Algebraic & Geometric Topology 16 (2016) 1453–1472

Centralizers in good groups are good

TOBIAS BARTHEL
NATHANIEL STAPLETON

We modify transchromatic character maps of the second author to land in a faithfully flat extension of Morava E–theory. Our construction makes use of the interaction between topological and algebraic localization and completion. As an application we prove that centralizers of tuples of commuting prime-power order elements in good groups are good and we compute a new example.

55N20

1 Introduction and outline

The character maps of the second author [25] suggest the intriguing possibility of approximating height n Morava E–theory by Morava E–theory of a lower height. In particular, it is easy to imagine that a character map from E_n to p–adic K–theory could have many applications because it could reduce height n problems to representation theory in the same way that the character map of Hopkins, Kuhn and Ravenel [10] reduces height n problems to combinatorial problems. However, the maps produced in [25] have a codomain which is an extension of the $K(t)$–localization of E_n for $t < n$. This cohomology theory is less familiar and presents some computational difficulties because its coefficients are not a complete local ring. In this paper, we present a modification of the character maps that, for good groups, land in a faithfully flat extension of E_t. This work grew out of work of the second author with Tomer Schlank [21] on Strickland's theorem, where this modified character map from height n to height 1 plays a critical role.

A finite group is good (at a prime p) if $E_n^*(BG)$ is free and evenly concentrated. There are a variety of classes of groups that are known to be good. These include finite abelian groups, symmetric groups, finite general linear groups away from the characteristic, wreath products and products of these groups, as well as groups of order p^3; see [10, Theorem E] and Tanabe [26, Proposition 7.10]. It was a conjecture of Hopkins, Kuhn and Ravenel that all groups are good; however this was disproved by Kriz [15]. A corollary of the construction of this modified character map is that

Published: 1 July 2016 DOI: 10.2140/agt.2016.16.1453

centralizers of abelian p–groups in good groups are good. This enlarges the class of groups known to be good in a very different way than the results above.

Our approach relies on a variety of facts concerning completion and localization of flat modules in stable homotopy theory. In order to put these into a more abstract context, we review the relationship between chromatic localization functors on the category of MU–modules and certain arithmetic localizations and completions, as described in Greenlees and May [8]. As an immediate consequence we obtain that the coefficients of the $K(t)$–localization of a flat E_n–module M are given by the simple formula

$$\pi_*(L_{K(t)}M) \cong (\pi_* M)[u_t^{-1}]^\wedge_{(p,\dots,u_{t-1})}.$$

We also provide a proof of a mild generalization of Hovey's unpublished theorem that the I_n–completion of a flat E_n^*–module is flat.

We then use these methods to analyze the spectrum $\bar{C}_t = L_{K(t)}(C_t \wedge E_t)$. Here C_t is an E_∞–ring with coefficient ring C_t^*, from [25], with the following properties: let $\mathbb{G}_{E_n}$ be the formal group associated to E_n viewed as a p–divisible group. The ring C_t^* is an E_n^*–algebra with the property that $\mathbb{G}_{E_n}$ decomposes as a sum of a height t formal group with a height $n-t$ constant étale p–divisible group after base change to C_t^*.

After proving that $\bar{C}_t^*$ is faithfully flat as an E_t^*–module we construct the modified character maps. For H the centralizer of a tuple of commuting elements in G, we show that there is an isomorphism

$$\bar{C}_t^* \otimes_{L_{K(t)}E_n^*} L_{K(t)}E_n^*(BH) \cong \bar{C}_t^* \otimes_{E_t^*} E_t^*(BH).$$

Now let $\mathcal{L}(-) = \mathrm{hom}(B\mathbb{Z}_p, -)$ be the p–adic free loop space functor; note that this is the space of unpointed maps. The main object of study is the composite of the character map from [25] with the isomorphism above

$$E_n^*(BG) \to \bar{C}_t^* \otimes_{L_{K(t)}E_n^*} L_{K(t)}E_n^*(\mathcal{L}^{n-t}BG) \cong \bar{C}_t^* \otimes_{E_t^*} E_t^*(\mathcal{L}^{n-t}BG).$$

The codomain of this character map is just the E_t–cohomology of $\mathcal{L}^{n-t}BG$ base-changed to a faithfully flat extension. Morava E_t is certainly more computable and more familiar than $L_{K(t)}E_n$. Our main result gives a condition for when this map induces an isomorphism after base change to $\bar{C}_t^*$; see Theorem 6.9.

Main theorem *For a good group G, the map above, base-changed to $\bar{C}_t^*$, gives an isomorphism*

$$\bar{C}_t^* \otimes_{E_n^*} E_n^*(BG) \xrightarrow{\cong} \bar{C}_t^* \otimes_{E_t^*} E_t^*(\mathcal{L}^{n-t}BG).$$

This allows us to reduce certain height n problems to height t problems without introducing more exotic cohomology theories. For instance, an argument using faithfully flat descent for finitely generated projective modules proves Corollary 7.1 (centralizers in good groups are good) from this result. The paper ends with a brief summary of what is known about good groups and a new example.

Acknowledgements We thank Omar Antolín Camarena, Mark Behrens, Tyler Lawson, Eric Peterson, Tomer Schlank, and Björn Schuster for many helpful conversations. We also thank the referee for useful comments on an earlier version of this paper. The second author was partially supported by NSF grant DMS-0943787.

2 Arithmetic localization and completion

In this section we summarize the Greenlees–May theory of localization and completion in topology, as developed in [8], using some insights from [17].

2.1 Construction of localization and completion

Let R be an E_∞–ring spectrum, so that the category Mod_R of R–modules has a symmetric monoidal structure, where the smash product over R will be denoted $\wedge$. Note that most of the arguments work as well for E_2–ring spectra, but we will not need this extra generality here.

Let $I \subseteq \pi_* R$ be a finitely generated ideal with a minimal set of generators $\{x_1, \dots, x_n\}$. It is possible to weaken this hypothesis as in [7], but for simplicity we will restrict ourselves to the finitely generated situation.

Definition 2.2 A module $M \in \mathrm{Mod}_R$ is called *I–nilpotent* if $I \subseteq \mathrm{supp}(m) = \{r \in \pi_0 R \mid \text{there exists } n \text{ with } r^n m = 0\}$ for all $m \in \pi_* M$. This condition is equivalent to $M[1/x] = 0$ for all $x \in I$.

If we define the Koszul complex as $\mathrm{Kos}(I) = \bigwedge_{i=1}^n \mathrm{Kos}(x_i)$ with $\mathrm{Kos}(x_i) = \mathrm{fib}(R \to R[1/x_i])$, then we can construct the fundamental cofiber sequence

$$\mathrm{Kos}(I) \to R \to \check{C}(I).$$

Lemma 2.3 *The inclusion functor* $\mathrm{Mod}_R^{I\text{–}nil} \hookrightarrow \mathrm{Mod}_R$ *has a right adjoint* Γ_I *which is given by* $\Gamma_I = - \wedge \mathrm{Kos}(I)$.

