Swiss-cheese action on the totalization of action-operads

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We prove that, if a pair of semicosimplicial spaces $(X_c^\bullet; X_o^\bullet)$ arises from a colored operad, then the semitotalization $\mathrm{sTot}(X_c^\bullet)$ has the homotopy type of a relative double loop space and the pair $(\mathrm{sTot}(X_c^\bullet); \mathrm{sTot}(X_o^\bullet))$ is weakly equivalent to an explicit algebra over the two dimensional Swiss-cheese operad $\mathcal{SC}_2$.

Introduction

A multiplicative operad O is an operad under the associative operad As. In [15], McClure and Smith build a cosimplicial space $O^\bullet$ from the multiplicative operad O and show that, under some conditions, its homotopy totalization is a double loop space. V Turchin in [19] and, independently, Dwyer and Hess in [7] are able to identify the space of double delooping and prove, under the assumption $O(0) \simeq O(1) \simeq *$, that

$$\mathrm{hoTot}(O^\bullet) \simeq \Omega^2 \, \mathrm{Operad}^h(\mathrm{As}; O),$$

where $\mathrm{Operad}^h(\mathrm{As}; O)$ is the space of derived maps from the associative operad to O.

In order to prove this statement, Turchin introduces the categories of bimodules and infinitesimal bimodules over an operad O, denoted respectively by Bimod_O and Ibimod_O, such that $\mathrm{hoTot}(O^\bullet)$ is weakly equivalent to $\mathrm{Ibimod}_{\mathrm{As}}^h(\mathrm{As}; O)$. Then he proves the following two weak equivalences:

$$\mathrm{Ibimod}_{\mathrm{As}}^h(\mathrm{As}; O) \simeq \Omega \, \mathrm{Bimod}_{\mathrm{As}}^h(\mathrm{As}; O),$$
$$\mathrm{Bimod}_{\mathrm{As}}^h(\mathrm{As}; O) \simeq \Omega \, \mathrm{Operad}^h(\mathrm{As}; O).$$

This result was motivated by the following theorem of D Sinha: the space of long knots $\mathrm{Emb}_c(\mathbb{R}; \mathbb{R}^d)$ has the homotopy type of $\mathrm{hoTot}(\mathcal{K}_d^\bullet)$, where $\mathcal{K}_d$ is a multiplicative operad weakly equivalent to the little d–disk operad. The *Swiss-cheese* operad $\mathcal{SC}_d$ is a relative version of the little disc operad. It is a two-colored topological operad with set of colors $S = \{o; c\}$ that has been introduced by A Voronov in [21]. In particular, if $f: A \to X$ is a pointed continuous map, then the following pair is an $\mathcal{SC}_d$–space:

$$(\Omega^d X; \Omega^d(X; A)) := (\Omega^d X; \mathrm{hofib}(\Omega^{d-1} A \to \Omega^{d-1} X)).$$

In this paper we make great use of the operad $\pi_0(\mathcal{SC}_1)$, which is the operad of monoid actions $\mathcal{Act}$: it is a 2–colored operad whose algebras are the pairs of spaces $(X; A)$, where X is a monoid and A a left X–module. The operad $\mathcal{Act}_{>0}$ is the nonunital version of $\mathcal{Act}$. Similarly to the uncolored case, there is a notion of an $\mathcal{Act}_{>0}$–bimodule and an $\mathcal{Act}_{>0}$–infinitesimal bimodule. We prove that, if O is an operad under $\mathcal{Act}$, then it gives rise to a pair of semicosimplicial spaces $(O_c; O_o)$ such that the pair $(\mathrm{sTot}(O_c); \mathrm{sTot}(O_o))$ is weakly equivalent to

$$\left(\Omega^2 \operatorname{Operad}^h(\mathrm{As}_{>0}; O_c); \Omega^2\left(\operatorname{Operad}^h(\mathrm{As}_{>0}; O_c); \operatorname{Operad}^h(\mathcal{Act}_{>0}; O)\right)\right);$$

that is, it is an $\mathcal{SC}_2$–space.

Organization of the paper The paper is divided into six sections. Section 1 is an introduction. It describes the categories of colored operads, bimodules and infinitesimal bimodules over an operad. An explicit description of a point X in $\mathrm{Bimod}_{\mathcal{Act}_{>0}}$ and $\mathrm{Ibimod}_{\mathcal{Act}_{>0}}$ in terms of pairs of semicosimplicial spaces $(X_c; X_o)$ is given. We insist on the link between bimodule structures over $\mathcal{Act}_{>0}$ and monoidal structures on semicosimplicial spaces introduced by McClure and Smith in [15].

Section 2 introduces the left adjoint functors to the forgetful functors from the categories of bimodules and infinitesimal bimodules over an S–colored operad to the category of S–sequences. These adjunctions will be used in Section 3 in order to define a model category structure on Bimod_O and Ibimod_O. We also determine an explicit cofibrant replacement of $\mathcal{Act}$ (resp. $\mathcal{Act}_{>0}$) in the model category $\mathrm{Ibimod}_{\mathcal{Act}_{>0}}$ (resp. $\mathrm{Bimod}_{\mathcal{Act}_{>0}}$) and prove the weak equivalence

$$\mathrm{Ibimod}^h_{\mathcal{Act}_{>0}}(\mathcal{Act}; M) \simeq \mathrm{Ibimod}^h_{\mathrm{As}_{>0}}(\mathrm{As}; M_c),$$

where M is an $\mathcal{Act}_{>0}$–infinitesimal bimodule and M_c is its closed part.

In Section 4 we prove the first relative delooping theorem. From an $\mathcal{Act}_{>0}$–bimodule map $\eta\colon \mathcal{Act} \to M$, we extract two semicosimplicial spaces $(M_c; M_o)$. We prove, under some conditions, the following weak equivalence of pairs:

$$(\mathrm{sTot}(M_c); \mathrm{sTot}(M_o)) \simeq$$
$$\left(\Omega \operatorname{Bimod}^h_{\mathrm{As}_{>0}}(\mathrm{As}_{>0}; M_c); \Omega\left(\operatorname{Bimod}^h_{\mathrm{As}_{>0}}(\mathrm{As}_{>0}; M_c); \operatorname{Bimod}^h_{\mathcal{Act}_{>0}}(\mathcal{Act}_{>0}; M)\right)\right).$$

Section 5 consists in considering a particular case where a double relative delooping theorem holds. Namely let $\alpha\colon \mathrm{As} \to O$ be a map of operads and $\beta\colon O \to B$ be a map of O–bimodules. The two objects O and B are equipped with semicosimplicial structures. Under some conditions, we prove the weak equivalence of pairs

$$(\mathrm{sTot}(O); \mathrm{sTot}(B)) \simeq$$
$$\left(\Omega^2 \operatorname{Operad}^h(\mathrm{As}_{>0}; O); \Omega^2\left(\operatorname{Operad}^h(\mathrm{As}_{>0}; O); \operatorname{Operad}^h_{\{o;c\}}(\mathcal{Act}_{>0}; X)\right)\right),$$

where X is a colored operad build out of O and B.

Section 6 is devoted to the proof of the main theorem: if O is an $\{o; c\}$–operad under $\mathcal{A}ct$ such that $O(0; c) \simeq O(1; c) \simeq O(1; o) \simeq *$, then we have the weak equivalence of pairs

$$(\mathrm{sTot}(O_c); \mathrm{sTot}(O_o)) \simeq$$
$$\left(\Omega^2 \operatorname{Operad}^h(\mathrm{As}_{>0}; O_c); \Omega^2\left(\operatorname{Operad}^h(\mathrm{As}_{>0}; O_c); \operatorname{Operad}^h_{\{o;c\}}(\mathcal{A}ct_{>0}; O)\right)\right).$$

Convention By space we mean compactly generated Hausdorff space, and by abuse of notation we denote by Top this category; see eg Section 2.4 in Hovey [14]. If X, Y and Z are spaces, then $\operatorname{Top}(X; Y)$ is equipped with the compact-open topology in order to have a homeomorphism $\operatorname{Top}(X; \operatorname{Top}(Y; Z)) \cong \operatorname{Top}(X \times Y; Z)$.

A *semicosimplicial space* $X^\bullet$ is a family of topological spaces $\{X^n \mid n \geq 0\}$ endowed with operations

$$d^i \colon X^n \to X^{n+1} \quad \text{for } i \in \{0, \ldots, n+1\},$$

satisfying the cosimplicial relations $d^j d^i = d^i d^{j-1}$ for $0 \leq i < j$. By *semitotalization* $\mathrm{sTot}(X^\bullet)$ we mean the space of natural transformations from the semicosimplicial space $\Delta^\bullet$ to $X^\bullet$. From a cosimplicial space, the semitotalization of the underlying semicosimplicial space is also called the fat-totalization, which is a homotopy invariant. Since the homotopy totalization of a cosimplicial space is weakly equivalent to the semi-totalization (see Lemma 3.8 in Dror and Dwyer [6]), we will ignore the codegeneracies in the present work. We denote weak equivalences by the symbol $\simeq$.

1 Bimodules and infinitesimal bimodules over a colored operad

In what follows we introduce the category of colored operads as well as the categories of bimodules and infinitesimal bimodules over a colored operad. We focus on the operads with two colors $\{o; c\}$, called $\{o; c\}$–operads. In particular, we define the $\{o; c\}$–operad $\mathcal{A}ct_{>0}$ of monoid actions as in [13]. Besides, we characterize the bimodules and infinitesimal bimodules over this operad in terms of semicosimplicial spaces.

1A The operad of (unital) monoid actions

Definition 1.1 Let S be a set. An S–*sequence* is a collection of topological spaces $\{O(s_1, \ldots, s_n; s_{n+1})\}_{s_i \in S}^{n \in \mathbb{N}}$. The set S is called the *set of colors*. A map between two S–sequences O_1 and O_2 is a collection of continuous maps

$$\{f_{s_1, \ldots, s_n; s_{n+1}} \colon O_1(s_1, \ldots, s_n; s_{n+1}) \to O_2(s_1, \ldots, s_n; s_{n+1})\}_{s_i \in S}^{n \in \mathbb{N}}.$$

We denote by $\mathrm{Coll}(S)$ the category of S–sequences.

Notation 1.2 If M is an $\{o; c\}$–sequence, then we use the following notation in the rest of the text:

$$M_c^n = M(n; c) = M(\underbrace{c, \ldots, c}_{n}; c) \quad \text{and} \quad M_o^n = M(n+1; o) = M(\underbrace{c, \ldots, c}_{n}, o; o).$$

We denote by M_c the family $\{M_c^n\}_{n \geq 0}$ and by M_o the family $\{M_o^n\}_{n \geq 0}$.

Definition 1.3 An S–*operad* is an S–sequence O endowed with operations

$$\circ_i \colon O(s_1, \ldots, s_n; s_{n+1}) \times O(s_1', \ldots, s_m'; s_i)$$
$$\to O(s_1, \ldots, s_{i-1}, s_1', \ldots, s_m', s_{i+1}, \ldots, s_n; s_{n+1}),$$

for $1 \leq i \leq n$, and distinguished elements $\{*_s \in O(s; s)\}_{s \in S}$ satisfying associativity and unit axioms [1]. We denote by $x \circ_i y$ the operation $\circ_i(x; y)$ for $x, y \in O$. Define Operad_S to be the category of S–operads, where a map of S–operads is an S–sequence map that preserves the operadic structure.

Let O be an S–operad and $A = \{A_s\}_{s \in S}$ be a family of topological spaces. The *endomorphism* S–*operad* End_A (see [4]) is the family of spaces of continuous maps defined by

$$\mathrm{End}_A(s_1, \ldots, s_n; s_{n+1}) = \mathrm{Top}(A_{s_1} \times \cdots \times A_{s_n}; A_{s_{n+1}}).$$

The family A is called an O–*space* if there exists a map of S–operads $O \to \mathrm{End}_A$.

Definition 1.4 [13] Let $S = \{o, c\}$. The S–*operad of monoid actions* $\mathcal{A}ct_{>0}$ is given by the S–sequence

$$\mathcal{A}ct_{>0}(n; c) = *_{n;c} \quad \text{for } n > 0, \qquad \mathcal{A}ct_{>0}(n; o) = *_{n+1;o} \quad \text{for } n > 0,$$

and the empty set otherwise, with $*_{n;c}$ and $*_{n;o}$ each being the one-point topological space. The compositions are as follows:

$$(1) \qquad \begin{cases} *_{n;c} \circ_i *_{m;c} = *_{n+m-1;c}, \\ *_{n;o} \circ_i *_{m;c} = *_{n+m-1;o} \quad \text{for } i \neq n, \\ *_{n;o} \circ_n *_{m;o} = *_{n+m-1;o}, \end{cases}$$

Similarly, *the S–operad of unital monoid actions* $\mathcal{A}ct$ is given by the S–sequence

$$\mathcal{A}ct_{>0}(n; c) = *_{n;c} \quad \text{for } n \geq 0, \qquad \mathcal{A}ct_{>0}(n; o) = *_{n;o} \quad \text{for } n > 0,$$

and the empty set otherwise, with the same compositions. Consequently, the S–operad $\mathcal{A}ct$ (resp. $\mathcal{A}ct_{>0}$) is generated by $*_{0;c}$, $*_{2;c}$ and $*_{2;o}$ (resp. $*_{2;c}$ and $*_{2;o}$).

An *Act–space* is a pair of topological spaces $(X; A)$ with X a topological monoid with unit and A a left module over X.

The $\{c\}$–sequence given by the restriction of $\mathcal{Act}$ (resp. $\mathcal{Act}_{>0}$) to the color $\{c\}$ is the *associative operad* As (resp. the *strict associative operad* $\mathrm{As}_{>0}$). We use the notation $*_n$ to refer to the one-point topological space $\mathrm{As}(n)$.

The operad of monoid actions has been introduced by Hoefel, Livernet and Stasheff [13] in the context of the recognition principle for relative loop spaces.

Definition 1.5 We define an *action-operad* to be an $\{o; c\}$–operad O equipped with a map $\eta\colon \mathcal{Act} \to O$.

1B Infinitesimal bimodules over a colored operad

Definition 1.6 Let O be an S–operad. An *infinitesimal bimodule* over the operad O (or O–infinitesimal bimodule) is an S–sequence M endowed with operations

$$\circ_i\colon O(s_1, \ldots, s_n; s_{n+1}) \times M(s'_1, \ldots, s'_m; s_i)$$
$$\to M(s_1, \ldots, s_{i-1}, s'_1, \ldots, s'_m, s_{i+1}, \ldots, s_n; s_{n+1}),$$
$$\circ^i\colon M(s_1, \ldots, s_n; s_{n+1}) \times O(s'_1, \ldots, s'_m; s_i)$$
$$\to M(s_1, \ldots, s_{i-1}, s'_1, \ldots, s'_m, s_{i+1}, \ldots, s_n; s_{n+1}),$$

for $1 \leq i \leq n$, satisfying associativity and unit relations [1]. A map between O–infinitesimal bimodules is given by an S–sequence map preserving this structure. Let Ibimod_O be the category of infinitesimal bimodules over O. We denote by $x \circ_i y$ (resp. $x \circ^i y$) the operation $\circ_i(x; y)$ (resp. $\circ^i(x; y)$) with $x \in O$ and $y \in M$ (resp. $x \in M$ and $y \in O$).

Example 1.7 If $\eta\colon O_1 \to O_2$ is any S–operad map, then O_2 is endowed with the following O_1–infinitesimal bimodule structure:

$$\circ_i\colon O_1 \times O_2 \xrightarrow{\eta \times \mathrm{id}} O_2 \times O_2 \xrightarrow{\circ_i} O_2,$$

and

$$\circ^i\colon O_2 \times O_1 \xrightarrow{\mathrm{id} \times \eta} O_2 \times O_2 \xrightarrow{\circ_i} O_2.$$

Consequently, if A is an O–space, then End_A is an O–infinitesimal bimodule.

Definition 1.8 Let N and M be two S–sequences. The sequence M is *of type N* if

$$N(s_1, \ldots, s_n; s_{n+1}) = \varnothing \quad \Rightarrow \quad M(s_1, \ldots, s_n; s_{n+1}) = \varnothing.$$

Proposition 1.9 *Let M be an $\{o;c\}$–sequence of type $\mathcal{A}ct$. The following assertions are equivalent:*

(i) *M is an $\mathcal{A}ct_{>0}$–infinitesimal bimodule;*

(ii) *the families M_c and M_o are semicosimplicial spaces, and there exists a semicosimplicial map $h\colon M_c \to M_o$.*

Moreover, (i) $\Rightarrow$ (ii) even if M is not of type $\mathcal{A}ct$.

Proof Let M be an $\mathcal{A}ct_{>0}$–infinitesimal bimodule. For $n \in \mathbb{N}$, let $h\colon M_c^n \to M_o^n$ be defined by $h(x) = *_{2;o} \circ_1 x$. The semicosimplicial structure is given as follows (see eg [1; 15; 17]):

$$d^i\colon M_c^n \to M_c^{n+1}; \quad x \mapsto \begin{cases} *_{2;c} \circ_2 x & \text{if } i = 0, \\ x \circ^i *_{2;c} & \text{if } i \in \{1, \ldots, n\}, \\ *_{2;c} \circ_1 x & \text{if } i = n+1, \end{cases}$$

$$d^i\colon M_o^n \to M_o^{n+1}; \quad x \mapsto \begin{cases} *_{2;o} \circ_2 x & \text{if } i = 0, \\ x \circ^i *_{2;c} & \text{if } i \in \{1, \ldots, n\}, \\ x \circ^{n+1} *_{2;o} & \text{if } i = n+1. \end{cases}$$

The reader can check that the relations (1) of Definition 1.4 and Definition 1.6 induce the semicosimplicial relations.

