i_s=M} \Lambda^{i_1}(\mathbb{C}_q^N) \otimes \cdots \otimes \Lambda^{i_s}(\mathbb{C}_q^N),$$

where the summands on the right-hand side are weight spaces $V(\lambda)$ associated to weights $\lambda = \sum_{j=1}^{s-1}(i_{j+1} - i_j)\alpha_j$, and Chevalley generators act as follows:

$$\cdots \otimes \Lambda^{i_k}(\mathbb{C}_q^N) \otimes \Lambda^{i_{k+1}}(\mathbb{C}_q^N) \otimes \cdots \underset{F_k}{\overset{E_k}{\rightleftarrows}} \cdots \otimes \Lambda^{i_k-1}(\mathbb{C}_q^N) \otimes \Lambda^{i_{k+1}+1}(\mathbb{C}_q^N) \otimes \cdots.$$

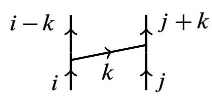
From the $U_q(\mathfrak{sl}_s)$–action on $\Lambda^M(\mathbb{C}_q^s \otimes \mathbb{C}_q^N)$ we immediately get a map

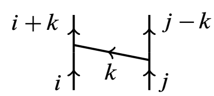
$$1_{\lambda'} {}_{\mathcal{A}}\dot{U}(\mathfrak{sl}_s) 1_\lambda \to \mathrm{Hom}_{U_q(\mathfrak{sl}_N)}(V(\lambda), V(\lambda')).$$

The weight spaces $V(\lambda)$ are exactly the possible (co-)domains for the Reshetikhin–Turaev invariants of tangles with at most s top and bottom boundary points and a labeling by fundamental representations. Furthermore, the basic building blocks of these invariants — the maps associated to crossings, caps and cups — are in the image of the above map and hence can be lifted to ${}_{\mathcal{A}}\dot{U}(\mathfrak{sl}_s)$. This is due to Cautis, Kamnitzer and Licata [7].

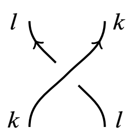
We now provide a dictionary to translate between the two descriptions of maps of $U_q(\mathfrak{sl}_N)$ representations, given by the action of ${}_{\mathcal{A}}\dot{U}(\mathfrak{sl}_s)$ on $\Lambda^M(\mathbb{C}_q^s \otimes \mathbb{C}_q^N)$ on the one hand and by webs on the other hand. In doing this we omit the weight space idempotents 1_λ and the subscript r of $E_r^{(k)}$ and $F_r^{(k)}$ from the notation.

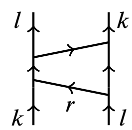
The action of Chevalley generators $E = E^{(1)}$ and $F = F^{(1)}$ and divided powers $E^{(k)}$ and $F^{(k)}$ can be written in the following way:

$E^{(k)} \mapsto$ 

and $F^{(k)} \mapsto$ 

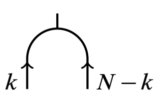
The crossing formulas can be written in terms of Es and Fs. For example, in the case $l \leq k$ we have

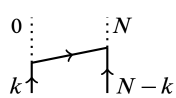
(2-6) 

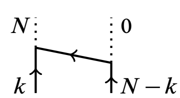
$\mapsto \sum_{r=0}^{l} (-1)^{r+(k+1)l} q^r$ 

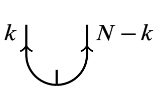
$$= \sum_{r=0}^{l} (-1)^{r+(k+1)l} q^r E^{(k-l+r)} F^{(r)}.$$

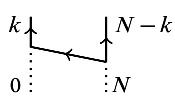
Caps and cups can be described as:

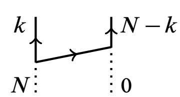
(2-7) 

$\mapsto$ 

$= 1_N E^{(k)} \cong 1_{-N} F^{(N-k)} =$ 

(2-8) 

$\mapsto$ 

$= F^{(k)} 1_N \cong E^{(N-k)} 1_{-N} =$ 

Here an issue has to be addressed. The two supposedly isomorphic ways of writing caps and cups as divided powers of E or F have different domain or target. This problem is solved by showing that all weight spaces $\Lambda^{i_1}(\mathbb{C}_q^N) \otimes \cdots \otimes \Lambda^{i_s}(\mathbb{C}_q^N)$ are canonically isomorphic to the weight space where all tensor factors $\Lambda^0(\mathbb{C}_q^N)$ are permuted to the left and all factors $\Lambda^N(\mathbb{C}_q^N)$ are permuted to the right. The isomorphism is realized by braiding the 0– and N–labeled strands (which are indicated by dotted lines) to the outside, using the standard crossing formula. The isomorphism is canonical because it does not depend on how the braiding is realized, as long as all involved crossings have at least a 0– or N–colored strand in them. This is [6, Lemma 7.1].

This realization of Reshetikhin–Turaev tangle invariants as images of elements of ${}_{\mathcal{A}}\dot{U}_q(\mathfrak{sl}_s)$ under the skew Howe map has the great advantage that many relations necessary for regular isotopy invariance already hold in the quantum group; see [6, Section 7]. In fact, quantum skew Howe duality is the main ingredient in the proof that characterizes the representation category of $U_q(\mathfrak{sl}_N)$ as the $\mathfrak{sl}_N$ spider category, as given by Cautis, Kamnitzer and Licata [9]. We summarize the situation:

Theorem 2.14 [6, Sections 6–7] *The maps associated to projections of framed oriented tangles colored with irreducible $U_q(\mathfrak{sl}_N)$–representations via the $U_q(\mathfrak{sl}_s)$–action on the $U_q(\mathfrak{sl}_N)$–representation $\Lambda(\mathbb{C}_q^s \otimes \mathbb{C}_q^N)$, for a suitable $s \in \mathbb{N}$, are invariants of such tangles and agree with the maps defined by Reshetikhin and Turaev up to non-zero scalars and powers of q.*

2.5 Categorification

Simulating skew Howe duality on a categorical level is an attractive approach to defining categorified Reshetikhin–Turaev invariants, because one can get many necessary relations for the invariance proof from the categorified quantum group in a similar way as one gets relations of the $U_q(\mathfrak{sl}_N)$ representation category from the ordinary quantum group. Categorified quantum groups were defined and studied by Lauda in the case of $\mathfrak{sl}_2$ (see [29; 30] and the survey [31]) and Khovanov and Lauda for the $\mathfrak{sl}_N$ case (see [23; 25; 24]).

Once one knows categorified quantum groups, it is another problem to define what it means to have an action of a categorified quantum group that categorifies a quantum group representation, eg the $U_q(\mathfrak{sl}_s)$ representation $\Lambda(\mathbb{C}_q^s \otimes \mathbb{C}_q^N)$. Yet another problem is to show that such categorical actions exist at all. We follow Cautis' approach from [5; 6] where he addresses the latter two problems. He axiomatically defines what it means to have a $(\mathfrak{g}, \theta)$–action on a 2–category that is the categorical analogue of a $U_q(\mathfrak{g})$–action. This is a fairly minimal definition and because it does not mention a priori the full higher representation-theoretic structure of the categorified quantum group, existence of such actions is easier to check in practice. However, in [5] Cautis shows that a $(\mathfrak{sl}_s, \theta)$–action always extends to a categorical $\mathfrak{sl}_s$–action in the stronger sense of Khovanov and Lauda [24]. This guarantees that 2–categories with $(\mathfrak{sl}_s, \theta)$–actions satisfy a number of further relations which allow the construction of categorical tangle and link invariants via a categorical analogue of skew Howe duality. To keep the exposition compact, we only present Cautis' definition of a $(\mathfrak{sl}_s, \theta)$–action and the additional relations necessary for the definition of categorical tangle and link invariants.

We want to mention that there also exists a categorification of skew Howe duality in a strong sense due to Ehrig and Stroppel [11]. It consists of commuting 2–actions of quantum groups on derived categories of abelian categories with Koszul self-duality intertwining the actions. Furthermore, the involved categories restrict to the ones used in the Mazorchuk–Stroppel construction of $\mathfrak{sl}_N$ link homology [39].

In the following definition we allow the case $s = \infty$, in which we assume that the Lie algebra $\mathfrak{sl}_\infty$ has roots and fundamental weights indexed by $\mathbb{Z}$; X still denotes the weight lattice of $\mathfrak{sl}_s$.

Definition 2.15 [6, Section 2.2] A $(\mathfrak{g}, \theta)$–action consists of a target graded, additive, $\mathbb{C}$–linear idempotent complete 2–category $\mathfrak{K}$ with

(1) objects indexed by $\lambda \in X$,

(2) 1–morphisms including $E_i 1_\lambda = 1_{\lambda+\alpha_i} E_i$ and $F_i 1_{\lambda+\alpha_i} = 1_\lambda F_i$, where 1_λ is the identity 1–morphism of λ, and

(3) 2–morphisms including a linear map $Y_\mathbb{C} \to \mathrm{Hom}(1_\lambda, 1_\lambda\langle 2\rangle)$ for each $\lambda \in X$.

Here $\langle l\rangle$ is the auto-equivalence given by shifting grading down by l. By abuse of notation we denote by θ the image of $\theta \in Y_\mathbb{C}$ under such a map.

These data are required to satisfy the following relations:

(i) $\mathrm{Hom}(1_\lambda, 1_\lambda\langle l\rangle)$ is zero if $l < 0$ and one-dimensional if $l = 0$ and $1_\lambda \neq 0$. Moreover, the space of maps between any two 1–morphisms is finite-dimensional.

(ii) E_i and F_i are left and right adjoints of each other up to shifts:
  (a) $(E_i 1_\lambda)_R \cong 1_\lambda F_i \langle \langle \lambda, \alpha_i\rangle + 1\rangle$.
  (b) $(E_i 1_\lambda)_L \cong 1_\lambda F_i \langle -\langle \lambda, \alpha_i\rangle - 1\rangle$.

(iii) We have

$$E_i F_i 1_\lambda \cong F_i E_i 1_\lambda \bigoplus_{[\langle\lambda,\alpha_i\rangle]} 1_\lambda \quad \text{if } \langle \lambda, \alpha_i\rangle \geq 0,$$

$$F_i E_i 1_\lambda \cong E_i F_i 1_\lambda \bigoplus_{[-\langle\lambda,\alpha_i\rangle]} 1_\lambda \quad \text{if } \langle \lambda, \alpha_i\rangle \leq 0,$$

where $\bigoplus_{p(q)} B$ for $p(q) = \sum_{i\in\mathbb{Z}} a_i q^i \in \mathbb{Z}[q^{\pm 1}]$ means $\bigoplus_{i\in\mathbb{Z}} (B\langle -i\rangle)^{\oplus a_i}$, and we contract $A \oplus \oplus_{p(q)} B$ to $A \oplus_{p(q)} B$.

(iv) If $i \neq j$ then $F_j E_i 1_\lambda \cong E_i F_j 1_\lambda$.

(v) If $\langle \lambda, \alpha_i \rangle \geq 0$ then the map $(I\theta I) \in \mathrm{Hom}(E_i 1_\lambda F_i, E_i 1_\lambda F_i \langle 2 \rangle)$ induces an isomorphism between $\langle \lambda, \alpha_i \rangle + 1$ (resp. zero) of the $\langle \lambda, \alpha_i \rangle + 2$ summands $1_{\lambda+\alpha_i}$ when $\langle \theta, \alpha_i \rangle \neq 0$ (resp. $\langle \theta, \alpha_i \rangle = 0$).

If $\langle \lambda, \alpha_i \rangle \leq 0$, the same holds for $(I\theta I) \in \mathrm{Hom}(F_i 1_\lambda E_i, F_i 1_\lambda E_i \langle 2 \rangle)$.

(vi) If $\alpha = \alpha_i$ or $\alpha = \alpha_i + \alpha_j$ for some roots with $\langle \alpha_i, \alpha_j \rangle = -1$ then $1_{\lambda + r\alpha} = 0$ for $r \gg 0$ or $r \ll 0$.

(vii) Suppose $i \neq j$ and $\lambda \in X$. If $1_{\lambda+\alpha_i}$ and $1_{\lambda+\alpha_j}$ are non-zero then 1_λ and $1_{\lambda+\alpha_i+\alpha_j}$ are also non-zero.

In [5] Cautis proves that a $(\mathfrak{g}, \theta)$–action carries an action of the quiver Hecke algebras and, in the case of $\mathfrak{g} = \mathfrak{sl}_s$, such an action extends to a categorical $\mathfrak{sl}_s$–action in the sense of Khovanov and Lauda [24]. As a consequence, $\mathfrak{K}$ also contains the divided powers $E_i^{(r)}$ or $F_i^{(r)}$, which are adjoint 1–morphisms up to shifts:

(ix) $(E_i^{(r)} 1_\lambda)_R \cong 1_\lambda F_i^{(r)} \langle r(\langle \lambda, \alpha_i \rangle + r) \rangle$.

(x) $(E_i^{(r)} 1_\lambda)_L \cong 1_\lambda F_i^{(r)} \langle -r(\langle \lambda, \alpha_i \rangle + r) \rangle$.

The following relations in $\mathfrak{K}$ then follow from work of Khovanov, Lauda, Mackaay and Stošić [26, Theorem 5.2.8 and Theorem 5.1.1]:

(xi) We have

$$E_i^{(a)} F_i^{(b)} 1_\lambda \cong \bigoplus_{j \geq 0} \bigoplus_{\left[\langle \lambda, \alpha_i \rangle + a - b \atop j\right]} F_i^{(b-j)} E_i^{(a-j)} 1_\lambda \qquad \text{if } \langle \lambda, \alpha_i \rangle + a - b \geq 0,$$

$$F_i^{(b)} E_i^{(a)} 1_\lambda \cong \bigoplus_{j \geq 0} \bigoplus_{\left[-\langle \lambda, \alpha_i \rangle - a + b \atop j\right]} E_i^{(a-j)} F_i^{(b-j)} 1_\lambda \qquad \text{if } \langle \lambda, \alpha_i \rangle + a - b \leq 0.$$

(xii) $E^{(a)} E^{(b)} \cong \left[{a+b \atop a}\right] E^{(a+b)}$, and analogously for the Fs.

