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On p–almost direct products and residual properties of pure braid groups of nonorientable surfaces

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We prove that the n^{th} pure braid group of a nonorientable surface (closed or with boundary, but different from $\mathbb{RP}^2$) is residually 2–finite. Consequently, this group is residually nilpotent. The key ingredient in the closed case is the notion of p–almost direct product, which is a generalization of the notion of almost direct product. We also prove some results on lower central series and augmentation ideals of p–almost direct products.

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1 Introduction

Let M be a compact, connected surface (orientable or not, possibly with boundary) and $F_n(M) = \{(x_1, \dots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}$ its n^{th} configuration space. The fundamental group $\pi_1(F_n(M))$ is called the n^{th} *pure braid group* of M and shall be denoted by $P_n(M)$.

The mapping class group $\Gamma(M)$ of M is the group of isotopy classes of homeomorphisms $h\colon M \to M$ which act as the identity on the boundary. Let $\mathcal{X}_n = \{z_1, \dots, z_n\}$ be a set of n distinguished points in the interior of M; the pure mapping class group $\mathrm{P}\Gamma(M, \mathcal{X}_n)$ relative to $\mathcal{X}_n$ is the group of isotopy classes of homeomorphisms $h\colon M \to M$ satisfying $h(z_i) = z_i$ for all i: since this group does not depend on the choice of the set $\mathcal{X}_n$ but only on its cardinality we can write $\mathrm{P}_n\Gamma(M)$ instead of $\mathrm{P}\Gamma(M, \mathcal{X}_n)$. Forgetting the marked points, we get a morphism $\mathrm{P}_n\Gamma(M) \to \Gamma(M)$ whose kernel is known to be isomorphic to $P_n(M)$ when M is not a sphere, a torus, a projective plane or a Klein bottle (see Scott [28] and Guaschi and Juan-Pineda [20]).

Now, recall that if $\mathcal{P}$ is a group-theoretic property, then a group G is said to be *residually* $\mathcal{P}$ if, for all $g \in G$, $g \neq 1$, there exists a group homomorphism $\varphi\colon G \to H$ such that H satisfies $\mathcal{P}$ and $\varphi(g) \neq 1$. We are interested in the following properties: to be nilpotent, to be free and to be a finite p–group for a prime number p (mostly $p = 2$). Recall that, if for subgroups H and K of G, $[H, K]$ is the subgroup generated

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by $\{[h,k] \mid (h,k) \in H \times K\}$ where $[h,k] = h^{-1}k^{-1}hk$, the lower central series of G, $(\Gamma_k G)_{k\geq 1}$, is defined inductively by $\Gamma_1 G = G$ and $\Gamma_{k+1} G = [G, \Gamma_k G]$. It is well known that G is residually nilpotent if and only if $\bigcap_{k=1}^{\infty} \Gamma_k G = \{1\}$. From the lower central series of G one can define another filtration $D_1(G) \supseteq D_2(G) \supseteq \cdots$ setting $D_1(G) = G$, and for $i \geq 2$ defining

$$D_i(G) = \{x \in G \mid \exists\, n \in \mathbb{N}^* \text{ with } x^n \in \Gamma_i(G)\}.$$

After Garoufalidis and Levine [13], this filtration is called the *rational lower central series* of G, and a group G is residually torsion-free nilpotent if and only if $\bigcap_{i=1}^{\infty} D_i(G) = \{1\}$.

When M is an orientable surface of positive genus (possibly with boundary) or a disc with holes, it is proved in Bellingeri, Gervais and Guaschi [6] and Bardakov and Bellingeri [1] that $P_n(M)$ is residually torsion-free nilpotent for all $n \geq 1$. The fact that a group is residually torsion-free nilpotent has several important consequences, notably that the group is bi-orderable (see Botto Mura and Rhemtulla [8]) and residually p–finite (see Gruenberg [19]). The goal of this article is to study the nonorientable case and, more precisely, to prove the following:

Theorem 1.1 *The n^{th} pure braid group of a nonorientable surface different from $\mathbb{RP}^2$ is residually 2–finite.*

In the case of $P_n(\mathbb{RP}^2)$ we give some partial results at the end of Section 4. Since a finite 2–group is nilpotent, a residually 2–finite group is residually nilpotent. Thus, we have:

Corollary 1.2 *The n^{th} pure braid group of a nonorientable surface different from $\mathbb{RP}^2$ is residually nilpotent.*

In González-Meneses [17] it was shown that the n^{th} pure braid group of a nonorientable surface is not bi-orderable and therefore it is not residually torsion-free nilpotent. Our technique doesn't extend to $p \neq 2$; therefore the question if pure braid groups of nonorientable surfaces are residually p for some $p \neq 2$ is still open (recall that there are groups residually p for infinitely many primes p which are not residually torsion-free nilpotent; see Hartley [21]).

One can prove that finite-type invariants separate classical braids using the fact that the pure braid group P_n is residually nilpotent without torsion (see Papadima [25]). Moreover, using the residual properties discussed above it is possible to construct algebraically a universal finite-type invariant over $\mathbb{Z}$ for the classical braid group B_n (see [25]). Similar constructions were afterwards proposed for braids on orientable surfaces (see Bellingeri and Funar [5] and González-Meneses and Paris [18]): in a further paper we will explore the relevance of Theorem 1.1 in the realm of finite-type invariants over $\mathbb{Z}/2\mathbb{Z}$ for braids on nonorientable surfaces.

From now on, $M = N_{g,b}$ is a nonorientable surface of genus g with b boundary components, simply denoted by N_g when $b = 0$. We will see N_g as a sphere S^2 with g open discs removed and g Möbius strips glued on each circle (see Figure 4, where each crossed disc represents a Möbius strip). The surface $N_{g,b}$ is obtained from N_g by removing b open discs. The mapping class groups $\Gamma(N_{g,b})$ and pure mapping class group $P_n\Gamma(N_{g,b})$ will be denoted by $\Gamma_{g,b}$ and $\Gamma^n_{g,b}$, respectively.

The paper is organized as follows. In Section 2, we prove Theorem 1.1 for surfaces with boundary: following what the authors did in the orientable case (see [6]), we embed $P_n(N_{g,b})$ in a Torelli group. The difference here is that we must consider mod 2 Torelli groups. In Section 3 we introduce the notion of p–almost direct product, which generalizes the notion of almost direct product (see Definition 3.1) and we prove some results on lower central series and augmentation ideals of p–almost direct products (Theorems 3.3 and 3.6) that can be compared with similar results on almost direct products (Theorem 3.1 in Falk and Randell [12] and Theorem 3.1 in Papadima [25]).

In Section 4, the existence of a split exact sequence

$$1 \longrightarrow P_{n-1}(N_{g,1}) \longrightarrow P_n(N_g) \longrightarrow \pi_1(N_g) \longrightarrow 1$$

and results from Sections 2 and 3 are used to prove Theorem 1.1 in the closed case (Theorem 4.5). The method is similar to the one developed for orientable surfaces in [1]: the difference will be that the semi-direct product $P_{n-1}(N_{g,1}) \rtimes \pi_1(N_g)$ is a 2–almost-direct product (and not an almost-direct product as in the case of closed oriented surfaces).

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2 The case of non-empty boundary

In this section, $N = N_{g,b}$ is a nonorientable surface of genus $g \geq 1$ with boundary (ie $b \geq 1$). In this case, one has $P_n(N) = \mathrm{Ker}(\Gamma^n_{g,b} \to \Gamma_{g,b})$ for all $n \geq 1$.

2.1 Notation

We will follow notation from [27]. A simple closed curve in N is an embedding $\alpha\colon S^1 \to N \setminus \partial N$; with a usual abuse of notation, we will call the image of a simple closed curve a simple closed curve also. Such a curve is said to be two-sided or one-sided if it admits a regular neighborhood homeomorphic to an annulus or a Möbius strip, respectively. We shall consider the following elements in $\Gamma_{g,b}$:

- If α is a two-sided simple closed curve in N with a given orientation, τ_α is a Dehn twist along α.
- Let μ and α be two simple closed curves such that μ is one-sided, α is oriented and two-sided, and such that $|\alpha \cap \mu| = 1$. A regular neighborhood K of $\alpha \cup \mu$ is diffeomorphic to a Klein bottle with one hole, and a regular neighborhood M of μ is diffeomorphic to a Möbius strip. Pushing M once along α, we get a diffeomorphism of K fixing the boundary (see Figure 1): it can be extended to N by the identity. Such a diffeomorphism is called a crosscap slide, and denoted by $Y_{\mu,\alpha}$.