Conversely, if $h\colon M_c \to M_o$ is a semicosimplicial map, then let $M(n;c) = M_c^n$, $M(n+1;o) = M_o^n$ and the empty set otherwise. The left and right infinitesimal module structures are defined by the above construction since $\mathcal{A}ct_{>0}$ is generated by $*_{2;c}$ and $*_{2;o}$ as a colored operad. $\square$

It is proved in [19] that the category of semicosimplicial spaces is equivalent to the category of $\mathrm{As}_{>0}$–infinitesimal bimodules. Consequently, the collection $M_o = \{M_o^n\}_{n \geq 0}$ is an infinitesimal bimodule over $\mathrm{As}_{>0}$. Since $\mathrm{As}_{>0}$ is generated by $*_2$ as an operad, the structure of M_o is given by:

$$\text{(2)} \quad \begin{cases} *_2 \circ_2 x = *_{2;o} \circ_2 x, & \text{for } x \in M_o^n, \\ *_2 \circ_1 x = x \circ^{n+1} *_{2;o}, & \text{for } x \in M_o^n, \\ x \circ^i *_2 = x \circ^i *_{2;c}, & \text{for } x \in M_o^n \text{ and } i \in \{1, \ldots, n\}. \end{cases}$$

1C   Bimodules over a colored operad

Definition 1.10 Let O be an S–operad. An S–sequence M is an O–bimodule if it

is endowed with operations

$$\gamma_l \colon O(s_1, \ldots, s_n; s_{n+1}) \times M(s_1^1, \ldots, s_{p_1}^1; s_1) \times \cdots \times M(s_1^n, \ldots, s_{p_n}^n; s_n)$$
$$\to M(s_1^1, \ldots, s_{p_n}^n; s_{n+1}),$$

$$\circ^i \colon M(s_1, \ldots, s_n; s_{n+1}) \times O(s_1', \ldots, s_m'; s_i)$$
$$\to M(s_1, \ldots, s_{i-1}, s_1', \ldots, s_m', s_{i+1}, \ldots, s_n; s_{n+1}),$$

for $1 \leq i \leq n$, satisfying associativity and unit axioms [1]. A map between O–bimodules is an S–sequence map that preserves the bimodule structure. Let Bimod_O be the category of O–bimodules. We denote by $x(y_1, \ldots, y_n)$ the operation $\gamma_l(x, y_1, \ldots, y_n)$ with $x \in O$ and $y_i \in M$.

Example 1.11 If $\eta \colon O_1 \to O_2$ is any S–operad map, then O_2 is endowed with the following O_1–bimodule structure:

$$\gamma_l \colon O_1 \times O_2 \times \cdots \times O_2 \xrightarrow{\eta \times \mathrm{id} \times \cdots \times \mathrm{id}} O_2 \times \cdots \times O_2 \to O_2$$

and

$$\circ^i \colon O_2 \times O_1 \xrightarrow{\mathrm{id} \times \eta} O_2 \times O_2 \to O_2.$$

Consequently, if A is an O–algebra, then End_A is an O–bimodule.

A priori there is no relation between an O–bimodule structure and an O–infinitesimal bimodule structure because the left operations differ. However, if $\eta \colon O \to M$ is a morphism of O–bimodules, then M is an O–infinitesimal bimodule, and the left infinitesimal bimodule structure is given by

$$\circ_i \colon O(s_1, \ldots, s_n; s_{n+1}) \times M(s_1', \ldots, s_m'; s_i)$$
$$\to M(s_1, \ldots, s_{i-1}, s_1', \ldots, s_m', s_{i+1}, \ldots, s_n; s_{n+1}),$$
$$(o; m) \mapsto o(\eta(*_{s_1}), \ldots, \eta(*_{s_{i-1}}), m, \eta(*_{s_{i+1}}), \ldots, \eta(*_{s_n})),$$

where $*_s$ is the distinguished element in $O(s; s)$.

In [15], McClure and Smith define a monoidal structure on the category of semicosimplicial spaces in order to recognize loop spaces. More precisely, they prove that the group completion of the semitotalization of a monoid in this category has the homotopy type of a loop space. We recall this construction since we need it to describe $\mathcal{Act}_{>0}$–bimodules under $\mathcal{Act}$.

Proposition 1.12 [15, Proposition 2.2] *Let $X^\bullet$ and $Y^\bullet$ be two semicosimplicial spaces, and let $X \boxtimes Y$ be the semicosimplicial space whose m^{th} space is given by*

$$\left(\coprod_{p+q=m} X^p \times Y^q \right) \Big/ \sim,$$

where $\sim$ is the equivalence relation generated by $(x, d^0 y) \sim (d^{|x|+1}x, y)$. The semicosimplicial structure is the following:

$$d^i(x, y) = \begin{cases} (d^i x, y) & \text{if } 0 \leq i \leq |x|, \\ (x, d^{i-|x|} y) & \text{if } |x| < i \leq |x| + |y| + 1. \end{cases}$$

The category of semicosimplicial spaces equipped with $\boxtimes$ is a monoidal category denoted by $(\mathrm{Top}^{\Delta_{\mathrm{inj}}}, \boxtimes)$, with unit e being the constant semicosimplicial one-point space.

Proposition 1.13 Let M be an $\{o; c\}$–sequence of type $\mathcal{A}ct$. The following assertions are equivalent:

(i) M is an $\mathcal{A}ct_{>0}$–bimodule under $\mathcal{A}ct$;

(ii) in $(\mathrm{Top}^{\Delta_{\mathrm{inj}}}, \boxtimes)$, the family M_c is a monoid with unit, the family M_o is an M_c–left module and there exists a morphism of M_c–left modules $h \colon M_c \to M_o$.

Moreover (i) $\Rightarrow$ (ii) even if M is not of type $\mathcal{A}ct$.

Proof Let M be an $\mathcal{A}ct_{>0}$–bimodule equipped with an $\mathcal{A}ct_{>0}$–bimodule map $\eta \colon \mathcal{A}ct \to M$. Let $M_c^n = M(n; c)$ and $M_o^n = M(n + 1; o)$ for $n \in \mathbb{N}$. The bimodule structure induces, for $i \in \{1, \dots, n\}$, the cofaces

(3)
$$\begin{aligned} d^i &\colon M_c^n \to M_c^{n+1}, & x &\mapsto x \circ^i *_{2;c}, \\ d^i &\colon M_o^n \to M_o^{n+1}, & x &\mapsto x \circ^i *_{2;c}, \\ d^{n+1} &\colon M_o^n \to M_o^{n+1}, & x &\mapsto x \circ^{n+1} *_{2;o}, \end{aligned}$$

satisfying the semicosimplicial relations and the following two operations:

(4)
$$\begin{aligned} M_c^j \times M_c^l &\to M_c^{j+l}, & (x; y) &\mapsto *_{2;c}(x; y), \\ M_c^j \times M_o^l &\to M_o^{j+l}, & (x; y) &\mapsto *_{2;o}(x; y). \end{aligned}$$

The map $\eta \colon \mathcal{A}ct \to M$ gives us the missing cofaces

(5)
$$\begin{aligned} d^0 &\colon M_c^n \to M_c^{n+1}, & x &\mapsto *_{2;c}(\eta(*_{1;c}), x), \\ d^{n+1} &\colon M_c^n \to M_c^{n+1}, & x &\mapsto *_{2;c}(x, \eta(*_{1;c})), \\ d^0 &\colon M_o^n \to M_o^{n+1}, & x &\mapsto *_{2;o}(\eta(*_{1;c}), x), \end{aligned}$$

inducing a semicosimplicial structure on M_c and M_o such that the two operations defined in (3) make M_c into a monoid with unit and M_o into an M_c–left module. The map

$$h \colon M_c^n \to M_o^n, \quad x \mapsto *_{2;o}(x, \eta(*_{1;o}))$$

is a left M_c–module map.

Conversely, let (M_c, M_o, h) be a triple satisfying the conditions of the proposition. By the same argument as in Proposition 1.9, the constructions (3) and (4) define an $Act_{>0}$–bimodule structure on M. In particular, if M_c and M_o coincide with the unit e, then the corresponding $Act_{>0}$–bimodule is Act. There exists a map η_c from the unit to M_c, for M_c is a monoid with unit. Let η_o be the map from the unit to M_o given by $\eta_o = h \circ \eta_c$. The map $\eta: Act \to M$ so obtained is an $Act_{>0}$–bimodule map.   □

Example 1.14 This proposition implies that the category whose objects are monoids in $(\mathrm{Top}^{\Delta_{\mathrm{inj}}}, \boxtimes)$ is equivalent to the category of $As_{>0}$–bimodules under As considered by Turchin. Furthermore, if we substitute $Act_{>0}$–bimodule by Act–bimodule and semicosimplicial space by cosimplicial space, Proposition 1.13 is still true. From now on, $(X; *)$ is a pointed topological space, and A is a subspace of X containing the basepoint $*$. Let $\Omega X^\bullet$ and $\Omega(X; A)^\bullet$ be the two cosimplicial spaces defined respectively by

$$\Omega X^n := X^{\times n} \quad \text{and} \quad \Omega(X; A)^n := X^{\times n} \times A \quad \text{for } n \in \mathbb{N},$$

and

$$d^i: \Omega X^n \to \Omega X^{n+1}, \quad (x_1, \dots, x_n) \mapsto \begin{cases} (*, x_1, \dots, x_n) & \text{if } i = 0, \\ (x_1, \dots, x_i, x_i, \dots, x_n) & \text{if } i \in \{1, \dots, n\}, \\ (x_1, \dots, x_n, *) & \text{if } i = n+1, \end{cases}$$

$$d^i: \Omega(X; A)^n \to \Omega(X; A)^{n+1},$$

$$(x_1, \dots, x_n, a) \mapsto \begin{cases} (*, x_1, \dots, x_n, a) & \text{if } i = 0, \\ (x_1, \dots, x_i, x_i, \dots, x_n, a) & \text{if } i \in \{1, \dots, n\}, \\ (x_1, \dots, x_n, a, a) & \text{if } i = n+1. \end{cases}$$

The codegeneracies consist in forgetting a point, and the concatenation makes $\Omega X^\bullet$ into a monoid with unit in $(\mathrm{Top}^{\Delta_{\mathrm{inj}}}, \boxtimes)$, and $\Omega(X; A)^\bullet$ into a left $\Omega X^\bullet$–module. The left $\Omega X^\bullet$–module map is defined by

$$h: \Omega X^n \to \Omega(X; A)^n, \quad (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, *).$$

Proposition 1.13 states that these data are equivalent to an Act–bimodule map. So the pair $(\Omega X^\bullet; \Omega(X; A)^\bullet)$ is the prototypical example of an Act–bimodule. The evaluation maps,

$$\Omega X \to \mathrm{Tot}(\Omega X^\bullet), \qquad f \mapsto \{f_n: (t_1; \cdots; t_n) \mapsto (f(t_1), \dots, f(t_n))\}_n,$$

$$\Omega(X; A) \to \mathrm{Tot}(\Omega(X; A)^\bullet), \quad f \mapsto \{f_n: (t_1; \cdots; t_n) \mapsto (f(t_1), \dots, f(t_n), f(1))\}_n,$$

induce homeomorphisms, due to the codegeneracies. Here ΩX and $\Omega(X; A)$ are the space of loops based on $*$ and the space of paths with endpoint in the subspace

A, respectively, and $(t_1; \cdots; t_n)$ is a nondecreasing sequence; that is, $t_i \leq t_{i+1}$ for $1 \leq i < n$. As seen in the introduction, these two spaces are models for the homotopy fiber $* \to X$ and $A \to X$ over the basepoint $*$, respectively.

It provides an example of an $\mathcal{Act}$–bimodule map $\eta \colon \mathcal{Act} \to M$ such that the totalization of M_c (resp. M_o) can be described as a loop space ΩX (resp. a relative loop space $\Omega(X; A)$) with explicit topological spaces X and A. We will prove that we can generalize this result for any $\mathcal{Act}_{>0}$–bimodule map $\eta \colon \mathcal{Act} \to M$ using the semitotalization. Let us notice that, in this particular case, the sequence of cosimplicial spaces $\Omega A^\bullet \to \Omega X^\bullet \to \Omega(X; A)^\bullet \to A \to X$ gives rise, under totalization, to the dual of the Barratt–Puppe sequence for the map $A \to X$.

2 The free (infinitesimal) bimodule generated by an S–sequence

In what follows, S is a set, O is an S–operad and M is an S–sequence. In order to prove that $\mathrm{sTot}(M_o)$ has the homotopy type of a relative loop space and to explicitly identify this space, we have to introduce a model category structure on the categories Ibimod_O and Bimod_O. The easiest way is to use a transfer theorem (see eg Theorem 3.4), which needs a left adjoint to the forgetful functor from the category of (infinitesimal) bimodules over O to $\mathrm{Coll}(S)$. In both cases, the first step consists in introducing the category of trees, which encodes the (infinitesimal) bimodule structure. Then we label the vertices by points in M or O. Similar constructions have been considered in [5] and, more recently, [20].

By a tree we mean a planar rooted tree with an orientation towards the root, and let t be such a tree.

- The set of its vertices is denoted by $V(t)$ and the set of its edges by $E(t)$.

- For a vertex v, the ordered set of its input edges is denoted by $\mathrm{in}(v)$ and its cardinality by $|v|$ such that $\mathrm{in}(v) = \{e_1(v), \ldots, e_{|v|}(v)\}$. The output edge of v is denoted by $e_0(v)$, and a vertex without input edge is called univalent.

- The edges connecting two vertices are called *inner edges*, and the set of inner edges is denoted by $E^{\mathrm{int}}(t)$.

- An element $e \in E^{\mathrm{int}}(t)$ is determined by a source vertex $s(e)$ and a target vertex $t(e)$ induced by the orientation of the tree.

- An edge with no source is called a *leaf*, and the ordered set of leaves is denoted by $\{l_1, \ldots, l_n\}$.

- The edge with no target is called the *trunk*, denoted by e_0, and its source, the *root*, is denoted by r.

- Each leaf is connected to the trunk by a unique path composed of edges.

- An S–*tree* is a pair (t, f) where t is a planar tree and $f: E(t) \to S$ is called an S–*labeling* of t.
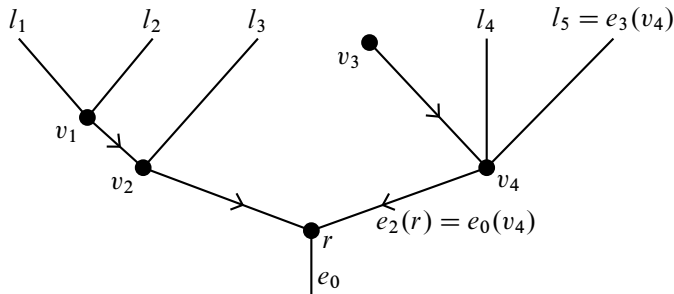

Figure 1: A planar tree: r is the root, e_0 is the trunk, l_1 is a leaf.

2A   The free infinitesimal bimodule

Definition 2.1   The trees encoding the infinitesimal bimodule structure are constructed as follows.

- The *join* $j(v_1; v_2)$ of two vertices v_1 and v_2 is the first common vertex shared by the two paths joining v_1 and v_2 to the root. If $j(v_1; v_2) = r$, then v_1 and v_2 are said to be *connected to the root*, and if $j(v_1; v_2) \in \{v_1; v_2\}$, then they are said to be *connected*. In Figure 1, the vertices v_1 and v_2 are connected whereas the vertices v_1 and v_3 are connected to the root.

- Let $d: V(T) \times V(T) \to \mathbb{N}$ be the distance defined as follows. The integer $d(v_1; v_2)$ is the number of edges connecting v_1 to v_2 if they are connected, otherwise $d(v_1; v_2) = d(v_1; v_3) + d(v_3; v_2)$ with $v_3 = j(v_1; v_2)$. In Figure 1, $d(v_1; r) = 2$, $d(v_3; v_4) = 1$ and $d(v_1; v_3) = 4$.

- A *pearl tree* (or ptree) is a pair (t, p) where t is a planar tree and $p \in V(t)$ is called the *pearl*, satisfying the property: if $v \in V(t) \setminus \{p\}$, then $d(v; p) = 1$. An S–ptree is a pearl tree t together with an S–labeling of t; see Figure 2.
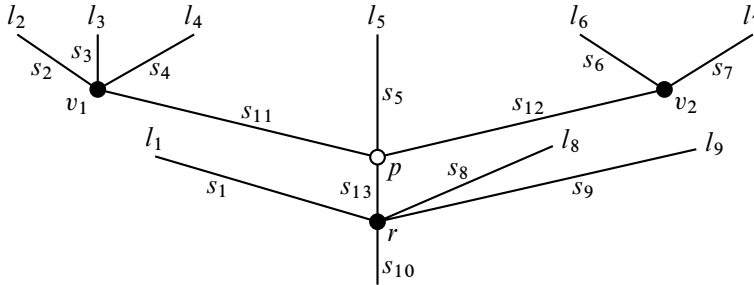
Figure 2: An S–ptree.