The definition of categorical knot and tangle invariants is now very similar to the construction of Reshetikhin–Turaev invariants via quantum skew Howe duality. The main ingredient is an $(\mathfrak{sl}_s, \theta)$–action on a 2–category $\mathfrak{K}$ which is a lift of the $U_q(\mathfrak{sl}_s)$–action on $\Lambda(\mathbb{C}_q^s \otimes \mathbb{C}_q^N)$. We now explain what this means.

Definition 2.16 Let M be a weight module of $U_q(\mathfrak{sl}_s)$, or equivalently a $\dot{U}(\mathfrak{sl}_s)$–module. Then we can consider M as a $\mathbb{C}(q)$–linear category as follows:

- Objects: weight spaces $M_\lambda = 1_\lambda M 1_\lambda$ indexed by $\lambda \in X$.
- Morphisms: $\mathrm{Hom}(M_\lambda, M_\mu)$ is given by the image of $1_\mu \dot{U}(\mathfrak{sl}_s) 1_\lambda$ under the action.

Let $\mathfrak{K}$ be a 2–category with a $(\mathfrak{sl}_s, \theta)$–action. Then we form the Grothendieck category $K(\mathfrak{K})$ as follows:

- Objects: the same as $\mathfrak{K}$.
- Morphisms: split Grothendieck groups of morphism categories of $\mathfrak{K}$, tensored with $\mathbb{C}(q)$.

Here the split Grothendieck group $K(C)$ of an additive, $\mathbb{C}$–linear category C is the free abelian group generated by isomorphism classes of objects of C modulo the relations $[A] = [B_1] + [B_2]$ for all triples $A \cong B_1 \oplus B_2$ of objects. If C is graded, $K(C)$ can be regarded as a $\mathbb{Z}[q^{\pm 1}]$–module with the autoequivalence $\langle -1 \rangle$ acting by multiplication by q.

We say the $(\mathfrak{sl}_s, \theta)$–action on $\mathfrak{K}$ lifts the $\dot{U}(\mathfrak{sl}_s)$–action on M if its Grothendieck category is isomorphic to M considered as a $\mathbb{C}(q)$–linear category. In particular, this means that the non-zero objects in $\mathfrak{K}$ are in bijection with the non-zero weight spaces of M and that via $K(\cdot)$ the 1–morphisms E_i and F_i in $\mathfrak{K}$ are sent to the images of E_i and F_i in $\mathrm{End}(M)$ under the $\dot{U}(\mathfrak{sl}_s)$–action.

From now on we assume that $\mathfrak{K}$ is a 2–category with a $(\mathfrak{sl}_s, \theta)$–action that lifts the $\dot{U}(\mathfrak{sl}_s)$–action on $\Lambda(\mathbb{C}_q^s \otimes \mathbb{C}_q^N)$. As Cautis explains in [6], this is sufficient to define categorical link invariants that lift the Reshetikhin–Turaev invariants. For the definition of categorical tangle invariants we further require that the objects of $\mathfrak{K}$ are categories themselves — with 1–morphisms and 2–morphisms in $\mathfrak{K}$ acting as functors and natural transformations, respectively, such that on K–theory we recover $\Lambda(\mathbb{C}_q^s \otimes \mathbb{C}_q^N)$. This means that the Grothendieck groups of the objects λ, tensored with $\mathbb{C}(q)$, are isomorphic as vector spaces to the weight spaces $\Lambda(\mathbb{C}_q^s \otimes \mathbb{C}_q^N)_\lambda$ and the 1–morphisms in $\mathfrak{K}$ decategorify to the corresponding linear maps between weight spaces that come from the $\dot{U}(\mathfrak{sl}_s)$–action on $\Lambda(\mathbb{C}_q^s \otimes \mathbb{C}_q^N)$.

The categorified invariants considered in this paper live in $\mathrm{Kom}^b(\mathfrak{K})$, the bounded homotopy 2–category of $\mathfrak{K}$.

Definition 2.17 Let $\mathfrak{K}$ be as above. Then $\mathrm{Kom}^b(\mathfrak{K})$, the bounded homotopy 2–category of $\mathfrak{K}$, consists of:

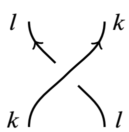
- Objects: the same as in $\mathfrak{K}$.
- 1–morphisms: bounded complexes of 1–morphisms in $\mathfrak{K}$ with differentials built out of 2–morphisms in $\mathfrak{K}$.
- 2–morphisms: chain maps built out of 2–morphisms in $\mathfrak{K}$ between bounded complexes of 1–morphisms, with homotopy equivalent chain maps identified.

$\mathrm{Kom}^b(\mathfrak{K})$ is $\mathbb{Z} \oplus \mathbb{Z}$–graded, the second $\mathbb{Z}$–grading being of homological nature. For the homological grading shift, denoted by $[l]$ in [6], we write the coefficient t^{-l} and for a q–grading shift $\langle k \rangle$ we write q^{-k}.

Given a projection of a framed oriented tangle T colored by irreducible $U_q(\mathfrak{sl}_N)$ representations and supposing that the Reshetikhin–Turaev invariant of T is a map $V(\lambda) \to V(\lambda')$, the categorified invariant is a chain complex with chain spaces built out of 1–morphisms in $\mathrm{Mor}(\lambda, \lambda')$ in $\mathfrak{K}$ and with differentials built out of 2–morphisms in–$\mathfrak{K}$. Different tangle projections produce different complexes; however, they are chain homotopy equivalent via chain maps built out of 2–morphisms.

We now describe the construction of the complex associated to a tangle projection. Again we prepare the tangle diagram by replacing downward-oriented strands by upward-oriented strands of complementary labeling and mark all critical points for the height function by tags.

Caps and cups can be described via (2-7) and (2-8) by the 1–morphisms replacing the divided powers in the quantum group. The main difference to the decategorified setting is that crossings do not get replaced by linear combination of webs as in (2-6), but by (grading-shifted) complexes, eg



$$\mapsto q^{k-l} E^{(k)} F^{(l)} \to \cdots \to q^1 E^{(k-l+1)} F^{(1)} \to E^{(k-l)} \quad \text{if } l \le k.$$

The differential in the first complex is given by the following composition of 2–morphisms:

$$\begin{aligned} q^r E^{(k-l+r)} F^{(r)} 1_{l-k} &\to q^{k-l+3r-2} E^{(k-l+r-1)} E 1_{l-k-2r} F F^{(r-1)} 1_{l-k} \\ &\cong q^{r-1} E^{(k-l+r-1)} E E_R F^{(r-1)} 1_{l-k} \\ &\to q^{r-1} E^{(k-l+r-1)} F^{(r-1)} 1_{l-k}. \end{aligned}$$

Here the first map is inclusion into the lowest q–grading summand and the last map is adjunction. The differential in the second complex is similar. In fact, the maps in the differential are unique as grading-preserving maps up to non-zero scalar multiple; see [6, Lemma 4.3]. These complexes are versions of the Chuang–Rouquier (or Rickard) complex [10].

As usual for link homology theories, planar composition of diagrams translates into taking formal tensor products of complexes under the replacement described above.

Cautis, Kamnitzer and Licata prove in [8] that the complexes associated to positive crossings are invertible in the homotopy category. An inspection of their proof (Proposition 5.4) shows that the inverse is the left (or right) adjoint of this complex. Informally speaking this means interchanging Es and Fs and inverting both the homological and the q–grading. The differentials in the resulting complex are again uniquely determined up to non-zero scalar multiple. In order to get invariance under Reidemeister moves of type 2, the inverse complex is used to replace negative crossings. Since taking the mirror image of tangle diagrams or webs also interchanges positive and negative crossings and Es and Fs, we have:

Proposition 2.18 *The complexes assigned to a tangle diagram and its mirror image have chain spaces that differ by interchanging Es and Fs and inverting all gradings.*

This matches well with the crossing formulas in the decategorified setting. Finally we get the categorified analogue of Theorem 2.14:

Theorem 2.19 [6, Proposition 7.9] *The complex associated to a framed oriented tangle diagram colored by fundamental representations via categorical skew Howe duality, as described above, is an invariant of the tangle in the homotopy 2–category* $\mathrm{Kom}^b(\mathfrak{K})$.

Proof The proof is similar to the proof in the decategorified setting via skew Howe duality; see [6, Proposition 7.4]. □

Remark 2.20 Cautis uses this setup to define categorifications of Reshetikhin–Turaev invariants for labelings by arbitrary irreducible representations. Every irreducible representation W of $U_q(\mathfrak{sl}_N)$ is the highest weight irreducible summand in some $\Lambda^{i_1}(\mathbb{C}_q^N) \otimes \cdots \otimes \Lambda^{i_l}(\mathbb{C}_q^N)$. The main idea is to replace a strand labeled by W by l parallel strands labeled by $\Lambda^{i_1}, \dots, \Lambda^{i_l}$ and to include a highest weight projector, also called a clasp, somewhere along the cable. Following ideas of Rozansky [45], Cautis shows that clasps can be realized as infinite twists in the cable. For details see [6, Sections 5–6].

In this paper, we are only interested in invariants of links colored by fundamental representations and thus we can get away with working with bounded complexes. In the more general case one allows complexes to be infinite in one direction and makes additional assumptions on the asymptotic behavior of the q–grading.

2.6 Link homology

To get from the categorified colored $\mathfrak{sl}_N$ invariant of a link that lives in a homotopy 2–category to computationally useful link invariants, an additional step is necessary. We recall that the categorified colored $\mathfrak{sl}_N$ invariant of a link L is a complex $\Psi(L)$ of endomorphisms of the highest weight object λ_h which corresponds to the sequence $(0,\dots,0,N,\dots,N)$ of labels on the vertical strands under the skew Howe map. Applying the functor $\mathrm{Hom}(1_{\lambda_h},\cdot)$ to this complex gives a complex of finite-dimensional graded vector spaces with grading-preserving differentials whose homology is the required bi-graded link invariant.

In computational practice, however, it is better to find a complex isomorphic to $\Psi(L)$ whose terms are grading shifts of the identity 1–morphism 1_{λ_h}. For chain spaces this is equivalent to evaluating a closed web to an element of the ground ring. We demonstrate this in Section 5. That the relations in $\mathfrak{K}$, imposed by the existence of a categorical $\mathfrak{sl}_N$–action, are sufficient to simplify the complex follows from the proof of the main theorem of Cautis, Kamnitzer and Morrison in [9], which is called Theorem 2.5 here. An example of a computation simplifying chain spaces and differentials can be found in Cautis [6, Section 10].

If the categorical 2–representation moreover satisfies that $\mathrm{Hom}(1_{\lambda_h},1_{\lambda_h})$ is one-dimensional and concentrated in q–grading 0, then the resulting link homology theory is completely determined by the defining relations of the categorified quantum group; see Cautis [6, Section 7.5] and Lauda, Queffelec and Rose [32, Section 4B]. Otherwise one can get deformed theories as in Lee [34], Gornik [17], and Rose and Wedrich [44].

3 The colored HOMFLY complexes of rational tangles

In this section we use categorical skew Howe duality to compute explicit twist rules that determine the chain spaces of the colored HOMFLY complexes of positive rational tangles labeled by fundamental representations of $\mathfrak{sl}_N$.

3.1 Rational tangles

All two-strand tangles that we consider have boundary points lying on the corners of the unit square in $\mathbb{R}^2$, which we denote by NE, SE, SW and NW. Such tangles can be vertically stacked and horizontally composed by gluing N$*$ to S$*$ and $*$E to $*$W labeled boundary points of the respective tangles. Consider the special two-strand tangle $T(1,1)$ that is given by a single crossing with the SW–NE strand lying on top. One can act on two-strand tangles by stacking $T(1,1)$ on top or by composing with

$T(1,1)$ on the right; we refer to these operations as *top twist* and *right twist* respectively. The two trivial tangles consisting of the obvious crossing-less matchings in N–S and W–E directions are called $T(0,1)$ and $T(1,0)$.

The closure of the set containing the trivial tangles $T(0,1)$ and $T(1,0)$ under the operations of top and right twisting is known as the set of *positive rational tangles*. Positive rational tangles are either trivial or can be described by a sequence of positive natural numbers $[a_1,\dots,a_r]$ which describes the construction process: start with $T(1,0)$ and add a_r top twists, then add a_{r-1} right twists, then again a_{r-2} top twists and so on. We can label such a tangle by the rational number $p/q>1$ with

$$\frac{p}{q}=a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots+\frac{1}{a_r}}}=[a_1,a_2,\dots,a_r].$$

More generally, a rational tangle is defined to be a proper embedding of two arcs α_1,α_2 in a three-ball B^3 with ends of the arcs lying on the boundary of B^3, such that there is a homeomorphism of pairs

$$(B^3,\alpha_1\sqcup\alpha_2)\to(D^2\times I,\{x,y\}\times I).$$

It is well-known that all rational tangles are equivalent (up to isotopy) to a (possibly trivial) rotation of either a positive rational tangle, as described above, or the mirror image of a positive rational tangle. For a proof of this fact see Kauffman and Lambropoulou [20]. We restrict our attention to positive rational tangles.

We consider positive rational tangles with additional data. Both arcs of the tangle are equipped with an orientation and a labeling by a fundamental representation Λ^k of $U_q(\mathfrak{sl}_N)$ for which we only record the natural number k. If we start with an orientation and a labeling on $T(0,1)$ or $T(1,0)$, this induces compatible data on tangles obtained by adding top and right twists. Unless explicitly stated otherwise we assume that the SW boundary point of a tangle is incoming and the corresponding strand is labeled by the color i while the other strand is labeled by a color j with $i\geq j$. Clearly this is preserved by adding top or right twists.

3.2 Objects in the chain complex

We now introduce basic webs that appear as objects in the colored HOMFLY complex of a positive rational tangle.