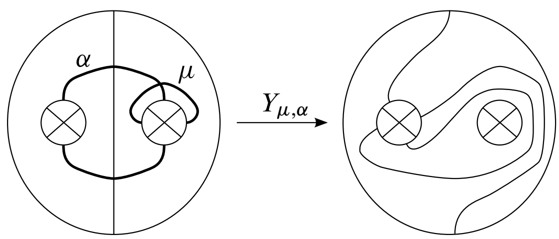

Figure 1: Crosscap slide.

2.2 Blowup homomorphism

Here we recall the construction of the blowup homomorphism $\eta^n_{g,b}\colon \Gamma^n_{g,b} \to \Gamma_{g+n,b}$ given in [30; 31] and [27].

Let $U = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and, for $i = 1, \dots, n$, fix an embedding $e_i\colon U \to N$ such that $e_i(0) = z_i$, $e_i(U) \cap e_j(U) = \varnothing$ if $i \neq j$ and $e_i(U) \cap \partial N = \varnothing$ for all i. If we remove the interior of each $e_i(U)$ (thus getting the surface $N_{g,b+n}$) and identify, for each $z \in \partial U$, $e_i(z)$ with $e_i(-z)$, we get a nonorientable surface of genus $g+n$ with b boundary components, that is to say a surface homeomorphic to $N_{g+n,b}$. Let us denote by $\gamma_i = e_i(S^1)$ the boundary of $e_i(U)$, and by μ_i its image in $N_{g+n,b}$; it is a one-sided simple closed curve.

Now, let h be an element of $\Gamma^n_{g,b}$. It can be represented by a homeomorphism $N_{g,b} \to N_{g,b}$, still denoted h, such that:

(1) $h(e_i(z)) = e_i(z)$ if h preserves local orientation at z_i.

(2) $h(e_i(z)) = e_i(\bar{z})$ if h reverses local orientation at z_i.

Such a homeomorphism h commutes with the identification leading to $N_{g+n,b}$ and thus induces an element $\eta(h) \in \Gamma_{g+n,b}$. It is proved in [31] that the map

$$\eta^n_{g,b} = \eta\colon \Gamma^n_{g,b} \to \Gamma_{g+n,b}, \quad h \longmapsto \eta(h)$$

is well defined for $n = 1$, but the proof also works for $n > 1$. This homomorphism is called the *blowup* homomorphism.

Proposition 2.1 *The blowup homomorphism* $\eta^n_{g,b}\colon \Gamma^n_{g,b} \to \Gamma_{g+n,b}$ *is injective if* $(g+n,b) \neq (2,0)$.

Remark 2.2 This result is proved in [30] for $(g,b) = (0,1)$, but the proof can be adapted in our case as follows.

Proof Suppose that $h\colon N_{g,b} \to N_{g,b}$ is a homeomorphism satisfying $h(z_i) = z_i$ for all i and that $\eta(h)\colon N_{g+n,b} \to N_{g+n,b}$ is isotopic to the identity. Then h is isotopic to a map equal to the identity on $e_i(U)$ for all i. If not, h reverses local orientation at z_i and $h(\gamma_i)$ is isotopic to γ_i^{-1}. Then $\eta(h)(\gamma_i)$ is isotopic to μ_i and μ_i^{-1} and we get $2[\mu_i] = 0$ in $H_1(N_{g+n,b};\mathbb{Z})$, which is a contradiction. Consequently, h lies in the kernel of the natural map $\Gamma_{g,b+n} \to \Gamma_{g+n,b}$ induced by gluing a Möbius strip onto n boundary components. However, this kernel is generated by the Dehn twists along the curves γ_i (see [29, Theorem 3.6[1]]). Now, any γ_i bounds a disc with one marked point in $N_{g,b}$: the corresponding Dehn twist is trivial in $\Gamma_{g,b}$ and therefore h is isotopic to the identity. □

2.3 Embedding $P_n(N_{g,b})$ in $\Gamma_{g+n+2(b-1),1}$

Since $b \geq 1$, we'll view $N_{g,b}$ as a disc D^2 with $g+b-1$ open discs removed and g Möbius strips glued on g boundary components so obtained (see Figure 2).

Proposition 2.3 *For* $g \geq 1$, $b \geq 1$ *and* $n \geq 1$, $P_n(N_{g,b})$ *has the following complete set of generators (depicted in Figures 2 and 4):*

$$(B_{i,j})_{1 \leq i < j \leq n}, \quad (\rho_{k,l})_{\substack{1 \leq k \leq n \\ 1 \leq l \leq g}} \quad \text{and} \quad (x_{u,t})_{\substack{1 \leq u \leq n \\ 1 \leq t \leq b-1}}.$$

[1] This result is wrong when $(g+n,b) = (2,0)$.

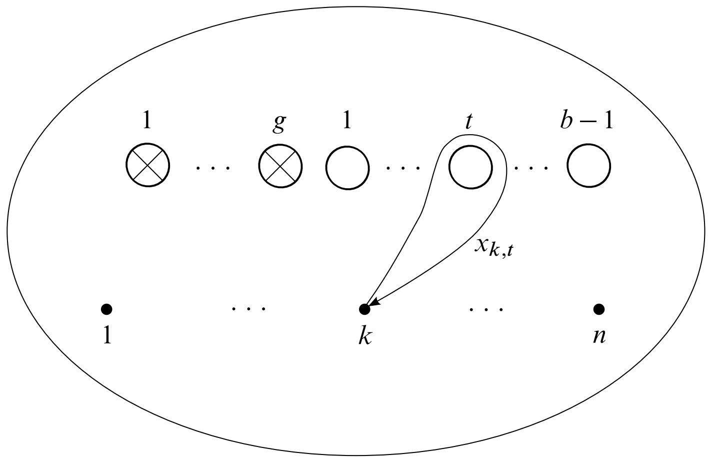

Figure 2: Generators $x_{k,t}$ for $P_n(N_{g,b})$, $b \geq 1$. See Figure 4 for a picture of generators $B_{i,j}$ and $\rho_{k,l}$.

Proof The proof works by induction and generalizes those of [16] (closed nonorientable case) and [4] (orientable case, possibly with boundary components). It uses the following short exact sequence obtained by forgetting the last strand (see [11]):

$$1 \longrightarrow \pi_1(N_{g,b} \setminus \{z_1, \dots, z_n\}, z_{n+1}) \xrightarrow{\alpha} P_{n+1}(N_{g,b}) \xrightarrow{\beta} P_n(N_{g,b}) \longrightarrow 1.$$

The set of generators is complete for $n = 1$: $P_1(N_{g,b}) = \pi_1(N_{g,b})$ is free on the $\rho_{1,l}$ and $x_{1,t}$ for $1 \leq l \leq g$ and $1 \leq t \leq b-1$. Suppose inductively that $P_n(N_{g,b})$ has the given complete set of generators. Then observe that

$$\{B_{i,n+1} \mid 1 \leq i \leq n\} \cup \{\rho_{n+1,l} \mid 1 \leq l \leq g\} \cup \{x_{n+1,t} \mid 1 \leq t \leq b-1\}$$

is a free generators set of $\mathrm{Im}(\alpha)$ and

$$(B_{i,j})_{1 \leq i < j \leq n}, \quad (\rho_{k,l})_{\substack{1 \leq k \leq n \\ 1 \leq l \leq g}} \quad \text{and} \quad (x_{u,t})_{\substack{1 \leq u \leq n \\ 1 \leq t \leq b-1}}$$

are coset representatives for the considered generators of $P_n(N_{g,b})$; this is a complete set of generators for $P_{n+1}(N_{g,b})$; see for instance [22, Theorem 1, Chapter 13]. Let us also remark that the above exact sequence could be used, as in [4] and [16], to find a complete set of relations for the group $P_n(N_{g,b})$. □

Gluing a one-holed torus onto $b-1$ boundary components of $N_{g,b}$ (recall that $b \geq 1$ in this second section), we get $N_{g,b}$ as a subsurface of $N_{g+2(b-1),1}$. This inclusion induces a homomorphism $\chi_{g,b}\colon \Gamma_{g,b} \to \Gamma_{g+2(b-1),1}$ which is injective (see [29]).

Thus, the composed map

$$\lambda^n_{g,b} = \chi_{g+n,b} \circ \eta^n_{g,b} \colon \Gamma^n_{g,b} \to \Gamma_{g+n+2(b-1),1}$$

is also injective.

Recall that the mod p Torelli group $I_p(N_{g,1})$ is the subgroup of $\Gamma_{g,1}$ defined as the kernel of the action of $\Gamma_{g,1}$ on $H_1(N_{g,1}; \mathbb{Z}/p\mathbb{Z})$. In the following we will consider in particular the case of the mod 2 Torelli group $I_2(N_{g,1})$.