Construction 2.2 The S–sequence $\mathrm{Ib}_O(M)$ is defined as follows:

$$\mathrm{Ib}_O(M)(s_1,\ldots,s_n;s_{n+1})$$

$$= \coprod_{\substack{(t,f,p)\in S\text{–ptree} \\ f(l_i)=s_i,\, f(e_0)=s_{n+1}}} \left[M\big(f(e_1(p)),\ldots,f(e_{|p|}(p)); f(e_0(p))\big) \right.$$

$$\left. \times \prod_{v\in V(t)\setminus\{p\}} O\big(f(e_1(v)),\ldots,f(e_{|v|}(v)); f(e_0(v))\big) \right] \Big/ \sim$$

Here, $\sim$ is the equivalence relation generated by

$$\begin{matrix} s_1 \\ \bullet \\ s_1 \end{matrix} \!*_{s_1} \;\; \sim \;\; \Big|_{s_1}$$

Let x be a point in the space $\mathrm{Ib}_O(M)(s_1,\ldots,s_n;s_{n+1})$ indexed by an S–ptree (t,f,p), and let $y\in O(s_1',\ldots,s_m';s_i)$. The right infinitesimal module structure consists of grafting the m–corolla indexed by y to the ith input of t, and contracting the inner edge so obtained if its target does not coincide with the pearl, by using the operadic structure of O, as in Figure 3.
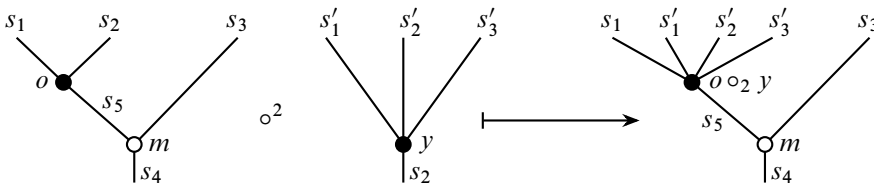

Figure 3: The right infinitesimal module structure.

Similarly, let x be a point in the space $\mathrm{Ib}_O(M)(s_1',\ldots,s_m';s_i)$ indexed by an S–ptree (t,f,p) and let $y\in O(s_1,\ldots,s_n;s_{n+1})$. The left infinitesimal module structure consists in grafting the tree t to the ith input of the n–corolla indexed by y, and

contracting the inner edge so obtained if its source does not coincide with the pearl, by using the operadic structure of O. These maps pass to the quotient and are continuous.

There exists a map from the S–sequence M to $\mathrm{Ib}_O(M)$ that maps a point $m \in M(s_1, \ldots, s_n; s_{n+1})$ to the pearl n–corolla whose leaves are labeled by $s_1, \ldots, s_n$ and the trunk by s_{n+1}; the pearl is indexed by m.

We denote by (t, f, p, g) a point in $\mathrm{Ib}_O(M)$ indexed by (t, f, p) and labeled by $g\colon V(t) \to O \sqcup M$.

Proposition 2.3 *The functor Ib_O is the left adjoint to the forgetful functor:*

$$\mathrm{Ib}_O(-)\colon \mathrm{Coll}(S) \leftrightarrows \mathrm{Ibimod}_O :U.$$

Proof Given an O–infinitesimal bimodule N and a map of S–sequences $h\colon M \to N$, we prove that there exists a unique map $\widetilde{h}\colon \mathrm{Ib}_O(M) \to N$ of O–infinitesimal bimodules such that the following diagram commutes:

$$
\begin{array}{ccc}
M & \xrightarrow{\ h\ } & N \\
\downarrow & \nearrow & \\
\mathrm{Ib}_O(M) & {}_{!\widetilde{h}} &
\end{array}
$$

Let (t, f, p, g) be a point in $\mathrm{Ib}_O(M)$. The map $\widetilde{h}$ is defined by induction on $|V(t)|$ as follows. If $|V(t)| = 1$, then the pearl p is the only vertex and t is a corolla. In this case we define $\widetilde{h}((t, f, p, g)) = h(g(p))$. Hence the commutativity of the previous diagram is guaranteed.

If t has two vertices, then there exists a unique edge e connecting the pearl p to the other vertex v. There are two cases to consider:

- if $s(e) = p$ and e is the ith input of v, then let $\widetilde{h}((t, f, p, g)) = g(v) \circ_i h(g(p))$;
- if $t(e) = p$ and e is the ith input of p, then let $\widetilde{h}((t, f, p, g)) = h(g(p)) \circ^i g(v)$.

Assume $\widetilde{h}$ has been defined for $|V(t)| = n \geq 2$. Let $(t, f, p, g) \in \mathrm{Ib}_O(M)$ such that t has $n + 1$ vertices. There exists an inner edge e connecting the pearl p to another vertex v such that $t(e) = p$. Let (t', f', p, g') be the tree obtained by cutting off the corolla corresponding to the vertex v (t' has only n vertices). We define

$$\widetilde{h}((t, f, p, g)) = \widetilde{h}((t', f', p, g')) \circ^i g(v).$$

Due to the associativity axioms of the infinitesimal bimodule structure of N, $\widetilde{h}$ does not depend on the choice of v, and $\widetilde{h}$ is an infinitesimal bimodule map. The uniqueness follows from the construction. $\qquad\square$

2B   The free bimodule

Definition 2.4   A *tree with section* (or stree) is a pair $(t, V^p(t))$ where t is a planar tree and $V^p(t)$ is a subset of $V(t)$, called the set of pearls, such that each path connecting a leaf or an univalent vertex to the trunk passes by a unique pearl, and

$$j(v; p) \in \{v; p\} \quad \Rightarrow \quad d(v; p) = 1$$

for all $v \in V(t) \setminus V^p(t)$ and all $p \in V^p(t)$. An S–*tree* with section (or S–*tree*) is given by a triple $(t, V^p(t), f)$ such that (t, f) is an S–tree and $(t, V^p(t))$ is a tree with section.
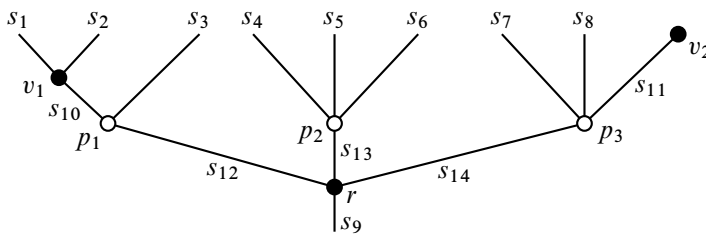

Figure 4: A tree with section.

Construction 2.5   The S–sequence $B_O(M)$ is defined as follows:

$$B_O(M)(s_1, \ldots, s_n; s_{n+1})$$
$$= \coprod_{\substack{(t, f, V^p(t)) \in S\text{–stree} \\ f(l_i) = s_i, f(e_0) = s_{n+1}}} \left[\prod_{v \in V^p(t)} M\big(f(e_1(v)), \ldots, f(e_{|v|}(v)); f(e_0(v))\big) \right.$$
$$\left. \times \prod_{v \in V(t) \setminus V^p(t)} O\big(f(e_1(v)), \ldots, f(e_{|v|}(v)); f(e_0(v))\big) \right] \Big/ \sim$$

with $\sim$ the equivalence relation generated by

$$\begin{matrix} s_1 \\ \bullet *_{s_1} \\ s_1 \end{matrix} \sim \begin{matrix} \\ \big| s_1 \\ \end{matrix}$$

Let $x \in B_O(M)(s_1, \ldots, s_n; s_{n+1})$ be indexed by a tree with section $(t, f, V^p(t))$, and let $y \in O(s_1, \ldots, s_n; s_i)$. The right module structure consists of grafting the m–corolla indexed by y to the ith input of t, and contracting the inner edge so obtained if its target does not coincide with a pearl, by using the operadic structure of O.

Let y be a point in $O(s_1, \ldots, s_n; s_{n+1})$ and let $x_i \in B_O(M)(s_1^i, \ldots, s_{n_i}^i; s_i)$ be indexed by $(t_i, f_i, V_i^p(t))$ for $1 \le i \le n$. The left module structure consists of grafting each

tree t_i to the ith input of the n–corolla indexed by y, and contracting the inner edges whose source is not a pearl by using the operadic structure of O, as in Figure 5.
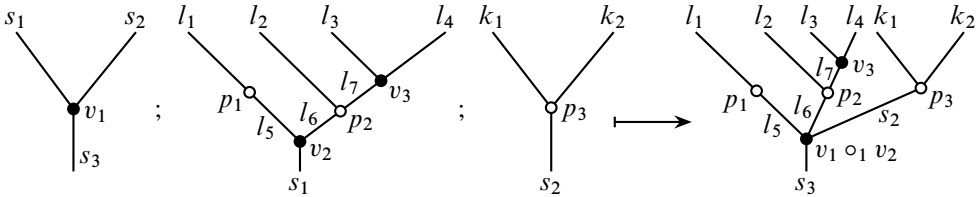

Figure 5: The left module structure.

These maps pass to the quotient and are continuous. Furthermore, there exists a map from the S–sequence M to $B_O(M)$ which maps a point $m \in M(s_1, \ldots, s_n; s_{n+1})$ to the pearl n–corolla whose leaves are labeled by $s_1, \ldots, s_n$ and the trunk by s_{n+1}; the pearl is indexed by m. We denote by $(t, f, V^P(t), g)$ a point in $B_O(M)$ indexed by $(t, f, V^P(t))$ and labeled by $g: V(t) \to O \sqcup M$.

Proposition 2.6 *The functor B_O is the left adjoint to the forgetful functor:*

$$B_O(-): \mathrm{Coll}(S) \leftrightarrows \mathrm{Bimod}_O : U.$$

Proof Given an O–bimodule N and $h: M \to N$ a map of S–sequences, we prove that there exists a unique map $\widetilde{h}: B_O(M) \to N$ of O–bimodules such that the following diagram commutes:

$$
\begin{array}{ccc}
M & \xrightarrow{\ h\ } & N \\
\downarrow & \nearrow & \\
B_O(M) & {}_{!\widetilde{h}} &
\end{array}
$$

Let $(t, V^P(t), f, g)$ be a point in $B_O(M)$ and let $\mathrm{nb}(t)$ be the cardinality of the set $V(t) \setminus V^P(t)$. The map $\widetilde{h}$ is defined by induction on $\mathrm{nb}(t)$. If $\mathrm{nb}(t) = 0$, then the pearl p is the only vertex, and t is a corolla. In this case, $\widetilde{h}((t, V^P(t), f, g)) = h(g(p))$. If $\mathrm{nb}(t) = 1$, we denote by v the unique element of $V(t) \setminus V^P(t)$. There are two cases to consider.

- If v is the source of an edge e that is connected to a pearl p, and e is the ith input of p, then

$$\widetilde{h}((t, V^P(t), f, g)) = h(g(p)) \circ^i g(v).$$

- If v coincides with the root, then all the pearls are connected to v. Let $p_1, \ldots, p_k$ be the set of ordered pearls. We define $\widetilde{h}$ by

$$\widetilde{h}((t, V^P(t), f, g)) = g(v)(h(g(p_1)), \ldots, h(g(p_k))).$$

Assume $\widetilde{h}$ has been defined for $\mathrm{nb}(t) = n \geq 1$. Let $(t, V^p(t), f, g) \in B_O(M)$ such that $\mathrm{nb}(t) = n + 1$. There exists an inner edge e whose target is a pearl p_i. Let $v = s(e)$ and let $(t', V^p(t), f', g')$ be the tree obtained from $(t, V^p(t), f, g)$ by cutting off the corolla corresponding to the vertex v. Consequently, $\mathrm{nb}(t') = n$ and $\widetilde{h}$ can be defined by induction as

$$\widetilde{h}((t, V^p(t), f, g)) = \widetilde{h}((t', V^p(t), f', g')) \circ^i g(v).$$

Due to the associativity axioms of the bimodule structure of N, $\widetilde{h}$ does not depend on the choice of v, and $\widetilde{h}$ is a map of O–bimodules. The uniqueness follows from the construction. $\qquad\square$

3 Cofibrant replacement of the operad of monoid actions in the category of (infinitesimal) bimodules over $\mathcal{A}ct_{>0}$

3A Model category structure on Bimod_O and Ibimod_O

In this section we define a model category structure on Bimod_O and Ibimod_O by using the previous adjunctions. The references used for model categories are [8; 12; 14]. These structures have been considered by many authors in the context of operads (symmetric or nonsymmetric), algebras over an operad, or left-right modules over operads, most of them in the uncolored case; see eg Fresse [9], Berger and Moerdijk [2] and Harper [10]. In order to be precise, we prefer to give the details of the model category structure in our context, and we make use of this section to state lemmas that will be useful for the sequel.

Theorem 3.1 [14, Theorem 2.4.24] *The category* Top *is equipped with the following model category structure.*

> **Weak equivalences** *are the continuous maps* $f\colon X \to Y$ *such that* $f_0^*\colon \pi_0(X) \to \pi_0(Y)$ *is a bijection and* $f_n^*\colon \pi_n(X; x) \to \pi_n(Y; f(x))$ *is an isomorphism, for all* $x \in X$ *and for all* $n > 0$.

> **Serre fibrations** *are the continuous maps* $f\colon X \to Y$ *having the homotopy lifting property, ie for every CW-complex* A, *a lift exists in every commutative diagram of the following form:*

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & X \\ \downarrow & \nearrow^{\exists} & \downarrow \\ A \times [0,1] & \longrightarrow & Y \end{array}$$

> **Cofibrations** *are the continuous maps having the left lifting property with respect to the acyclic Serre fibrations.*

Moreover, this model category is cofibrantly generated. The cofibrations are generated by the inclusions $\partial\Delta^n \to \Delta^n$ for $n > 0$, whereas the acyclic cofibrations are generated by the inclusions of the horns $\Lambda_k^n \to \Delta^n$ for $n > 0$ and $n \geq k \geq 0$. We call this model category the Serre model category.

Corollary 3.2 *The category $\mathrm{Coll}(S)$ inherits a cofibrantly generated model category structure from the Serre model category in which a map is a cofibration, a fibration or a weak equivalence if each of its components is.*

Lemma 3.3 [12] *Let $A \hookrightarrow B$ be a cofibration in the Serre model category. For every space Y, the induced map $\mathrm{Top}(B; Y) \to \mathrm{Top}(A; Y)$ is a fibration.*

Theorem 3.4 [2, Section 2.5] *Let $\mathcal{C}_1$ be a cofibrantly generated model category and let I (resp. J) be the set of generating cofibrations (resp. acyclic cofibrations). Let $L\colon \mathcal{C}_1 \leftrightarrows \mathcal{C}_2 \colon R$ be a pair of adjoint functors. Assume that $\mathcal{C}_2$ has small colimits and small limits. Define a map f in $\mathcal{C}_2$ to be a weak equivalence (resp. a fibration) if $R(f)$ is a weak equivalence (resp. fibration). If we have that*

(i) *the functor R preserves filtered colimits,*

(ii) *$\mathcal{C}_2$ has a functorial fibrant replacement, and*

(iii) *for each fibrant object $X \in \mathcal{C}_2$ we have a functorial path object $\mathrm{Path}(X)$ with $X \xrightarrow{\simeq} \mathrm{Path}(X) \twoheadrightarrow X \times X$ (a weak equivalence followed by a fibration) a factorization of the diagonal map,*

then $\mathcal{C}_2$ is equipped with a cofibrantly generated model category (LI, LJ) with $LI = \{L(u) \mid u \in I\}$ and $LJ = \{L(v) \mid v \in J\}$. Furthermore, (L, R) is a Quillen pair.

Application 3.5 For the adjunction $\mathrm{Ib}_O(-)\colon \mathrm{Coll}(S) \rightleftarrows \mathrm{Ibimod}_O \colon U$, the identity induces a functorial fibrant replacement since all the objects of $\mathrm{Coll}(S)$ are fibrants. From an O–infinitesimal bimodule M, a functorial path object $\mathrm{Path}(M)$ is given by the following S–sequence:

$$\mathrm{Path}(M)(s_1, \ldots, s_n; s_{n+1}) = \mathrm{Top}([0, 1]; M(s_1, \ldots, s_n; s_{n+1})).$$

The O–infinitesimal bimodule structure and the functoriality of $\mathrm{Path}(-)$ are induced by that of M. The factorization of the diagonal map is given pointwise:

$$M \xrightarrow{f_1} \mathrm{Path}(M) \xrightarrow{f_2} M \times M.$$

The map f_1 maps a point $m \in M(s_1, \ldots, s_n; s_{n+1})$ to the constant path in m. Due to the homotopy between a path h and the constant path in $h(0)$, the map f_1 is a weak equivalence. The map f_2 maps a point $h \in \text{Path}(M)(s_1, \ldots, s_n; s_{n+1})$ to the pair $(h(0); h(1)) \in (M \times M)(s_1, \ldots, s_n; s_{n+1})$. This map is a fibration since $\text{Path}(M)(s_1, \ldots, s_n; s_{n+1})$ is a path object in the Serre model category.

Similarly, the adjunction $B_O(-) \colon \text{Coll}(S) \leftrightarrows \text{Bimod}_O : U$ induces a cofibrantly generated model category on Bimod_O.