It follows easily that $\mathrm{Mod}_R^{I-\mathrm{nil}}$ is compactly generated by $\bigwedge_{i=1}^{n} \mathrm{cofib}(R \xrightarrow{x_i} R)$. For example, $\mathrm{Mod}_{S^0_{(p)}}^{(p)-\mathrm{nil}} \simeq \mathrm{Mod}_{S^0_{(p)}}^{\mathbb{Q}-\mathrm{acyclic}}$ is generated as a localizing subcategory by S^0/p.

If $\mathcal{C}$ is a full stable subcategory of Mod_R, its left orthogonal is defined as the full subcategory of Mod_R on those objects N for which $\mathrm{Hom}(M,N)=0$ for all $M \in \mathcal{C}$.

Definition 2.4 $\mathrm{Mod}_R^{I-\mathrm{loc}}$ is the left orthogonal to $\mathrm{Mod}_R^{I-\mathrm{nil}}$, ie $N \in \mathrm{Mod}_R^{I-\mathrm{loc}}$ if and only if $\mathrm{Hom}_R(M,N)=0$ for all $M \in \mathrm{Mod}_R^{I-\mathrm{nil}}$.

Lemma 2.5 *The inclusion functor* $\mathrm{Mod}_R^{I-\mathrm{loc}} \hookrightarrow \mathrm{Mod}_R$ *admits a left adjoint, given by* I*–localization* $L_I = - \wedge \check{C}(I)$*, also written as* $(-)[I^{-1}]$*. In particular, this gives rise to a fiber sequence of functors*

$$\Gamma_I \to \mathrm{id} \to L_I.$$

Definition 2.6 $\mathrm{Mod}_R^{I-\mathrm{comp}}$ is the left orthogonal to $\mathrm{Mod}_R^{I-\mathrm{loc}}$.

Equivalently, an R–module M is I–complete if and only if $\lim(\cdots \xrightarrow{x} M \xrightarrow{x} M) = 0$ for all $x \in I$, as shown in [17, Corollary 4.2.8].

Lemma 2.7 *The* I*–adic completion functor is defined as* $(-)^{\wedge}_I = \mathrm{Hom}_R(\mathrm{Kos}(I), -)$*, and it is left-adjoint to the inclusion functor* $\mathrm{Mod}_R^{I-\mathrm{comp}} \hookrightarrow \mathrm{Mod}_R$.

As a special case, $(-)^{\wedge}_{(x)} \simeq \lim_s(-/x^s)$, which coincides with the familiar construction of completion.

The following diagram summarizes our discussion of arithmetic localization and completion,

$$\begin{tikzcd}
 & \mathrm{Mod}_R^{I-\mathrm{loc}} \arrow[d, shift left, hook] \arrow[ddr, dashed, bend left] & \\
 & \mathrm{Mod}_R \arrow[u, shift left, "L_I"] \arrow[dl, shift left, "\Gamma_I"] \arrow[dr, shift left, "(-)^{\wedge}_I"] & \\
\mathrm{Mod}_R^{I-\mathrm{nil}} \arrow[ur, shift left] \arrow[uur, dashed, bend left] \arrow[rr, "\sim"'] & & \mathrm{Mod}_R^{I-\mathrm{comp}} \arrow[ul, shift left]
\end{tikzcd}$$

where the dotted arrows indicate left orthogonality. The bottom triangle of the diagram commutes, and the bottom map is an equivalence by [13, Theorem 3.3.5(g)]. Moreover, this reduces to the usual derived functors of localization and completion upon specialization to Eilenberg–Mac Lane spectra.

2.8 Bousfield localization

Following [6], we will work in the category of Mod_R of modules over an E_∞–ring spectrum R. Let E be an R–module.

Definition 2.9 An R–module X is called *E–acyclic* if $X \wedge E = 0$, and $Y \in \mathrm{Mod}_R$ is called *E–local* if $\mathrm{Hom}(X, Y) = 0$ for all E–acyclic X. A morphism f is an *E–equivalence* if $f \wedge E$ is an equivalence.

The following fundamental result was proven by Bousfield [4].

Theorem 2.10 *There exists a functor $L_E\colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ together with a natural transformation $\mathrm{id} \to L_E$ such that $X \to L_E X$ is an E–equivalence with E–local target for all X. Equivalently, $X \to L_E X$ is the initial map from X into an E–local object.*

Recall also that a localization functor L is called smashing if for all $M \in \mathrm{Mod}_R$ $LM = M \wedge LR$. As in [8, Theorems 4.2 and 5.1], we now identify the arithmetic localization and completion functors encountered earlier as special cases of Bousfield localization.

Proposition 2.11 *Let R be an E_∞–ring spectrum with Noetherian coefficients and let I be an ideal in $\pi_* R$, then the following hold:*

1. *L_I is the smashing Bousfield localization with respect to $\check{C}(I)$.*
2. *There is a spectral sequence $E^2_{p,q} = \check{C}H_I^{-p,-q}(\pi_* M) \Rightarrow \pi_{p+q}(L_I M)$. Here, $\check{C}H_I^*$ denotes Čech cohomology with respect to I as defined in [8].*

We will be mainly interested in completion. Recall that algebraic I–completion is not exact on the category of all R_*–modules, but we can consider its left derived functors $L^I_s = \mathbb{L}_s(-)^\wedge_I$. These will be studied in more detail in Section 3.9.

Proposition 2.12 *Let R be an E_∞–ring spectrum with Noetherian coefficients and let I be an ideal in $\pi_* R$, then the following hold:*

1. *$(-)^\wedge_I$ is Bousfield localization with respect to $\mathrm{Kos}(I)$. In general, $(-)^\wedge_I$ is not smashing.*
2. *There is a spectral sequence $E^2_{s,t} = L_s(\pi_* M)_t \Rightarrow \pi_{s+t}(M^\wedge_I)$, where $L_s = L^I_s$ denotes the s^{th} left derived functor of ordinary I–adic completion.*

Remark 2.13 More generally, the E^2 term of the above spectral sequence can be identified with the local homology of groups of $\pi_* M$ with respect to I; $E^2_{s,t} = H^I_{s,t}(\pi_* M)$.

3 Localization and completion of MU–modules

The goal of this section is to show that the restrictions of certain Bousfield localization functors appearing in chromatic homotopy theory to MU–modules can be expressed as combinations of the arithmetic functors of Section 2. This is certainly well known to experts, but since there is no published reference for these results, we include the proofs.

Moreover, the same techniques allow us to study the effect of $K(n)$–localization on coefficients, which admits an explicit description for flat modules.