Proposition 2.4 *If $b \geq 1$, $\lambda^n_{g,b}(P_n(N_{g,b}))$ is a subgroup of the mod 2 Torelli subgroup $I_2(N_{g+n+2(b-1),1})$.*

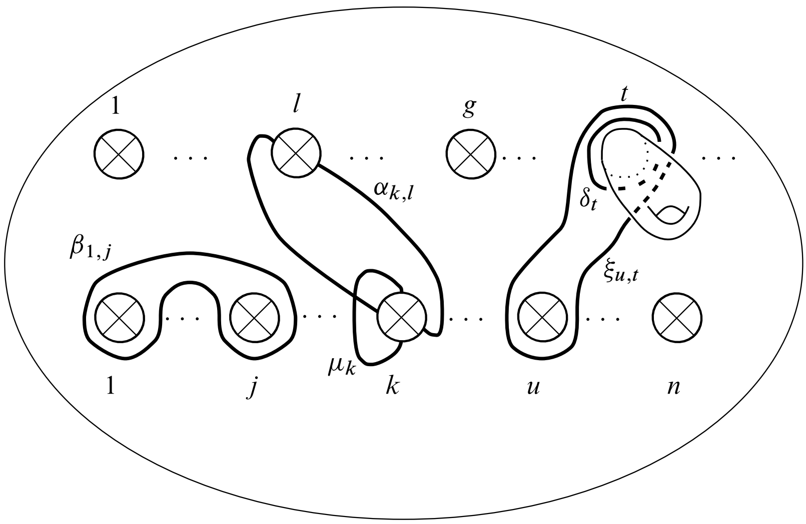

Figure 3: Image of the generators of $P_n(N_{g,b})$ in $\Gamma_{g+n+2(b-1),1}$.

Proof The image of the generators (see Figures 2, 4 and Proposition 2.3)

$$(B_{i,j})_{1\leq i<j\leq n}, \quad (\rho_{k,l})_{\substack{1\leq k\leq n\\ 1\leq l\leq g}} \quad \text{and} \quad (x_{u,t})_{\substack{1\leq u\leq n\\ 1\leq t\leq b-1}}$$

of $P_n(N_{g,b})$ under $\lambda^n_{g,b}$ are, respectively (see Figure 3):

- Dehn twists along curves $\beta_{i,j}$ which bound a subsurface homeomorphic to $N_{2,1}$.
- Crosscap slides $Y_{\mu_k,\alpha_{k,l}}$.
- The product $\tau_{\xi_{u,t}}\tau^{-1}_{\delta_t}$ of Dehn twists along the bounding curves $\xi_{u,t}$ and δ_t.

According to [31], all of these elements are in $I_2(N_{g+n+2(b-1),1})$. □

Remark 2.5 The embedding from Proposition 2.4 is invalid for $I_p(N_{g+n+2(b-1),1})$ when $p \neq 2$: for example, the crosscap slide $Y_{\mu_k,\alpha_{k,l}}$ is not in the mod p Torelli subgroup since it sends μ_k to μ_k^{-1}.

2.4 Conclusion of the proof

We shall use the following result, which is a straightforward consequence of a similar result for mod p Torelli groups of orientable surfaces due to L Paris [26]:

Theorem 2.6 *Let $g \geq 1$. The mod p Torelli group $I_p(N_{g,1})$ is residually p–finite.*

Proof We use the Dehn–Nielsen–Baer theorem (see for instance [32, Theorem 5.15.3]), which states that $\Gamma_{g,1}$ embeds in $\mathrm{Aut}(\pi_1(N_{g,1}))$. Since $\pi_1(N_{g,1})$ is free we can apply [26, Theorem 1.4] which claims that, if G is a free group, its mod p Torelli group (ie the kernel of the canonical map from $\mathrm{Aut}(G)$ to $\mathrm{GL}(H_1(G, \mathbb{F}_p))$) is residually p–finite. Therefore $I_p(N_{g,1})$ is residually p–finite. □

Theorem 2.7 *Let $g \geq 1$, $b > 0$, $n \geq 1$. Then $P_n(N_{g,b})$ is residually 2–finite.*

Proof The group $P_n(N_{g,b})$ is a subgroup of $I_2(N_{g+n+2(b-1),1})$ by Proposition 2.4 and by injectivity of the map $\lambda^n_{g,b}$. Then by Theorem 2.6 it follows that $P_n(N_{g,b})$ is residually 2–finite. □

3 *p*–almost direct products

3.1 On residually *p*–finite groups

Let p be a prime number and G a group. If H is a subgroup of G, we denote by H^p the subgroup generated by $\{h^p \mid h \in H\}$. Following [26], we define the lower $\mathbb{F}_p$–linear central filtration $(\gamma_n^p G)_{n\in\mathbb{N}^*}$ of G as follows: $\gamma_1^p G = G$ and, for $n \geq 1$, $\gamma_{n+1}^p G$ is the subgroup of G generated by $[G, \gamma_n^p G] \cup (\gamma_n^p G)^p$. Note that the subgroups $\gamma_n^p G$ are characteristic in G and that the quotient group $G/\gamma_2^p G$ is nothing but the first homology group $H_1(G; \mathbb{F}_p)$. The following are proved in [26]:

- $[\gamma_m^p G, \gamma_n^p G] \subset \gamma_{m+n}^p G$ for $m, n \geq 1$.
- A finitely generated group G is a finite p–group if and only if there exists some $N \geq 1$ such that $\gamma_N^p G = \{1\}$.
- A finitely generated group G is residually p–finite if and only if $\bigcap_{n=1}^{\infty} \gamma_n^p G = \{1\}$.

Clearly, if $f\colon G \to G'$ is a group homomorphism, then $f(\gamma_n^p G) \subset \gamma_n^p G'$ for all $n \geq 1$.

Definition 3.1 Let

$$1 \longrightarrow A \hookrightarrow B \overset{\lambda}{\underset{\sigma}{\rightleftarrows}} C \longrightarrow 1$$

be a split exact sequence.

- If the action of C induced on $H_1(A;\mathbb{Z})$ is trivial (ie the action is trivial on $A^{\mathrm{Ab}} = A/[A,A]$), we say that B is an almost direct product of A and C.
- If the action of C induced on $H_1(A;\mathbb{F}_p)$ is trivial (ie the action is trivial on $A/\gamma_2^p A$), we say that B is a p–almost direct product of A and C.

Let us remark that, as in the case of almost direct products [7, Proposition 6.3], the property of being a p–almost direct product does not depend on the choice of section.

Proposition 3.2 *Let $1 \longrightarrow A \hookrightarrow B \overset{\lambda}{\longrightarrow} C \longrightarrow 1$ be a split exact sequence of groups. Let σ, σ' be sections for λ, and suppose that the induced action of C on A via σ on $H_1(A;\mathbb{F}_p)$ is trivial. Then the same is true for the section σ'.*

Proof Let $a \in A$ and $c \in C$. By hypothesis, $\sigma(c)a(\sigma(c))^{-1} \equiv a \bmod \gamma_2^p A$. Since σ' is also a section for λ, we have $\lambda \circ \sigma'(c) = \lambda \circ \sigma(c)$, and so $\sigma'(c)(\sigma(c))^{-1} \in \mathrm{Ker}(\lambda)$. Thus there exists $a' \in A$ such that $\sigma'(c) = a'\sigma(c)$, and hence

$$\sigma'(c)a(\sigma'(c))^{-1} \equiv a'\sigma(c)a(\sigma(c))^{-1}a'^{-1} \equiv a'aa'^{-1} \equiv a \bmod \gamma_2^p A.$$

Thus the induced action of C on $H_1(A;\mathbb{F}_p)$ via σ' is also trivial. □

The first goal of this section is to prove the following theorem (see [12, Theorem 3.1] for an analogous result for almost direct products).

Theorem 3.3 *Let*

$$1 \longrightarrow A \hookrightarrow B \overset{\lambda}{\underset{\sigma}{\rightleftarrows}} C \longrightarrow 1$$

be a split exact sequence where B is a p–almost direct product of A and C. Then, for all $n \geq 1$, one has a split exact sequence

$$1 \longrightarrow \gamma_n^p A \hookrightarrow \gamma_n^p B \overset{\lambda_n}{\underset{\sigma_n}{\rightleftarrows}} \gamma_n^p C \longrightarrow 1,$$

where λ_n and σ_n are restrictions of λ and σ.

We shall need the following preliminary result.

Lemma 3.4 *Under the hypotheses of Theorem 3.3, one has*

$$[\gamma_m^p C', \gamma_n^p A] \subset \gamma_{m+n}^p A \quad \text{for all } m, n \geq 1,$$

where C' denotes $\sigma(C)$.