Definition 3.6 The O–infinitesimal bimodule M is *obtained from the O–infinitesimal bimodule N by attaching cells* if M is obtained as a pushout diagram of the form

(6)
$$
\begin{array}{ccc}
\text{Ib}_O(A) & \xhookrightarrow{\ \text{Ib}_O(i)\ } & \text{Ib}_O(B) \\
{\scriptstyle \widetilde{f}} \downarrow & & \downarrow \\
N & \longrightarrow & M
\end{array}
$$

with i a cofibration in $\text{Coll}(S)$, $f \colon A \to N$ an S–sequence map called the *attaching map* and $\widetilde{f}$ the O–infinitesimal bimodule map induced by f; see Proposition 2.3.

Similarly, an O–bimodule M *is obtained from an O–bimodule N by attaching cells* if M is obtained as a pushout diagram of the form

(7)
$$
\begin{array}{ccc}
B_O(A) & \xhookrightarrow{\ B_O(i)\ } & B_O(B) \\
{\scriptstyle \widetilde{f}} \downarrow & & \downarrow \\
N & \longrightarrow & M
\end{array}
$$

with i a cofibration in $\text{Coll}(S)$, $f \colon A \to N$ an S–sequence map called the *attaching map* and $\widetilde{f}$ the O–bimodule map induced by f; see Proposition 2.6. In both cases, the map $N \to M$ so defined is a cofibration.

Definition 3.7 Let A, B and C be three topological spaces and $f \colon A \to B$ be a continuous map. The space of continuous maps $g \colon C \to B$ such that $g_{|A} = f$ is denoted by $\text{Top}^f((C, A), B)$.

Lemma 3.8 [19] *Let M and N be two O–infinitesimal bimodules. If M is obtained from N by attaching cells as in* (6), *then one has the homeomorphism*

$$
\text{Ibimod}_O^g((M, N); Y) \cong \text{Coll}(S)^{g \circ f}((B, A); Y),
$$

with f the attaching map and $g \colon N \to Y$ an O–infinitesimal bimodule map.

Similarly, let M and N be two O–bimodules. If M is obtained from N by attaching cells as in (7), then one has the homeomorphism

$$\mathrm{Bimod}_O^g((M, N); Y) \cong \mathrm{Coll}(S)^{g \circ f}((B, A); Y),$$

with f the attaching map and $g\colon N \to Y$ an O–bimodule map.

Definition 3.9 (i) As in [19] (see also [8, Lemma 4.24]), if A and B are O–infinitesimal bimodules (resp. O–bimodules), and A^c is a cofibrant replacement of A, then $\mathrm{Ibimod}_O(A^c; B)$ (resp. $\mathrm{Bimod}_O(A^c; B)$) is independent, up to weak equivalences, of the choice of a cofibrant replacement of A since every O–infinitesimal bimodule (resp. O–bimodule) B is fibrant. This space is called the space of derived O–infinitesimal bimodule (resp. O–bimodule) maps from A to B and is denoted by

$$\mathrm{Ibimod}_O^h(A; B) \quad (\text{resp. } \mathrm{Bimod}_O^h(A; B)).$$

(ii) Similarly, Berger and Moerdijk define a model category structure on the category of S–colored operads in [2], and $\mathrm{Operad}_S^h(A; B)$ denotes the space of derived S–operad maps from A to B.

(iii) If $\mathcal{C}$ is the category $\mathrm{Bimod}_{\mathcal{A}ct_{>0}}$ (resp. $\mathrm{Operad}_{\{o;c\}}$), then for any cofibrant model A of $\mathcal{A}ct_{>0}$, the family A_c gives rise to a cofibrant replacement of $\mathrm{As}_{>0}$ in the category $\mathrm{Bimod}_{\mathrm{As}_{>0}}$ (resp. Operad). As a consequence, the homotopy fiber of the projection onto the closed part is independent (up to weak equivalences) of the choice of a cofibrant model. By abuse of notation, we denote by

$$(8) \qquad p_1^h\colon \mathrm{Bimod}_{\mathcal{A}ct_{>0}}^h(\mathcal{A}ct_{>0}; M) \to \mathrm{Bimod}_{\mathrm{As}_{>0}}^h(\mathrm{As}_{>0}; M_c),$$

$$(9) \qquad p_2^h\colon \mathrm{Operad}_{\{o;c\}}^h(\mathcal{A}ct_{>0}; O) \to \mathrm{Operad}^h(\mathrm{As}_{>0}; O_c),$$

the projections onto the closed part whenever a cofibrant model of $\mathcal{A}ct_{>0}$ is fixed. Furthermore, if the $\mathcal{A}ct_{>0}$–bimodule M and the $\{o; c\}$–operad O are endowed with a map from $\mathcal{A}ct$, then all the spaces and maps are pointed. In this case, define

$$\Omega(\mathrm{Bimod}_{\mathrm{As}_{>0}}^h(\mathrm{As}_{>0}; M_c); \mathrm{Bimod}_{\mathcal{A}ct_{>0}}^h(\mathcal{A}ct_{>0}; M))$$

and

$$\Omega(\mathrm{Operad}^h(\mathrm{As}_{>0}; O_c); \mathrm{Operad}_{\{o;c\}}^h(\mathcal{A}ct_{>0}; O))$$

to be respectively the homotopy fiber of the projection p_1^h and p_2^h. They are called relative loop spaces.

Hence, in order to describe the spaces of derived maps and the relative loop spaces, we need to understand specific cofibrant replacement in the different categories involved. This is the aim of the two following subsections.

3B Cofibrant replacement of $\mathcal{A}ct$ in $\mathrm{Ibimod}_{\mathcal{A}ct_{>0}}$

Proposition 3.10 *A cofibrant replacement of the $\mathcal{A}ct_{>0}$–infinitesimal bimodule $\mathcal{A}ct$ is the $\mathcal{A}ct_{>0}$–infinitesimal bimodule $\blacktriangle$ given by*

$$\blacktriangle(n;c) = \Delta^n \quad \text{for } n \geq 0 \qquad \text{and} \qquad \blacktriangle(n;o) = \Delta^{n-1} \times [0,1] \quad \text{for } n > 0,$$

where the structure is defined, for $1 \leq i \leq n$ (resp. $1 \leq i \leq n-1$) in the first (resp. second) map, by

$$
\begin{aligned}
-\circ^i *_{2;c} &: \blacktriangle(n;c) \to \blacktriangle(n+1;c), & (t_1;\cdots;t_n) & \mapsto (t_1;\cdots;t_i;t_i;\cdots;t_n), \\
-\circ^i *_{2;c} &: \blacktriangle(n;o) \to \blacktriangle(n+1;o), & (t_1;\cdots;t_{n-1}) \times t & \mapsto (t_1;\cdots;t_i;t_i;\cdots;t_{n-1}) \times t, \\
-\circ^n *_{2;o} &: \blacktriangle(n;o) \to \blacktriangle(n+1;o), & (t_1;\cdots;t_{n-1}) \times t & \mapsto (t_1;\cdots;t_{n-1};1) \times t, \\
*_{2;c} \circ_2 - &: \blacktriangle(n;c) \to \blacktriangle(n+1;c), & (t_1;\cdots;t_n) & \mapsto (0;t_1;\cdots;t_n), \\
*_{2;c} \circ_1 - &: \blacktriangle(n;c) \to \blacktriangle(n+1;c), & (t_1;\cdots;t_n) & \mapsto (t_1;\cdots;t_n;1), \\
*_{2;o} \circ_2 - &: \blacktriangle(n;o) \to \blacktriangle(n+1;o), & (t_1;\cdots;t_{n-1}) \times t & \mapsto (0;t_1;\cdots;t_{n-1}) \times t, \\
*_{2;o} \circ_1 - &: \blacktriangle(n;c) \to \blacktriangle(n+1;o), & (t_1;\cdots;t_n) & \mapsto (t_1;\cdots;t_n) \times 1.
\end{aligned}
$$

Here, $(t_1;\cdots;t_m)$ is a nondecreasing sequence; that is, $t_i \leq t_{i+1}$ for $1 \leq i < m$.

Proof Since $\mathcal{A}ct_{>0}$ is generated as a colored operad by $*_{2;c}$ and $*_{2;o}$ with the relations (1) of Definition 1.4, the previous structure makes $\blacktriangle$ into an $\mathcal{A}ct_{>0}$–infinitesimal bimodule. Let $\blacktriangle_N$ be the sub-$\mathcal{A}ct_{>0}$–infinitesimal bimodule of $\blacktriangle$ generated by

$$\{\blacktriangle(n;c)\}_{n=0}^{N} \sqcup \{\blacktriangle(n;o)\}_{n=1}^{N} \quad \text{with } N \in \mathbb{N}.$$

That is, $\blacktriangle_{N-1}(N;c)$ is the boundary of the space $\blacktriangle(N;c)$, whereas $\blacktriangle_{N-1}(N;o)$ is composed of all the faces of dimension $N-1$ in $\blacktriangle(N;o)$ except $\Delta^{N-1} \times \{0\}$. By convention, $\blacktriangle_{-1}$ is the infinitesimal bimodule $\mathrm{Ib}_{\mathcal{A}ct_{>0}}(\varnothing)$, and $\partial\Delta^0 = \varnothing$. The space $\blacktriangle_0$ is obtained from $\blacktriangle_{-1}$ by the attaching cells

$$
\begin{array}{ccc}
\mathrm{Ib}_{\mathcal{A}ct_{>0}}(\partial A) & \longrightarrow & \mathrm{Ib}_{\mathcal{A}ct_{>0}}(A) \\
\downarrow & & \downarrow \\
\blacktriangle_{-1} & \longrightarrow & \blacktriangle_0
\end{array}
$$

with $A(0;c) = \Delta^0$ and the empty set otherwise.

Let B and C be the $\{o;c\}$–sequences given by $B(N;o) = \Delta^{N-1} \times \{0\}$, $C(N;o) = \Delta^{N-1} \times [0,1]$, $C(N;c) = \Delta^N$ and the empty set otherwise. For $N \in \mathbb{N}_{>0}$, the
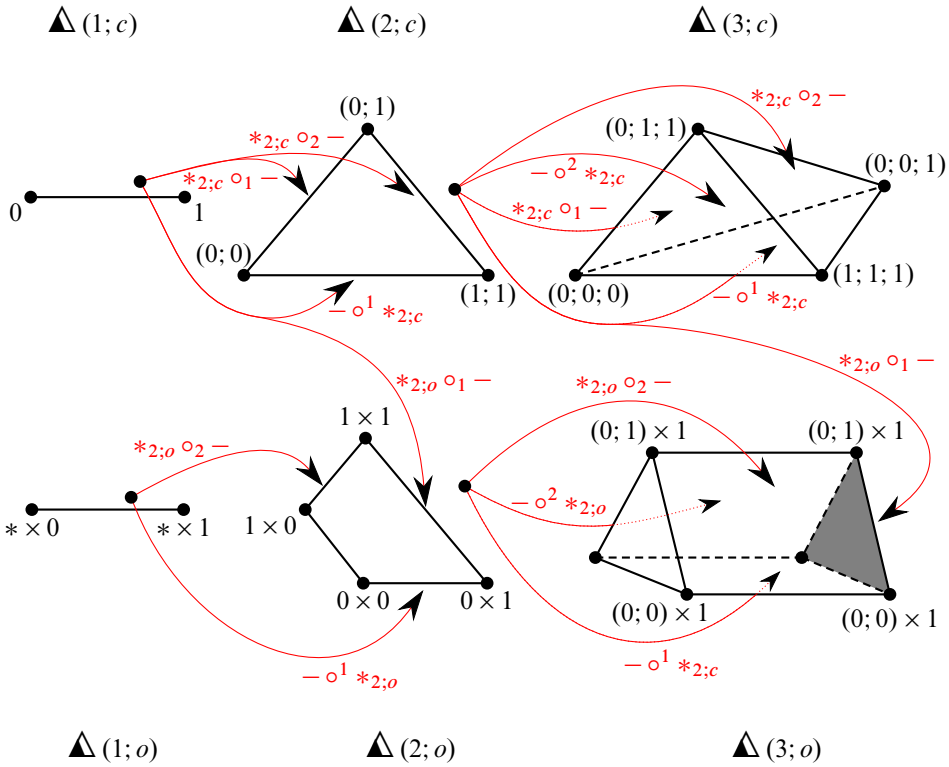
Figure 6: The structure of $\blacktriangle$.

infinitesimal bimodule $\blacktriangle_N$ is obtained from $\blacktriangle_{N-1}$ by the sequence of attaching cells:

$$
\begin{array}{ccc}
\mathrm{Ib}_{\mathcal{A}ct_{>0}}(\partial B) \longrightarrow \mathrm{Ib}_{\mathcal{A}ct_{>0}}(B) & & \mathrm{Ib}_{\mathcal{A}ct_{>0}}(\partial C) \longrightarrow \mathrm{Ib}_{\mathcal{A}ct_{>0}}(C) \\
\downarrow \qquad\qquad \downarrow & \text{and} & \downarrow \qquad\qquad \downarrow \\
\blacktriangle_{N-1} \longrightarrow \Lambda & & \Lambda \longrightarrow \blacktriangle_N
\end{array}
$$

The attaching map $\partial B \to \blacktriangle_{N-1}$ is the restriction to the boundary of the map

$$
i\colon \Delta^{N-1} \to \blacktriangle(N;o), \quad (t_1;\cdots;t_{N-1}) \mapsto (t_1;\cdots;t_{N-1}) \times 0.
$$

This step consists in adding the missing face of dimension $N-1$. The homeomorphisms $\blacktriangle_{N-1}(N;o) \to \partial(\Delta^{N-1} \times [0,1]) \setminus \mathrm{Int}(\Delta^{N-1} \times \{0\})$ and $B(N;o) \to \Delta^{N-1} \times \{0\}$ give rise to a homeomorphism from $\Lambda(N;o)$ to $\partial\blacktriangle(N;o) = \partial C(N;o)$, yielding the right hand side attaching map. For $N \geq n$, $\blacktriangle_N(n;k) = \blacktriangle(n;k)$ with $k \in \{o;c\}$. Consequently, $\lim_N \blacktriangle_N = \blacktriangle$, and $\blacktriangle$ is cofibrant. The weak equivalence between $\blacktriangle$ and $\mathcal{A}ct$ is due to the convexity of $\blacktriangle$ in each degree. $\qquad\square$

Remark 3.11 According to Definition 3.9, the sequence given by $\Delta(n) = \blacktriangle(n; c) = \Delta^n$ inherits an $\mathrm{As}_{>0}$–infinitesimal bimodule structure, and it is a cofibrant replacement of As in the model category $\mathrm{Ibimod}_{\mathrm{As}_{>0}}$; see also [19, Proposition 3.2].

Theorem 3.12 *Let M be an $\mathcal{A}ct_{>0}$–infinitesimal bimodule. One has*

$$\mathrm{Ibimod}^h_{\mathcal{A}ct_{>0}}(\mathcal{A}ct; M) \simeq \mathrm{Ibimod}^h_{\mathrm{As}_{>0}}(\mathrm{As}; M_c) \simeq \mathrm{sTot}(M_c).$$

Proof From Proposition 3.10 and the previous remark, a cofibrant replacement of $\mathcal{A}ct$ in the model category $\mathrm{Ibimod}_{\mathcal{A}ct_{>0}}$ is given by $\blacktriangle$, and a cofibrant replacement of the associative operad As in the model category $\mathrm{Ibimod}_{\mathrm{As}_{>0}}$ is given by Δ. Since M_c is an infinitesimal bimodule over $\mathrm{As}_{>0}$ (see Proposition 1.9), Definition 3.9 induces

$$\mathrm{Ibimod}^h_{\mathrm{As}_{>0}}(\mathrm{As}; M_c) \simeq \mathrm{Ibimod}_{\mathrm{As}_{>0}}(\Delta; M_c)$$

and

$$\mathrm{Ibimod}^h_{\mathcal{A}ct_{>0}}(\mathcal{A}ct; M) \simeq \mathrm{Ibimod}_{\mathcal{A}ct_{>0}}(\blacktriangle; M).$$

Let i be the inclusion defined by

$$i\colon \mathrm{Ibimod}_{\mathrm{As}_{>0}}(\Delta; M_c) \hookrightarrow \mathrm{Ibimod}_{\mathcal{A}ct_{>0}}(\blacktriangle; M),$$

which sends a point $f := \{f_{n;c}\colon \Delta^n \to M(n; c)\}_{n\in\mathbb{N}}$ to the map g defined by

$$g_{n;c}\colon \Delta^n \to M(n; c), \qquad (t_1; \cdots; t_n) \mapsto f_{n;c}(t_1; \cdots; t_n),$$
$$g_{n;o}\colon \Delta^{n-1} \times I \to M(n; o), \quad (t_1; \cdots; t_{n-1}) \times t \mapsto *_{2;o} \circ_1 f_{n-1;c}(t_1; \cdots; t_{n-1}).$$

The space $\mathrm{Ibimod}_{\mathrm{As}_{>0}}(\Delta; M_c)$ is a deformation retract of $\mathrm{Ibimod}_{\mathcal{A}ct_{>0}}(\blacktriangle; M)$ with the homotopy

$$H\colon \mathrm{Ibimod}_{\mathcal{A}ct_{>0}}(\blacktriangle; M) \times [0, 1] \to \mathrm{Ibimod}_{\mathcal{A}ct_{>0}}(\blacktriangle; M)$$

sending a point $(f \times u)$ to the map $H(f; u)$ given by

$$H(f; u)_{n;c}\colon \Delta^n \to M(n; c), \qquad (t_1; \cdots; t_n) \mapsto f_{n;c}(t_1; \cdots; t_n),$$
$$H(f; u)_{n;o}\colon \Delta^{n-1} \times I \to M(n; o), \quad (t_1; \cdots; t_{n-1}) \times t$$
$$\mapsto f_{n;o}\big((t_1; \cdots; t_{n-1}) \times (tu + (1 - u))\big).$$