Definition 3.1 The first two rows in Figure 1 show our notation for the basic webs that arise in the colored HOMFLY complex of a positive rational tangle labeled by fundamental representations Λ^i and Λ^j with $i\geq j$. The six variants UP, UPs, OP, OPs, RI and RIs correspond to the six possible patterns of boundary data. The first two arguments, eg i and j in $\mathrm{UP}[i,j,k]$, refer to the colors on the boundary and the third is an index k, which we call the *weight* of the web. These webs form bases for the vector spaces of webs with matching boundary data, although we do not immediately use this fact. In intermediate steps we will also use the more general web $\mathrm{UP}[i,j,k,l]$ and the operation $\cdot \mapsto (\cdot)^r$ that rotates webs by π around the vertical axis, as shown in Figure 1.

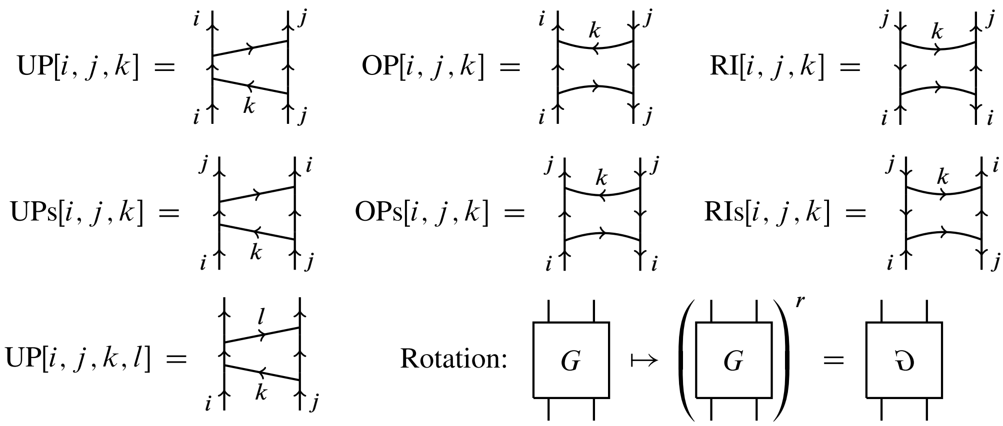


Figure 1

In Lemma 2.7 we have seen that two adjacent tags cancel. This shows that the operation of adding a tag close to the end of a strand in a web is an involution and we consider webs that are related by such tag additions as isomorphic. Because we work in the projective setting, it does not matter on which side the tag is placed. Furthermore, the relations (2-3) in the spider category say that tags slide past trivalent vertices. In particular, we obtain isomorphisms between the special webs introduced above.

Lemma 3.2 *In the projective $U_q(\mathfrak{sl}_N)$ representation category we have:*

(3-1) $$\mathrm{OPs}[i,j,k] \cong \mathrm{UP}[N-i,i,i-j+k,k]^r.$$

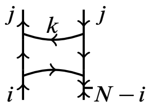
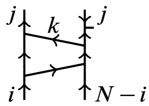
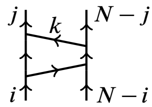
(3-2) $$\mathrm{RI}[i,j,k] \cong \mathrm{UP}[i,N-i,N-j-i+k,k].$$

(3-3) $$\mathrm{OP}[i,j,k] \cong \mathrm{UP}[N-j,i,k]^r.$$

(3-4) $$\mathrm{RIs}[i,j,k] \cong \mathrm{UPs}[N-j,i,k]^r.$$

(3-5) $$\mathrm{OPs}[i,j,k] \cong \mathrm{RI}[N-i,j,k]^r.$$

Proof For (3-1) we compute:

$\mathrm{OPs}[i,j,k] \cong$  $=$  $\cong$  $= \mathrm{UP}[N-i, i, i-j+k, k]^r.$

The other cases are very similar. □

3.3 Computation of the twist rules

In order to determine the Poincaré polynomial associated to a colored positive rational tangle (or alternatively the chain spaces in the colored HOMFLY complex), we compute the result of applying a top or right twist to one of the basic webs from Definition 3.1. It turns out that the dependence of these twist rules on $N \gg 0$ and i for $i \geq j$ can be hidden by introducing the variables $a = q^N$ and $s = q^{i-j}$. Furthermore, we omit global grading shifts in the formulation of the twist rules.

Definition 3.3 The quantum binomial coefficient $\begin{bmatrix} h \\ k \end{bmatrix}$ is a Laurent polynomial in q that is supported in degrees $-k(h-k)$ up to $k(h-k)$. We thus define the variants

$$\begin{bmatrix} h \\ k \end{bmatrix}^+ := q^{k(h-k)} \begin{bmatrix} h \\ k \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} h \\ k \end{bmatrix}^- := q^{-k(h-k)} \begin{bmatrix} h \\ k \end{bmatrix},$$

which have lowest and highest degree 0, respectively.

The following proposition relies heavily on results of Cautis [6, Section 4] that describe how (generalizations of) crossing complexes can absorb Es and Fs. To be more precise, the result of composing a crossing complex with a divided power 1–morphism is a complex τ or τ' which again has chain spaces that are products of Es and Fs and q–grading-preserving differentials that are uniquely determined up to a non-zero scalar. However, in general they are not invertible in the homotopy 2–category.

Proposition 3.4 *Adding top twists to* $\mathrm{UP}[i,j,k]$, $\mathrm{UPs}[i,j,k]$ *and* $\mathrm{OPs}[i,j,k]$ *has the following effect on the level of Poincaré polynomials:*

$$\mathrm{TUP}[i,j,k] \cong \sum_{h=k}^{j} t^{-h} s^k q^{k^2+h} \begin{bmatrix} h \\ k \end{bmatrix}^+ \mathrm{UPs}[i,j,h]. \tag{3-6}$$

$$\mathrm{TUPs}[i,j,k] \cong \sum_{h=k}^{j} t^{-h} s^h q^{k^2+h} \begin{bmatrix} h \\ k \end{bmatrix}^+ \mathrm{UP}[i,j,h]. \tag{3-7}$$

$$\mathrm{TOPs}[i,j,k] \cong \sum_{h=k}^{j} t^{-h} a^k s^{h-k} q^{k(k-2j)+h} \begin{bmatrix} h \\ k \end{bmatrix}^+ \mathrm{RI}[i,j,h]. \tag{3-8}$$

Proof For the first isomorphism we just translate into Cautis' notation and apply [6, Proposition 4.5]:

$$\begin{aligned}\mathrm{TUP}[i,j,k] &= 1_{i-j}\tau_{i-j}1_{j-i}E^{(k)}F^{(k)}\\ &\cong t^{-k}q^{k(i-j+k+1)}1_{i-j}\tau_{i-j+k}1_{j-i-2k}F^{(k)}\\ &\cong \sum_{r=0}^{j-k} t^{-(k+r)}q^{k(i-j+k+1)+r(k+1)}1_{i-j}E^{(i-j+k+r)}F^{(r)}F^{(k)}1_{j-i}.\end{aligned}$$

After replacing $F^{(r)}F^{(k)}$ by

$$\begin{bmatrix} r+k \\ k \end{bmatrix} F^{(r+k)} = q^{-kr}\begin{bmatrix} r+k \\ k \end{bmatrix}^{+} F^{(r+k)}$$

via relation (xii) and re-parametrizing the summation, this yields (3-6). Next, we compute

$$\begin{aligned}\mathrm{TUPs}[i,j,k] &= 1_{j-i}\tau'_{i-j}1_{i-j}E^{(i-j+k)}F^{(k)}\\ &\cong 1_{j-i}\tau'_{i-j}1_{i-j}F^{(k)}1_{i-j+2k}E^{(i-j+k)}\\ &\cong t^{-k}q^{k(i-j+k+1)}1_{j-i}E^{(k)}1_{j-i-2k}\tau'_{i-j+2k}1_{i-j+2k}E^{(i-j+k)}\\ &\cong t^{-k}q^{k(i-j+k+1)}1_{j-i}E^{(k)}1_{j-i-2k}\tau'_{k}1_{j-i}\\ &= \sum_{r=0}^{j-k} t^{-(k+r)}q^{(k+r)(i-j+k+1)}1_{j-i}E^{(k)}E^{(r)}F^{(k+r)}1_{j-i}\\ &\cong \sum_{h=k}^{j} t^{-h}q^{h(i-j)}q^{k^2+h}\begin{bmatrix} h \\ k \end{bmatrix}^{+} E^{(h)}F^{(h)}1_{j-i},\end{aligned}$$

which gives (3-7). Here we have used relations (xi), [6, Corollary 4.6 and Proposition 4.5] and relation (xii).

For (3-8) we first use isomorphism (3-1) together with the fact that tags slide through crossing complexes. Then we apply [6, Corollary 4.6 and Proposition 4.5] and finally isomorphism (3-2) and relation (xii):

$$\begin{aligned}&\mathrm{TOPs}[i,j,k]\\ &\quad\cong \mathrm{TUP}[N-i,i,i-j+k,k]^{r}\\ &\quad= 1_{2j-N}\tau'_{N-2j}F^{(k)}E^{(i-j+k)}1_{N-2i}\\ &\quad\cong t^{-k}q^{k(N-2j+k+1)}1_{2j-N}E^{(k)}1_{2j-N-2k}\tau'_{N-2j+2k}E^{(i-j+k)}1_{N-2i}\\ &\quad\cong t^{-k}q^{k(N-2j+k+1)}1_{2j-N}E^{(k)}1_{2j-N-2k}\tau'_{N-j-i+k}1_{N-2i}\end{aligned}$$

$$= \sum_{h=k}^{j} t^{-h} a^k q^{(h-k)(i-j)} q^{k(k-2j)+h} \begin{bmatrix} h \\ k \end{bmatrix}^+ 1_{2j-N} E^{(h)} F^{(N-j-i+h)} 1_{N-2i}$$

$$\cong \sum_{h=k}^{j} t^{-h} a^k q^{(h-k)(i-j)} q^{k(k-2j)+h} \begin{bmatrix} h \\ k \end{bmatrix}^+ \mathrm{RI}[i,j,h]. \qquad \square$$

Corollary 3.5 *Adding top twists to* $\mathrm{OP}[i,j,k]$, $\mathrm{RI}[i,j,k]$ *and* $\mathrm{RIs}[i,j,k]$ *has the following effect on the level of Poincaré polynomials:*

$$\mathrm{TOP}[i,j,k] \cong \sum_{h=k}^{j} t^{-h} a^k s^{-k} q^{k(k-2j)+h} \begin{bmatrix} h \\ k \end{bmatrix}^+ \mathrm{RIs}[i,j,h]. \tag{3-9}$$

$$\mathrm{TRI}[i,j,k] \cong \sum_{h=k}^{j} t^{-h} a^h s^{k-h} q^{k^2+h(1-2j)} \begin{bmatrix} h \\ k \end{bmatrix}^+ \mathrm{OPs}[i,j,h]. \tag{3-10}$$

$$\mathrm{TRIs}[i,j,k] \cong \sum_{h=k}^{j} t^{-h} a^h s^{-h} q^{k^2+h(1-2j)} \begin{bmatrix} h \\ k \end{bmatrix}^+ \mathrm{OP}[i,j,h]. \tag{3-11}$$

Proof For (3-9):

$$\begin{aligned} \mathrm{TOP}[i,j,k] &\cong \mathrm{TUP}[N-j,i,k]^r \\ &\cong \sum_{h=k}^{i} t^{-h} q^{k(N-i-j)} q^{k^2+h} \begin{bmatrix} h \\ k \end{bmatrix}^+ \mathrm{UPs}[N-j,i,h]^r \\ &\cong \sum_{h=k}^{i} t^{-h} a^k q^{-k(i-j)} q^{k(k-2j)+h} \begin{bmatrix} h \\ k \end{bmatrix}^+ \mathrm{RIs}[i,j,h]. \end{aligned}$$

Here we have used isomorphism (3-3) and (3-4) and (3-6). The sum gets truncated because $\mathrm{UPs}[N-j,i,h]^r \cong 0 \cong \mathrm{RIs}[i,j,h]$ for $h > j$.

For (3-10):

$$\begin{aligned} \mathrm{TRI}[i,j,k] &\cong \mathrm{TOPs}[N-i,j,k]^r \\ &= \sum_{h=k}^{j} t^{-h} a^h q^{(k-h)(i-j)} q^{k^2+h(1-2j)} \begin{bmatrix} h \\ k \end{bmatrix}^+ \mathrm{RI}[N-i,j,h]^r \\ &\cong \sum_{h=k}^{j} t^{-h} a^h q^{(k-h)(i-j)} q^{k^2+h(1-2j)} \begin{bmatrix} h \\ k \end{bmatrix}^+ \mathrm{OPs}[i,j,h]. \end{aligned}$$

Here we have used isomorphism (3-5) and (3-8).