3.1 Recollections

Fix a prime p, and let E_n and $K(n)$ denote Morava E–theory and Morava K–theory at height n, respectively. Recall that E_n is a Landweber exact E_∞–ring spectrum with coefficients $E_n^* = \mathrm{W}k[\![u_1, \dots, u_{n-1}]\!][u^{\pm 1}]$, where $\mathrm{W}k$ is the ring of Witt vectors of a perfect field k of characteristic p and u has degree 2. The spectrum representing Morava K–theory is a complex orientable A_∞–ring spectrum with $K(n)_* = \mathbb{F}_{p^n}[v_n^{\pm 1}]$ with v_n of degree $2(p^n - 1)$.

These spectra come with associated Bousfield localization functors L_n and $L_{K(n)}$ that play a fundamental role in the chromatic approach to stable homotopy theory. We recall two important relations between these functors:

- $L_n = L_{\bigvee_{i=0}^n K(i)} = L_{E(n)}$, where $E(n)$ is height n Johnson–Wilson theory; see [19, Theorem 2.1].
- There is a homotopy pullback square of functors on spectra

$$\begin{tikzcd} L_n \arrow[r] \arrow[d] & L_{K(n)} \arrow[d] \\ L_{n-1} \arrow[r] & L_{n-1}L_{K(n)} \end{tikzcd}$$

  usually called the chromatic fracture square.

The n^{th} monochromatic layer $M_n\colon \mathrm{Sp} \to \mathrm{Sp}$ is defined as the fiber $M_n = \mathrm{fib}(L_n \to L_{n-1})$. By the smash product theorem of Hopkins and Ravenel, L_n and hence M_n are smashing for all n, whereas $L_{K(n)}$ does not have this property unless $n = 0$. Moreover, Hovey and Strickland provide a convenient description of $K(n)$–localization.

Proposition 3.2 *For any spectrum X and $n \geq 0$, there is an equivalence*

$$L_{K(n)}X = \mathrm{Hom}_{S^0}(M_n S^0, L_n X).$$

3.3 Identification of chromatic functors

Greenlees and May [8] provide the starting point of a dictionary between arithmetic and chromatic localization and completion functors on the category of MU–modules. Since BP is known to be E_4 by [3], we could work with BP as well.

Proposition 3.4 *For $N \in \mathrm{Mod}_{MU}$ and any $t \geq 0$:*

(1) $L_t N = N[I_{t+1}^{-1}] \simeq N \wedge \check{C}(I_{t+1})$.

(2) $N^\wedge_{I_t} \simeq \mathrm{Hom}_{S^0}(\mathrm{colim}_i\, M_i, N)$, *where the M_i form a cofinal sequence of generalized type t Moore spectra.*

Here, I_t denotes the ideal $(p, v_1, \dots, v_{t-1})$.

Remark 3.5 The obvious analogue of this result hold for the category of E_n–modules as well.

Lemma 3.6 *For $N \in \mathrm{Mod}_{MU}$, $M_t(N) \simeq \Gamma_{I_t}(N[v_t^{-1}])$.*

Proof Using the octahedral axiom, the following commutative diagram in which all rows and columns are fiber sequences shows that $M_t(-) = (-) \wedge \mathrm{Kos}(I_t) \wedge \check{C}(v_t)$, since $L_t N \simeq N \wedge \check{C}(I_{t+1})$.

$$\begin{tikzcd}
\mathrm{Kos}(I_{t+1}) \arrow[r] \arrow[d] & MU \arrow[r] \arrow[d, "\simeq"] & \check{C}(I_{t+1}) \arrow[d] \\
\mathrm{Kos}(I_t) \arrow[r] \arrow[d] & MU \arrow[r] \arrow[d] & \check{C}(I_t) \arrow[d] \\
\mathrm{Kos}(I_t) \wedge \check{C}(v_t) \arrow[r] & 0 \arrow[r] & \Sigma \mathrm{Kos}(I_t) \wedge \check{C}(v_t)
\end{tikzcd}$$

□

Proposition 3.7 *If $N \in \mathrm{Mod}_{MU}$, then $L_{K(t)} N \simeq (N[v_t^{-1}])^\wedge_{I_t}$.*

Proof By Proposition 3.2 and Proposition 3.4, we have

$$L_{K(t)} N \simeq (N[I_{t+1}^{-1}])^\wedge_{I_t} \simeq \mathrm{Hom}(\mathrm{Kos}(I_t), N \wedge \check{C}(I_{t+1})).$$

Consider the following commutative diagram of fiber sequences:

$$\begin{tikzcd}
\mathrm{Hom}(\mathrm{Kos}(I_t), N \wedge \mathrm{Kos}(I_{t+1})) \arrow[r] \arrow[d, "\phi"'] & \mathrm{Hom}(\mathrm{Kos}(I_t), N) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}(\mathrm{Kos}(I_t), N \wedge \check{C}(I_{t+1})) \arrow[d] \\
\mathrm{Hom}(\mathrm{Kos}(I_t), N \wedge \mathrm{Kos}(v_t)) \arrow[r] & \mathrm{Hom}(\mathrm{Kos}(I_t), N) \arrow[r] & \mathrm{Hom}(\mathrm{Kos}(I_t), N \wedge \check{C}(v_t)).
\end{tikzcd}$$

We claim that the first vertical map ϕ is an equivalence, hence so is the last one and the result follows. To see the claim, note that ϕ fits into a cofiber sequence

$$\begin{tikzcd} \mathrm{Hom}(\mathrm{Kos}(I_t), N \wedge \mathrm{Kos}(I_t) \wedge \mathrm{Kos}(v_t)) \arrow[r] & \mathrm{Hom}(\mathrm{Kos}(I_t), N \wedge \mathrm{Kos}(v_t)) \arrow[d] \\ & \mathrm{Hom}(\mathrm{Kos}(I_t), N \wedge \check{C}(I_t) \wedge \mathrm{Kos}(v_t)), \end{tikzcd}$$

where the cofiber is contractible by adjunction, as there are no non-trivial maps from an I_t–local module to an I_t–complete module. $\square$

Remark 3.8 The spectral sequence of Proposition 2.11

$$E^2_{s,t} = \check{C}H^{-p,-q}_{I_{t+1}}(\pi_* N) \Rightarrow \pi_{p+q}(L_t N)$$

corresponds to the geometric decomposition induced by the chromatic fracture cube. It accounts for the existence of odd-dimensional classes in the homotopy of $L_t E_n$, $0 < t < n$.

The following table summarizes the identifications of the chromatic functors on the category of MU–modules; here, the right column should really be interpreted as derived arithmetic functors.