Proof First, we prove by induction on n that $[C', \gamma_n^p A] \subset \gamma_{n+1}^p A$ for all $n \geq 1$. The case $n = 1$ corresponds to the hypotheses: the action of C on $H_1(A; \mathbb{F}_p) = A/\gamma_2^p A$ is trivial if and only if $[C', A] \subset \gamma_2^p A$. Thus, suppose that $[C', \gamma_n^p A] \subset \gamma_{n+1}^p A$ for some $n \geq 1$ and let us prove that $[C', \gamma_{n+1}^p A] \subset \gamma_{n+2}^p A$. In view of the definition of $\gamma_{n+1}^p A$, we have to prove that

$$\big[C', [A, \gamma_n^p A]\big] \subset \gamma_{n+2}^p A \quad \text{and} \quad [C', (\gamma_n^p A)^p] \subset \gamma_{n+2}^p A.$$

For the first case, we use a classical result (see [24, Theorem 5.2]) which says

$$\big[C', [A, \gamma_n^p A]\big] = \big[\gamma_n^p A, [C', A]\big]\big[A, [\gamma_n^p A, C']\big].$$

We have just seen that $[C', A] \subset \gamma_2^p A$, thus

$$\big[\gamma_n^p A, [C', A]\big] \subset [\gamma_n^p A, \gamma_2^p A] \subset \gamma_{n+2}^p A.$$

Then, the induction hypotheses says that $[\gamma_n^p A, C'] \subset \gamma_{n+1}^p A$, thus

$$\big[A, [\gamma_n^p A, C']\big] \subset [A, \gamma_{n+1}^p A] \subset \gamma_{n+2}^p A.$$

The second case works as follows: for $c \in C'$ and $x \in \gamma_n^p A$, one has, using the fact that $[u, vw] = [u, w][u, v]\big[[u, v], w\big]$ (see [24]),

$$\begin{aligned} [c, x^p] &= [c, x][c, x^{p-1}]\big[[c, x^{p-1}], x\big] \\ &= \cdots = [c, x]^p\big[[c, x], x\big]\big[[c, x^2], x\big] \cdots \big[[c, x^{p-1}], x\big]. \end{aligned}$$

Since $c \in C'$ and $x \in \gamma_n^p A$, one has $[c, x^i] \in [C', \gamma_n^p A] \subset \gamma_{n+1}^p A$ for all i, $1 \leq i \leq p-1$, which leads to

$$[c, x]^p \in (\gamma_{n+1}^p A)^p \subset \gamma_{n+2}^p A \quad \text{and} \quad \big[[c, x^i], x\big] \in [\gamma_{n+1}^p A, A] \subset \gamma_{n+2}^p A.$$

Now, we suppose that $[\gamma_m^p C', \gamma_n^p A] \subset \gamma_{m+n}^p A$ for some $m \geq 1$ and all $n \geq 1$ and prove that $[\gamma_{m+1}^p C', \gamma_n^p A] \subset \gamma_{m+n+1}^p A$. As above, there are two cases which work in the same way:

(i) $$\big[[C',\gamma_m^pC'],\gamma_n^pA\big]=\big[[\gamma_n^pA,C'],\gamma_m^pC'\big]\big[[\gamma_m^pC',\gamma_n^pA],C'\big] \subset \big[\gamma_{n+1}^pA,\gamma_m^pC'\big]\big[\gamma_{m+n}^pA,C'\big] \subset \gamma_{m+n+1}^pA.$$

(ii) For $c\in\gamma_m^pC'$ and $x\in\gamma_n^pA$, one has

$$[c^p,x]=\big[c,[x,c^{p-1}]\big][c^{p-1},x][c,x]=\cdots=\big[c,[x,c^{p-1}]\big]\cdots\big[c,[x,c]\big][c,x]^p,$$

which is an element of γ_{m+n+1}^pA by induction hypotheses. □

Proof of Theorem 3.3 The restrictions of λ and σ give rise to morphisms

$$\lambda_n\colon \gamma_n^pB\to\gamma_n^pC \quad\text{and}\quad \sigma_n\colon \gamma_n^pC\to\gamma_n^pB$$

such that $\lambda_n\circ\sigma_n=\mathrm{Id}_{\gamma_n^pC}$, λ_n is onto and $\mathrm{Ker}(\lambda_n)=A\cap\gamma_n^pB$. Thus, we need to prove that $A\cap\gamma_n^pB=\gamma_n^pA$ for all $n\geq 1$. Clearly one has $\gamma_n^pA\subset A\cap\gamma_n^pB$. In order to prove the reverse inclusion, we follow the method developed in [12] for almost semi-direct products and define $\tau\colon B\to B$ by $\tau(b)=(\sigma\lambda(b))^{-1}b$. This map has the following properties:

(i) Since $\lambda\sigma=\mathrm{Id}_C$, $\tau(B)\subset A$.

(ii) For $x\in B$, $\tau(x)=x$ if and only if $x\in A$.

(iii) For $(b_1,b_2)\in B^2$, $\tau(b_1b_2)=[\sigma\lambda(b_2),\tau(b_1)^{-1}]\tau(b_1)\tau(b_2)$.

(iv) For $b\in B$, setting $a=\tau(b)$ and $c=\sigma\lambda(b)$, we get a unique decomposition $b=ca$ with $c\in C'=\sigma(C)$ and $a\in A$.

We claim that $\tau(\gamma_n^pB)\subset\gamma_n^pA$ for all $n\geq 1$. From this, we easily conclude the proof: if $x\in A\cap\gamma_n^pB$, then $x=\tau(x)\in\gamma_n^pA$.

One has $\tau(\gamma_1^pB)\subset\gamma_1^pA$. Suppose inductively that $\tau(\gamma_n^pB)\subset\gamma_n^pA$ for some $n\geq 1$, and let us prove that $\tau(\gamma_{n+1}^pB)\subset\gamma_{n+1}^pA$. Suppose first that x is an element of γ_n^pB. Then using (iii) we get

$$\begin{aligned}\tau(x^p)&=[\sigma\lambda(x),\tau(x^{p-1})^{-1}]\tau(x^{p-1})\tau(x)\\ &\;\;\vdots\\ &=[\sigma\lambda(x),\tau(x^{p-1})^{-1}][\sigma\lambda(x),\tau(x^{p-2})^{-1}]\cdots[\sigma\lambda(x),\tau(x)^{-1}]\tau(x)^p.\end{aligned}$$

Since $\sigma\lambda(x)\in\gamma_n^pC'$, and since $\tau(x^i)\in\gamma_n^pA$ for $1\leq i\leq p-1$ by the induction hypothesis, we get

$$\tau(x^p)\in[\gamma_n^pC',\gamma_n^pA]\cdot(\gamma_n^pA)^p\subset\gamma_{n+1}^pA$$

by Lemma 3.4: this proves that $\tau((\gamma_n^p B)^p) \subset \gamma_{n+1}^p A$. Next, let $b \in B$ and $x \in \gamma_n^p B$. Setting $a = \tau(b) \in A$, $y = \tau(x) \in \gamma_n^p A$ by the induction hypothesis, $c = \sigma\lambda(b) \in C'$ and $z = \sigma\lambda(x) \in \gamma_n^p C'$, we get

$$\begin{aligned}
\tau([b,x]) &= (\sigma\lambda([b,x]))^{-1}[b,x] \\
&= [\sigma\lambda(b), \sigma\lambda(x)]^{-1}[b,x] \\
&= [c,z]^{-1}[ca,zy] = [z,c]a^{-1}c^{-1}y^{-1}z^{-1}cazy \\
&= [z,c](a^{-1}c^{-1}y^{-1}cya)(a^{-1}y^{-1}c^{-1}z^{-1}czya)(a^{-1}y^{-1}z^{-1}azy) \\
&= [z,c](a^{-1}[c,y]a)(a^{-1}y^{-1}[c,z]ya)(a^{-1}y^{-1}ay)(y^{-1}a^{-1}z^{-1}azy) \\
&= [z,c](a^{-1}[c,y]a)(a^{-1}y^{-1}[c,z]ya)[a,y](y^{-1}[a,z]y) \\
&= \big[[c,z],(a^{-1}[y,c]a)\big](a^{-1}[c,y]a)[z,c](a^{-1}y^{-1}[c,z]ya)[a,y](y^{-1}[a,z]y) \\
&= \big[[c,z],(a^{-1}[y,c]a)\big](a^{-1}[c,y]a)\big[[c,z],ya\big][a,y](y^{-1}[a,z]y).
\end{aligned}$$

Now,

$$[c,z] \in [C', \gamma_n^p C'] \subset \gamma_{n+1}^p C' \quad \text{and} \quad [y,c] \in [\gamma_n^p A, C'] \subset \gamma_{n+1}^p A$$

(Lemma 3.4), thus $\big[[c,z],(a^{-1}[y,c]a)\big] \in \gamma_{n+1}^p A$. Then

$$\begin{aligned}
\big[[c,z],ya\big] &\in [\gamma_{n+1}^p C', A] \subset \gamma_{n+1}^p A, \\
[a,y] &\in [A, \gamma_n^p A] \subset \gamma_{n+1}^p A, \\
[a,z] &\in [A, \gamma_n^p C'] \subset \gamma_{n+1}^p A.
\end{aligned}$$

Thus, $\tau([b,x]) \in \gamma_{n+1}^p A$ and $\tau([B, \gamma_n^p B]) \subset \gamma_{n+1}^p A$. □

Corollary 3.5 *Let*

$$1 \longrightarrow A \hookrightarrow B \underset{\sigma}{\overset{\lambda}{\rightleftarrows}} C \longrightarrow 1$$

be a split exact sequence such that B is a p–almost direct product of A and C. If A and C are residually p–finite, then B is residually p–finite.