For (3-11):

$$\begin{aligned}\mathrm{TRIs}[i,j,k] &\cong \mathrm{TUPs}[N-j,i,k]^r\\ &\cong \sum_{h=k}^{i} t^{-h} q^{h(N-j-i)} q^{k^2+h} \begin{bmatrix} h \\ k \end{bmatrix}^+ \mathrm{UP}[N-j,i,h]^r\\ &\cong \sum_{h=k}^{i} t^{-h} a^h q^{-h(i-j)} q^{k^2+h(1-2j)} \begin{bmatrix} h \\ k \end{bmatrix}^+ \mathrm{OP}[i,j,h].\end{aligned}$$

Here we have used isomorphisms (3-4) and (3-3) from Lemma 3.2 and (3-7). The sum gets truncated because $\mathrm{UP}[N-j,i,h]^r \cong 0 \cong \mathrm{OP}[i,j,h]$ for $h > j$. □

Corollary 3.6 *We compute the right twist rules by reflecting top twist rules:*

$$\mathrm{RUP}[i,j,k] \cong \sum_{h=0}^{k} t^{-h} a^h s^{k-h} q^{k(2j-k)+h(1-2j)} \begin{bmatrix} j-h \\ k-h \end{bmatrix}^- \mathrm{OP}[i,j,h]. \tag{3-12}$$

$$\mathrm{RUPs}[i,j,k] \cong \sum_{h=0}^{k} t^{-h} a^h s^{-h} q^{k(2j-k)+h(1-2j)} \begin{bmatrix} j-h \\ k-h \end{bmatrix}^- \mathrm{OPs}[i,j,h]. \tag{3-13}$$

$$\mathrm{ROP}[i,j,k] \cong \sum_{h=0}^{k} t^{-h} a^k s^{h-k} q^{-k^2+h} \begin{bmatrix} j-h \\ k-h \end{bmatrix}^- \mathrm{UP}[i,j,h]. \tag{3-14}$$

$$\mathrm{ROPs}[i,j,k] \cong \sum_{h=0}^{k} t^{-h} a^k s^{-k} q^{-k^2+h} \begin{bmatrix} j-h \\ k-h \end{bmatrix}^- \mathrm{UPs}[i,j,h]. \tag{3-15}$$

$$\mathrm{RRI}[i,j,k] \cong \sum_{h=0}^{k} t^{-h} s^k q^{-k(k-2j)+h} \begin{bmatrix} j-h \\ k-h \end{bmatrix}^- \mathrm{RIs}[i,j,h]. \tag{3-16}$$

$$\mathrm{RRIs}[i,j,k] \cong \sum_{h=0}^{k} t^{-h} s^h q^{-k(k-2j)+h} \begin{bmatrix} j-h \\ k-h \end{bmatrix}^- \mathrm{RI}[i,j,h]. \tag{3-17}$$

Proof Note that after reflecting across the plane spanned by the SW–NE diagonal and the normal to the blackboard, the problem of adding a right twist transforms into adding a top twist to the reflected web. Proposition 2.18 extends to the case of complexes associated to knotted webs and thus under reflection t, q, a and s get replaced by their inverses and $\mathrm{UP}(s)[i,j,k]$ and $\mathrm{RI}(s)[i,j,j-k]$ as well as $\mathrm{OP}[i,j,k]$ and $\mathrm{OPs}[i,j,j-k]$ are interchanged. □

Remark 3.7 An essential feature of these rules is that their dependence on N and the higher color i can be hidden by introducing $a = q^N$ and $s = q^{i-j}$. This proves Theorem 1.7.

4 A geometric model for the colored HOMFLY complex of a rational tangle

In this section we introduce a geometric algorithm for computing the chain spaces in the colored HOMFLY complex of a positive rational tangle $T(p,q)$ labeled with fundamental $\mathfrak{sl}_N$ representations Λ^i and Λ^j with $i \geq j$.

4.1 The geometric setup

We start by drawing a picture that resembles a 2–bridge diagram of the denominator closure of the tangle. Let $\sigma \in \{1,-1\}$ be the parity of the length of the continued fraction expansion of p/q and $\operatorname{par}(p) \in \{1,-1\}$ the parity of p.

Draw the intervals $[-2,-1]$ and $[1,2]$ on the real axis in $\mathbb{C}$ on a piece of paper, partition them into p parts of equal size and mark the divisions between the parts by small vertical line segments. For both divided intervals, number the $2p$ endpoints of line segments by $0,\dots,2p-1 \in \mathbb{Z}/(2p)$, starting from the points 1 and -1 respectively and proceeding clockwise or anticlockwise, depending on $\sigma = \mp 1$. Next, draw an arc α starting at the 0–labeled point $x = -\operatorname{par}(p)\sigma$, then proceeding to the point labeled q on the other interval, intersecting the interval transversely, proceeding to the point labeled $2q$ and so on until it hits the set $\{-2,-1,1,2\}$ again. We require α to have no self-intersections and the minimal possible number of intersections with the two intervals. Further, we fix the picture uniquely up to isotopy by requiring that α has no intersections with the interval $(-\infty,-2]$ on the real axis if $\sigma = -1$ and no intersections with the interval $[2,\infty)$ on the real axis if $\sigma = 1$.

Next we draw the imaginary axis, labeled l_p, and a parallel of it labeled l_q that intersects one of the intervals $[1,2]$ and $[-2,-1]$ on the real axis, depending on $\sigma = \mp 1$, and has the minimal possible number of intersections with the arc α. The two vertical lines l_p and l_q then have p and q intersections with α, respectively. We call these intersections with the left or right vertical *left* and *right primary intersections*. Figure 2 shows the case $\frac{p}{q} = \frac{5}{2}$ and how this picture can be interpreted as a projection of the tangle.

The diagrams we draw are essentially 2–bridge diagrams of the denominator closure of the positive rational tangle, with one over-bridge erased. We distinguish the cases according to σ and $\operatorname{par}(p)$ and choose a more complicated way of relating the drawing

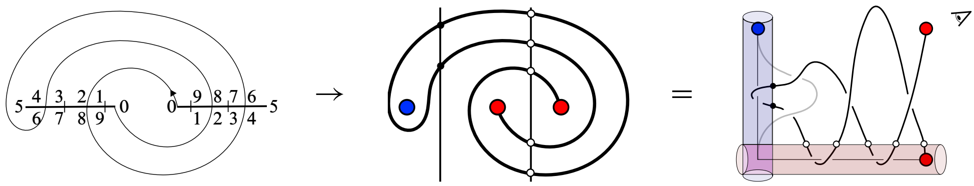
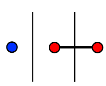
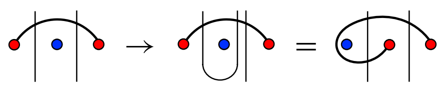


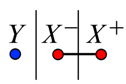
Figure 2

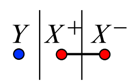
to the rational p/q in order to get a simpler relation between the drawing and the continued fraction expansion of p/q. We explain this in the following lemma, whose proof is left to the reader.

Lemma 4.1 *The picture can inductively be constructed in the following way. Start with the trivial diagram. For a top twist, bend the left vertical towards the right one, then flip over the left-hand side of the diagram into the middle to straighten the left vertical again. For a right twist, do the analogue for the right vertical. The trivial diagram and a top twist are illustrated below:*

Trivial:  *Top twist:* 

Definition 4.2 We begin the computation of the colored HOMFLY complex by starting with $\mathrm{UP}[i,j,0]$ or $\mathrm{OP}[i,j,0]$, depending on the orientation of the tangle, and then applying a sequence of top and right twists. In these two cases we label the three distinguished points by X^+, X^- and Y as in the following figures:

Start configuration for UP: 

Start configuration for OP: 

Corollary 4.3 *Via the interpretation of top and right twisting as bending verticals, the six permutations of the labels X^+, X^- and Y that arise in tangle diagrams correspond to the six types of webs introduced above. This is shown in Figure 3.*

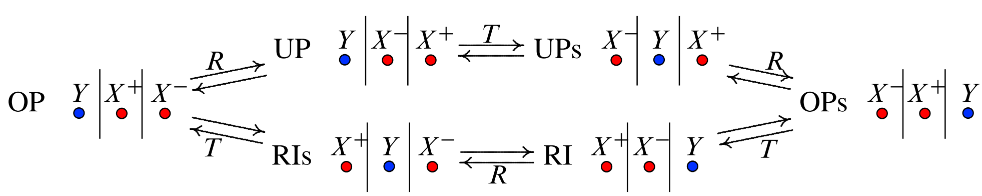


Figure 3

4.2 The main theorem

Definition 4.4 In order to find the generators of weight h for the colored HOMFLY complex with respect to colors i and j in the geometric picture, we replace the left vertical by h parallel copies and the right vertical by $j-h$ parallel copies. We call the first kind of parallels, $w_1, \dots, w_h$, the *weighted verticals* and the others, $u_1, \dots u_{j-h}$, *unweighted verticals*.

Theorem 4.5 *The generators of weight h in the colored HOMFLY complex with respect to colors i and j are in bijection with j–tuples*

$$(x_1, \dots, x_j) \in (w_1 \cap \alpha) \times \dots \times (w_h \cap \alpha) \times (u_1 \cap \alpha) \times \dots \times (u_{j-h} \cap \alpha)$$

of intersections points of verticals with α. To find the relative gradings of these generators, it is sufficient to determine:

(1) *The grading difference of two generators with equal weight that differ only in one coordinate.*

(2) *For each h, the grading difference of some special generators of weights h and $h-1$.*

By iteration, rule (1) *determines the relative gradings of all generators of the same weight. Rule* (2) *then fixes the relative gradings between weight groups.*

Ad (1) *The following determines the grading difference between two generators of weight h that differ only in one coordinate. Let $\overline{x} = (x_1, \dots, x_k, \dots, x_j)$ and $\overline{x}' = (x_1, \dots, x'_k, \dots, x_j)$ be the two generators with $x_k \neq x'_k$. Let α' be the segment of α starting at x_k and ending at x'_k, β the segment of the k^{th} vertical starting at x_k and ending at x'_k and D the domain enclosed by α' and β in $\mathbb{R}^2$. D is a singular 2–chain, ie a formal sum of closed discs with multiplicities in $\mathbb{Z}$ determined by the winding of $-\beta\alpha'$ around interior points. In particular, D can be written as a $\mathbb{Z}$–linear combination of* simple discs *which we define to be the closures of bounded path-components of the complement of the union of α with one vertical, equipped with the standard orientation. The grading difference between $\overline{x}$ and $\overline{x}'$ is the product of two terms:*

- *(Additive part Q) The graded intersection number of D with the set of distinguished points $\{X^+, X^-, Y\}$, where non-empty intersections with a simple disc D_s count as*

$$D_s \cdot X^+ = \frac{a^2}{tq^{4j-2}s^2}, \quad D_s \cdot X^- = \frac{q^2 s^2}{t}, \quad D_s \cdot Y = \frac{q^2}{t}.$$

- *(Non-additive part $q-Q$) The graded intersection number of D with the other intersection points $\{x_1,\dots,x_{k-1},x_{k+1},\dots x_j\}$, with a simple disc D_s contributing*
$$D_s\cdot x_r=\begin{cases} q^4 & \textit{if} \quad x_r\in D_s^\circ,\\ q^2 & \textit{if} \quad x_r\in \partial D_s,\\ 1 & \textit{else.}\end{cases}$$

Ad (2) *The following determines the grading shift between two generators of weight h and $h-1$ that agree outside the h^{th} coordinate and whose h^{th} coordinates are related by sliding the h^{th} vertical (which is the innermost vertical) from the weighted to the unweighted side. We can think of this sliding as a geometric realization of the differential tying together the two generators. The grading of this differential depends on which distinguished point the vertical crosses while sliding:*

$$\textit{Vertical sliding across}\begin{cases}Y\\ X^-\\ X^+\end{cases}\textit{as in}\begin{cases}\text{UPs } \textit{and} \text{ RIs}\\ \text{UP } \textit{and} \text{ RI}\\ \text{OP } \textit{and} \text{ OPs}\end{cases}\textit{causes shift by}\begin{cases}t/q,\\ t/qs,\\ tq^{2j-1}s/a.\end{cases}$$

If two generators are related by sliding over the innermost vertical, we say that the generators are related by a simple slide*.*

The proof of this theorem occupies the rest of this section and is split into several lemmata. Then Theorem 1.9 follows immediately from the fact that the a–, s–, t– and Q–gradings are additive.

Lemma 4.6 *The grading differences stated in the theorem define a $\mathbb{Z}^4$–grading on generators with shifts denoted by powers of a, q, t and s.*

Proof While rule (2) does not cause any ambiguities, there are several ways to compute the grading difference of two generators of equal weight by repeated application of rule (1). We need to check that all of these ways yield the same result. This is easily seen for the additive part. It remains to check the non-additive part.

Consider first the case of generators that differ in exactly two coordinates, say $\overline{x}=(x_1,x_2,\dots)$ and $\overline{y}=(y_1,y_2,\dots)$, where the two domains D_1 and D_2 connecting x_1 to y_1 and x_2 to y_2, respectively, are simple discs, possibly with the opposite orientation. We need to check that the non-additive part of the grading that we get from $D_2\circ D_1\colon \overline{x}\to(y_1,x_2,\dots)\to\overline{y}$ is the same as the one from $D_1\circ D_2\colon \overline{x}\to(x_1,y_2,\dots)\to\overline{y}$. It is sufficient to consider the non-additive component that comes from intersections with the first two coordinates, since all other contributions will be equal a priori. If $D_1\cap D_2=\varnothing$, then there are no non-additive grading shifts and we are done. Otherwise we distinguish two groups of cases, depending on whether the simple discs are of shape D or $\mathrm{Ɑ}$:

(1) D_1 and D_2 intersect and are both of shape D (or of shape ᗡ). If $x_2, y_2 \in D_1$ or $x_1, y_1 \in D_2$, then both ways of composing D_1 and D_2 produce non-additive grading shifts at different intersection points, but of equal value $q^{\pm 2}$ or $q^{\pm 4}$. Otherwise there are no non-additive shifts at all

(2) D_1 and D_2 intersect and one is of shape D while the other one is of shape ᗡ. Here we have three different cases, which are easily checked:
- One disc D_i contains both intersection points x_j, y_j (which are linked by the other disc D_j) as interior points.
- One disc D_i contains x_j, y_j, exactly one in its interior, the other one on its boundary.
- D_1 contains exactly one of x_2, y_2 and D_2 contains exactly one of x_1, y_1.

In the general case of weight-h generators that might differ in several coordinates, every way of getting from one generator to the other by transforming one coordinate at a time can be described as a sequence of changes via simple discs. By the previous argument, we can permute these changes and cancel redundant pairs of inverse changes without altering the grading shift. This way one can reach a unique reduced expression which consists of the minimal number of changes via simple discs and which is ordered by the indices of the coordinates in which the changes take place. □

It is clear that our graphical method of calculating the chain spaces in the colored HOMFLY complex produces only one generator in the case of the trivial tangle. With this as the start of an induction proof it suffices to show that the geometric algorithm observes the twist rules from Section 3.3.

To prepare some notation for the following definition, we look at a top twist applied to a diagram P_1, producing a diagram P_2. If we only draw the verticals l_p and l_q for the moment, we can distinguish three sets of primary intersections: L_1, the intersections with the left vertical in P_1; R, the intersections with the right vertical in P_1; and $L_2 = L_1 \sqcup R'$, the set of intersections with the left vertical in P_2. Note that R' is in bijection with R and R also labels the intersections with the right vertical in P_2.