Chromatic functor	Arithmetic functor
$L_t(-)$	$(-)[I_{t+1}^{-1}]$
$C_t(-)$	$\Gamma_{I_{t+1}}(-)$
$M_t(-)$	$\Gamma_{I_t}(-)[v_t^{-1}]$
$L_{K(t)}(-)$	$(-)[v_t^{-1}]^{\wedge}_{I_t}$

In particular, we have a commutative diagram

$$\begin{tikzcd} & \mathrm{Mod}_{MU} \arrow[dl, bend right, "M_t"'] \arrow[d, "{[v_t^{-1}]}"] \arrow[dr, bend left, "L_{K(t)}"] & \\ & \mathrm{Mod}_{v_t^{-1}MU} \arrow[dl, shift left, "\Gamma_{I_t}"] \arrow[dr, shift left, "(-)^{\wedge}_{I_t}"] & \\ \mathrm{Mod}^{I_t-\mathrm{nil}}_{v_t^{-1}MU} \arrow[ur, shift left] \arrow[rr, "\sim"'] & & \mathrm{Mod}^{I_t-\mathrm{comp}}_{v_t^{-1}MU} \arrow[ul, shift left] \end{tikzcd}$$

where the bottom horizontal equivalence is the well-known equivalence between the height t monochromatic category and the $K(t)$–local category when restricted to MU–modules; see [13, Theorem 3.3.5(g)].

3.9 $K(n)$–localization and flatness

Proposition 3.7 can be used to compute the homotopy groups of localizations. Recall that a module spectrum M over an E_1–ring spectrum R is said to be flat if and only if $\pi_* M$ is flat as a graded module over $\pi_* R$.

Corollary 3.10 *If $N \in \mathrm{Mod}_{E_n}$ is flat, then $\pi_* L_{K(t)} N \cong ((\pi_* N)[v_t^{-1}])^\wedge_{I_t}$.*

Proof By Proposition 3.7, $L_{K(t)} N \simeq (N[v_t^{-1}])^\wedge_{I_t}$. Since N is flat, so is $N[v_t^{-1}]$, hence the spectral sequence of Proposition 2.12 computing the completion collapses by [14, Theorem A.2(b)]. Therefore

$$\pi_*(N[v_t^{-1}])^\wedge_{I_t} \cong ((\pi_* N)[v_t^{-1}])^\wedge_{I_t},$$

since π_* preserves filtered colimits. □

More generally, Proposition 2.12 gives a natural strongly convergent spectral sequence

$$E^2_{s,t} = (L_s \pi_* M)_t \Rightarrow \pi_{s+t} L_{K(n)} M$$

with $E^2_{s,*} = 0$ if $s > n$ and differentials $d^r \colon E^r_{s,t} \to E^r_{s-r,t+r-1}$. Using this spectral sequence, it is not hard to see [2, Corollary 3.14] that $M \in \mathrm{Mod}_{E_n}$ is $K(n)$–local if and only if $\pi_* M$ is isomorphic to $L_0(\pi_* M)$. In fact this holds more generally for any completion functor over a connective E_2–ring spectrum; see [17, Theorem 4.2.13].

Remark 3.11 This corollary complements Hovey's result for ring spectra, [11, Theorem 1.5.4.].

For the rest of this section, let R be a regular complete local Noetherian commutative ring of dimension n, and let $I = (x_1, \dots, x_t)$ be an ideal in R with a chosen minimal regular sequence of generators. The main example of interest to us is $E_n^0 = \mathrm{W}k[\![u_1, \dots u_{n-1}]\!]$ with its maximal ideal $\mathfrak{m} = (p, u_1, \dots, u_{n-1})$.

By the Artin–Rees lemma, the algebraic completion functor $(-)^\wedge_I$ is exact when restricted to finitely generated modules, but it is neither left nor right exact in general. Therefore, for general R–modules, we have to consider the left derived functors L_s of I–adic completion. However, L_0 coincides with ordinary I–adic completion for flat modules, so we may restrict ourselves to this case here. An overview of the construction and properties of these functors relevant to topology can be found in [14; 2; 20].

Remark 3.12 The assumption that R is Noetherian can be weakened. In particular, the theory applies as well to the non-Noetherian but coherent ring $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$; see [7].

The following result was proven in the special case of $R = E_{n*}$ and $I = \mathfrak{m}$ by Hovey [12]; the arguments easily generalize to give the following flatness criterion.

Proposition 3.13 *If M is a flat R–module such that M/I is projective over R/I, then $M_I^\wedge$ is also flat over R.*

Sketch of proof By [27, Tag 05D3], the hypotheses imply that $M_I^\wedge$ is a retract of a pro-free module, ie a module of the form $F_I^\wedge$ with $F \in \mathrm{Mod}_R$ free. Since pro-free objects are retracts of products of R by [14, Proposition A.13][1], hence flat as R is Noetherian, it follows that $M_I^\wedge$ is also flat over R. □

Remark 3.14 In fact, in case R is local and $I = \mathfrak{m}$ is the maximal ideal, the class of pro-free objects coincides with the collection of flat R–modules which are I–complete. This characterization does not generalize to arbitrary finitely generated ideals I, as the example $I = (0)$ shows.

4 A short digression on Landweber exact theories

We include a short digression on Landweber exact cohomology theories. This is partly to set up some technicalities that will be of use later and partly to clarify the relation between Landweber exactness and Brown representability.

Assume that E is a Landweber exact spectrum and R is a flat E–module. It is always the case that

$$R_*(X) \cong R_* \otimes_{E_*} E_*(X)$$

defines a homology theory on all spaces X. However, we prefer to work cohomologically so that our theories are naturally ring-valued; here, things are a bit more complicated. Base change provides a cohomology theory defined on finite spaces (spaces equivalent to finite CW–complexes) $R^* \otimes_{E^*} E^*(X)$ and on these spaces this is the same as $R^*(X)$.

We may extend this to finite G–CW complexes Borel equivariantly: $R^* \otimes_{E^*} E^*(EG \times_G X)$, but there can be a large difference between $R^*(Y)$ and $R^* \otimes_{E^*} E^*(Y)$ for infinite Y.

Example 4.1 Let $E = E_n$, $R = p^{-1}E_n$, and $Y = B\mathbb{Z}/p$. Then $E_n^*(B\mathbb{Z}/p)$ is a free E_n^*–module of rank p^n. Thus we see that $p^{-1}E_n^* \otimes_{E_n^*} E_n^*(B\mathbb{Z}/p)$ is a free module of rank p^n over $p^{-1}E_n^*$. However, $(p^{-1}E_n)^*(B\mathbb{Z}/p) \cong p^{-1}E_n^*$ as $p^{-1}E_n$ is a rational cohomology theory.

[1] Note that the proof of [14, Proposition A.13] generalizes to any finitely generated ideal I.

The key observation is that, in general, $R^* \otimes_{E^*} E^*(-)$ does not satisfy the infinite wedge axiom. However, Brown representability (in the form of [1]) applied to this theory defined on finite spaces produces a spectrum R'.