The map H is continuous, and $H(f; 1) = f$. Furthermore,

$$H(f; 0)_{n;o}\big((t_1; \cdots; t_{n-1}) \times t\big) = f_{n;o}\big((t_1; \cdots; t_{n-1}) \times 1\big)$$
$$= f_{n;o}\big(*_{2;o} \circ_1 (t_1; \cdots; t_{n-1})\big)$$
$$= *_{2;o} \circ_1 f_{n-1;c}(t_1; \cdots; t_{n-1}).$$

So $H(f; 0)$ is in the image of the inclusion map i; also, for all $f \in \mathrm{Ibimod}_{\mathrm{As}_{>0}}(\Delta; M_c)$ and for all $u \in [0, 1]$, we have $H(i(f); u) = i(f)$. Indeed,

$$
\begin{aligned}
\big(H(i(f); u)\big)_{n;o}((t_1; \cdots ; t_{n-1}) \times t) &= i(f)_{n;o}\big((t_1; \cdots ; t_{n-1}) \times (tu + (1-u))\big) \\
&= (*_{2;o} \circ_1 f_{n-1;c})(t_1; \cdots ; t_{n-1}) \\
&= i(f)_{n;o}((t_1; \cdots ; t_{n-1}) \times t). \qquad \square
\end{aligned}
$$

Corollary 3.13 *Let M denote an $\mathcal{A}ct_{>0}$–bimodule such that $M(0; c) \simeq *$ and let $\eta: \mathcal{A}ct \to M$ be a map of $\mathcal{A}ct_{>0}$–bimodules. The following weak equivalences hold:*

$$
\mathrm{Ibimod}^h_{\mathcal{A}ct_{>0}}(\mathcal{A}ct; M) \simeq \mathrm{Ibimod}^h_{\mathrm{As}_{>0}}(\mathrm{As}; M_c) \simeq \Omega\, \mathrm{Bimod}^h_{\mathrm{As}_{>0}}(\mathrm{As}_{>0}; M_c).
$$

Similarly, let O denote an action-operad such that $O(0; c) \simeq O(1; c) \simeq *$. The following weak equivalences hold:

$$
\mathrm{Ibimod}^h_{\mathcal{A}ct_{>0}}(\mathcal{A}ct; O) \simeq \mathrm{Ibimod}^h_{\mathrm{As}_{>0}}(\mathrm{As}; O_c) \simeq \Omega^2\, \mathrm{Operad}^h_{\{o;c\}}(\mathrm{As}_{>0}; O_c).
$$

Proof It is a consequence of Theorem 3.12 together with [19, Theorem 6.2] and [19, Theorem 7.2]. $\qquad \square$

3C Cofibrant replacement of $\mathcal{A}ct_{>0}$ in $\mathbf{Bimod}_{\mathcal{A}ct_{>0}}$

Proposition 3.14 *A cofibrant replacement of the $\mathcal{A}ct_{>0}$–bimodule $\mathcal{A}ct_{>0}$ is the $\mathcal{A}ct_{>0}$–bimodule $\square$ defined, for $n > 0$, by*

$$
\square(n; c) = [0; 1]^{n-1} \quad \text{and} \quad \square(n; o) = [0; 1]^{n-1},
$$

whose bimodule structure is given, for $1 \le i \le n$ (resp. $1 \le i \le n-1$) in the first (resp. second) map, by

$$
\begin{aligned}
-\circ^i *_{2;c}: \square(n; c) &\to \square(n+1; c), & (t_1, \ldots, t_{n-1}) &\mapsto (t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}), \\
-\circ^i *_{2;c}: \square(n; o) &\to \square(n+1; o), & (t_1, \ldots, t_{n-1}) &\mapsto (t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}), \\
-\circ^n *_{2;o}: \square(n; o) &\to \square(n+1; o), & (t_1, \ldots, t_{n-1}) &\mapsto (t_1, \ldots, t_{n-1}, 0),
\end{aligned}
$$

$$
\begin{aligned}
*_{2;c}(-; -): \square(n; c) \times \square(m; c) &\to \square(n+m; c), \\
(t_1, \ldots, t_{n-1}); (t'_1, \ldots, t'_{m-1}) &\mapsto (t_1, \ldots, t_{n-1}, 1, t'_1, \ldots, t'_{m-1}), \\
*_{2;o}(-; -): \square(n; c) \times \square(m; o) &\to \square(n+m; o), \\
(t_1, \ldots, t_{n-1}); (t'_1, \ldots, t'_{m-1}) &\mapsto (t_1, \ldots, t_{n-1}, 1, t'_1, \ldots, t'_{m-1}).
\end{aligned}
$$

Proof Since $\mathcal{A}ct_{>0}$ is generated as a colored operad by $*_{2;c}$ and $*_{2;o}$ with the relations (1) of Definition 1.4, the previous structure induces an $\mathcal{A}ct_{>0}$–bimodule structure on $\square$. For $N > 0$ let $\square_N$ be the sub-$\mathcal{A}ct_{>0}$–bimodule of $\square$ generated

by $\{\square(n;k) \mid n \in \{1, \dots, N\}, k \in \{o;c\}\}$. In other words, the spaces $\square_{N-1}(n;c)$ and $\square_{N-1}(n;o)$ are the $(N-2)$–skeletons of $\square(n;c)$ and $\square(n;o)$, respectively. By convention, $\square_0$ is the $\mathcal{Act}_{>0}$–bimodule $B_{\mathcal{Act}_{>0}}(\varnothing)$. The bimodule $\square_N$ is obtained from $\square_{N-1}$ by the attaching cells

$$
\begin{array}{ccc}
B_{\mathcal{Act}_{>0}}(\partial A) & \longrightarrow & B_{\mathcal{Act}_{>0}}(A) \\
\downarrow & & \downarrow \\
\square_{N-1} & \longrightarrow & \square_N
\end{array}
$$

with A the $\{o;c\}$–sequence defined by $A(N;c) = A(N;o) = [0;1]^{N-1}$ and the empty set otherwise.

For $N \geq n$, $\square_N(n;k) = \square(n;k)$ with $k \in \{o;c\}$. Consequently, $\lim_N \square_N = \square$, and thus $\square$ is cofibrant. The weak equivalence between $\square$ and $\mathcal{Act}_{>0}$ is due to the convexity of $\square$ in each degree. $\square$

Remark 3.15 According to Definition 3.9, the sequence given by $\square_c(n) = \square(n;c)$ inherits an $\text{As}_{>0}$–bimodule structure, and it is a cofibrant replacement of $\text{As}_{>0}$ in the model category $\text{Bimod}_{\text{As}_{>0}}$; see [19, Proposition 4.1].

4 Relative delooping of sTot(M_o)

Let M be an $\mathcal{Act}_{>0}$–bimodule endowed with a map $\eta: \mathcal{Act} \to M$. Since the semicosimplicial space M_o is not a monoid in $(\text{Top}^{\Delta_{\text{inj}}}, \boxtimes)$ (see Proposition 1.12), M_o is not a bimodule over $\text{As}_{>0}$, and we can not expect that its semitotalization has the homotopy type of a loop space. However, we will use the left module structure on M_o to prove that the pair $(\text{sTot}(M_c); \text{sTot}(M_o))$ has the homotopy type of an $\mathcal{SC}_1$–space. The first step consists in showing that $\text{sTot}(M_o)$ is weakly equivalent to the homotopy fiber of the map (8) of Definition 3.9. The next definition gives a model of this homotopy fiber using the cofibrant replacement $\square$ of $\mathcal{Act}_{>0}$.

Definition 4.1 Let $\eta: \mathcal{Act} \to M$ be an $\mathcal{Act}_{>0}$–bimodule map, and let $\square \times I$ be the $\{o;c\}$–sequence defined by

$$(\square \times I)(n;c) = \square(n;c) \times [0, 1] \quad \text{and} \quad (\square \times I)(n;o) = \square(n;o) \times \{1\} \quad \text{for } n > 0.$$

A *relative loop* in M is an $\{o;c\}$–sequence map g from $\square \times I$ to M defined by

$$g_{n;c}: \square(n;c) \times [0,1] \to M(n;c) \quad \text{and} \quad g_{n;o}: \square(n;o) \times \{1\} \to M(n;o) \quad \text{for } n > 0,$$

satisfying

$$g_{n;c}(x \circ^i *_{2;c}; t) = g_{n-1;c}(x; t) \circ^i *_{2;c} \qquad \text{for } x \in \square(n-1; c) \text{ and } 1 \leq i \leq n-1,$$

$$g_{n;c}(*_{2;c}(x; y); t)$$
$$= *_{2;c}(g_{l;c}(x; t); g_{n-l;c}(y; t)) \quad \text{for } x \in \square(l; c) \text{ and } y \in \square(n-l; c),$$

$$g_{n;o}(x \circ^i *_{2;c}; 1) = g_{n-1;o}(x; 1) \circ^i *_{2;c} \qquad \text{for } x \in \square(n-1; o) \text{ and } 1 \leq i \leq n-2,$$

$$g_{n;o}(x \circ^{n-1} *_{2;o}; 1) = g_{n-1;o}(x; 1) \circ^{n-1} *_{2;o} \quad \text{for } x \in \square(n-1; o),$$

$$g_{n;o}(*_{2;o}(x; y); 1)$$
$$= *_{2;o}(g_{l;c}(x; 1); g_{n-l;o}(y; 1)) \quad \text{for } x \in \square(l; c) \text{ and } y \in \square(n-l; o),$$

with the boundary conditions $g_{n;c}(x; 0) = \eta(*_{n;c})$ for $x \in \square(n; c)$. This model for the space of relative loops is denoted by $\Omega(\mathrm{Bimod}_{\mathcal{As}_{>0}}(\square_c; M_c); \mathrm{Bimod}_{\mathcal{Act}_{>0}}(\square; M))$.

Theorem 4.2 *If M is an $\mathcal{Act}_{>0}$–bimodule endowed with a map of $\mathcal{Act}_{>0}$–bimodules $\eta\colon \mathcal{Act} \to M$, then*

$$\mathrm{sTot}(M_o) \simeq \Omega(\mathrm{Bimod}^h_{\mathcal{As}_{>0}}(\mathrm{As}_{>0}; M_c); \mathrm{Bimod}^h_{\mathcal{Act}_{>0}}(\mathcal{Act}_{>0}; M)).$$

Proof It is a consequence of Proposition 4.4 and Proposition 4.5. □

Notation 4.3 Let M be an $\mathcal{Act}_{>0}$–bimodule endowed with a map $\eta\colon \mathcal{Act} \to M$. The $\{o; c\}$–sequence M^* given by

$$M^*(n; c) = \eta(*_{n;c}) \quad \text{for } n \geq 0, \quad M^*(n; o) = M(n; o) \quad \text{for } n > 0,$$

and the empty set otherwise, inherits from M an $\mathcal{Act}_{>0}$–bimodule structure with a map $\eta\colon \mathcal{Act} \to M^*$.

Proposition 4.4 *We have the following weak equivalence:*

$$\mathrm{sTot}(M_o) \simeq \mathrm{Bimod}^h_{\mathcal{Act}_{>0}}(\mathcal{Act}_{>0}; M^*).$$

Proof As seen in the first section, $\mathrm{sTot}(M_o) \simeq \mathrm{Ibimod}^h_{\mathrm{As}_{>0}}(\mathrm{As}; M_o)$ using the structure (2). The first step of the proof consists in building an alternative cofibrant replacement $\tilde{\square}$ of As in the category of infinitesimal bimodules over $\mathrm{As}_{>0}$ such that there exists a map $\xi\colon \mathrm{Bimod}_{\mathcal{Act}_{>0}}(\square; M^*) \to \mathrm{Ibimod}_{\mathrm{As}_{>0}}(\tilde{\square}; M_o)$. Let us recall that a point $g \in \mathrm{Bimod}_{\mathcal{Act}_{>0}}(\square; M^*)$ is described by

$$g_{n;c}\colon \square(n; c) \to M^*(n; c), \quad x \mapsto \eta(*_{n;c}) \quad \text{for } n > 0,$$
$$g_{n;o}\colon \square(n; o) \to M^*(n; o) \qquad\qquad\qquad \text{for } n > 0,$$

satisfying

$$g_{n;o}(x \circ^i *_{2;c}) = g_{n-1;o}(x) \circ^i *_{2;c} \qquad \text{for } x \in \square(n-1;o) \text{ and } i \neq n-1,$$

$$g_{n;o}(x \circ^{n-1} *_{2;o}) = g_{n-1;o}(x) \circ^{n-1} *_{2;o} \qquad \text{for } x \in \square(n-1;o),$$

$$g_{n;o}(*_{2;o}(x; y)) = *_{2;o}(g_{l;c}(x); g_{n-l;o}(y))$$
$$= *_{2;o}(\eta(*_{l;c}); g_{n-l;o}(y)) \quad \text{for } x \in \square(l;c) \text{ and } y \in \square(n-l;o).$$

Define $\sim$ to be the equivalence relation on $[0,1]^n$ generated by

$$(t_1, \ldots, t_n) \sim (t_1', \ldots, t_n') \quad \Longleftrightarrow \quad \text{there exists } i \in \{1, \ldots, n\}$$

such that

$$t_i = t_i' = 1 \quad \text{and} \quad t_j = t_j' \quad \text{for } j > i.$$

We denote by $\widetilde{\square}$ the sequence $\{\widetilde{\square}(n) = [0,1]^n / \sim\}_{n \geq 0}$.

The map g induces a sequence map $\widetilde{g} := \{\widetilde{g}_{n+1} \colon \widetilde{\square}(n) \to M^*(n+1;o) = M_o^n\}_{n \geq 0}$. Indeed, if $(t_1, \ldots, t_n) \sim (t_1', \ldots, t_n')$, then there exists i such that $t_i = t_i' = 1$ and $t_j = t_j'$ for $j > i$. So the following equalities hold:

$$g_{n+1;o}(t_1, \ldots, t_n) = g_{n+1;o}(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n)$$
$$= g_{n+1;o}(*_{2;o}((t_1, \ldots, t_{i-1}); (t_{i+1}, \ldots, t_n)))$$
$$= *_{2;o}(g_{i;c}(t_1, \ldots, t_{i-1}); g_{n-i;o}(t_{i+1}, \ldots, t_n))$$
$$= g_{n+1;o}(t_1', \ldots, t_n').$$

This equivalence relation is the usual way to describe a simplex as a quotient of the cube. So there exists an homeomorphism between $\widetilde{\square}(n)$ and Δ^n. For this work we will just prove that $\widetilde{\square}$ is a cofibrant replacement of As as an $As_{>0}$–infinitesimal bimodule. The infinitesimal bimodule structure is given, for $1 \leq i \leq n$, by

(i) $\quad - \circ^i *_2 \colon \widetilde{\square}(n) \to \widetilde{\square}(n+1), \quad (t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_n),$

(ii) $\quad *_2 \circ_1 - \colon \widetilde{\square}(n) \to \widetilde{\square}(n+1), \quad (t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_n, 0),$

(iii) $\quad *_2 \circ_2 - \colon \widetilde{\square}(n) \to \widetilde{\square}(n+1), \quad (t_1, \ldots, t_n) \mapsto (1, t_1, \ldots, t_n).$

This structure satisfies the infinitesimal bimodule axioms over $As_{>0}$, and it makes $\widetilde{g}$ into an $As_{>0}$–infinitesimal bimodule map. Furthermore, $\widetilde{\square}$ is a cofibrant replacement of the $As_{>0}$–infinitesimal bimodule As, as we will now show.

Cofibrant   Let $\widetilde{\square}_n$ be the sub-$As_{>0}$–infinitesimal bimodule of $\widetilde{\square}$ generated by $\{\widetilde{\square}(i) \mid 0 \leq i \leq n\}$ for $n \in \mathbb{N}$. By convention, $\widetilde{\square}_{-1}$ is the $As_{>0}$–infinitesimal bimodule $Ib_{As_{>0}}(\varnothing)$. Let us notice that the boundary of $\widetilde{\square}(n)$ is determined by $\widetilde{\square}(n-1)$ and

its infinitesimal bimodule structure. Indeed, the map $[0,1]^n \to \tilde{\square}(n)$ preserves the boundary, and by definition, a point in $\partial[0,1]^n$ has one of the following forms:

$$(t_1, \ldots, t_{l-1}, 0, t_{l+1}, \ldots, t_n) \quad \text{or} \quad (t_1, \ldots, t_{l-1}, 1, t_{l+1}, \ldots, t_n).$$

In the first case, the class of such a point lies in $\tilde{\square}_{n-1}$ by the axioms (i) and (ii). In the second case, we have the following identification:

$$[(t_1, \ldots, t_{l-1}, 1, t_{l+1}, \ldots, t_n)] = [(\underbrace{1, \ldots, 1}_{l}, t_{l+1}, \ldots, t_n)]$$

$$= *_2 \circ_2 [(\underbrace{1, \ldots, 1}_{l-1}, t_{l+1}, \ldots, t_n)].$$

Consequently, $\tilde{\square}_n$ is obtained from $\tilde{\square}_{n-1}$ by the pushout diagram

(10)
$$\begin{array}{ccc}
\mathrm{Ib}_{\mathrm{As}_{>0}}(\partial A) & \longrightarrow & \mathrm{Ib}_{\mathrm{As}_{>0}}(A) \\
\tilde{q} \downarrow & & \downarrow \\
\tilde{\square}_{n-1} & \longrightarrow & \tilde{\square}_n
\end{array}$$

where A is the sequence given by $A(n) = [0,1]^n$ and the empty set otherwise. The attaching map is the restriction of the quotient map $q \colon [0,1]^n \to [0,1]^n/\sim$ to the boundary. Moreover, if $n \geq i$, then $\tilde{\square}_n(i) = \tilde{\square}(i)$, and the map $\partial A \to A$ is a cofibration. So $\lim_n \tilde{\square}_n = \tilde{\square}$, and thus $\tilde{\square}$ is cofibrant.