Generators $(x_1, \dots, x_j) \in (w_1 \cap \alpha) \times \cdots \times (w_h \cap \alpha) \times (u_1 \cap \alpha) \times \cdots \times (u_{j-h} \cap \alpha)$ of the colored HOMFLY complex have an equivalent description by $(x'_1, \dots, x'_j)$ with $x'_1, \dots, x'_h$ being left primary intersections and $x'_{h+1}, \dots, x'_j$ right primary intersections, where the order remembers the order in which these primary intersections are placed on parallels of the two verticals.

Definition 4.7 (Top twist case) Let $\overline{x} = (x'_1, \dots, x'_j)$ be a generator of the colored HOMFLY complex of P_2, which is obtained from P_1 by a top twist. Partition the left primary intersections $x'_1, \dots, x'_h$ into two subsequences according to whether they belong to L_1 or R'. Next, reverse the subsequence in L_1 to get l, concatenate it with the sequence r_1 in R corresponding to the sequence in R' and finally append the sequence $r_2 = x'_{h+1}, \dots, x'_j$. The sequence $l \cdot r_1 \cdot r_2$ represents a generator for the complex of P_1 and we call it the *parent* of its *child* $\overline{x}$. The definition of the relation between child and parent is analogous in the case of a right twist.

One can think of the top twisting process as bending over the left verticals towards the right and allowing them to steal verticals and intersection points from the right verticals. Similarly, in a right twist, the right verticals bend over to the left and steal verticals and intersection points from the left verticals. The twist rules describe how a generator — the parent — in P_1 gives rise to several generators — its children — in P_2.

The relative gradings of generators in P_2 are influenced by three factors in the geometric picture, which we will treat in this order:

(1) How the new verticals that are stolen from the other side get sorted in between the old verticals. This corresponds to the quantum binomial coefficients in the twist rules.

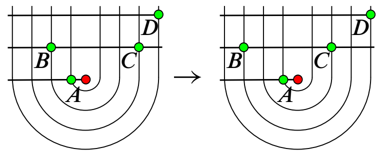
(2) How many verticals are stolen. This corresponds to the coefficients depending on $h-k$ in the twist rules.

(3) The weight of the parent in P_1. This corresponds to coefficients depending on k in the twist rules.

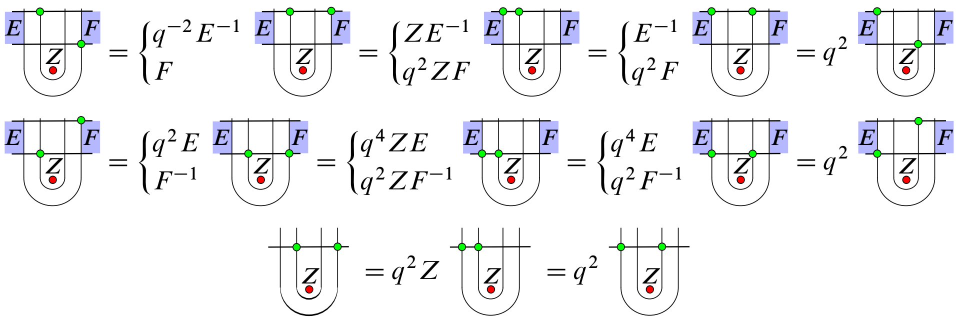
Lemma 4.8 *The generators corresponding to the possible ways of sorting $h-k$ stolen verticals (with intersections) in between k existing verticals have relative q–gradings as described by the quantum binomial coefficient $\left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right]$. In the picture with bent verticals, the lowest q–grading configuration has all intersection points as far right as possible and the highest q–grading configuration has all intersection points as far left as possible.*

Proof We first look at the relevant region where the weighted verticals are bent over. The figure below shows a typical situation. If we assume that it is the result of a top twist, then we see a generator represented by a tuple $(\dots, A, B, C, D, \dots)$ on the left-hand side.

To prove the statement of the lemma, it suffices to show that if an intersection B on the left of the bend and an intersection C on the right of the bend, lying on adjacent verticals, swap verticals such that C moves left, this causes a grading shift of q^2. The result of such a swap is shown on the right-hand side of the figure. If it is the result of a top twist, the generator is represented by the tuple $(\ldots, A, C, B, D, \ldots)$.



We distinguish three cases, two of which have two sub-cases each, depending on which domain (E or F) outside the local picture determines relative gradings of generators in the local picture:



□

Lemma 4.9 *The simple slides in the geometric picture induce the grading shifts required by the twist rules.*

Proof First note that it is sufficient to check this for simple slides between lowest q–grading configurations in the case of top twists and between highest q–grading configurations in the case of right twists. In writing the twist rules, we have renormalized the quantum binomial coefficient such that this relevant grading is 0. Given this, it is straightforward to check that simple slides induce the same grading shifts as described by the twist rules. For example, in

$$\mathrm{TUP}[i,j,k] \cong \sum_{h=k}^{j} t^{-h} s^k q^{k^2+h} \begin{bmatrix} h \\ k \end{bmatrix}^+ \mathrm{UPs}[i,j,h]$$

we see the dependence on h is in a shift of q/t when passing from a weight-$(h-1)$ generator to the weight-h generator which is related by a simple slide, as expected for generators of type UPs. □

So far we have shown that the geometric algorithm accurately reproduces the relative gradings between the children of each parent. It remains to understand the relative gradings between children of different parents.

Definition 4.10 Each parent has a distinguished child that we call its *clone*. If the parent is represented by the sequences l, r of left and right primary intersections, then the clone is the child that is represented by the reverse of l concatenated with r. It is the child that arises by stealing zero verticals.

Lemma 4.11 *Clones of equal weight have the same relative gradings as their parents.*

Proof Clones of equal weight have parents of equal weight. Their relative grading is computed by domains in P_1 which survive the twisting to P_2. The same domains thus compute the same relative gradings between clones. □

Combining the statement of the lemma with previous results we see that the geometric algorithm correctly computes the relative gradings between all children of parents of a certain weight. The last step in the proof of the main theorem is, therefore, to find the geometrically determined relative gradings of a set of children of parents of all possible weights and compare them with the twist rules.

Lemma 4.12 *Let p_0 be an arbitrary weight-0 parent and c_0 its clone. Write p_k for the weight-k parent that is related to p_0 by simple slides and c_k for its clone. Then the geometric algorithm correctly computes the relative gradings of the generators $c_0, \dots, c_j$.*

Proof We choose temporary absolute gradings on the complexes associated to P_1 and P_2 and align them by requiring that passing from p_0 to its clone c_0 shifts grading by the amount described by the twist rules. We then check for each of the twelve cases (corresponding to the twelve rules) inductively on k that the known shift from p_{k-1} to c_{k-1} together with rules (1) and (2) from the statement of Theorem 4.5 correctly compute the shift from p_k to c_k. We give two examples, one for a top twist and one for a right twist, and omit ten very similar cases.

Case TUP
$$\mathrm{TUP}[i,j,k] \cong \sum_{h=k}^{j} t^{-h} s^k q^{k^2+h} \begin{bmatrix} h \\ k \end{bmatrix}^+ \mathrm{UPs}[i,j,h].$$

In this case the parent is $p_k = \mathrm{UP}[i,j,k]$ and the clone is the summand for $h = k$ on the right-hand side of the twist rule above. In Figure 4 we have drawn the parents p_{k-1} and p_k in the first row, their clones c_{k-1} and c_k on the left and right in the second row, and intermediate diagrams, computing the grading shift from c_{k-1} to c_k, in the middle. Green dots on thick verticals represent tuples of intersections on tuples of verticals with size written above. The lower left shift comes from sliding the thin vertical to the left. The central lower shift comes from re-ordering the bent verticals. The lower right shift comes from the small disc containing X^-. The left vertical and the upper central grading shifts are known and the right vertical grading shift is determined by the commutativity of the diagram. It agrees with the shift described by the rule for TUP.

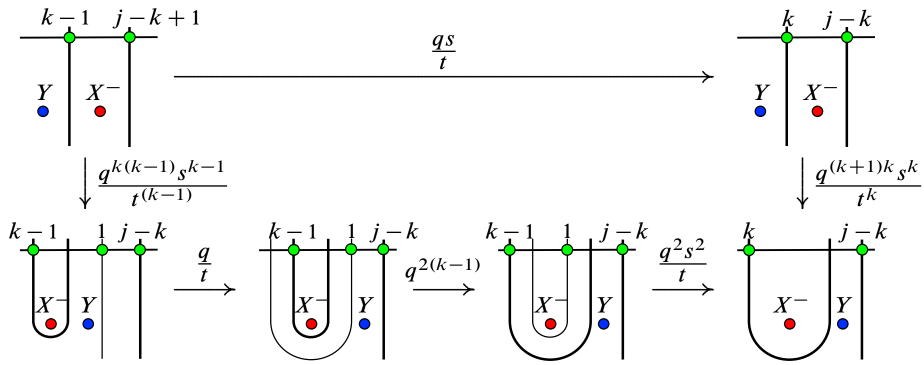


Figure 4

Case ROP For this situation see Figure 5. The known shift is on the right vertical arrow, the new shift is on the left vertical arrow and it is correctly computed by commutativity of the diagram. □

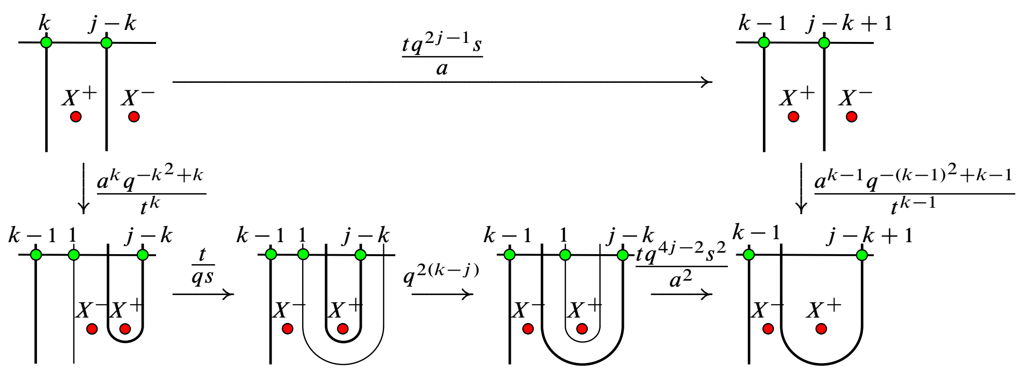


Figure 5

4.3 Differentials

We have mentioned in Section 2.5 that the differentials in the crossing complex are essentially uniquely determined as the composition of an inclusion and the adjunction between Es and Fs. The same holds for all complexes in the twist rules of Section 3.3 by [6, Lemma 4.3]. The colored HOMFLY complexes are simplifications of tensor products of crossing complexes. In our approach, these simplifications are computed iteratively by adding a crossing at a time via the twist rules.

It follows from the proof of Theorem 4.5 that the component of the differential coming from the last crossing added corresponds to sliding the innermost vertical from left to right, as described in the statement of the theorem. However, this has to be understood as a differential that ties together the groups of generators that differ only by re-ordering verticals as in Lemma 4.8. The actual differential between generators then can be computed as a composition of an inclusion, the differential on groups of generators and a projection.

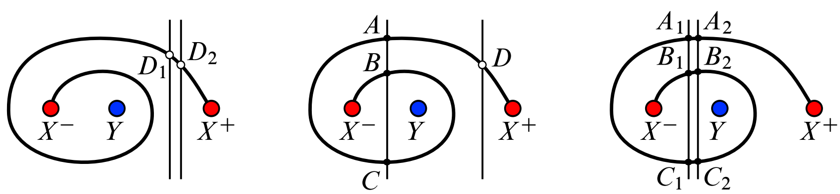
The differentials coming from previous crossings can be identified with oppositely oriented simple discs in the picture, with boundary on α and verticals and containing one of the points X^+, X^- and Y. As before, these differentials map between groups of generators that are related by re-ordering verticals. We have only managed to compute these differentials explicitly in simple cases, but we expect that there are essentially only three types of morphisms involved, which depend on the special point X^+, X^- or Y that is contained in the corresponding simple disc.

4.4 Examples

In this section we give two example computations. First, to demonstrate the geometric algorithm we compute the chain groups in the colored HOMFLY complex of the rational tangle $T(3,1)$ with respect to colors (Λ^i, Λ^2) with $i \geq 2$.

Second, we compute the Poincaré polynomial of the $\mathfrak{sl}_N$ link homology of the (Λ^i, Λ^j)–colored Hopf link with $i > j$, reduced with respect to Λ^i.

Example 4.13 The following figure shows the intersection points used for generators of weight 0, 1 and 2 in the colored HOMFLY complex of the rational tangle $T(3,1)$:



The generators of the colored HOMFLY complex of this tangle are pairs of intersection points, one taken from each vertical. Here we have 13 generators, which are shown in Figure 6 together with the expected differentials between them. Vertical and horizontal arrows indicate differentials coming from the first and second crossings, respectively, which correspond to the oppositely oriented simple discs containing the special points Y and X^-. The diagonal arrows represent the differential coming from the last crossing; in the geometric picture this corresponds to sliding a left vertical to the right across Y.

$$
\begin{tikzcd}
B_1B_2 \arrow[d] & & & BD \arrow[d] & & D_1D_2 \\
\begin{matrix} B_1C_2 \\ C_1B_2 \end{matrix} \arrow[r] \arrow[d] & \begin{matrix} A_1B_2 \\ B_1A_2 \end{matrix} \arrow[d] \arrow[urr] & & CD \arrow[r] & AD \arrow[ur] & \\
C_1C_2 \arrow[r] & \begin{matrix} A_1C_2 \\ C_1A_2 \end{matrix} \arrow[r] \arrow[urr] & A_1A_2 \arrow[urr] & & &
\end{tikzcd}
$$

Figure 6

We apply the rules of Theorem 4.5 in order to find the relative gradings of these generators. The non-additive part of the q–grading comes from re-ordering verticals, as shown in Lemma 4.8. For example, there is a shift of q^2 from C_1A_2 to A_1C_2. In the figure we have written the lower q–grading configurations C_1A_2, B_1A_2 and C_1B_2 below the corresponding other configurations.