Lemma 4.2 *With the above notation, $R \simeq R'$.*

Proof The cohomology theory associated to R' must take the same value as the cohomology theory associated to R on finite spaces. Now this lemma is an immediate consequence of [14, Theorem 2.8], which says that Landweber exact spectra are determined by their coefficients. □

In the following, we will also need the fact that the smash product of even Landweber exact theories is again even. Note that this fails for general spectra, as the example $H\mathbb{F}_p \wedge H\mathbb{F}_p$ shows.

Lemma 4.3 *If E and F are Landweber exact theories, then so is $E \wedge F$. Additionally, if E and F are even, then $E \wedge F$ is even as well.*

Proof The first part of the claim follows from Hopkins discussion in [9]. Indeed, there is a pullback diagram

$$\begin{tikzcd} \mathrm{Spec}(\pi_* E \wedge F) \arrow[r] \arrow[d] & \mathrm{Spec}(F_*) \arrow[d, "v"] \\ \mathrm{Spec}(E_*) \arrow[r, "u"'] & \mathcal{M}_{\mathrm{FG}} \end{tikzcd}$$

where $\mathcal{M}_{fg}$ denotes the stack of one-dimensional formal groups. Since u and v are flat, the composite $\mathrm{Spec}(\pi_* E \wedge F) \to \mathcal{M}_{fg}$ is flat by base change. To show that $E \wedge F$ is even, recall that

$$\pi_*(E \wedge F) \cong E_* \otimes_{MU_*} MU_*MU \otimes_{MU_*} F_*.$$

The claim follows since MU_*MU is concentrated in even degrees. □

5 Some spectra related to character theory

5.1 Recollections

We recall the character maps of [25]. For the rest of the paper fix a prime p. Let E_n be Morava E–theory and $L_{t,n} = L_{K(t)} E_n$ be the localization of E_n by Morava $K(t)$. By Corollary 3.10 there is an isomorphism

$$\pi_0 L_{t,n} \cong \mathrm{W}k[\![u_1, \ldots, u_{n-1}]\!][u_t^{-1}]^{\wedge}_{I_t}.$$

Let $\mathbb{G}_{E_n}$ be the p–divisible group associated to E_n and $\mathbb{G}_{L_{t,n}}$ the p–divisible group associated to $L_{t,n}$. In [25] a flat extension C_t^* of $L_{t,n}^*$ is constructed with the following property.

Let $\mathbb{G} := L_{t,n}^0 \otimes \mathbb{G}_{E_n}$ so that $\mathbb{G}_{L_{t,n}}$ is the connected component of the identity of $\mathbb{G}$ [25, Proposition 2.4]. Note that this means that there is a canonical map $\mathbb{G}_{L_{t,n}} \to \mathbb{G}$. For an $L_{t,n}^0$–algebra R, let

$$\mathrm{Iso}_{\mathbb{G}_{L_{t,n}}/}(R \otimes \mathbb{G}_{L_{t,n}} \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-t}, R \otimes \mathbb{G})$$

be the set of isomorphisms of p–divisible groups under $\mathbb{G}_{L_{t,n}}$.

Recall the following proposition.

Proposition 5.2 [25, Proposition 2.17] *The functor from $L_{t,n}^0$–algebras to sets*

$$\mathrm{Iso}_{\mathbb{G}_{L_{t,n}}/}(\mathbb{G}_{L_{t,n}} \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-t}, \mathbb{G})\colon\ R \mapsto \mathrm{Iso}_{\mathbb{G}_{L_{t,n}}/}(R \otimes \mathbb{G}_{L_{t,n}} \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-t}, R \otimes \mathbb{G})$$

is representable by C_t^0.

C_t^* is constructed as a localization of the ring $\mathrm{colim}_k\, L_{t,n}^* \otimes_{E_n^*} E_n^*(B(\mathbb{Z}/p^k)^{n-t})$. The following result should be compared to [10, Proposition 6.5].

Proposition 5.3 *The ring C_t^0 is a faithfully flat $L_{t,n}^0$–algebra.*

Proof Note that C_t^0 is a flat $L_{t,n}^0$–algebra since it is constructed as a filtered colimit of algebras, each of which is a localization of a finitely generated free module.

Recall that a map of commutative rings is faithfully flat if and only if it is flat and surjective on $\mathrm{Spec}(-)$ [27, Tag 00HQ]. Let $P \subset L_{t,n}^0$ be a prime ideal, we must produce a prime ideal in C_t^0 that restricts to P. Let i be the smallest natural number such that $u_i \notin P$. Note that $i \le t$. Now consider the algebraic closure of the fraction field

$$K := \overline{(L_{t,n}^0/P)_{(0)}}.$$

There is an isomorphism [5, page 34]

$$K \otimes \mathbb{G} \cong \mathbb{G}_{\mathrm{for}} \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-i}.$$

The formal part has height i because u_i has been inverted. Now since

$$\mathbb{G}_{\mathrm{for}} \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-i} \cong \mathbb{G}_{\mathrm{for}} \oplus \mathbb{Q}_p/\mathbb{Z}_p^{t-i} \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-t},$$

Proposition 5.2 implies that this is classified by a map $C_t^0 \xrightarrow{q} K$ that extends the canonical map $L_{t,n}^0 \to K$. Now the kernel of q is a prime ideal and must restrict to P, thus C_t^0 is a faithfully flat $L_{t,n}^0$–algebra. □

The ring C_t^* is used in the construction of the transchromatic generalized character maps of [25]. For a finite G–CW complex X, let

$$\mathrm{Fix}_h(X) = \coprod_{\alpha \in \hom(\mathbb{Z}_p^h, G)} X^{\mathrm{im}\,\alpha}.$$

This is a finite G–CW complex with G–action given by $x \in X^{\mathrm{im}\,\alpha} \mapsto gx \in X^{\mathrm{im}\,g\alpha g^{-1}}$. The character map is a map $E_n^*(EG \times_G X) \to C_t^* \otimes_{L_{t,n}^*} L_{t,n}^*(EG \times_G \mathrm{Fix}_{n-t}\, X)$ with the following property.

Theorem 5.4 [25] *The character map has the property that the map induced by tensoring the domain up to C_t*

$$C_t^* \otimes_{E_n^*} E_n^*(EG \times_G X) \xrightarrow{\cong} C_t^* \otimes_{L_{t,n}^*} L_{t,n}^*(EG \times_G \mathrm{Fix}_{n-t}\, X)$$

is an isomorphism.