3.2 Augmentation ideals

Given a group G and $\mathbb{K} = \mathbb{Z}$ or $\mathbb{F}_2$, we will denote by $\mathbb{K}[G]$ the group ring of G over $\mathbb{K}$ and by $\overline{\mathbb{K}[G]}$ the augmentation ideal of G. The group ring $\mathbb{K}[G]$ is filtered by the powers $\overline{\mathbb{K}[G]}^j$ of $\overline{\mathbb{K}[G]}$, and we can define the associated graded algebra

$$\operatorname{gr}\mathbb{K}[G] = \bigoplus \overline{\mathbb{K}[G]}^j / \overline{\mathbb{K}[G]}^{j+1}.$$

The following theorem provides a decomposition formula for the augmentation ideal of a 2–almost direct product (see [25, Theorem 3.1] for an analogous result in the case of almost direct products).

Let $A \rtimes C$ be a semi-direct product between two groups A and C. It is a classical result that the map $a \otimes c \mapsto ac$ induces a $\mathbb{K}$–isomorphism from $\mathbb{K}[A] \otimes \mathbb{K}[C]$ to $\mathbb{K}[A \rtimes C]$. Identifying these two $\mathbb{K}$–modules, we have the following:

Theorem 3.6 *If $A \rtimes C$ is a 2–almost direct product, then*

$$\overline{\mathbb{F}_2[A \rtimes C]}^k = \sum_{i+h=k} \overline{\mathbb{F}_2[A]}^i \otimes \overline{\mathbb{F}_2[C]}^h \quad \textit{for all } k.$$

Proof We sketch the proof, which is almost verbatim the same as the proof of [25, Theorem 3.1]. Let

$$R_k = \sum_{i+h=k} \overline{\mathbb{F}_2[A]}^i \otimes \overline{\mathbb{F}_2[C]}^h;$$

R_k is a descending filtration on $\mathbb{F}_2[A] \otimes \mathbb{F}_2[C]$, and with the above identification, we get that $R_k \subset \overline{\mathbb{F}_2[A \rtimes C]}^k$. To verify the other inclusion we have to check that $\prod_{j=1}^k (a_j c_j - 1) \in R_k$ for every $a_1, \dots, a_k$ in A and $c_1, \dots, c_k$ in C. Actually it is enough to verify that $e = \prod_{j=1}^k (e_j - 1) \in R_k$ where either $e_j \in A$ or $e_j \in C$ (see [25, Theorem 3.1] for a proof of this fact); we call e a *special* element. We associate to a special element e an element in $\{0,1\}^k$: let $\mathrm{type}(e) = (\delta(e_1), \dots, \delta(e_k))$, where $\delta(e_j) = 0$ if $e_j \in A$ and $\delta(e_j) = 1$ if $e_j \in C$. We will say that the special element e is *standard* if

$$\mathrm{type}(e) = (\overbrace{0, \dots, 0}^{i}, \overbrace{1, \dots, 1}^{h}).$$

In this case $e \in \overline{\mathbb{F}_2[A]}^i \otimes \overline{\mathbb{F}_2[C]}^h \subset R_k$ and we are done. We claim that we can reduce all special elements to linear combinations of standard elements. If e is not standard, then it must be of the form

$$e = \prod_{i=1}^{r} (a_i - 1) \prod_{j=1}^{s} (c_j - 1)(c-1)(a-1) \prod_{l=1}^{t} (e_l - 1),$$

where $a_1, \dots, a_r, a \in A$, $c_1, \dots, c_s, c \in C$, the element $\tilde{e} = \prod_{l=1}^t (e_l - 1)$ is special and $r + s + t + 2 = k$. Therefore

$$\mathrm{type}(e) = (\overbrace{0, \dots, 0}^{r}, \overbrace{1, \dots, 1}^{s}, 1, 0, \delta(e_1), \dots, \delta(e_t)).$$

Now we can use the assumption that $A \rtimes C$ is a 2–almost direct product to claim that one has commutation relations in $\mathbb{Z}[A \rtimes C]$ expressing the difference

$$(c-1)(a-1)-(a-1)(c-1)$$

as a linear combination of terms of the form

$$(a'-1)(a''-1)c \quad \text{with } a', a'' \in A,$$

for any $a \in A$ and $c \in C$. In fact,

$$(c-1)(a-1)-(a-1)(c-1) = ca - ac = (cac^{-1}a^{-1}-1)ac = (f-1)ac,$$

where $f = [c^{-1}, a^{-1}] \in [C, A] \subset \gamma_2^2(A)$ by Lemma 3.4. We can decompose f as $f = h_1 k_1 \cdots h_m k_m$, where, for $j = 1, \dots, m$, h_j belongs to $[A, A]$ and $k_j = (k_j')^2$ for some $k_j' \in A$. One knows (see for instance [10, page 194]) that, for $j = 1, \dots, m$, $(h_j - 1)$ is a linear combination of terms of the form

$$(h_j'-1)(h_j''-1)\alpha_j \quad \text{with } h_j', h_j'', \alpha_j \in A.$$

On the other hand, for $j = 1, \dots, m$, we have also that

$$(k_j-1) = (k_j'-1)(k_j'-1) \quad \text{with } k_j' \in A, \qquad \text{since the coefficients are in } \mathbb{F}_2.$$

Then, recalling that $(hk-1) = (h-1)k + (k-1)$ for any $h, k \in A$, we can conclude that $f-1$ can be rewritten as a linear combination of terms of the form

$$(f'-1)(f''-1)\alpha \quad \text{with } f', f'', \alpha \in A$$

and that $(c-1)(a-1)-(a-1)(c-1)$ is a linear combination of terms of the form

$$(f'-1)(f''-1)\alpha c \quad \text{with } f', f'', \alpha \in A.$$

Rewriting $(f''-1)\alpha$ as $(f''\alpha-1)-(\alpha-1)$, we obtain that the difference

$$(c-1)(a-1)-(a-1)(c-1)$$

can be seen as a linear combination of terms of the form

$$(a'-1)(a''-1)c \quad \text{with } a', a'' \in A.$$

Therefore e can be rewritten as a sum whose first term is the special element

$$e' = \prod_{i=1}^{r}(a_i-1)\prod_{j=1}^{s}(c_j-1)(a-1)(c-1)\prod_{l=1}^{t}(e_l-1)$$

and whose second term is a linear combination of elements of the form $e''c$, where

$$e'' = \prod_{i=1}^{r}(a_i - 1)\prod_{j=1}^{s}(c_j - 1)(a' - 1)(a'' - 1)\prod_{l=1}^{t}(ce_l c^{-1} - 1).$$

Using the lexicographic order from the left, one has $\mathrm{type}(e) > \mathrm{type}(e')$ and $\mathrm{type}(e) > \mathrm{type}(e'')$.

By induction on the lexicographic order, we infer that e' and e'' belong to R_k; since $R_k \cdot c \subset R_k$ for any $c \in C$, it follows that e belongs to R_k and we are done. □

4 The closed case

4.1 A presentation of $P_n(N_g)$ and induced identities

We recall a group presentation of $P_n(N_g)$ given in [16]; the geometric interpretation of generators is provided in Figure 4.