This construction implies that $\tilde{\square}(m)$ is a CW-complex. We recall that if $A(n) = [0,1]^n$ and the empty set otherwise, then the points in $\mathrm{Ib}_{\mathrm{As}_{>0}}(A)(m)$ are the pairs $(t; x)$ with $x \in A(n)$ and t a $\{c\}$–tree satisfying the following conditions:

(11)
$$\begin{cases} t \text{ has } m \text{ leaves,} \\ \text{for all } v \in V(t) \setminus \{p\}, \text{ we have } |v| > 1, \text{ and} \\ |p| = n. \end{cases}$$

We denote by tr_m^n the number of $\{c\}$–trees satisfying the conditions in (11). The space $\tilde{\square}_0(m)$ is the disjoint union of tr_m^0 points, that is, a CW-complex. Assume $\tilde{\square}_{n-1}(m)$ is a CW-complex for all $m \geq 0$. For $m \leq n-1$, we have that $\tilde{\square}_n(m) = \tilde{\square}_{n-1}(m) = \tilde{\square}(m)$ is a CW-complex. The pushout (10) implies that $\tilde{\square}_n(n) = \tilde{\square}(n)$ is a CW-complex. Finally, for $m > n$, the space $\tilde{\square}_n(m)$ is obtained from the CW-complex $\tilde{\square}_{n-1}(m)$ by attaching tr_m^n cells of dimension n according to the infinitesimal bimodule structure over $\mathrm{As}_{>0}$, and is thus a CW-complex.

Contractible The map $q \colon [0,1]^n \to \tilde{\square}(n)$ is a continuous map between compact CW-complexes. Since the fiber of q over a point $(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n)$, with

$t_j \neq 1$ for $j > i$, is homeomorphic to the contractible space $[0, 1]^{i-1}$, the map q is a weak equivalence [18, Main Theorem]. Hence $\widetilde{\Box}(n)$ is contractible.

Since $\widetilde{\Box}$ is a cofibrant replacement of As as an infinitesimal bimodule over $As_{>0}$, the semitotalization $sTot(M_o)$ is weakly equivalent to $Ibimod_{As_{>0}}(\widetilde{\Box}; M_o)$, and we have a map

$$\xi\colon Bimod_{Act_{>0}}(\Box; M^*) \to Ibimod_{As_{>0}}(\widetilde{\Box}; M_o), \quad g \mapsto \widetilde{g}.$$

In order to prove that ξ is a weak equivalence, we will introduce two towers of fibrations. For $k \geq 0$, define A_k and B_k to be the subspaces

$$A_k \subset \underbrace{\prod_{i=1}^{k+1} Top(\Box(i; c); M^*(i; c))}_{\text{reduced to a point}} \times \prod_{i=1}^{k+1} Top(\Box(i; o); M^*(i; o))$$

and

$$B_k \subset \prod_{i=0}^{k} Top(\widetilde{\Box}(i); M_o^i),$$

with A_k satisfying the $Act_{>0}$–bimodule relations and B_k the $As_{>0}$–infinitesimal bimodule relations. In other words, A_k and B_k are the spaces $Bimod_{Act_{>0}}(\Box_{k+1}; M^*)$ and $Ibimod_{As_{>0}}(\widetilde{\Box}_k; M_o)$, respectively, where $\Box_{k+1}$ is the sub-$Act_{>0}$–bimodule introduced in the proof of Proposition 3.14. The projection

$$\prod_{i=1}^{k+1} Top(\widetilde{\Box}(i); M_o^i) \to \prod_{i=1}^{k} Top(\widetilde{\Box}(i); M_o^i)$$

induces a map $B_{k+1} \to B_k$. From Lemma 3.3, the following map is a fibration:

$$Top(\widetilde{\Box}(k+1); M_o^{k+1}) \to Top(\partial\widetilde{\Box}(k+1); M_o^{k+1}).$$

The space B_{k+1} is obtained from B_k by the following pullback diagram:

$$\begin{array}{ccc} B_{k+1} & \longrightarrow & Top(\widetilde{\Box}(k+1); M_o^{k+1}) \\ \downarrow & & \downarrow \\ B_k & \longrightarrow & Top(\partial\widetilde{\Box}(k+1); M_o^{k+1}) \end{array}$$

Since the fibrations are preserved by pullbacks, $B_{k+1} \to B_k$ is a fibration. Similarly, the next pullback square makes the map $A_{k+1} \to A_k$ induced by the projection into

a fibration:

$$
\begin{array}{ccc}
A_{k+1} & \longrightarrow & \mathrm{Top}(\square(k+2;c); M^*(k+2;c)) \times \mathrm{Top}(\square(k+2;o); M^*(k+2;o)) \\
\downarrow & & \downarrow \\
A_k & \longrightarrow & \mathrm{Top}(\partial\square(k+2;c); M^*(k+2;c)) \times \mathrm{Top}(\partial\square(k+2;o); M^*(k+2;o))
\end{array}
$$

So we consider the two towers of fibrations

$$
A_0 \longleftarrow A_1 \longleftarrow \cdots \longleftarrow A_k \longleftarrow A_{k+1} \longleftarrow \cdots \longleftarrow,
$$
$$
B_0 \longleftarrow B_1 \longleftarrow \cdots \longleftarrow B_k \longleftarrow B_{k+1} \longleftarrow \cdots \longleftarrow
$$

such that

$$
A_\infty = \lim_k A_k \simeq \mathrm{holim}_k A_k \simeq \mathrm{Bimod}_{\mathcal{A}ct_{>0}}(\square; M^*),
$$
$$
B_\infty = \lim_k B_k \simeq \mathrm{holim}_k B_k \simeq \mathrm{Ibimod}_{\mathrm{As}_{>0}}(\widetilde{\square}; M_o).
$$

By restriction, the map ξ induces a map between the two towers

$$
\begin{array}{ccccccccc}
A_0 & \longleftarrow & A_1 & \longleftarrow & \cdots & \longleftarrow & A_k & \longleftarrow & A_{k+1} & \longleftarrow & \cdots \\
\downarrow{\scriptstyle\xi_0} & & \downarrow{\scriptstyle\xi_1} & & & & \downarrow{\scriptstyle\xi_k} & & \downarrow{\scriptstyle\xi_{k+1}} & & \\
B_0 & \longleftarrow & B_1 & \longleftarrow & \cdots & \longleftarrow & B_k & \longleftarrow & B_{k+1} & \longleftarrow & \cdots
\end{array}
$$

with $\xi = \lim_k \xi_k = \mathrm{holim}_k \xi_k$. Consequently, ξ is a weak equivalence if each ξ_k is a weak equivalence. We will prove this result by induction on k.

First, note that ξ_0 and ξ_1 coincide with the identity. They are weak equivalences. Now assume that ξ_{k-1} is a weak equivalence. We consider the following diagram, where g is a point in A_{k-1}, F_A is the fiber over g and F_B the fiber over $\xi_{k-1}(g)$:

$$
\begin{array}{ccccc}
A_{k-1} & \longleftarrow & A_k & \longleftarrow & F_A \\
\downarrow{\scriptstyle\xi_{k-1}} & & \downarrow{\scriptstyle\xi_k} & & \downarrow{\scriptstyle\xi_g} \\
B_{k-1} & \longleftarrow & B_k & \longleftarrow & F_B
\end{array}
$$

Since the two left horizontal arrows are fibrations, the map ξ_k is a weak equivalence if the induced map ξ_g is a weak equivalence.

The fiber F_A is homeomorphic to the space $\mathrm{Top}^{g_{k+1};o}\big(([0,1]^k; \partial[0,1]^k); M(k+1;o)\big)$ by Lemma 3.8. Similarly, $\widetilde{\square}_k$ is obtained from $\widetilde{\square}_{k-1}$ by the pushout diagram (10). So the fiber F_B is homeomorphic to $\mathrm{Top}^{\xi_{k-1}(g)_k \circ q}\big(([0,1]^k; \partial[0,1]^k); M(k+1;o)\big)$, and we have the following commutative square:

$$
\begin{array}{ccc}
F_A & \longrightarrow & \mathrm{Top}^{g_{k+1};o}\big(([0,1]^k; \partial[0,1]^k); M(k+1;o)\big) \\
{\scriptstyle\xi_g}\downarrow & & \Big\| {\scriptstyle\mathrm{id}} \\
F_B & \longrightarrow & \mathrm{Top}^{\xi_{k-1}(g)_k \circ q}\big(([0,1]^k; \partial[0,1]^k); M(k+1;o)\big)
\end{array}
$$

Consequently, ξ_k is a weak equivalence. $\qquad\square$

Proposition 4.5 *The space* $\Omega(\mathrm{Bimod}^h_{\mathrm{As}_{>0}}(\mathrm{As}_{>0}; M_c)); \mathrm{Bimod}^h_{\mathcal{A}ct_{>0}}(\mathcal{A}ct_{>0}; M))$ *is weakly equivalent to the space* $\mathrm{Bimod}^h_{\mathcal{A}ct_{>0}}(\mathcal{A}ct_{>0}; M^*)$.

Proof In this proof, $\square$ will serve as a cofibrant model of the $\mathcal{A}ct_{>0}$–bimodule $\mathcal{A}ct_{>0}$. Through the following inclusion, we can consider $\mathrm{Bimod}_{\mathcal{A}ct_{>0}}(\square; M^*)$ as a subspace of $\Omega(\mathrm{Bimod}_{\mathrm{As}_{>0}}(\square_c; M_c)); \mathrm{Bimod}_{\mathcal{A}ct_{>0}}(\square; M))$:

$$i\colon \mathrm{Bimod}_{\mathcal{A}ct_{>0}}(\square; M^*) \to \Omega(\mathrm{Bimod}_{\mathrm{As}_{>0}}(\square_c; M_c)); \mathrm{Bimod}_{\mathcal{A}ct_{>0}}(\square; M)),$$

$$g \mapsto \begin{cases} \widetilde{g}_{n;c}\colon \square(n;c) \times [0,1] \to M(n;c); (x;t) \mapsto \eta(*_{n;c}), \\ \widetilde{g}_{n;o}\colon \square(n;o) \times \{1\} \to M(n;o); (x;1) \mapsto g_{n;o}(x). \end{cases}$$

In order to show that i is a weak equivalence, we introduce two towers of fibrations. One of them is the tower A_k of Proposition 4.4. The second one is defined by

$$C_k \subset \prod_{i=1}^{k+1} \mathrm{Top}(\square(i;c) \times [0,1]; M(i;c)) \times \prod_{i=1}^{k+1} \mathrm{Top}(\square(i;o); M(i;o)),$$

satisfying the relations of Definition 4.1. The map $C_{k+1} \to C_k$ induced by the projection is a fibration due to Lemma 3.3 and the following pullback diagram:

$$
\begin{array}{ccc}
C_{k+1} & \longrightarrow & \mathrm{Top}\big(\square(k+2;c) \times [0,1]; M(k+2;c)\big) \times \\
 & & \mathrm{Top}\big(\square(k+2;o); M(k+2;o)\big) \\
\downarrow & & \downarrow \\
C_k & \longrightarrow & \mathrm{Top}\big(\partial'(\square(k+2;c) \times [0,1]); M(k+2;c)\big) \times \\
 & & \mathrm{Top}\big(\partial\square(k+2;o); M(k+2;o)\big)
\end{array}
$$

Here,

$$\partial'(\square(k+2;c) \times [0,1]) = \square(k+2;c) \times \{0\} \cup \partial\square(k+2;c) \times [0,1].$$

The restriction of the inclusion i induces a map between the two towers:

$$
\begin{array}{ccccccccc}
A_0 & \longleftarrow & A_1 & \longleftarrow & \cdots & \longleftarrow & A_k & \longleftarrow & A_{k+1} & \longleftarrow & \cdots \\
\downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_1} & & & & \downarrow{\scriptstyle i_k} & & \downarrow{\scriptstyle i_{k+1}} & & \\
C_0 & \longleftarrow & C_1 & \longleftarrow & \cdots & \longleftarrow & C_k & \longleftarrow & C_{k+1} & \longleftarrow & \cdots
\end{array}
$$

We will prove that i is a weak equivalence by induction on k. If $k = 0$, a point in C_0 is a pair $(g_{1;c}; g_{1;o})$, and the points in the image of i_0 are the pairs satisfying

$$g_{1;c} \colon \square(1;c) \times [0,1] \to M(1;c), \quad (*;t) \mapsto \eta(*_{1;c}).$$

Since $g_{1;c}(*;0) = \eta(*_{1;c})$ for any pair in C_0, the inclusion i_0 induces the following deformation retract:

$$H \colon C_0 \times [0,1] \to C_0, \quad y = ((g_{1;c}; g_{1;o}); t_1) \mapsto \begin{cases} H(y)_{1;c}(*;t) = g_{1;c}(*; t(1-t_1)), \\ H(y)_{1;o}(*;1) = g_{1;o}(*;1). \end{cases}$$

From now on, we assume that i_{k-1} is a weak equivalence for $k \geq 1$. We consider the following diagram, where g is a point in A_{k-1}, F_A is the fiber over g and F_C the fiber over $i_{k-1}(g)$:

$$
\begin{array}{ccccc}
A_{k-1} & \longleftarrow & A_k & \longleftarrow & F_A \\
\downarrow{\scriptstyle i_{k-1}} & & \downarrow{\scriptstyle i_k} & & \downarrow{\scriptstyle i_g} \\
C_{k-1} & \longleftarrow & C_k & \longleftarrow & F_C
\end{array}
$$

Since the two left horizontal arrows are fibrations, the map i_k is a weak equivalence if the induced map i_g is a weak equivalence.

A point in F_C is defined by a pair $(g_{k+1;c}; g_{k+1;o})$ satisfying the relations seen in Definition 4.1. Since $g_{k+1;c}$ is in the fiber over $i_{k-1}(g)$, the map sends all the faces of $\square(k+1;c) \times [0,1]$ to $\eta(*_{k+1;c})$, except for the face $\square(k+1;c) \times \{1\}$. Furthermore, they are no interaction between $g_{k+1;c}$ and $g_{k+1;o}$.

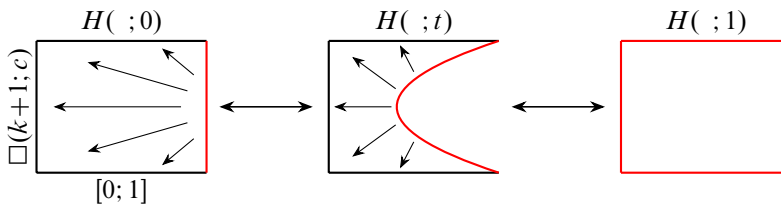On the other hand, the points in the image of i_g coincide with the pair $(g_{k+1;c}; g_{k+1;o})$ such that

$$g_{k+1;c} \colon \square(k+1;c) \times [0,1] \to M(k+1;c), \quad (x;t) \mapsto \eta(*_{k+1;c}).$$

In order to prove that i_g induces a deformation retract, we introduce the homotopy (also described in [11, Proposition 0.16])

$$H \colon \big(\square(k+1;c) \times [0,1]\big) \times [0,1] \to \square(k+1;c) \times [0,1]$$

illustrated by the following picture:

In other words, the points in the image of i_g coincide with the pairs such that

$$g_{k+1;c}(x;t) = g_{k+1;c}\big(H((x;t);1)\big) \quad \text{for } x \in \square(k+1;c) \text{ and } t \in [0,1].$$

Finally, the deformation retract $H_2\colon F_C \times [0,1] \to F_C$ is given by

$$y = ((g_{k+1;c}; g_{k+1;o}); t_1)$$
$$\mapsto \begin{cases} H_2(y)_{k+1;c}(x;t) = g_{k+1;c}\big(H((x;t);t_1)\big) & \text{for } x \in \square(k+1;c) \text{ and } t \in [0,1], \\ H_2(y)_{k+1;o}(x;1) = g_{k+1;o}(x;1) & \text{for } x \in \square(k+1;o). \end{cases}$$

Thus, the space $\Omega(\mathrm{Bimod}_{\mathrm{As}_{>0}}(\square_c; M_c); \mathrm{Bimod}_{\mathcal{A}ct_{>0}}(\square; M))$ is weakly equivalent to $\mathrm{Bimod}_{\mathcal{A}ct_{>0}}(\square; M^*)$. $\qquad\square$

5 Double relative delooping: a particular case

First of all we recall that, for any pointed continuous map $f\colon A \to X$, the homotopy fiber $\mathrm{hofib}(f)$ and the loop space ΩX based at $*$ are weakly equivalent to the pullback diagrams (I) and (II):

$$
\begin{array}{ccc}
& \mathrm{Top}\big([0,1];X\big) & \\
{\scriptstyle (ev_0;ev_1)}\Big\downarrow & & (\mathrm{I}) \\
*\times A \xrightarrow[\mathrm{id}\times f]{} X \times X &
\end{array}
\qquad
\begin{array}{ccc}
& \mathrm{Top}\big([0,1];X\big) & \\
{\scriptstyle (ev_0;ev_1)}\Big\downarrow & & (\mathrm{II}) \\
*\times * \longrightarrow X \times X &
\end{array}
$$

By the double relative loop space $\Omega^2(X; A)$, we mean the loop space of the homotopy fiber $\mathrm{hofib}(f)$. Since finite colimits commute, the double loop space can also be defined by the homotopy fiber of the continuous map Ωf.