5.5 Some spectra related to character theory

Let $\Lambda_k = (\mathbb{Z}/p^k)^{n-t}$ and let C_t be the spectrum

$$S^{-1} \operatorname{colim}_k \; L_{t,n} \wedge_{E_n} E_n^{B\Lambda_k}.$$

It is clear that C_t is an E_∞–ring and that the coefficients of C_t is the ring C_t^* from the previous section. Let $E(t)$ be the height t Johnson–Wilson spectrum.

Proposition 5.6 *The spectrum C_t is $E(t)$–local.*

Proof Note that $L_{t,n} \wedge_{E_n} E_n^{B\Lambda_k}$ is $E(t)$–local. In fact, it is $K(t)$–local as it is equivalent to $L_{K(t)}(E_n^{B\Lambda_k})$. This follows from the fact that $E_n^{B\Lambda_k}$ is a free E_n–module spectrum. Now colimits in the $E(t)$–local category may be computed in the category of spectra (since localization with respect to $E(t)$ is smashing) so C_t is $E(t)$–local. □

While $C_t \wedge E_t$ is $E(t)$–local and flat as a C_t–module and as an E_t–module, it is not $K(t)$–local and thus the argument of Proposition 6.2 cannot be used. For that reason we introduce a variant $\overline{C}_t$, which allows us to exploit the good finiteness properties of the $K(t)$–local category.

Definition 5.7 We define $\overline{C}_t := L_{K(t)}(C_t \wedge E_t)$.

Proposition 5.8 *The spectrum $\overline{C}_t$ is even periodic, and $\overline{C}_t^*$ is faithfully flat as an E_t^*–module.*

Proof As seen in Lemma 4.3, the smash product of even periodic Landweber exact spectra is even periodic and this cannot be changed by completion (which is all that $K(t)$–localization is for $E(t)$–local BP–modules). By even periodicity it suffices to prove that $\bar{C}_t^0$ is faithfully flat over E_t^0. Note that $\pi_0(C_t \wedge E_t)$ is flat as a C_t^0–module and as an E_t^0–module by Lemma 4.3. By Proposition 3.13, the completion of a flat E_t^0–module at I_t is flat. Thus $\bar{C}_t^0$ is flat as an E_t^0–module. For faithful flatness it suffices to prove that $\bar{C}_t^0/I_t$ is nonzero [27, Tag 00HP]. But Hovey–Strickland implies that it is $\pi_0(K(t) \wedge C_t)$ and since C_t^0/I_t is nonzero [25, Proposition 2.17], we know that the $K(t)$–localization of C_t is nonzero. □

6 From E–theory to E–theory

In this section we present a modification of the character maps of [25]. We begin by recalling the character maps. An upshot of the presentation here is that the character map is a map of E_∞–rings. We then analyze the modification of the character map applied to good groups and use this to show that centralizers of tuples of commuting elements in good groups are good.

6.1 Character maps written spectrally

The character maps of [25] admit an obvious spectral interpretation. For a finite group G and a finite G–CW complex X we may consider the evaluation map $B\Lambda_k \times EG \times_G \mathrm{Fix}_{n-t}(X) \to EG \times_G X$. For X a point this is just the evaluation map

$$B\Lambda_k \times \hom(B\Lambda_k, BG) \to BG,$$

for X not a point this can be interpreted as an evaluation map by using topological groupoids and the inertia groupoid construction. This induces the map of spectra

$$E_n^{EG\times_G X} \to E_n^{B\Lambda_k} \wedge_{E_n} E_n^{EG\times_G \mathrm{Fix}_{n-t}(X)}.$$

Now the canonical maps $E_n^{B\Lambda_k} \to C_t$ and $E_n \to L_{t,n}$ induce

$$E_n^{B\Lambda_k} \wedge_{E_n} E_n^{EG\times_G \mathrm{Fix}_{n-t}(X)} \to C_t \wedge_{L_{t,n}} L_{t,n}^{EG\times_G \mathrm{Fix}_{n-t}(X)}.$$

After extending coefficients in the domain the composite induces an equivalence

$$C_t \wedge_{E_n} E_n^{EG\times_G X} \xrightarrow{\simeq} C_t \wedge_{L_{t,n}} L_{t,n}^{EG\times_G \mathrm{Fix}_{n-t}(X)}.$$

In all of this discussion we are merely using the flatness of C_t^* over E_n^* and $L_{t,n}^*$ to translate the algebraic results of [25] to these spectral statements. It is worth noting that all of the maps above are E_∞ (after choosing an E_∞–inverse to the Künneth map). Thus it is clear that the character map is an equivalence of E_∞–rings.

Now we present a modification of the above map that has some desirable properties. In particular, the codomain is related to E_t in the same way that the above map is related to $L_{t,n}$. It seems to suffer in two respects though. It does not seem to induce an equivalence for all spaces after base change of the domain to $\bar{C}_t$ and it is not as computable as the above map.

The modification is the composite of two maps. The first is the character map above. The second is the canonical map of E_∞–rings

$$C_t \wedge_{L_{t,n}} L_{t,n}^{EG\times_G \mathrm{Fix}_{n-t}(X)} \to \bar{C}_t^{EG\times_G \mathrm{Fix}_{n-t}(X)}.$$

Proposition 6.2 *For any finite G–CW complex X the canonical map*

$$\bar{C}_t \wedge_{E_t} E_t^{EG\times_G X} \xrightarrow{\simeq} \bar{C}_t^{EG\times_G X}$$

is an equivalence.

Proof Because $\bar{C}_t^*$ is flat over E_t^* there is an isomorphism

$$\bar{C}_t^* \otimes_{E_t^*} E_t^*(EG\times_G X) \cong \pi_{-*}(\bar{C}_t \wedge_{E_t} E_t^{EG\times_G X}).$$

It is clear that $\bar{C}_t \wedge_{E_t} E_t^{EG\times_G X}$ is $E(t)$–local. However, since $E_t^*(EG\times_G X)$ is finitely generated and E_t^* is Noetherian, the above isomorphism implies that

$$\bar{C}_t^* \otimes_{E_t^*} E_t^*(EG\times_G X)$$

is I_t–complete and thus $\bar{C}_t \wedge_{E_t} E_t^{EG\times_G X}$ is $K(t)$–local. Let $D(EG\times_G X) = F(EG\times_G X, L_{K(t)}S)$. Now we have equivalences

$$\begin{aligned}
\bar{C}_t \wedge_{E_t} E_t^{EG\times_G X} &\simeq L_{K(t)}(\bar{C}_t \wedge_{E_t} E_t^{EG\times_G X}) \\
&\simeq L_{K(t)}(\bar{C}_t \wedge_{E_t} L_{K(t)}(E_t \wedge D(EG\times_G X))) \\
&\simeq L_{K(t)}(\bar{C}_t \wedge_{E_t} E_t \wedge D(EG\times_G X)) \\
&\simeq L_{K(t)}(\bar{C}_t \wedge D(EG\times_G X)) \\
&\simeq \bar{C}_t^{EG\times_G X}.
\end{aligned}$$