Theorem 4.1 [16] *For $g \geq 2$ and $n \geq 1$, $P_n(N_g)$ has a presentation with generators*

$$(B_{i,j})_{1\leq i<j\leq n} \quad \text{and} \quad (\rho_{k,l})_{\substack{1\leq k\leq n\\ 1\leq l\leq g}}$$

and relations of the following four types:

(a) *For $1 \leq i < j \leq n$ and $1 \leq r < s \leq n$,*

$$B_{r,s}B_{i,j}B_{r,s}^{-1} = \begin{cases} B_{i,j} & \text{if } i<r<s<j \text{ or } r<s<i<j, & (a_1)\\ B_{i,j}^{-1}B_{r,j}^{-1}B_{i,j}B_{r,j}B_{i,j} & \text{if } r<i=s<j, & (a_2)\\ B_{s,j}^{-1}B_{i,j}B_{s,j} & \text{if } i=r<s<j, & (a_3)\\ B_{s,j}^{-1}B_{r,j}^{-1}B_{s,j}B_{r,j}B_{i,j}B_{r,j}^{-1}B_{s,j}^{-1}B_{r,j}B_{s,j} & \text{if } r<i<s<j. & (a_4)\end{cases}$$

(b) *For $1 \leq i < j \leq n$ and $1 \leq k, l \leq g$,*

$$\rho_{i,k}\rho_{j,l}\rho_{i,k}^{-1} = \begin{cases} \rho_{j,l} & \text{if } k<l. & (b_1)\\ \rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}^{2} & \text{if } k=l. & (b_2)\\ \rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}B_{i,j}^{-1}\rho_{j,l}B_{i,j}\rho_{j,k}^{-1}B_{i,j}\rho_{j,k} & \text{if } k>l. & (b_3)\end{cases}$$

(c) *For $1 \leq i \leq n$,*

$$\rho_{i,1}^2 \cdots \rho_{i,g}^2 = T_i, \quad \text{where } T_i = B_{1,i}\cdots B_{i-1,i}B_{i,i+1}\cdots B_{i,n}. \qquad (c)$$

(d) *For* $1 \le i < j \le n$, $1 \le k \le n$, $k \ne j$ *and* $1 \le l \le g$,

$$\rho_{k,l} B_{i,j} \rho_{k,l}^{-1} = \begin{cases} B_{i,j} & \text{if } k < i \text{ or } j < k, \quad (d_1) \\ \rho_{j,l}^{-1} B_{i,j}^{-1} \rho_{j,l} & \text{if } k = i, \quad (d_2) \\ \rho_{j,l}^{-1} B_{k,j}^{-1} \rho_{j,l} B_{k,j}^{-1} B_{i,j} B_{k,j} \rho_{j,l}^{-1} B_{k,j} \rho_{j,l} & \text{if } i < k < j. \quad (d_3) \end{cases}$$

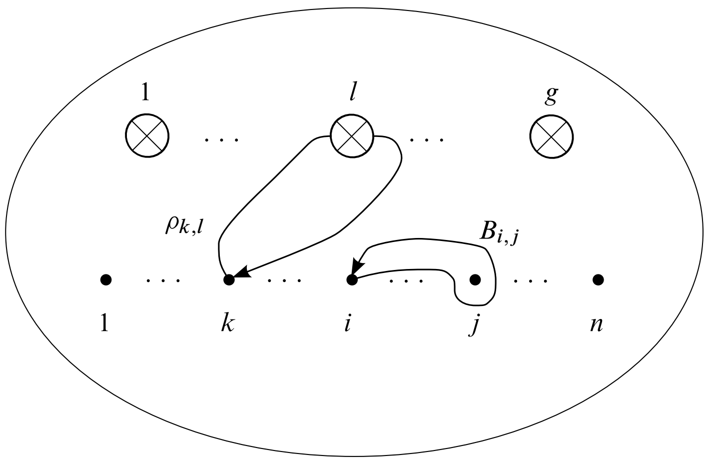

Figure 4: Generators of $P_n(N_g)$.

Let us denote by U the element $\rho_{n,1}\rho_{n-1,1}\cdots\rho_{2,1}$ of $P_n(N_g)$.

Lemma 4.2 *The following relations hold in* $P_n(N_g)$*:*

(1) $[\rho_{i,k}, \rho_{j,k}^{-1}] = B_{i,j}^{-1}$ *for* $1 \le i < j \le n$ *and* $1 \le k \le g$. (e)

(2) $[\rho_{k,1}, \rho_{1,1}\rho_{1,1}\rho_{1,2}T_1^{-1}] = 1$ *for* $2 \le k \le n$. (f)

(3) $[U, \rho_{1,1}\rho_{1,1}\rho_{1,2}T_1^{-1}] = 1$. (g)

(4) $U\rho_{1,1}U^{-1} = \rho_{1,1}T_1^{-1}$. (h)

Proof The first and second identities can be verified by drawing the corresponding braids (see Figure 5 and 6). The third one is a direct consequence of the second one and the definition of U. We prove the last one as follows:

$$\begin{aligned} \rho_{1,1}^{-1}U\rho_{1,1} &= (\rho_{1,1}^{-1}\rho_{n,1}\rho_{1,1})\cdots(\rho_{1,1}^{-1}\rho_{2,1}\rho_{1,1}) \\ &= (B_{1,n}^{-1}\rho_{n,1})\cdots(B_{1,2}^{-1}\rho_{2,1}) \quad \text{by (e)} \\ &= B_{1,n}^{-1}\cdots B_{1,2}^{-1}\rho_{n,1}\cdots\rho_{2,1} \quad \text{by } (d_1) \\ &= T_1^{-1}U. \end{aligned}$$

□

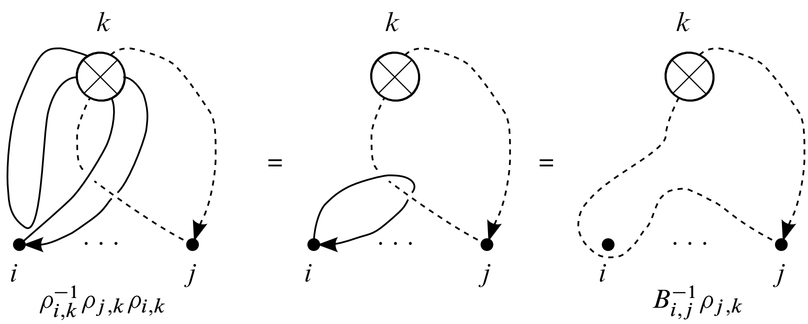

Figure 5: Identity (e).

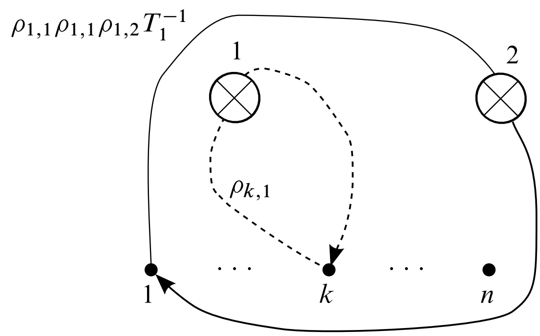

Figure 6: Identity (f).

4.2 The pure braid group $P_n(N_g)$ is residually 2–finite

Following [14], one has, for $g \geq 2$, a split exact sequence

$$(1)\qquad 1 \longrightarrow P_{n-1}(N_{g,1}) \xrightarrow{\ \mu\ } P_n(N_g) \xrightarrow{\ \lambda\ } P_1(N_g) = \pi_1(N_g) \longrightarrow 1,$$

where λ is induced by the map which forgets all strands except the first one, and μ is defined by capping the boundary component by a disc with one marked point (the first strand in $P_n(N_g)$). According to the definition of μ and to Proposition 2.3, $\mathrm{Im}(\mu)$ is generated by $\{\rho_{i,k} \mid 2 \leq i \leq n,\ 1 \leq k \leq g\} \cup \{B_{i,j} \mid 2 \leq i < j \leq n\}$.