The additive part of the grading can be read off from the type of arrows in the figure. Vertical, horizontal and diagonal arrows induce grading shifts of t/q^2, t/q^2s^2 and t/q, respectively. Here one has to be careful about identifying the generators between which the simple disc determines the grading shift without non-additive component. For example, in the lowest row in the figure the left arrow induces a shift of t/q^2s^2 between C_1C_2 and A_1C_2 while the right arrow induces a shift of equal magnitude between C_1A_2 and A_1A_2.

If we normalize the invariant by requiring D_1D_2 to lie in grading $a^0q^0s^0t^0$, then the other generators have gradings as shown in Table 1. The non-additive part of the grading is written in bold font.

	D	A_2	C_2	B_2
A_1	q/t	q^2/t^2	$\boldsymbol{q}^{\mathbf{2}}q^4s^2/t^3$	$\boldsymbol{q}^{\mathbf{2}}q^6s^2/t^4$
C_1	q^3s^2/t^2	q^4s^2/t^3	$\boldsymbol{q}^{\mathbf{2}}q^6s^4/t^4$	$\boldsymbol{q}^{\mathbf{2}}q^8s^4/t^5$
B_1	q^5s^2/t^3	q^6s^2/t^4	$\boldsymbol{q}^{\mathbf{4}}q^8s^4/t^5$	$\boldsymbol{q}^{\mathbf{4}}q^{10}s^4/t^6$

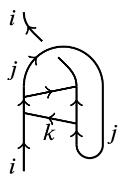
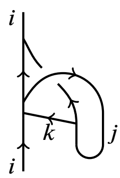
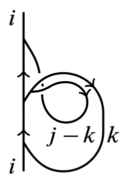
Table 1

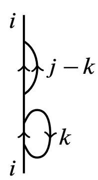
Example 4.14 As a second example we compute the Poincaré polynomial of the $\mathfrak{sl}_N$ link homology of the (Λ^i, Λ^j)–colored Hopf link with $i > j$, reduced with respect to Λ^i. We first consider the colored HOMFLY complex of the tangle $T(2,1)$ but suppress grading shifts.

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathrm{UP}[i,j,2] \arrow[d] & & \\
\cdots \arrow[r] & \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]\mathrm{UP}[i,j,2] \arrow[r] \arrow[d] & \mathrm{UP}[i,j,1] \arrow[d] & \\
\cdots \arrow[r] & \mathrm{UP}[i,j,2] \arrow[r] & \mathrm{UP}[i,j,1] \arrow[r] & \mathrm{UP}[i,j,0]
\end{tikzcd}
\qquad
\begin{tikzcd}
\vdots \arrow[d] \\
\mathrm{TUPs}[i,j,2] \arrow[d] \\
\mathrm{TUPs}[i,j,1] \arrow[d] \\
\mathrm{TUPs}[i,j,0]
\end{tikzcd}
$$

Figure 7

The k^{th} row from the bottom in the left diagram in Figure 7 is the complex $\mathrm{TUPs}[i,j,k]$. One way to compute the Λ^i–reduced $\mathfrak{sl}_N$ link homology of the colored Hopf link would be to close off the j–colored strand in all webs in the colored HOMFLY complex on the left side of Figure 7, replace each of these webs with the isomorphic direct sum of copies of the i–colored strand $\mathrm{UP}[i,0,0]$, compute the induced differentials and apply Gaussian elimination to cancel all null-homotopic summands. However, it is much easier to do this one row at a time. For this we close off the j–colored strand in the knotted webs $\mathrm{TUPs}[i,j,k]$ in the complex shown in the right column in Figure 7 to get a complex of knotted webs $\mathrm{ClTUPs}[i,j,k]$. For each of the knotted webs in this chain complex, the following isomorphisms hold up to grading shifts:[2]

$\mathrm{ClTUPs}[i,j,k] =$  $\cong$  $\cong$ 

$\cong$  $\cong \begin{bmatrix} N-i \\ k \end{bmatrix}\begin{bmatrix} i \\ j-k \end{bmatrix}$  $= \begin{bmatrix} N-i \\ k \end{bmatrix}\begin{bmatrix} i \\ j-k \end{bmatrix}\mathrm{UP}[i,0,0].$

Thus, each of the complexes $\mathrm{ClTUPs}[i,j,k]$ is homotopy equivalent to a complex concentrated in one homological degree only. We next replace each row in the (closed-off) colored HOMFLY complex by the corresponding 1–term complex computed above. So far we have disregarded grading shifts, but from the decategorified invariant it is

[2]Here we assume that the categorical tangle invariants extend to invariants of knotted webs. This is known for the matrix factorization construction (see Wu [52]) and is expected to extend to the general setting via the uniqueness results of Cautis [6].

easy to work out that the remaining terms are concentrated in the highest homological grading in each row (see Figure 8) and that all differentials thus must be trivial.

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[rr] & & {\left[\begin{smallmatrix} N-i \\ 2 \end{smallmatrix}\right]\left[\begin{smallmatrix} i \\ j-2 \end{smallmatrix}\right] \mathrm{UP}[i,0,0]} \arrow[d] & & & \\
\cdots \arrow[rr] & & 0 \arrow[r] \arrow[d] & {\left[\begin{smallmatrix} N-i \\ 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} i \\ j-1 \end{smallmatrix}\right] \mathrm{UP}[i,0,0]} \arrow[d] & & \\
\cdots \arrow[rr] & & 0 \arrow[r] & 0 \arrow[r] & {\left[\begin{smallmatrix} N-i \\ 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} i \\ j-0 \end{smallmatrix}\right] \mathrm{UP}[i,0,0]}
\end{tikzcd}$$

Figure 8

Tracking the q–grading through the above computation or comparing with the decategorified invariant proves the following proposition.

Proposition 4.15 *The Poincaré polynomial of $\mathfrak{sl}_N$ link homology of the (Λ^i, Λ^j)–colored Hopf link with $i > j$, reduced with respect to Λ^i, is, up to multiplication by a monomial,*

$$\mathcal{P}^N_{i,j}(\mathrm{Hopf}, \Lambda^i) = \sum_{k=0}^{j} t^{-2k} q^{k(2+N)} \begin{bmatrix} N-i \\ k \end{bmatrix} \begin{bmatrix} i \\ j-k \end{bmatrix}.$$

4.5 Comparison with Bigelow's and Manolescu's constructions

In this section we explain similarities between the geometric picture described in Section 4 and Bigelow's geometric model for the Jones polynomial [3] (see also earlier work of Lawrence [33]) and Manolescu's extension [37] to a model for the generators of a chain complex computing Seidel–Smith homology (symplectic Khovanov homology) [46].

Before going into details about the similarities, we want to mention the most visible differences between the construction in this paper and Bigelow and Manolescu's construction. First of all, they work with arbitrary knots and links that are presented as closures of braids, while our geometric algorithm is (so far) restricted to rational tangles. Manolescu's picture computes generators for uncolored (ie Λ^1–colored) $\mathfrak{sl}_2$ chain complexes and it says very little about differentials. Our picture, on the other hand, computes invariants for arbitrary fundamental $\mathfrak{sl}_N$ representations and at least some components of the differential can be read off. Note that Bigelow [4] and Manolescu [38] also have corresponding theories for $\mathfrak{sl}_N$, but whether they are related to each other as in the $\mathfrak{sl}_2$ case or to the construction here is unclear.

We now rephrase our geometric algorithm in the language of [3]. Let

$$M := \mathbb{C} \setminus \{X^+, X^-, Y\}$$

be $\mathbb{C}$ with the three special points X^+, X^- and Y removed. The weight-h part of the colored HOMFLY complex of a colored rational tangle can be interpreted as a graded intersection number of the submanifolds

$$A := \mathrm{Sym}^j(\alpha) \setminus \Delta \quad \text{and} \quad V_h := w_1 \times \cdots w_h \times u_1 \times \cdots \times u_{j-h}$$

in $\mathrm{Conf}^j(M) := \mathrm{Sym}^j(M) \setminus \Delta$. Here Δ denotes the appropriate big diagonal. Similarly as in Bigelow's setting, the graded intersection number can be described as the algebraic intersection number of lifts of A and V_h to a covering space specified by a surjective homomorphism $\Phi\colon \pi_1(\mathrm{Conf}^j(M)) \to \mathbb{Z}^4$. Relative gradings of intersection points in this picture can be computed by taking a loop γ in $\mathrm{Conf}^j(M)$ that starts at one intersection point, travels to the second intersection point along A and returns along V_h, and the grading difference is $\Phi([\gamma])$. Such a loop γ connecting intersection points $(x_1, \dots, x_j)$ and $(y_1, \dots, y_j)$ of A and V_h can be represented by a j–tuple of paths γ_k starting at x_k, proceeding to $y_{\sigma(k)}$ along α and further to $x_{\sigma(k)}$ along the $\sigma(k)^{\text{th}}$ vertical, where $\sigma \in S_j$ is a permutation. In this picture, Φ computes a linear combination of winding numbers of the paths γ_k around the points X^+, X^-, Y and around each other.

The homomorphism Φ can be constructed in a similar way as in Bigelow [3] and Manolescu [37] to reproduce exactly the behavior described in Theorem 4.5. As an example we explain how to count the winding of arcs around X^+ and around each other.

Example 4.16 Define the two homomorphisms

$$\Phi_1\colon \pi_1(\mathrm{Conf}^j(M)) \to \pi_1(\mathrm{Conf}^j(\mathbb{C})) = \mathrm{Br}_j \to \mathbb{Z},$$
$$\Phi_2\colon \pi_1(\mathrm{Conf}^j(M)) \to \pi_1(\mathrm{Conf}^j(\mathbb{C} \setminus X^+)) \to \pi_1(\mathrm{Conf}^{j+1}(\mathbb{C})) = \mathrm{Br}_{j+1} \to \mathbb{Z}.$$

The first map in each line is induced by inclusion. The second map in the second line comes from adding the point X^+ to the unordered j–tuple. The last maps in each line is the natural abelianization map. Then $(\Phi_2 - \Phi_1)/2\colon \pi_1(\mathrm{Conf}^j(M)) \to \mathbb{Z}$ is a homomorphism and it counts the winding of the arcs γ_k around X^+, while Φ_1 alone counts twice the winding of arcs around each other.

An alternative description is that $(\Phi_2 - \Phi_1)/2$ and Φ_1 count the winding of γ around the divisor $X^+ \times \mathrm{Sym}^{j-1}(\mathbb{C}) \subset \mathrm{Sym}^j(\mathbb{C})$ and around the big diagonal $\Delta \subset \mathrm{Sym}^j(\mathbb{C})$, respectively.

Once one knows how to count winding around the divisors of the special points and the diagonal, it is easy to assemble the correct $\Phi\colon \pi_1(\mathrm{Conf}^j(M)) \to \mathbb{Z}^4$. For example, the contribution coming from winding around the diagonal, which is exactly the non-additive part $q - Q$ of the q–grading, contributes $2\Phi_1$ to the q–coordinate of Φ.

Example 4.17 The main step in the proof of Lemma 4.8 describes a half-twist of arcs around each other, which causes a shift of q^2.

Example 4.18 An oppositely oriented simple disc intersecting just one other primary intersection in its interior corresponds to a full twist of arcs around each other and causes a non-additive q–grading shift of q^4, in agreement with the statement of Theorem 4.5.

Remark 4.19 We require the path γ that is used to compute the winding number around certain divisors to lie in $\mathrm{Conf}^j(M)$. In particular, the part of γ that lies on $\mathrm{Sym}^j(\alpha)$ is required to be disjoint from the diagonal. In the statement of Theorem 4.5 and the interpretation of differentials in the colored HOMFLY complex, on the other hand, we do use paths γ that can have intersections with the diagonal.

The problematic cases are exactly the ones where a simple disc D_s representing a differential between two generators has an intersection with another primary intersection along its boundary $\gamma = \partial D_s$. Theorem 4.5 tells us there should be a contribution of q^2 for each such intersection.

Before we can verify that this contribution comes from winding around the diagonal Δ, we have to push γ off Δ. Here (and in the generic case) γ only hits the multiplicity-two part of the diagonal and, hence, we can restrict to the model case $\mathrm{Sym}^2(\alpha) \subset \mathrm{Sym}^2(\mathbb{C})$, where we can explicitly describe the canonical push-off γ' of γ. Figure 9 shows the typical situation of a simple disc D_s (shaded) intersecting another primary intersection (blue dot) along its boundary.

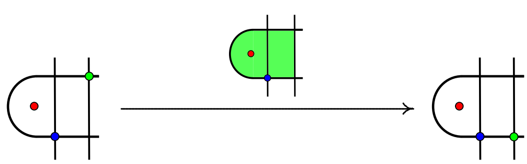


Figure 9

Theorem 4.5 tells us that the grading difference of the generators shown on both sides of the figure consists of a contribution from winding around the special point (red dot) and a contribution of q^2 from the intersection of the simple disc D_s with the other primary intersection (blue dot). The standard loop $\gamma = \partial D_s$ is given by moving the green dot from its position in the left image around the left bend (producing the right image) and back along its vertical while keeping the blue dot fixed. The loop γ intersects the diagonal when the two dots coincide.

The canonical push-off γ', which is disjoint from the diagonal, is given by the following two arcs. The green dot moves left from its position in the left image and around the bend to the original position of the blue dot while the blue dot itself moves right (which produces the right image with colors swapped) and then up along the vertical (which produces the left image again, but with colors swapped).

By forgetting the special point (red dot), it is easy to see that the arcs representing γ', viewed as a braid in Br_2, are just the braid group generator. Thus $\Phi_1(\gamma') = 1$, as required for the contribution q^2. Also, γ' still winds once around the divisor of the special point.

5 The color-stable HOMFLY polynomial

This section gives two proofs for Conjecture 1.1 on the decategorified level of polynomial HOMFLY invariants. The first one uses skew Howe duality and tries to stay as close to the categorified setting as possible. The second proof uses skein theory and proves Proposition 1.8 for arbitrary links with an unknot component. We define the color-stable HOMFLY polynomial of a link with an unknot component and prove that for 2–component links it specializes to the multivariable Alexander polynomial. Finally we compare the color-stable HOMFLY polynomial with multivariable link invariants arising from the Lie superalgebras $\mathfrak{sl}_{m|n}$ as described by Geer and Patureau-Mirand [14; 15] and Geer, Patureau-Mirand and Turaev [16].