The second and fifth equivalences follow from the $K(t)$–local duality of spaces of the form $EG\times_G X$ [14, Corollary 8.7]. □

Thus in the category of E_∞–rings we have the map

$$E_n^{EG\times_G X} \to \bar{C}_t \wedge_{E_t} E_t^{EG\times_G \mathrm{Fix}_{n-t}(X)}$$

that factors

$$E_n^{EG\times_G X} \to \bar{C}_t \wedge_{L_{t,n}} L_{t,n}^{EG\times_G \mathrm{Fix}_{n-t}(X)} \to \bar{C}_t^{EG\times_G \mathrm{Fix}_{n-t}(X)} \xleftarrow{\simeq} \bar{C}_t \wedge_{E_t} E_t^{EG\times_G \mathrm{Fix}_{n-t}(X)}.$$

The middle map is the most mysterious. The reason for this is that it is not clear at all that $\bar{C}_t^0$ is a flat $L_{t,n}^0$–module.

Proposition 6.3 *Let G be a finite group and let X be a finite G–CW complex with the property $L_{t,n}^*(EG\times_G X)$ finitely generated and projective as an $L_{t,n}^*$–module, then there is an isomorphism*

$$\bar{C}_t^* \otimes_{L_{t,n}^*} L_{t,n}^*(EG\times_G X) \cong \bar{C}_t^*(EG\times_G X).$$

Proof This proof is essentially the same as the proof of Proposition 6.2. Since $L_{t,n}^*(EG\times_G X)$ is projective we have an isomorphism

$$\bar{C}_t^* \otimes_{L_{t,n}^*} L_{t,n}^*(EG\times_G X) \cong \pi_{-*}(\bar{C}_t \wedge_{L_{t,n}} L_{t,n}^{EG\times_G X}).$$

Since $L_{t,n}^*(EG\times_G X)$ is finitely generated and $L_{t,n}^*$ is Noetherian, the smash product is $K(t)$–local. Now we have the same set of equivalences as in Proposition 6.2 with E_t replaced by $L_{t,n}$. □

Remark 6.4 When $G=\mathbb{Z}/p^k$ we recover the anticipated result that $\bar{C}_t^0\otimes\mathbb{G}_{L_{t,n}}[p^k]\cong \bar{C}_t^0\otimes\mathbb{G}_{E_t}[p^k]$.

Definition 6.5 A finite group G is *good* (at the fixed prime p) if $E_n^*(BG)$ is even and free for all n.

Remark 6.6 Our definition of a good group differs somewhat from the original [10, Definition 7.1]. They observe that their definition implies Definition 6.5.

Remark 6.7 Because the E_n–cohomology of a good group is even, it admits an algebro-geometric interpretation. Because the E_n–cohomology of a good group is free the character map of [10] is an embedding and so the ring can be attacked using character-theoretic methods.

Corollary 6.8 *Let G be a good group and let $\alpha\colon \mathbb{Z}_p^{n-t}\to G$, then*

$$\bar{C}_t^* \otimes_{L_{t,n}^*} L_{t,n}^*(BC(\operatorname{im}\alpha)) \cong \bar{C}_t^*(BC(\operatorname{im}\alpha)).$$

Proof We show that $L_{t,n}^*(BC(\operatorname{im}\alpha))$ is finitely generated and projective. The character map of Theorem 5.4 gives a factorization

$$C_t^*\otimes_{E_n^*} E_n^*(BG) \cong \prod_{[\alpha]\in\hom(\mathbb{Z}_p^{n-t},G)/\sim} C_t^*\otimes_{L_{t,n}^*} L_{t,n}^*(BC(\operatorname{im}\alpha)),$$

where $\hom(\mathbb{Z}_p^{n-t},G)/\sim$ is the set of continuous homomorphisms up to conjugation.

For a fixed α, this implies that $C_t^* \otimes_{L_{t,n}^*} L_{t,n}^*(BC(\operatorname{im}\alpha))$ is finitely generated projective because it is a summand of a finitely generated free module. Now since C_t^* is faithfully flat as an $L_{t,n}^*$–module, faithfully flat descent for finitely generated projective modules implies that $L_{t,n}^*(BC(\operatorname{im}\alpha))$ is finitely generated and projective. □

Together Proposition 6.2 and Corollary 6.8 give the main theorem.

Theorem 6.9 *For a good group G we have an equivalence*

$$\bar{C}_t \wedge_{E_n} E_n^{BG} \xrightarrow{\simeq} \bar{C}_t^{\mathcal{L}^{n-t} BG} \xleftarrow{\simeq} \bar{C}_t \wedge_{E_t} E_t^{\mathcal{L}^{n-t} BG},$$

where $\mathcal{L}^h BG = \hom(B\mathbb{Z}_p^h, BG) = EG \times_G \operatorname{Fix}_h(*)$.

Remark 6.10 When $n = 1$ we obtain a map from E–theory to p–adic K–theory. This seems like a useful tool. It allows one to reduce certain computations at height n to computations in representation theory. This is used by Tomer Schlank and the second author in [21] to give a new proof and generalization of Strickland's theorem regarding the E–theory of symmetric groups.

7 Examples of good groups

A comprehensive list of finite groups that are known to be good at a fixed prime p can be found in the habilitation thesis of Björn Schuster [22], to which we refer for the original references; these include

(1) abelian groups

(2) symmetric groups

(3) $GL_n(\mathbb{F}_q)$ with $p \nmid q$

(4) all groups of order p^3 and of order 32

(5) metacyclic groups

(6) the Mathieu group M_{12}; see [24].

The collection $\mathcal{G}$ of good groups is closed under products and also under wreath products with $\mathbb{Z}/p$; moreover, a group is good if its Sylow p–subgroup is good. If $G = H_1 \times H_2$, then H_1 is good if both G and H_2 are good because there is a Künneth isomorphism for good groups. Furthermore, given an extension of the form

$$* \to H \to G \to \mathbb{Z}/p \to *,$$

Kriz [15] gives conditions for when H good implies G is good, and conversely. In particular, semi-direct products of elementary abelian p–groups and $\mathbb{Z}/p$ are good.

The methods of the previous section allow us to deduce a new closure property of $\mathcal{G}$.