The section given in [14] is geometric, ie it is induced by a crossed section at the level of fibrations. In order to study the action of $\pi_1(N_g)$ on $P_{n-1}(N_{g,1})$, we need an algebraic section. Recall that $\pi_1(N_g)$ has a group presentation with generators $p_1, \dots, p_g$ and the single relation $p_1^2 \cdots p_g^2 = 1$. We define the set map $\sigma\colon \pi_1(N_g) \to P_n(N_g)$ by setting

$$\sigma(p_i) = \begin{cases} T_1^{-1} U \rho_{1,1} T_1 & \text{for } i = 1, \\ T_1^{-1} \rho_{1,1}^{-1} U^{-1} \rho_{1,1}\rho_{1,2} & \text{for } i = 2, \\ \rho_{1,i} & \text{for } 3 \leq i \leq g. \end{cases}$$

Proposition 4.3 *The map σ is a well-defined homomorphism satisfying $\lambda \circ \sigma = \mathrm{Id}_{\pi_1(N_g)}$.*

Proof Since $\lambda(\rho_{1,i}) = p_i$ for all $1 \le i \le g$ and $\lambda(U) = \lambda(T_1) = 1$, one clearly has $\lambda\sigma = \mathrm{Id}_{\pi_1(N_g)}$ if σ is a group homomorphism. Thus, we just have to prove that $\sigma(p_1)^2 \cdots \sigma(p_g)^2 = 1$:

$$\begin{aligned}
\sigma(p_1)^2 \cdots \sigma(p_g)^2 &= T_1^{-1} U \rho_{1,1} T_1 T_1^{-1} U \rho_{1,1} T_1 T_1^{-1} \rho_{1,1}^{-1} U^{-1} \rho_{1,1}\rho_{1,2} T_1^{-1} \\
&\quad \times \rho_{1,1}^{-1} U^{-1} \rho_{1,1}\rho_{1,2}\rho_{1,3}^2 \cdots \rho_{1,g}^2 \\
&= T_1^{-1} \underbrace{U\rho_{1,1}\rho_{1,1}\rho_{1,2}T_1^{-1}}\, \rho_{1,1}^{-1} U^{-1} \rho_{1,1}\rho_{1,2}\rho_{1,3}^2 \cdots \rho_{1,g}^2 \\
&= T_1^{-1} \rho_{1,1}\rho_{1,1}\rho_{1,2} T_1^{-1}\, \underbrace{U\rho_{1,1}^{-1}U^{-1}}\, \rho_{1,1}\rho_{1,2}\rho_{1,3}^2 \cdots \rho_{1,g}^2 \quad \text{by (g)} \\
&= T_1^{-1} \rho_{1,1}\rho_{1,1}\rho_{1,2} T_1^{-1} T_1 \rho_{1,1}^{-1} \rho_{1,1}\rho_{1,2}\rho_{1,3}^2 \cdots \rho_{1,g}^2 \quad \text{by (h)} \\
&= T_1^{-1} \rho_{1,1}\rho_{1,1}\rho_{1,2}\rho_{1,2}\rho_{1,3}^2 \cdots \rho_{1,g}^2 \\
&= 1 \quad \text{by (c).}
\end{aligned}$$

□

So, the exact sequence (1) splits. In order to apply Theorem 3.3, we have to prove that the action of $\pi_1(N_g)$ on $P_{n-1}(N_{g,1})$ is trivial on $H_1(P_{n-1}(N_{g,1}); \mathbb{F}_2)$. This is the claim of the following proposition.

Proposition 4.4 *For all $x \in \mathrm{Im}(\sigma)$ and $a \in \mathrm{Im}(\mu)$, one has*

$$[x^{-1}, a^{-1}] = xax^{-1}a^{-1} \in \gamma_2^2(\mathrm{Im}(\mu)).$$

Proof It is enough to prove the result for generators

$$a \in \{B_{j,k} \mid 2 \le j < k \le n\} \cup \{\rho_{j,l} \mid 2 \le j \le n \text{ and } 1 \le l \le g\}$$

and

$$x \in \{\sigma(p_1), \dots, \sigma(p_g)\}$$

of $\mathrm{Im}(\mu)$ and $\mathrm{Im}(\sigma)$, respectively. Suppose first that $2 \le j < k \le n$. One has:

- $[\sigma(p_i)^{-1}, B_{j,k}^{-1}] = [\rho_{1,i}^{-1}, B_{j,k}^{-1}] = 1$ for $3 \le i \le g$ by (d_1).
- Then, one has

$$\begin{aligned}
[\sigma(p_2)^{-1}, B_{j,k}^{-1}] &= [\rho_{1,2}^{-1}\rho_{1,1}^{-1}U\rho_{1,1}T_1, B_{j,k}^{-1}] \\
&= (\rho_{1,1}^{-1}U\rho_{1,1}T_1)^{-1}[\rho_{1,2}^{-1}, B_{j,k}^{-1}](\rho_{1,1}^{-1}U\rho_{1,1}T_1)[\rho_{1,1}^{-1}U\rho_{1,1}T_1, B_{j,k}^{-1}] \\
&= [\rho_{1,1}^{-1}U\rho_{1,1}T_1, B_{j,k}^{-1}] \quad \text{by } (\mathrm{d}_1).
\end{aligned}$$

But U and T_1 are elements of $\mathrm{Im}(\mu)$ and the latter being normal in $P_n(N_g)$, $\rho_{1,1}^{-1}U\rho_{1,1}T_1 \in \mathrm{Im}(\mu)$ thus $[\sigma(p_2)^{-1}, B_{j,k}^{-1}] \in \Gamma_2(\mathrm{Im}(\mu)) \subset \gamma_2^2(\mathrm{Im}(\mu))$.

- In the same way, one has

$$\begin{aligned}[\sigma(p_1)^{-1}, B_{j,k}^{-1}] &= [T_1^{-1}\rho_{1,1}^{-1}U^{-1}T_1, B_{j,k}^{-1}] = [\rho_{1,1}^{-1}\rho_{1,1}T_1^{-1}\rho_{1,1}^{-1}U^{-1}T_1, B_{j,k}^{-1}] \\ &= (\rho_{1,1}T_1^{-1}\rho_{1,1}^{-1}U^{-1}T_1)^{-1}[\rho_{1,1}^{-1}, B_{j,k}^{-1}](\rho_{1,1}T_1^{-1}\rho_{1,1}^{-1}U^{-1}T_1) \\ &\qquad \times [\rho_{1,1}T_1^{-1}\rho_{1,1}^{-1}U^{-1}T_1, B_{j,k}^{-1}] \\ &= [\rho_{1,1}T_1^{-1}\rho_{1,1}^{-1}U^{-1}T_1, B_{j,k}^{-1}] \quad \text{by } (\mathrm{d}_1),\end{aligned}$$

thus, as before, $[\sigma(p_1)^{-1}, B_{j,k}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu))$.

Now, let j and l be integers such that $2 \le j \le n$ and $1 \le l \le g$, and let us first prove that $[\rho_{1,i}^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu))$ for all i, $1 \le i \le g$:

- This is clear for $i < l$ by (b_1).
- For $i = l$, the relation (b_2) gives $[\rho_{1,l}^{-1}, \rho_{j,l}^{-1}] = \rho_{j,l}^{-1}B_{1,j}^{-1}\rho_{j,l}$. But

$$B_{1,j}^{-1} = B_{2,j}\cdots B_{j-1,j}B_{j,j+1}\cdots B_{j,n}\rho_{j,g}^{-2}\cdots\rho_{j,1}^{-2} \quad \text{(relation (c))}$$

is an element of $\gamma_2^2(\mathrm{Im}(\mu))$ by (e), thus we get $[\rho_{1,l}^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu))$.

- If $l < i$ then $[\rho_{1,i}^{-1}, \rho_{j,l}^{-1}] = [B_{1,j}\rho_{j,i}^{-1}B_{1,j}\rho_{j,i}, \rho_{j,l}^{-1}]$ by (b_3) so $[\rho_{1,i}^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu))$ since $\rho_{j,l}$, $\rho_{j,i}$ and $B_{1,j}$ are elements of $\mathrm{Im}(\mu)$.

From this, we deduce the following facts.

(1) $[\sigma(p_i)^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu))$ for $i \ge 3$ since $\sigma(p_i) = \rho_{1,i}$.

(2) Next, one has

$$\begin{aligned}[\sigma(p_2)^{-1}, \rho_{j,l}^{-1}] &= [\rho_{1,2}^{-1}\rho_{1,1}^{-1}U\rho_{1,1}T_1, \rho_{j,l}^{-1}] \\ &= (\rho_{1,1}^{-1}U\rho_{1,1}T_1)^{-1}[\rho_{1,2}^{-1}, \rho_{j,l}^{-1}](\rho_{1,1}^{-1}U\rho_{1,1}T_1)[\rho_{1,1}^{-1}U\rho_{1,1}T_1, \rho_{j,l}^{-1}].\end{aligned}$$

Since $\rho_{1,1}^{-1}U\rho_{1,1}T_1$ and $\rho_{j,l}$ are elements of $\mathrm{Im}(\mu)$ and $[\rho_{1,2}^{-1}, \rho_{j,l}^{-1}]$ is an element of $\gamma_2^2(\mathrm{Im}(\mu))$, we get $[\sigma(p_2)^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu))$.