From now on, let O be a multiplicative operad; that is, there exists an operad map $\alpha\colon \mathrm{As} \to O$. Let B be an O–bimodule equipped with an O–bimodule map $\beta\colon O \to B$. If we assume that $B(0) \simeq *$, we know from [19, Theorem 6.2] and the $\mathrm{As}_{>0}$–bimodule map $\beta \circ \alpha\colon \mathrm{As} \to B$ that $\mathrm{sTot}(B)$ is weakly equivalent to the loop space $\Omega\,\mathrm{Bimod}^h_{\mathrm{As}_{>0}}(\mathrm{As}_{>0}; B)$. Since B is not an operad, we can not expect that its semitotalization has the homotopy type of a double loop space. However, we will prove that $\mathrm{Bimod}^h_{\mathrm{As}_{>0}}(\mathrm{As}_{>0}; B)$ has the homotopy type of a relative loop space by building an $\{o; c\}$–operad X from the pair $(O; B)$ as follows:

$$(12) \qquad X(n; c) = O(n) \quad \text{for } n \geq 0, \quad X(n; o) = B(n-1) \quad \text{for } n > 0,$$

and the empty set otherwise. The operadic structure is defined by

$$\circ_i \colon X(n;c) \times X(m;c) \to X(n+m-1;c), \quad (x;y) \mapsto x \circ_i y,$$
$$\circ_i \colon X(n;o) \times X(m;c) \to X(n+m-1;o), \quad (x;y) \mapsto x \circ^i y,$$
$$\circ_n \colon X(n;o) \times X(m;o) \to X(n+m-1;o), \quad (x;y) \mapsto \alpha(*_2)(x;y),$$

using the operadic structure of O, the right O–bimodule structure of B, and the left O–bimodule structure of B, respectively. The operadic axioms are satisfied except the unit axiom. This axiom holds under the assumption

$$(13) \qquad \alpha(*_2)\big(\beta \circ \alpha(*_0); x\big) = \alpha(*_2)\big(x; \beta \circ \alpha(*_0)\big) = x \quad \text{for } x \in X(m;o).$$

Under this assumption, X is an action-operad with

$$\eta \colon \mathcal{A}ct \to X, \qquad \begin{aligned} \eta(*_{i;c}) &= \alpha(*_i), \\ \eta(*_{i;o}) &= \beta \circ \alpha(*_{i-1}). \end{aligned}$$

Theorem 5.1 *Under assumption* (13), $\mathrm{Bimod}^h_{\mathrm{As}_{>0}}(\mathrm{As}_{>0}; B)$ *is weakly equivalent to the relative loop space* $\Omega(\mathrm{Operad}^h(\mathrm{As}_{>0}; O); \mathrm{Operad}^h_{\{o;c\}}(\mathcal{A}ct_{>0}; X))$.

Proof It is a consequence of Propositions 5.5 and 5.6. □

Definition 5.2 In order to describe the homotopy fiber the map (9) of Definition 3.9, we need a cofibrant replacement of $\mathcal{A}ct_{>0}$ as a colored operad. Since $\mathcal{A}ct_{>0}$ is cofibrant as an $\{o;c\}$–sequence, we know from [3] that the Boardman–Vogt resolution of $\mathcal{A}ct_{>0}$, denoted by $BV(\mathcal{A}ct_{>0})$ or just $\mathcal{WA}$ in our case, is the object we are looking for. We recall the construction here.

- Let tree_n^o be the subset of $\{o;c\}$–trees consisting of trees (t, f) with n leaves, where f is an $\{o;c\}$–labeling of t with the trunk labeled by o and satisfying, for all $v \in V(t)$,

$$f(e_0(v)) = c \implies f(e_i(v)) = c \text{ for all } i \in \{1, \dots, |v|\}$$
$$f(e_0(v)) = o \implies |v| > 0, f(e_{|v|}(v)) = o \text{ and } f(e_i(v)) = c \text{ for } i \in \{1, \dots, |v|-1\}.$$

- The operad $\mathcal{WA}$ is the $\{o;c\}$–sequence given by

$$\mathcal{WA}(n;c) := \coprod_{t \in \{c\}\text{–tree}} \prod_{v \in V(t)} \mathcal{A}ct_{>0}\big(f(e_1(v)), \dots, f(e_{|v|}(v)); f(e_0(v))\big) \times \prod_{e \in E^{\mathrm{int}}(t)} [0,1] \Big/ \sim,$$

$$\mathcal{WA}(n;o) := \coprod_{t \in \mathrm{tree}_n^o} \prod_{v \in V(t)} \mathcal{A}ct_{>0}\big(f(e_1(v)), \dots, f(e_{|v|}(v)); f(e_0(v))\big) \times \prod_{e \in E^{\mathrm{int}}(t)} [0,1] \Big/ \sim,$$

and the empty set otherwise. The equivalence relation $\sim$ is generated by contracting the inner edges indexed by 0, using the operadic structure of $\mathcal{A}ct_{>0}$ and the following relation:

A point in $\mathcal{WA}(n;o)$ will be denoted by $[T;\{t_e\}]$, where T is an element in tree_n^o such that each vertex has at least two inputs and $t_e \in [0,1]$ for each inner edge $e \in E^{\mathrm{int}}(T)$. Similarly, a point in $\mathcal{WA}(n;c)$ is denoted by $[T;\{t_e\}]$ with T an element in $\{c\}$–tree. We will use the notation $v_1 < v_2$ if v_1 and v_2 are two distinct, connected vertices such that $d(v_1;r) < d(v_2;r)$.

The operadic composition $\circ_i$ of two points $[T;\{t_e\}]$ and $[T';\{t'_e\}]$ consists of grafting the tree T' to the ith leaf of T and labeling the new inner edge by 1.

- It is well known that the operad $\bigcirc := \{\bigcirc(n) = \mathcal{WA}(n;c)\}_{n>0}$ is a cofibrant replacement of $\mathrm{As}_{>0}$ as an operad. It is usually called *the Stasheff operad*.

The operad $\mathcal{WA}$ has been introduced in [13] so as to recognize A_∞–spaces and A_∞–maps. The next definition is a description, using the cofibrant replacement $\mathcal{WA}$, of the relative loop space $\Omega(\mathrm{Operad}^h(\mathrm{As}_{>0};O);\mathrm{Operad}^h_{\{o;c\}}(\mathcal{A}ct_{>0};X))$.

Definition 5.3 Define $\Omega(\mathrm{Operad}(\bigcirc;X_c);\mathrm{Operad}_{\{o;c\}}(\mathcal{WA};X))$ as the space of maps given, for $n > 0$, by

$$g_{n;c}\colon \mathcal{WA}(n;c) \times [0,1] \to X(n;c) \quad \text{and} \quad g_{n;o}\colon \mathcal{WA}(n;o) \times \{1\} \to X(n;o),$$

satisfying the relations

$$g_{n;c}(x \circ_i y; t) = g_{l+1;c}(x;t) \circ_i g_{n-l;c}(y;t)$$
$$\text{for } x \in \mathcal{WA}(l+1;c),\ y \in \mathcal{WA}(n-l;c) \text{ and } i \in \{1,\dots,l+1\},$$
$$g_{n;o}(x \circ_i y; 1) = g_{l+1;o}(x;1) \circ_i g_{n-l;c}(y;1)$$
$$\text{for } x \in \mathcal{WA}(l+1;o),\ y \in \mathcal{WA}(n-l;c) \text{ and } i \in \{1,\dots,l\},$$
$$g_{n;o}(x \circ_{l+1} y; 1) = g_{l+1;o}(x;1) \circ_{l+1} g_{n-l;o}(y;1)$$
$$\text{for } x \in \mathcal{WA}(l+1;o) \text{ and } y \in \mathcal{WA}(n-l;o),$$

and the boundary condition $g_{n;c}(x;0) = \eta(*_{n;c})$ for $x \in \mathcal{WA}(n;c)$.

Notation 5.4 Let $\alpha\colon \mathrm{As}_{>0} \to O$ be a map of operads, $\beta\colon O \to B$ be a map of O–bimodules and $\eta\colon \mathcal{A}ct_{>0} \to X$ the corresponding map of $\{o;c\}$–operads. The $\{o;c\}$–sequence X^* given by

$$X^*(n;c) = \eta(*_{n;c}) \quad \text{for } n \geq 0, \qquad X^*(n;o) = B(n-1) \quad \text{for } n > 0,$$

and the empty set otherwise, inherits from X an $\{o; c\}$–operadic structure endowed with a map $\eta\colon \mathcal{A}ct_{>0} \to X^*$.

Proposition 5.5 *Under assumption* (13), *the space* $\mathrm{Bimod}^h_{\mathrm{As}_{>0}}(\mathrm{As}_{>0}; B)$ *is weakly equivalent to* $\mathrm{Operad}^h_{\{o;c\}}(\mathcal{A}ct_{>0}; X^*)$.

Proof By assumption, B is an $\mathrm{As}_{>0}$–bimodule. The first step of the proof consists of building a cofibrant replacement $\widetilde{\bigcirc}$ of $\mathrm{As}_{>0}$ as an $\mathrm{As}_{>0}$–bimodule such that there exists a map $\xi\colon \mathrm{Operad}_{\{o;c\}}(\mathcal{WA}; X^*) \to \mathrm{Bimod}_{\mathrm{As}_{>0}}(\widetilde{\bigcirc}; B)$. Let us recall that a point $g \in \mathrm{Operad}(\mathcal{WA}; X^*)$ is described by

$$g_{n;c}\colon \mathcal{WA}(n; c) \to X^*(n; c), \quad x \mapsto \eta(*_{n;c}) \quad \text{for } n > 0,$$
$$g_{n;o}\colon \mathcal{WA}(n; o) \to X^*(n; o) \qquad\qquad\quad \text{for } n > 0,$$

satisfying, notably for $x \in \mathcal{WA}(l+1; o)$, $y \in \mathcal{WA}(n-l; c)$ and $1 \leq i \leq l$, the relation

(14) $\qquad g_{n;o}(x \circ_i y) = g_{l+1;o}(x) \circ_i g_{n-l;c}(y) = g_{l+1;o}(x) \circ_i \eta(*_{n-l;c})$.

Define $\approx$ to be the equivalence relation on $\mathcal{WA}(n; o)$ generated by

$$[T; \{t_e\}] \approx [T; \{l_e\}] \Longleftrightarrow \begin{cases} t_e = l_e & \forall e \in E^{\mathrm{int}}(T) \text{ with } f(e) = o, \text{ and} \\ t_e = l_e & \text{if } \nexists e_1 < e \text{ such that } t_{e_1} = l_{e_1} = 1 \text{ and } f(e_1) = c. \end{cases}$$

We will denote by $\widetilde{\bigcirc}$ the sequence $\{\widetilde{\bigcirc}(n) = \mathcal{WA}(n+1; o)/\approx\}_{n>0}$. By convention, $\widetilde{\bigcirc}(0)$ is the empty set.

Due to relation (14), the map g induces a sequence map $\widetilde{g} := \{\widetilde{g}_n\colon \widetilde{\bigcirc}(n) \to B(n)\}_{n>0}$.

Let us prove that $\widetilde{\bigcirc}$ is a cofibrant replacement of $\mathrm{As}_{>0}$ as an $\mathrm{As}_{>0}$–bimodule. The bimodule structure is given, for $1 \leq i \leq n$, by

(i) $\quad -\circ^i *_2\colon \widetilde{\bigcirc}(n) \to \widetilde{\bigcirc}(n+1)$, $\quad [T; \{t_e\}] \mapsto [T; \{t_e\}] \circ_i \delta_{2;c}$,

(ii) $\quad *_2(-; -)\colon \widetilde{\bigcirc}(n) \times \widetilde{\bigcirc}(m) \to \widetilde{\bigcirc}(n+m)$, $\quad ([T_1; \{t_e\}]; [T_2; \{l_e\}])$
$$\mapsto [T_1; \{t_e\}] \circ_{n+1} [T_2; \{l_e\}],$$

where $\delta_{n;c}$ is the n–corolla in $\{c\}$–trees and $\delta_{n;o}$ is the n–corolla in tree^o_n. This structure satisfies the bimodule axioms over $\mathrm{As}_{>0}$, and it makes $\widetilde{f}$ into an $\mathrm{As}_{>0}$–bimodule map. Furthermore $\widetilde{\bigcirc}$ is a cofibrant replacement.

Cofibrant Let $\widetilde{\bigcirc}_n$ be the $\mathrm{As}_{>0}$–bimodule generated by $\{\widetilde{\bigcirc}(i)\}_{i=1}^n$ for $n > 0$. By convention, $\widetilde{\bigcirc}_0$ is the $\mathrm{As}_{>0}$–bimodule $B_{\mathrm{As}_{>0}}(\varnothing)$. Let us notice that the map $\mathcal{WA}(n+1; o) \to \widetilde{\bigcirc}(n)$ preserves the boundary, and a point in $\partial \mathcal{WA}(n+1; o)$, by definition, has the form $[T; \{t_e\}]$ such that there exists $e_1 \in E^{\mathrm{int}}(T)$ with $t_{e_1} = 1$ and either $f(e_1) = o$ or $f(e_1) = c$. If $f(e_1) = o$, then $[T; \{t_e\}]$ has a decomposition

$[T_1; \{t_e^1\}] \circ_{|T_1|} [T_2; \{t_e^2\}]$. The image lies in $\widetilde{\bigcirc}_{n-1}$ by axiom (ii). If $f(e_1) = c$, then $[T; \{t_e\}]$ has a similar decomposition $[T_1; \{t_e^1\}] \circ_i [T_2; \{t_e^2\}]$ with $i < |T_1|$ and $|T_2| > 1$, and we have the identification

$$[T; \{t_e\}] = [(T_1; \{t_e^1\}) \circ_i (T_2; \{t_e^2\})] = [(T_1; \{t_e^1\}) \circ_i \delta_{|T_2|;c}] = [(T_1; \{t_e^1\}) \circ_i \delta_{|T_2|-1;c}]\circ^i *_2.$$

Hence $[T; \{t_e\}]$ lies in $\widetilde{\bigcirc}_{n-1}$. Consequently, $\widetilde{\bigcirc}_n$ is obtained from $\widetilde{\bigcirc}_{n-1}$ by the pushout diagram

(15)
$$
\begin{array}{ccc}
B_{As_{>0}}(\partial A) & \longrightarrow & B_{As_{>0}}(A) \\
\tilde{q} \downarrow & & \downarrow \\
\widetilde{\bigcirc}_{n-1} & \longrightarrow & \widetilde{\bigcirc}_n
\end{array}
$$

where A is the sequence given by $A(n) = \mathcal{WA}(n+1; o)$ and the empty set otherwise. The attaching map is the restriction of the quotient map $q: \mathcal{WA}(n+1; o) \to \widetilde{\bigcirc}(n)$ to the boundary. Furthermore, if $i \geq n$, then $\widetilde{\bigcirc}_i(n) = \widetilde{\bigcirc}(n)$, and the map $\partial A \to A$ is a cofibration. So $\lim_i \widetilde{\bigcirc}_i = \widetilde{\bigcirc}$, and thus $\widetilde{\bigcirc}$ is cofibrant. Like in the proof of Proposition 4.4, these sequences of pushout diagram imply that, for each n, the space $\widetilde{\bigcirc}(n)$ is a CW-complex.

Contractible The map $q: \mathcal{WA}(n+1; o) \to \widetilde{\bigcirc}(n)$ is a continuous map between compact CW-complexes. Since the fiber of q over a point is homeomorphic to a product of polytopes that is contractible, the map q is a weak equivalence; see [18, Main Theorem]. Hence $\widetilde{\bigcirc}(n)$ is contractible for $n > 0$.