5.1 First proof

We start by giving a proof for the decategorified Conjecture 1.1 for rational links that stays as close as possible to the categorified version.

Suppose we are given a (Λ^i, Λ^j)–colored rational two-component link that can be written as the closure of a positive rational tangle with all-upwards boundary orientations. Then the Poincaré polynomial of this tangle decategorifies by setting $t = -1$ to a $\mathbb{Z}[a^{\pm 1}, q^{\pm 1}, s^{\pm 1}]$–linear combination of webs $\mathrm{UP}[i, j, k]$.

The next step is to take the closures of the webs $\mathrm{UP}[i, j, k]$ and evaluate them to elements of the ground ring, for example by using relations (2-2) in the $\mathfrak{sl}_N$ spider category. We use this opportunity to demonstrate, as an alternative, the simplification process described in Section 2.6.

The closure of UP$[i,j,k]$ is the skew Howe image of

$$E_3^{(j)}F_1^{(i)}E_2^{(k)}F_2^{(k)}E_1^{(i)}F_3^{(j)}1_\lambda \in {}_{\mathcal{A}}\dot{U}(\mathfrak{sl}_4),$$

where λ corresponds to the sequence $(N,0,0,N)$. Using the commutation relations for Es and Fs and the fact that some weight spaces for the $\mathfrak{sl}_4$–action are trivial, we can simplify this expression:[3]

$$\begin{aligned}
&E_3^{(j)}F_1^{(i)}E_2^{(k)}F_2^{(k)}E_1^{(i)}F_3^{(j)}1_{(N,0,0,N)}\\
&\quad\cong E_3^{(j)}E_2^{(k)}F_1^{(i)}E_1^{(i)}1_{(N,k,j-k,N-j)}F_2^{(k)}F_3^{(j)}\\
&\quad\cong \begin{bmatrix}N-k\\ i\end{bmatrix}E_3^{(j)}E_2^{(k)}F_2^{(k)}1_{(N,0,j,N-j)}F_3^{(j)}\\
&\quad\cong \begin{bmatrix}N-k\\ i\end{bmatrix}\begin{bmatrix}j\\ k\end{bmatrix}E_3^{(j)}F_3^{(j)}1_{(N,0,0,N)}\\
&\quad\cong \begin{bmatrix}N-k\\ i\end{bmatrix}\begin{bmatrix}j\\ k\end{bmatrix}\begin{bmatrix}N\\ j\end{bmatrix}1_{(N,0,0,N)}\\
&\quad\cong \begin{bmatrix}N-k\\ i\end{bmatrix}\begin{bmatrix}j\\ k\end{bmatrix}\begin{bmatrix}N\\ j\end{bmatrix}1_{(0,0,N,N)}.
\end{aligned}$$

In the HOMFLY evaluation we replace $\left[\begin{smallmatrix}N+b\\ c\end{smallmatrix}\right]$ by

$$\begin{bmatrix}b\\ c\end{bmatrix}_a := \prod_{k=1}^{c}\frac{aq^{b-k+1}-a^{-1}q^{-b+k-1}}{q^k-q^{-k}}.$$

The closure of UP$[i,j,k]$ thus evaluates to

$$\begin{bmatrix}-k\\ i\end{bmatrix}_a\begin{bmatrix}j\\ k\end{bmatrix}\begin{bmatrix}0\\ j\end{bmatrix}_a = \begin{bmatrix}-k\\ j-k\end{bmatrix}_a\begin{bmatrix}-i\\ k\end{bmatrix}_a\begin{bmatrix}0\\ i\end{bmatrix}_a.$$

In order to get the behavior claimed in Conjecture 1.1 we have to reduce with respect to the higher color Λ^i. On the decategorified level, this just means dividing by $\left[\begin{smallmatrix}0\\ i\end{smallmatrix}\right]_a$, the invariant of the Λ^i–colored unknot. We can write the reduced evaluation of the closure of UP$[i,j,k]$ as

$$\begin{bmatrix}-k\\ j-k\end{bmatrix}_a\begin{bmatrix}-i\\ k\end{bmatrix}_a = \begin{bmatrix}-k\\ j-k\end{bmatrix}_a\prod_{l=1}^{k}\frac{as^{-1}q^{-j-l+1}-a^{-1}sq^{j+l-1}}{q^l-q^{-l}}.$$

This shows that, in the reduced case, it is not only the coefficients coming from the grading shifts in the colored HOMFLY complex that depend in a controlled way on the

[3] For clarity we use as subscript in $1_{(a,b,c,d)}$ the sequence (a,b,c,d) instead of the corresponding weight μ.

higher color i; the evaluation of $\mathrm{UP}[i,j,k]$ does also. More precisely, we have shown that there exists a three-variable invariant[4] of two-component rational links, which takes values in $\mathbb{Z}[a^{\pm 1}, s^{\pm 1}](q)$, and for any $i \geq j$ it specializes to the higher-color reduction of the (Λ^i, Λ^j)–colored HOMFLY polynomial upon setting $s = q^{i-j}$.

Remark 5.1 The method of computing colored HOMFLY polynomials of 2–bridge links explained in this subsection in fact provides explicit q–holonomic formulas in the sense of Garoufalidis [13]. For more details and a Wolfram Mathematica implementation of this algorithm see Wedrich [50].

5.2 Second proof

A different proof of Conjecture 1.1 on the decategorified level is possible via skein theory. In fact, this proof works for arbitrary colored links L' with a Λ^i–colored unknot component U. Let $L = L' \setminus U$ which we consider to live in a solid torus.

The idea is to compute the colored HOMFLY polynomial of L' in two steps. First one evaluates L in the skein of the annulus onto which the solid torus projects. Second one pairs this again with the Λ^i–colored unknot.

Definition 5.2 Let F be an open subset of $\mathbb{R}^2$. Then define $S(F)$ to be the free $\mathbb{Z}[a^{\pm 1}](q)$–module spanned by closed webs embedded in F, modulo web relations inside F. $S(F)$ is the HOMFLY skein of F.

If $G \subset F$ is an inclusion of open subsets of $\mathbb{R}^2$, then there is a canonical homomorphism $S(G) \to S(F)$ given by interpreting webs as lying in F. The homomorphism $S(F) \to S(\mathbb{R}^2)$ is called evaluation and is denoted by $\langle \cdot \rangle$.

Examples 5.3 (1) $S(\mathbb{R}^2)$ is free of rank 1 over $\mathbb{Z}[a^{\pm 1}](q)$ and is spanned by the empty diagram.

(2) The skein of the annulus $S(A)$ has the structure of a commutative algebra, where multiplication is given by stacking two annuli inside each other.

By projecting a link L lying in $F \times I$ onto F and replacing crossings via the formulas in Section 2.3, L can be regarded as an element of the skein $S(F)$. Up to multiplication by a scalar depending on framing, this is well defined.

Lemma 5.4 *As a commutative algebra, the skein of the annulus is freely generated by the set $\{d_j \mid j \in \mathbb{Z}\}$, where d_j is given by a $S^{|j|}$–colored longitudinal unknot*

[4] Up to multiplication by a monomial.

in counterclockwise (clockwise) orientation if $j > 0$ ($j < 0$), and d_0 is the empty diagram. Here S^k stands for the one-row Young diagram with k boxes; for $\mathfrak{sl}_N$ this corresponds to the k^{th} symmetric power of the standard representation.

Another free generating set is given by $\{\phi_j \mid j \in \mathbb{Z}\}$, where ϕ_j is the closure of the $|j|$–strand braid $\sigma_{|j|-1} \cdots \sigma_1$, written in standard braid group generators, with orientation counterclockwise (clockwise) if $j > 0$ ($j < 0$).

Proof The second generating set is due to Turaev [47]. This also shows that $S(A)$ splits as an algebra into a tensor product $S(A) = S(A)^+ \otimes S(A)^-$ of isomorphic algebras which are freely generated by $\{\phi_j \mid j \in \mathbb{N}\}$ and $\{\phi_{-j} \mid j \in \mathbb{N}\}$, respectively. Lukac [35] showed that $S(A)^{\pm}$ are isomorphic to the ring of symmetric functions on a countably infinite alphabet, with the i^{th} complete (elementary) symmetric function corresponding to a S^i–colored (Λ^i–colored) unknot d_i (c_i) with the appropriate orientation. See also Aiston [1]. □

Given an element X of $S(A)$ and $i \in \mathbb{Z}$ we can get a new element $\psi_i(X)$ of $S(A)$ by linking X with a meridional Λ^i–colored unknot c_i.

Lemma 5.5 *For $i \in \mathbb{Z}$ there exist algebra homomorphisms $t_i\colon S(A) \to \mathbb{Z}[a^{\pm 1}](q)$ given by*

$$t_i(X) = \frac{\langle \psi_i(X) \rangle}{\langle c_i \rangle}.$$

Proof See Morton and Lukac [41, Sections 1.5 and 1.6]. □

Recall that we have decomposed L' into a Λ^i–colored unknot U and some remainder L in a solid torus $A \times I$. By taking a projection of L onto the annulus and applying the crossing replacement rules, L evaluates to some element of $S(A)$, which we denote by $\pi(L)$. The colored HOMFLY polynomial of L' is then $\langle \psi_i(\pi(L)) \rangle$ and its reduction with respect to color Λ^i is $t_i(\pi(L))$. In order to prove Conjecture 1.1 it suffices to show that $S(A)$ has a generating set that behaves well under color shift.

Proposition 5.6 *There exist functions $p_j \in \mathbb{Z}[a^{\pm 1}, r^{\pm 1}](q)$ such that for the generating set $\{d_j \mid j \in \mathbb{Z}\}$ we have*

$$t_i(d_j) = p_j(a, r = q^i, q) \quad \text{for all } i \geq 0.$$

Proof This follows readily from [41, Lemma 3.1] where for $i, j > 0$ Morton and Lukac prove the first equality in the following computation (with different notation):

$$t_i(d_j) = \langle d_j \rangle \frac{a - a^{-1}(q^j - q^{j-i} + q^{-i})}{a - a^{-1}} = \langle d_j \rangle \frac{a - a^{-1}(q^j - q^j r^{-1} + r^{-1})}{a - a^{-1}}.$$

The right-hand side is really a Laurent polynomial in a because

$$\langle d_j \rangle = \prod_{k=0}^{|j|-1} \frac{aq^{-k} - a^{-1}q^k}{q^{k+1} - q^{-k-1}},$$

which is also proved in [41]. The cases for other signs of i or j are similar. $\square$

Remark 5.7 Alternatively, one can deduce the statement of the proposition for the generating set $\{\phi_j \mid j \in \mathbb{Z}\}$ from the already established decategorified Conjecture 1.1 for rational links. For this note that $t_i(\phi_j)$ is the Λ^i–reduced colored HOMFLY polynomial of the (Λ^i, Λ^1)–colored $(2, 2j)$ torus link.

5.3 The color-stable HOMFLY polynomial and the multivariable Alexander polynomial.

Let $L' = L \cup U$ be a link with an unknot component U and a fixed coloring on L.

Definition 5.8 The *color-stable HOMFLY polynomial* $P^{\mathrm{st}}(L', U) \in \mathbb{Z}[a^{\pm 1}, r^{\pm 1}](q)$ of L' with respect to the unknot component U is the unique element of $\mathbb{Z}[a^{\pm 1}, r^{\pm 1}](q)$ satisfying

$$P^{\mathrm{st}}(L', U)(a, r = q^i, q) = t_i(L) \quad \text{for all } i \geq 0.$$

Theorem 5.9 *Let $L' = K \cup U$ be a two-component link with an unknot component U, and suppose K is colored by Λ^1. Then we have*

$$P^{\mathrm{st}}(L', U)(1, u, v) = (1 - u^2)\Delta(L')(u^2, v^2),$$

where $\Delta(L')(x, y)$ is the multivariable Alexander polynomial of L' with U labeled by x.

We first prove this for rational links and show how the computation of the multivariable Alexander polynomial ties in with the geometric algorithm.

Lemma 5.10 *The theorem is true for rational links L'.*

Proof Actually we prove

$$(q - q^{-1}) P^{\mathrm{st}}(L', U) \mid_{a=1,\ r=u,\ q=v} = (1 - v^2)(1 - u^2)\Delta(L')(u^2, v^2).$$

We may assume that L' is the closure of a rational tangle whose continued fraction expansion has odd length. Since we work with a (Λ^j, Λ^1)–colored tangle, we only need one vertical, left or right, in the geometric picture. We assume that the tangle has all upward boundary orientation and the colored HOMFLY complex has objects UP$[j,1,1]$ and UP$[j,1,0]$. The case of the other orientation with objects OP$[j,1,1]$ and OP$[j,1,0]$ is analogous.

In Section 5.2 we have computed the reduced closures of the objects UP$[j,1,k]$. Their contributions to the left-hand side of the above equation are computed as follows:

$$\text{Reduced closure of} \begin{cases} \mathrm{UP}[j,1,0] \\ \mathrm{UP}[j,1,1] \end{cases} = \begin{cases} \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]_a \left[\begin{smallmatrix}-j\\0\end{smallmatrix}\right] \\ \left[\begin{smallmatrix}-1\\0\end{smallmatrix}\right]_a \left[\begin{smallmatrix}-j\\1\end{smallmatrix}\right] \end{cases} \to \begin{cases} 0, \\ u^{-1}(1-u^2). \end{cases}$$

Here the arrow indicates multiplication by $q-q^{-1}$ and subsequent substitution $a=1$, $s=u/v$, $q=v$.