(3) In the same way, one has

$$\begin{aligned}[\sigma(p_1)^{-1}, \rho_{j,l}^{-1}] &= [T_1^{-1}\rho_{1,1}^{-1}U^{-1}T_1, \rho_{j,l}^{-1}] = [T_1^{-1}\rho_{1,1}^{-1}U^{-1}T_1\rho_{1,1}\rho_{1,1}^{-1}, \rho_{j,l}^{-1}] \\ &= \rho_{1,1}[T_1^{-1}\rho_{1,1}^{-1}U^{-1}T_1\rho_{1,1}, \rho_{j,l}^{-1}]\rho_{1,1}^{-1}[\rho_{1,1}^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu)). \quad \square\end{aligned}$$

We are now ready to prove the main result of this section.

Theorem 4.5 *For all $g \geq 2$ and $n \geq 1$, the pure braid group $P_n(N_g)$ is residually 2–finite.*

Proof Proposition 4.3 says that the sequence

$$1 \longrightarrow P_{n-1}(N_{g,1}) \longrightarrow P_n(N_g) \longrightarrow \pi_1(N_g) \longrightarrow 1$$

splits. Now $P_{n-1}(N_{g,1})$ is residually 2–finite (Theorem 2.7). It is proved in [2] and [3] that $\pi_1(N_g)$ is residually free for $g \geq 4$, so it is residually 2–finite. This result is proved in [23, Lemma 8.9] for $g = 3$. When $g = 2$, $\pi_1(N_2)$ has presentation $\langle a, b \mid aba^{-1} = b^{-1} \rangle$ so is a 2–almost direct product of $\mathbb{Z}$ by $\mathbb{Z}$. Since $\mathbb{Z}$ is residually 2–finite, $\pi_1(N_2)$ is residually 2–finite by Corollary 3.5. So, using Proposition 4.4 and Corollary 3.5, we can conclude that $P_n(N_g)$ is residually 2–finite. □

4.3 The case $P_n(\mathbb{RP}^2)$

The main reason to exclude $N_1 = \mathbb{RP}^2$ in Theorem 4.5 is that the exact sequence (1) doesn't exist in this case, but forgetting at most $n-2$ strands we get the following exact sequence ($1 \leq m \leq n-2$; see [9]):

$$1 \longrightarrow P_m(N_{1,n-m}) \longrightarrow P_n(\mathbb{RP}^2) \longrightarrow P_{n-m}(\mathbb{RP}^2) \longrightarrow 1.$$

This sequence splits if and only if $n = 3$ and $m = 1$ (see [15]). Thus, what we know is the following:

- $P_1(\mathbb{RP}^2) = \pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$: $P_1(\mathbb{RP}^2)$ is a 2–group.
- $P_2(\mathbb{RP}^2) = Q_8$, the quaternion group (see [9]): $P_2(\mathbb{RP}^2)$ is a 2–group.
- One has the split exact sequence

  $$1 \longrightarrow P_1(N_{1,2}) \longrightarrow P_3(\mathbb{RP}^2) \longrightarrow P_2(\mathbb{RP}^2) \longrightarrow 1,$$

  where $P_1(N_{1,2}) = \pi_1(N_{1,2})$ is a free group of rank 2, thus is residually 2–finite. Since $P_2(\mathbb{RP}^2)$ is 2–finite, we can conclude that $P_3(\mathbb{RP}^2)$ is residually 2–finite using [19, Lemma 1.5].

References

[1] **V G Bardakov**, **P Bellingeri**, *On residual properties of pure braid groups of closed surfaces*, Comm. Algebra 37 (2009) 1481–1490 MR2526317

[2] **B Baumslag**, *Residually free groups*, Proc. London Math. Soc. 17 (1967) 402–418 MR0215903

[3] **G Baumslag**, *On generalised free products*, Math. Z. 78 (1962) 423–438 MR0140562

[4] **P Bellingeri**, *On presentations of surface braid groups*, J. Algebra 274 (2004) 543–563 MR2043362

[5] **P Bellingeri**, **L Funar**, *Braids on surfaces and finite type invariants*, C. R. Math. Acad. Sci. Paris 338 (2004) 157–162 MR2038286

[6] **P Bellingeri**, **S Gervais**, **J Guaschi**, *Lower central series of Artin–Tits and surface braid groups*, J. Algebra 319 (2008) 1409–1427 MR2383053

[7] **P Bellingeri**, **E Godelle**, **J Guaschi**, *Exact sequences, lower central series and representations of surface braid groups*, preprint (2011) `arXiv:1106.4982`

[8] **R Botto Mura**, **A Rhemtulla**, *Orderable groups*, Lecture Notes in Pure and Applied Mathematics 27, Marcel Dekker, New York (1977) MR0491396

[9] **J van Buskirk**, *Braid groups of compact 2–manifolds with elements of finite order*, Trans. Amer. Math. Soc. 122 (1966) 81–97 MR0189013

[10] **K-T Chen**, *Extension of C^∞ function algebra by integrals and Malcev completion of π_1*, Advances in Math. 23 (1977) 181–210 MR0458461

[11] **E Fadell**, **L Neuwirth**, *Configuration spaces*, Math. Scand. 10 (1962) 111–118 MR0141126

[12] **M Falk**, **R Randell**, *The lower central series of a fiber-type arrangement*, Invent. Math. 82 (1985) 77–88 MR808110

[13] **S Garoufalidis**, **J Levine**, *Finite type 3–manifold invariants and the structure of the Torelli group, I*, Invent. Math. 131 (1998) 541–594 MR1614551

[14] **D L Gonçalves**, **J Guaschi**, *On the structure of surface pure braid groups*, J. Pure Appl. Algebra 186 (2004) 187–218 MR2025598

[15] **D L Gonçalves**, **J Guaschi**, *The braid groups of the projective plane and the Fadell–Neuwirth short exact sequence*, Geom. Dedicata 130 (2007) 93–107 MR2365780

[16] **D L Gonçalves**, **J Guaschi**, *Braid groups of non-orientable surfaces and the Fadell–Neuwirth short exact sequence*, J. Pure Appl. Algebra 214 (2010) 667–677 MR2577674

[17] **J González-Meneses**, *Ordering pure braid groups on compact, connected surfaces*, Pacific J. Math. 203 (2002) 369–378 MR1897904

[18] **J González-Meneses**, **L Paris**, *Vassiliev invariants for braids on surfaces*, Trans. Amer. Math. Soc. 356 (2004) 219–243 MR2020030

[19] **K W Gruenberg**, *Residual properties of infinite soluble groups*, Proc. London Math. Soc. 7 (1957) 29–62 MR0087652

[20] **J Guaschi**, **D Juan-Pineda**, *A survey of surface braid groups and the lower algebraic K–theory of their group rings*, from: "Handbook of group actions, Vol. II", (L Ji, A Papadopoulos, S-T Yau, editors), Adv. Lect. Math. 32, Int. Press, Somerville, MA (2015) 23–75 MR3382024

[21] **B Hartley**, *On residually finite p–groups*, from: "Convegno sui Gruppi Infiniti, INDAM, Roma, 1973", Symposia Mathematica 17, Academic Press, London (1976) 225–234 MR0419611

[22] **D L Johnson**, *Presentations of groups*, London Math. Soc. Lect. Notes Ser. 22, Cambridge Univ. Press (1976) MR0396763

[23] **G Levitt**, **A Minasyan**, *Residual properties of automorphism groups of (relatively) hyperbolic groups*, Geom. Topol. 18 (2014) 2985–3023 MR3285227

[24] **W Magnus**, **A Karrass**, **D Solitar**, *Combinatorial group theory*, 2nd edition, Dover, New York (1976) MR0422434

[25] **Ş Papadima**, *The universal finite-type invariant for braids, with integer coefficients*, Topology Appl. 118 (2002) 169–185 MR1877723

[26] **L Paris**, *Residual p properties of mapping class groups and surface groups*, Trans. Amer. Math. Soc. 361 (2009) 2487–2507 MR2471926

[27] **L Paris**, **B Szepietowski**, *A presentation for the mapping class group of a nonorientable surface*, Bull. Soc. Math. France 143 (2015) 503–566 MR3417732

[28] **G P Scott**, *Braid groups and the group of homeomorphisms of a surface*, Proc. Cambridge Philos. Soc. 68 (1970) 605–617 MR0268889

[29] **M Stukow**, *Commensurability of geometric subgroups of mapping class groups*, Geom. Dedicata 143 (2009) 117–142 MR2576298

[30] **B Szepietowski**, *Embedding the braid group in mapping class groups*, Publ. Mat. 54 (2010) 359–368 MR2675928

[31] **B Szepietowski**, *Crosscap slides and the level 2 mapping class group of a nonorientable surface*, Geom. Dedicata 160 (2012) 169–183 MR2970047

[32] **H Zieschang**, **E Vogt**, **H-D Coldewey**, *Surfaces and planar discontinuous groups*, Lecture Notes in Mathematics 835, Springer, Berlin (1980) MR606743

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