Corollary 7.1 *Let G be a good group and let $\alpha\colon \mathbb{Z}_p^h \to G$, then $E_n^*(BC(\operatorname{im}\alpha))$ is finitely generated, free, and evenly generated as an E_n^*–module.*

Proof Corollary 6.8 implies that $\bar{C}_t^*(BC(\operatorname{im}\alpha))$ is finitely generated, projective, and even. Proposition 6.2 implies that

$$\bar{C}_t^*(BC(\operatorname{im}\alpha)) \cong \bar{C}_t^* \otimes_{E_t^*} E_t^*(BC(\operatorname{im}\alpha))$$

and Proposition 5.3 implies that C_t^* is faithfully flat as an E_t^*–modules. Faithfully flat descent for finitely generated projective modules implies that $E_t^*(BC(\operatorname{im}\alpha))$ is finitely generated, even, and projective. But now since E_t^* is complete local, projective implies free. □

Remark 7.2 Calling a group E_n–good if its E_n–cohomology is concentrated in even degrees and free, Corollary 7.1 applied to the identity element shows that E_{n+1}–good implies E_n–good. This is compatible with Minami's result [18].

Using the computer algebra system GAP, we can thus construct new examples of good groups.

Example 7.3 Let $p = 2$ and consider $G = GL_2(\mathbb{F}_3) \wr C_2$, a good group of order 4608. There exists an element $g \in G$ of order 4 with centralizer

$$C_G(g) = H \rtimes C_2,$$

where H is the binary octahedral group, ie a non-split extension of S_4 by C_2. The GAP ID of $C_G(g)$ in the Small Groups Library is [96, 192]. Since the Sylow 2–subgroup of $C_G(g)$ has order 32, [23] independently shows that this group has to be good.

However, the group $K = C_G(g) \wr C_2$ contains an element k of order 8 whose centralizer

$$C_K(k) = (((C_8 \times C_2) \rtimes C_2) \rtimes C_3) \rtimes C_2$$

has GAP ID [192, 963] and Sylow 2–subgroup $(C_8 \times C_4) \rtimes C_2$, which is therefore not covered by the previous list of examples. This process can be iterated, giving rise to other new examples.

For an odd prime p, [16] implies that the unipotent radical in $\mathrm{GL}_4(\mathbb{F}_p)$ is not good. So, as a curious consequence, we see that it cannot be obtained by iteratively applying the constructions that $\mathcal{G}$ is closed under to the above list of known good groups. We do not know how to show this using only algebraic methods.

References

[1] **J F Adams**, *A variant of E H Brown's representability theorem*, Topology 10 (1971) 185–198 MR0283788

[2] **T Barthel**, **M Frankland**, *Completed power operations for Morava E–theory*, Algebr. Geom. Topol. 15 (2015) 2065–2131

[3] **M Basterra**, **M A Mandell**, *The multiplication on BP*, J. Topol. 6 (2013) 285–310 MR3065177

[4] **A K Bousfield**, *The localization of spectra with respect to homology*, Topology 18 (1979) 257–281 MR551009

[5] **M Demazure**, *Lectures on p–divisible groups*, Lecture Notes in Mathematics 302, Springer, Berlin (1986) MR883960

[6] **A D Elmendorf**, **I Kriz**, **M A Mandell**, **J P May**, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs 47, Amer. Math. Soc., Providence, RI (1997) MR1417719

[7] **J P C Greenlees**, **J P May**, *Derived functors of I–adic completion and local homology*, J. Algebra 149 (1992) 438–453 MR1172439

[8] **J P C Greenlees**, **J P May**, *Completions in algebra and topology*, from: "Handbook of algebraic topology", (I M James, editor), North-Holland, Amsterdam (1995) 255–276 MR1361892

[9] **M Hopkins**, *Complex oriented cohomology theories and the language of stacks*, course notes (1999) Available at `http://www.math.rochester.edu/people/faculty/doug/otherpapers/coctalos.pdf`

[10] **M J Hopkins**, **N J Kuhn**, **D C Ravenel**, *Generalized group characters and complex oriented cohomology theories*, J. Amer. Math. Soc. 13 (2000) 553–594 MR1758754

[11] **M A Hovey**, *v_n–elements in ring spectra and applications to bordism theory*, Duke Math. J. 88 (1997) 327–356 MR1455523

[12] **M Hovey**, *Some spectral sequences in Morava E–theory* (2004) Available at `http://mhovey.web.wesleyan.edu/papers/morava-E-SS.pdf`

[13] **M Hovey**, **J H Palmieri**, **N P Strickland**, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. 610, Amer. Math. Soc., Providence, RI (1997) MR1388895

[14] **M Hovey**, **N P Strickland**, *Morava K–theories and localisation*, Mem. Amer. Math. Soc. 666, Amer. Math. Soc., Providence, RI (1999) MR1601906

[15] **I Kriz**, *Morava K–theory of classifying spaces: some calculations*, Topology 36 (1997) 1247–1273 MR1452850

[16] **I Kriz**, **K P Lee**, *Odd-degree elements in the Morava $K(n)$ cohomology of finite groups*, Topology Appl. 103 (2000) 229–241 MR1758436

[17] **J Lurie**, *DAG, XII: Proper morphisms, completions, and the Grothendieck existence theorem* (2011) Available at `http://www.math.harvard.edu/~lurie/papers/DAG-XII.pdf`

[18] **N Minami**, *From* $K(n+1)^*(X)$ *to* $K(n)^*(X)$, Proc. Amer. Math. Soc. 130 (2002) 1557–1562 MR1879983

[19] **D C Ravenel**, *Localization with respect to certain periodic homology theories*, Amer. J. Math. 106 (1984) 351–414 MR737778

[20] **C Rezk**, *Analytic completion* (2013) Available at `http://www.math.uiuc.edu/~rezk/analytic-paper.pdf`

[21] **T M Schlank**, **N Stapleton**, *A transchromatic proof of Strickland's theorem*, Adv. Math. 285 (2015) 1415–1447

[22] **B Schuster**, *Morava K–theory of classifying spaces*, Habilitationsschrift, Wuppertal (2006) Available at `http://www.ruhr-uni-bochum.de/imperia/md/content/mathematik/lehrstuhlxiii/s_knbg.pdf`

[23] **B Schuster**, *Morava K–theory of groups of order* 32, Algebr. Geom. Topol. 11 (2011) 503–521 MR2783236

[24] **B Schuster**, *On the Morava K–theory of* M_{12}, Proc. Amer. Math. Soc. 142 (2014) 1437–1445

[25] **N Stapleton**, *Transchromatic generalized character maps*, Algebr. Geom. Topol. 13 (2013) 171–203 MR3031640

[26] **M Tanabe**, *On Morava K–theories of Chevalley groups*, Amer. J. Math. 117 (1995) 263–278 MR1314467

[27] **The Stacks Project authors**, *The Stacks Project* (2014) Available at `http://stacks.math.columbia.edu`

Max-Planck-Institut für Mathematik
Vivatsgasse 7, D-53111 Bonn, Germany

Max-Planck-Institut für Mathematik
Vivatsgasse 7, D-53111 Bonn, Germany

`tbarthel@mpim-bonn.mpg.de, nstapleton@mpim-bonn.mpg.de`

Received: 4 September 2014 Revised: 31 July 2015