Hence, in the geometric algorithm we only need to count intersection points with the left vertical. Note that because the continued fraction expansion has odd length, the last twist applied to the diagram was a top twist. Thus the intersection points with the left vertical are paired up by simple discs and their relative grading is $-v^2$. We can replace two paired intersection points by the single intersection point of α with the real axis along the segment of α that joins the pair. The contribution of such a double intersection point is thus $u^{-1}(1-v^2)(1-u^2)$. The relative gradings of such double intersection points can be computed from winding numbers of connecting paths around the special points Y, X^-, X^+, which count as $-v^2$, $-u^2$ and $-u^{-2}$ under the substitution. Here a connecting path starts at one double intersection point on the real axis, travels along α to the second double intersection point and returns on the real axis. This shows that, up to multiplication by a monomial, the geometric algorithm for the modified colored HOMFLY polynomial outputs $(1-u^2)(1-v^2)P(u^2,v^2)$, where P is some two-variable polynomial. It remains to show that $P=\Delta(L')$.

A classical way of computing the multivariable Alexander polynomial is via Fox calculus on a presentation of the fundamental group of the link complement. Since our diagrams are essentially genus-two Heegaard diagrams of the link complement, we can extract a presentation $\langle u^2, v^2 \mid w=1\rangle$ for its fundamental group by the following procedure. First we have to replace the arc α by the embedded circle $\overline{\alpha}$ which is the boundary of a small neighborhood of α. Starting from any point on $\overline{\alpha}$, the word w is assembled from letters $\{u^2, u^{-2}, v^2, v^{-2}\}$ by appending a letter $v^{\pm 2}$ for every intersection with the segment $[-2,-1]$ on the real axis, where the exponent depends on whether α hits the real axis from above or below, and similarly $u^{\pm 2}$ for intersections with the segment $[1,2]$. Then the multivariable Alexander polynomial can be extracted from the presentation by taking the Fox derivative of w with respect to the variable v^2 and then dividing by $(1-u^2)$.

Claim The summands produced by the Fox derivative are in bijection with the intersection points of $\overline{\alpha}$ with the segment $[-2,-1]$ on the real axis and their relative gradings are determined by the winding of connecting paths around the special points Y, X^-, X^+, which count as $-v^2$, $-u^2$ and $-u^{-2}$. The connecting paths run along $\overline{\alpha}$ from one intersection point to the other and back on the real axis. The proof of this claim is an exercise for the reader who is familiar with the Fox derivative.

One can further simplify this picture by noting that in our case intersection points of $\overline{\alpha}$ with $[-2,-1]$ always come in pairs that correspond to an intersection of α with $[-2,-1]$. Furthermore, these pairs have relative grading $-u^2$ and hence we expect a factor of $(1-u^2)$ in the result of the Fox derivative — exactly the factor that has to be canceled in order to get the multivariable Alexander polynomial. This shows that $\Delta(L')(u^2, v^2)$ can be directly computed by counting intersections of α with $[-2,-1]$, where relative gradings are computed as winding numbers of connecting paths around the special points Y, X^-, X^+, which count as $-v^2$, $-u^2$ and $-u^{-2}$, exactly as described by the specialization of the geometric algorithm for the colored HOMFLY complex. Thus $P = \Delta(L')$ and we are done. □

Remark 5.11 The statement of the lemma can also be interpreted as saying that the geometric algorithm in Section 4 computes the link Floer homology of L, because for rational links it contains exactly as much information as its multivariable Alexander polynomial. It would be interesting to see if the color-stability of Conjecture 1.1 could be related to link Floer homology of a more general class of links with unknot components.

Proof of Theorem 5.9 We prove the theorem in two steps. First we compare the skein relations in the HOMFLY skein of the annulus and in an appropriate Alexander skein of the annulus. In the second step we use Lemma 5.10 in the case of $(2,2k)$ torus links to compare the two polynomials on a common basis for the skeins.

For the first step, pick a crossing c in a diagram of L' that does not involve strands in U and denote by L^+, L^- and L^0 the diagrams which have a positive crossing, a negative crossing and the oriented resolution of the crossing at position c, respectively. Because the involved strands are Λ^1–colored, the colored HOMFLY polynomial P satisfies

$$aP(L^+) - a^{-1}P(L^-) = (q-q^{-1})P(L^0).$$

After substituting variables $a=1$, $q=v$, $s=u/v$ we get exactly the skein relation for the multivariable Alexander polynomial for crossings with strands labeled by v^2:

$$\Delta(L^+) - \Delta(L^-) = (v-v^{-1})\Delta(L^0).$$

From now on we assume that we have made these substitutions in all expressions.

For a link X in the annulus, let $\Delta'(X)$ denote the evaluation of X linked with an unknot via the skein theory of the multivariable Alexander polynomial, with X labeled by v^2 and the unknot labeled by u^2. By virtue of the identical skein relation in the annulus, we can write

$$P^{\mathrm{st}}(L',U) = \sum_{I\in A} a_I t_i(\phi_I) \quad \text{and} \quad \Delta(L) = \sum_{I\in A} a_I \Delta'(\phi_I)$$

with the same coefficients $a_I \in \mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$ and with ϕ_I denoting a monomial in braid closures ϕ_i in the annulus.

Lemma 5.12 *Let $\phi_I = \prod_{k=1}^{n_I} \phi_{i_{I,k}}$. Then $\Delta'(\phi_I) = (1-u^2)^{n_I-1} \prod_{k=1}^{n_I} \Delta'(\phi_{i_{I,k}})$.*

Proof At the expense of perhaps changing the label v^2 into v^{-2} on some components $\phi_{i_{I,k}}$, we may assume that they are coherently oriented and hence ϕ_I can be written as a braid closure. The multivariable Alexander polynomial of a braid closure together with its axis can be computed via Theorem 1 of Morton [40]. There $\Delta'(\phi_I)$ is presented as the characteristic polynomial $\det(I - u^2 B(\phi_I))$ of a matrix $B(\phi_I)$ which is inductively built from a braid representative for ϕ_I. It is easy to see that since ϕ_I is a disjoint union of n_I braids, the matrix $B(\phi_I)$ has $n_I - 1$ rows containing a single entry 1 and 0 elsewhere. Removing all such rows (and the corresponding columns) via Laplace expansion, we get $\det(I - u^2 B(\phi_I)) = (1-u^2)^{n_I-1} \det(I - uB')$, where B' is of block diagonal form and the blocks are exactly the matrices $B(\phi_{i_{I,k}})$. □

Using the lemma we have

$$\begin{aligned}(1-u^2)\Delta(L') &= (1-u^2)\sum_{I\in A} a_I \Delta'(\phi_I) = \sum_{I\in A} a_I (1-u^2)^{n_I} \prod_{k=1}^{n_I} \Delta'(\phi_{i_{I,k}}) \\ &= \sum_{I\in A} a_I \prod_{k=1}^{n_I} (1-u^2)\Delta'(\phi_{i_{I,k}}) = \sum_{I\in A} a_I \prod_{k=1}^{n_I} t_i(\phi_{i_{I,k}}) \\ &= \sum_{I\in A} a_I t_i(\phi_I) = P^{\mathrm{st}}(L',U).\end{aligned}$$

Here the key step is that

$$(1-u^2)\Delta'(\phi_{i_{I,k}}) = P^{\mathrm{st}}(\phi_{i_{I,k}} \cup U, U) = t_i(\phi_{i_{I,k}})$$

since $\phi_{i_{I,k}}$ linked with a meridional unknot is a $(2, 2i_{I,k})$ torus link, for which theorem holds by Lemma 5.10. □

5.4 Comparison with multivariable link invariants from Lie superalgebras

Geer, Patureau-Mirand and Turaev [14; 15; 16] define multivariable polynomial link invariants using modified Reshetikhin–Turaev invariants for the Lie superalgebras $\mathfrak{sl}_{m|n}$. We give a brief review of this construction and how it is related to color-stability of colored HOMFLY polynomials.

Reshetikhin–Turaev $\mathfrak{sl}_{m|n}$ invariants are invariants of (framed) oriented tangles labeled by irreducible representations of the Lie superalgebra $\mathfrak{sl}_{m|n}$. They can be defined in a similar way as described in Section 2.1 by scanning the tangle in generic position from bottom to top and associating certain maps of $\mathfrak{sl}_{m|n}$ representations to cups, caps and crossings. While the representation theory of $\mathfrak{sl}_{m|n}$ is richer and more complicated than the representation theory of $\mathfrak{sl}_N$, it turns out that for most (to be precise: for so-called *typical*) colorings, the resulting colored link invariants are trivial. The reason for this is that the quantum dimensions of these $\mathfrak{sl}_{m|n}$ representations, and thus the invariants of unknots colored by such representations, are zero.

The solution to this problem, as described in detail in [15], is to cut one component of the link L open and consider it as an oriented two-ended tangle T_λ, where λ indicates the representation on the open strand. The Reshetikhin–Turaev invariant of T_λ is a multiple $x(T_\lambda)\,\mathrm{Id}_\lambda$ of the identity map on the representation λ. Here $x(T_\lambda)$ lives in the ground ring $\mathbb{C}[\![h]\!][h^{-1}]$ and h is related to the familiar variable q by $q=\exp(h/2)$. We can think of $x(T_\lambda)$ as an invariant of L that is reduced with respect to the opened link component. Geer and Patureau-Mirand then show that there exist "fake quantum dimensions" $d(\lambda)\in\mathbb{C}[\![h]\!][h^{-1}]$ of representations λ that can be used to "fake unreduce" the invariant in a non-trivial[5] way.

Theorem 5.13 [15, Theorem 0.1] *The map $L\mapsto F'(L):=d(\lambda)x(T_\lambda)$ is independent of the choice of cut component and, hence, is a well-defined framed colored link invariant.*

A great difference between the representation theories of $\mathfrak{sl}_{m|n}$ and $\mathfrak{sl}_N$ is that in the first case isomorphism classes of finite-dimensional irreducible representations come in continuous families. To be more precise, for $\mathfrak{sl}_{m|n}$ they are indexed by $(d,z)\in\mathbb{N}^{m+n-2}\times\mathbb{C}$. Supposing that all colors on a link live in the same continuous family, it turns out that the invariants $F'(L)$ change in a very predictable way under varying these colors in their family.

[5] As noted earlier, unreducing with respect to actual quantum dimensions produces trivial invariants.

Theorem 5.14 [15, part of Theorem 0.2] *Let L be a framed link with $k \geq 2$ components in some order, let $d \in \mathbb{N}^{m+n-2}$ and denote by $L(z_1, \dots, z_k)$ the link L with components colored by the $\mathfrak{sl}_{m|n}$–representations indexed by (d, z_i).*

Then there exists a framing-independent invariant $M^d_{\mathfrak{sl}_{m|n}}(L) \in \mathbb{Z}[q^{\pm 1}, q_1^{\pm 1}, \dots, q_k^{\pm 1}]$ of L such that (up to renormalization) the identity

$$F'(L(z_1, \dots, z_k)) = M^d_{\mathfrak{sl}_{m|n}}(L)(q, q_1 = q^{z_1}, \dots, q_k = q^{z_k})$$

of Laurent series in h holds for all z_i such that (d, z_i) are typical representations. Note that we identify $q = \exp(h/2)$ and $q^z = \exp(zh/2)$.

This theorem shows that the modified $\mathfrak{sl}_{m|n}$ Reshetikhin–Turaev invariants have very strong stability properties under shifting colors with respect to the continuous parameters z_i. In fact, Geer and Patureau-Mirand show that the color-stability captured in $M^d_{\mathfrak{sl}_{m|n}}(L)$ is determined by the color-stability of specializations of colored HOMFLY polynomials. In [14, Proposition 3.4] they prove that for certain integer values of z_i the invariant $F'(L(z_1, \dots, z_k))$ agrees up to renormalization with the $a = q^{m-n}$ specialization of the colored HOMFLY polynomial of L labeled by Young diagrams that are determined by the pairs (d, z_i); for details see [14, proof of Corollary 3.5]. This together with Theorem 5.14 shows the following proposition.

Proposition 5.15 [14, Corollary 3.5] *The multivariable link invariants $M^d_{\mathfrak{sl}_{m|n}}(L)$ are determined by and can in principle be computed from $a = q^{m-n}$ specializations of colored HOMFLY polynomials of L.*

We expect that for a link L with an unknot component the invariants $M^0_{\mathfrak{sl}_{m|1}}(L)$ are closely related to the color-stable HOMFLY polynomial. In the case of $\mathfrak{sl}_{m|1}$ and $d = 0$, the Young diagrams constructed in the proof of Proposition 5.15 represent exterior powers Λ^i.

Corollary 5.16 *Let L be a k–component link. Then*

$$M^0_{\mathfrak{sl}_{m|1}}(L)(q, q_1 = q^{1+m-i_1}, \dots, q_k = q^{1+m-i_k})$$

agrees with the $a = q^{m-1}$ specialization of the $(\Lambda^{i_1}, \dots, \Lambda^{i_k})$–colored HOMFLY polynomial of L up to renormalization.

The strong color-stability properties described by $M^d_{\mathfrak{sl}_{m|n}}$ come at the price that (unlike the color-stable HOMFLY polynomial) these invariants don't seem to be stable in super-rank $m-n$. In particular, we cannot expect colored HOMFLY polynomials of arbitrary links to be stable under changing color (in a simple way) without specializing $a = q^{m-n}$;

this is already indicated by the decategorification of Conjecture 1.2. However, it is a very interesting question what information about the large-color behavior of HOMFLY type invariants of general links can be inferred from the color-stability of related Lie superalgebra invariants.

Finally, we want to mention another parallel between $M^0_{\mathfrak{sl}_{m|1}}(L)$ and the color-stable HOMFLY polynomial. Geer and Patureau-Mirand prove that their invariants specialize to the multivariable Alexander polynomial in a similar way as the color-stable HOMFLY polynomial; see Theorem 5.9.

Theorem 5.17 [14, Theorem 3] *The invariants $M^0_{\mathfrak{sl}_{m|1}}(L)$ of a k–component link L specialize to the Conway potential function $\nabla(L) \in \mathbb{Q}(t_1, \dots, t_k)$ of L, which is a refinement of the multivariable Alexander polynomial:*

$$\nabla(L)(q_1^m, \dots, q_k^m) = e^{\sqrt{-1}(m-1)\pi/2} M^0_{\mathfrak{sl}_{m|1}}(L)(q = e^{\sqrt{-1}\pi/m}, q_1, \dots, q_k).$$

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