Since $\widetilde{\bigcirc}$ is a cofibrant replacement of $As_{>0}$ as a bimodule over itself, the space $\mathrm{Bimod}^h_{As_{>0}}(As_{>0}; B)$ is weakly equivalent to $\mathrm{Bimod}_{As_{>0}}(\widetilde{\bigcirc}; B)$, and the assignment $\xi(g) = \tilde{g}$ defines a map

$$\xi: \mathrm{Operad}_{\{o;c\}}(\mathcal{WA}; X^*) \to \mathrm{Bimod}_{As_{>0}}(\widetilde{\bigcirc}; B), \quad g \mapsto \tilde{g}.$$

In order to prove that ξ is a weak equivalence, we introduce two towers of fibrations. Define A'_k and B'_k to be the subspaces

$$A'_k \subset \underbrace{\prod_{i=1}^{k+1} \mathrm{Top}(\mathcal{WA}(i;c); X^*(i;c))}_{\text{reduced to a point}} \times \prod_{i=1}^{k+1} \mathrm{Top}(\mathcal{WA}(i;o); X^*(i;o))$$

and

$$B'_k \subset \prod_{i=1}^{k} \mathrm{Top}(\widetilde{\bigcirc}(i); B(i)),$$

with A_k' satisfying the operadic relations and B_k' the $As_{>0}$–bimodule relations for $k > 0$. In other words, A_k' and B_k' are respectively the space $\text{Operad}(\mathcal{WA}_{k+1}; X^*)$ and $\text{Bimod}_{As_{>0}}(\widetilde{\bigcirc}_k; B)$, where $\mathcal{WA}_{k+1}$ is the suboperad of $\mathcal{WA}$ generated by $\{\mathcal{WA}(i; c) \mid 1 \leq i \leq k + 1\}$ and $\{\mathcal{WA}(i; o) \mid 1 \leq i \leq k + 1\}$. Since $\mathcal{WA}(1; c)$ and $\mathcal{WA}(1; o)$ are reduced to the unit, the factors $\text{Top}\big(\mathcal{WA}(1; c); X^*(1; c)\big)$ and $\text{Top}\big(\mathcal{WA}(1; o); X^*(1; o)\big)$ are one point spaces and can be ignored. So we consider the two towers

$$A_1' \longleftarrow A_2' \longleftarrow \cdots \longleftarrow A_k' \longleftarrow A_{k+1}' \longleftarrow \cdots,$$
$$B_1' \longleftarrow B_2' \longleftarrow \cdots \longleftarrow B_k' \longleftarrow B_{k+1}' \longleftarrow \cdots$$

such that

$$A_\infty' = \lim_k A_k' \simeq \operatorname*{holim}_k A_k' \simeq \text{Operad}_{\{o; c\}}(\mathcal{WA}; X^*),$$
$$B_\infty' = \lim_k B_k' \simeq \operatorname*{holim}_k B_k' \simeq \text{Bimod}_{As_{>0}}(\widetilde{\bigcirc}; B).$$

By restriction, the map ξ induces a map between the two towers

$$
\begin{array}{ccccccccc}
A_1' & \longleftarrow & A_2' & \longleftarrow & \cdots & \longleftarrow & A_k' & \longleftarrow & A_{k+1}' & \longleftarrow & \cdots \\
\downarrow{\scriptstyle \xi_1} & & \downarrow{\scriptstyle \xi_2} & & & & \downarrow{\scriptstyle \xi_k} & & \downarrow{\scriptstyle \xi_{k+1}} & & \\
B_1' & \longleftarrow & B_2' & \longleftarrow & \cdots & \longleftarrow & B_k' & \longleftarrow & B_{k+1}' & \longleftarrow & \cdots
\end{array}
$$

with $\xi = \lim_k \xi_k = \operatorname{holim}_k \xi_k$. We will prove that ξ is a weak equivalence if each ξ_k is a weak equivalence by induction on k. First, note that ξ_1 coincides with the identity. It is a weak equivalence. Next, assume that ξ_{k-1} is a weak equivalence. We consider the following diagram, where g is a point in A_{k-1}', $F_{A'}$ is the fiber over g and $F_{B'}$ the fiber over $\xi_{k-1}(g)$:

$$
\begin{array}{ccccc}
A_{k-1}' & \longleftarrow & A_k' & \longleftarrow & F_{A'} \\
\downarrow{\scriptstyle \xi_{k-1}} & & \downarrow{\scriptstyle \xi_k} & & \downarrow{\scriptstyle \xi_g} \\
B_{k-1}' & \longleftarrow & B_k' & \longleftarrow & F_{B'}
\end{array}
$$

Since the two left horizontal arrows are fibrations, the map ξ_k is a weak equivalence if the induced map ξ_g is a weak equivalence.

From Lemma 3.8, the space $\text{Top}^{g_{k+1; o}}\big((\mathcal{WA}(k + 1; o); \partial\mathcal{WA}(k + 1; o)); B(k)\big)$ is homeomorphic to the fiber $F_{A'}$. Similarly, $\widetilde{\bigcirc}_k$ is obtained from $\widetilde{\bigcirc}_{k-1}$ by the pushout diagram (15). So the space $\text{Top}^{\xi_{k-1}(g)_k \circ q}\big((\mathcal{WA}(k + 1; o); \partial\mathcal{WA}(k + 1; o)); B(k)\big)$ is

homeomorphic to the fiber $F_{B'}$, and we have the following commutative square:

$$
\begin{array}{ccc}
F_{A'} & \longrightarrow & \mathrm{Top}^{g_{k+1;o}}\big((\mathcal{WA}(k+1;o);\partial\mathcal{WA}(k+1;o));B(k)\big) \\
\downarrow{\scriptstyle \xi_g} & & \parallel{\scriptstyle \mathrm{id}} \\
F_{B'} & \longrightarrow & \mathrm{Top}^{\xi_{k-1}(g)_k\circ q}\big((\mathcal{WA}(k+1;o);\partial\mathcal{WA}(k+1;o));B(k)\big)
\end{array}
$$

Consequently, ξ_k is a weak equivalence. $\qquad\square$

Proposition 5.6 *Under assumption (13), the space* $\mathrm{Operad}^{h}_{\{o;c\}}(\mathcal{A}ct_{>0};X^*)$ *is weakly equivalent to the relative loop space* $\Omega(\mathrm{Operad}^{h}(\mathrm{As}_{>0};X_c);\mathrm{Operad}^{h}_{\{o;c\}}(\mathcal{A}ct_{>0};X))$.

Proof Using the following inclusion, we can consider $\mathrm{Operad}_{\{o;c\}}(\mathcal{WA};X^*)$ as a subspace of $\Omega(\mathrm{Operad}(\bigcirc;X_c);\mathrm{Operad}_{\{o;c\}}(\mathcal{WA};X))$.

$$i\colon \mathrm{Operad}_{\{o;c\}}(\mathcal{WA};X^*) \to \Omega(\mathrm{Operad}(\bigcirc;X_c);\mathrm{Operad}_{\{o;c\}}(\mathcal{WA};X))$$

$$g \mapsto \begin{cases} \widetilde{g}_{n;c}\colon \mathcal{WA}(n;c)\times[0,1] \to X(n;c), & (x;t) \mapsto \eta(*_{n;c}) \\ \widetilde{g}_{n;o}\colon \mathcal{WA}(n;o)\times\{1\} \to X(n;o), & (x;1) \mapsto g_{n;o}(x) \end{cases}$$

In order to show that i is a weak equivalence, we introduce two towers of fibrations. One of them is the tower A'_k of Proposition 5.5. The second one is defined by

$$C'_k \subset \prod_{i=1}^{k+1}\mathrm{Top}(\mathcal{WA}(i;c)\times[0,1];X(i;c)) \times \prod_{i=1}^{k+1}\mathrm{Top}(\mathcal{WA}(i;o);X(i;o)),$$

satisfying the relations of Definition 5.3. Since $\mathcal{WA}(1;c)$ and $\mathcal{WA}(1;o)$ are reduced to the unit, the factors $\mathrm{Top}(\mathcal{WA}(1;c)\times[0,1];X(1;c))$ and $\mathrm{Top}(\mathcal{WA}(1;o);X(1;o))$ are the one point space and can be ignored. The restriction to the space A'_k of the inclusion i induces a map between the two towers as follows:

$$
\begin{array}{ccccccccc}
A'_1 & \longleftarrow & A'_2 & \longleftarrow & \cdots \longleftarrow & A'_k & \longleftarrow & A'_{k+1} & \longleftarrow \cdots \\
\downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & & \downarrow{\scriptstyle i_k} & & \downarrow{\scriptstyle i_{k+1}} & \\
C'_1 & \longleftarrow & C'_2 & \longleftarrow & \cdots \longleftarrow & C'_k & \longleftarrow & C'_{k+1} & \longleftarrow \cdots
\end{array}
$$

Since the space $\Omega(\mathrm{Operad}(\bigcirc;X_c);\mathrm{Operad}_{\{o;c\}}(\mathcal{WA};X))$ is weakly equivalent to the limit of C'_k, the map i is a weak equivalence if each i_k is a weak equivalence. We will prove this result by induction on k.

If $k=1$, a point in C'_1 is a pair $(g_{2;c};g_{2;o})$, whereas the points in the image of i_1 coincide with the pairs satisfying

$$g_{2;c}\colon \mathcal{WA}(2;c)\times[0,1] \to X(2;c), \quad (x;t) \mapsto \eta(*_{2;c}).$$

Since $g_{2;c}(x;0) = \eta(*_{1;c})$ for any pair in C_1', the inclusion i_1 induces the following deformation retract:

$$H\colon C_1' \times [0,1] \to C_1', \quad y = ((g_{2;c}; g_{2;o}); t_1) \mapsto \begin{cases} H(y)_{2;c}(\delta_{2;c}; t) = g_{2;c}(\delta_{2;c}; t(1-t_1)), \\ H(y)_{2;o}(\delta_{2;o}; 1) = g_{2;o}(\delta_{2;o}; 1). \end{cases}$$

From now on we assume that i_{k-1} is a weak equivalence for $k \geq 2$. We consider the following diagram, where g is a point in A_{k-1}', $F_{A'}$ the fiber over g and $F_{C'}$ the fiber over $i_{k-1}(g)$:

$$
\begin{array}{ccccc}
A_{k-1}' & \longleftarrow & A_k' & \longleftarrow & F_{A'} \\
\downarrow{\scriptstyle i_{k-1}} & & \downarrow{\scriptstyle i_k} & & \downarrow{\scriptstyle i_g} \\
C_{k-1}' & \longleftarrow & C_k' & \longleftarrow & F_{C'}
\end{array}
$$

Since the two left horizontal arrows are fibrations, the map i_k is a weak equivalence if the induced map i_g is a weak equivalence.

A point in the fiber $F_{C'}$ is defined by a pair $(g_{k+1;c}; g_{k+1;o})$ satisfying the relations of Definition 5.3. Since the pair is in the fiber over $i_{k-1}(g)$, the map $g_{k+1;c}$ sends all the faces of $\mathcal{WA}(k+1;c) \times [0,1]$ to $\eta(*_{k+1;c})$, except for the face $\mathcal{WA}(k+1;c) \times \{1\}$.

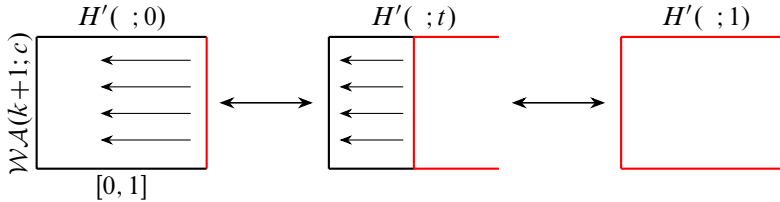On the other hand, the points in the image of i_g coincide with the pairs $(g_{k+1;c}; g_{k+1;o})$ such that

$$g_{k+1;c}\colon \mathcal{WA}(k+1;c) \times [0,1] \to X(k+1;c), \quad (x;t) \mapsto \eta(*_{k+1;c}).$$

In order to prove that i_g induces a deformation retract, we consider a lift H' in the following diagram:

$$
\begin{array}{ccc}
\begin{array}{c} \partial(\mathcal{WA}(k+1;c) \times [0,1]) \times [0,1] \\ \sqcup\, (\mathcal{WA}(k+1;c) \times [0,1]) \times \{0\} \end{array} & \longrightarrow & \mathcal{WA}(k+1;c) \times [0,1] \\
\downarrow & \nearrow{\scriptstyle H'} & \\
(\mathcal{WA}(k+1;c) \times [0,1]) \times [0,1] & &
\end{array}
$$

Here, the horizontal arrow is the inclusion on the factor $(\mathcal{WA}(k+1;c) \times [0,1]) \times \{0\}$ and sends a point $((x; t_1); t_2) \in \partial(\mathcal{WA}(k+1;c) \times [0,1]) \times [0,1]$ to $(x; (1-t_2)t_1)$. Such a lift exists since the vertical arrow is an acyclic cofibration and the space $\mathcal{WA}(k+1;c) \times [0,1]$

is fibrant. The homotopy H' is illustrated by the following picture:

In other words, the points in the image of i_g coincide with the pairs such that

$$g_{k+1;c}(x;t) = g_{k+1;c}(H'((x;t);1)) = \eta(*_{k+1;c})$$

for $x \in \mathcal{WA}(k+1;c)$ and $t \in [0,1]$. Finally, the deformation retract

$$H_2 \colon F_{C'} \times [0,1] \to F_{C'}$$

sends a point $y = ((g_{k+1;c}; g_{k+1;o}); t_1)$ to

$$\begin{cases} H_2(y)_{k+1;c}(x;t) = g_{k+1;c}(H'((x;t);t_1)) & \text{for } x \in \mathcal{WA}(k;c) \text{ and } t \in [0,1], \\ H_2(y)_{k+1;o}(x;1) = g_{k+1;o}(x;1) & \text{for } x \in \mathcal{WA}(k+1;o). \end{cases}$$

Consequently, we have that $\Omega(\text{Operad}(\Box; X_c); \text{Operad}_{\{o;c\}}(\mathcal{WA}; X))$ is weakly equivalent to $\text{Operad}_{\{o;c\}}(\mathcal{WA}; X^*)$. $\qquad\square$

Corollary 5.7 *Let $\alpha \colon \text{As} \to O$ be a map of operads and $\beta \colon O \to B$ be a map of O–bimodules. Under assumption (13), if $B(0) \simeq *$ and $O(0) \simeq O(1) \simeq *$, then the pair $(\text{sTot}(O); \text{sTot}(B))$ is weakly equivalent to the $\mathcal{SC}_2$–space*

$$\left(\Omega^2 \text{Operad}^h(\text{As}_{>0}; O); \Omega^2\left(\text{Operad}^h(\text{As}_{>0}; X_c); \text{Operad}^h_{\{o;c\}}(\mathcal{A}ct_{>0}; X)\right)\right),$$

where X is the operad given by relations (12).

6 Double relative delooping: general case

In this section, O is an $\{o;c\}$–operad endowed with a map of operads $\eta \colon \mathcal{A}ct \to O$ that makes O into an $\mathcal{A}ct_{>0}$–bimodule under $\mathcal{A}ct$. We denote by $(O_c; O_o)$ the pair of semicosimplicial spaces associated to O; see Proposition 1.13. In Section 4, we proved that the pair $(\text{sTot}(O_c); \text{sTot}(O_o))$ is weakly equivalent to

$$\left(\Omega \text{Bimod}^h_{\text{As}_{>0}}(\text{As}_{>0}; O_c); \Omega\left(\text{Bimod}^h_{\text{As}_{>0}}(\text{As}_{>0}; O_c); \text{Bimod}^h_{\mathcal{A}ct_{>0}}(\mathcal{A}ct_{>0}; O)\right)\right),$$

under the assumption $O(0;c) \simeq *$.

If we assume that $O(1;c) \simeq *$, then $\text{Bimod}^h_{\text{As}_{>0}}(\text{As}_{>0}; O_c) \simeq \Omega \text{Operad}^h(\text{As}_{>0}; O_c)$. Similarly, Marcy D Robertson shows in [16] that the derived space of bimodule maps

is weakly equivalent to the loop space of the derived space of operadic maps. More precisely, in our context, we have the following proposition.

Proposition 6.1 Let O be an action-operad with $O(1;c) \simeq O(1;o) \simeq *$. The space $\mathrm{Bimod}^h_{\mathcal{A}ct_{>0}}(\mathcal{A}ct_{>0}; O)$ is weakly equivalent to $\Omega\,\mathrm{Operad}^h_{\{o;c\}}(\mathcal{A}ct_{>0}; O)$.

Sketch of proof The proof is the same as in [19, Theorem 7.2]. According to the notation of Turchin, $\mathcal{B}\square$ is a cofibrant replacement of $\mathrm{As}_{>0}$ in the model category $\mathrm{Bimod}_{\mathrm{As}_{>0}}$ such that there exists a map

$$\xi_c\colon \Omega\,\mathrm{Operad}(\square; O_c) \to \mathrm{Bimod}_{\mathrm{As}_{>0}}(\mathcal{B}\square; O_c), \quad f \mapsto \xi_c^f.$$

Using towers of fibrations as we do in Sections 4 and 5, Turchin proves that ξ_c is a weak equivalence. The construction of the map ξ_c is obtained from a polytope subdivision $\mathcal{B}\square(n) = \{\mathcal{B}\square(T)\}_T$ indexed by $\{c\}$–trees with n leaves. More precisely, for any $\{c\}$–tree T with n leaves, the space $\mathcal{B}\square(T)$ is the product of the following two spaces:

$$\lambda_\square(T) = \prod_{v \in V(T)} \square(|v|),$$

$$\chi_\blacktriangle(T) = \{\{t_v\}_{v \in V(T)} \mid t_v \in [0,1] \text{ and } t_{v_1} \le t_{v_2} \text{ if } v_1 < v_2\} \subset [0,1]^{|V(T)|}.$$

A point in $\mathcal{B}\square(T)$ is denoted by $\{x_v; t_v\}$ with $\{x_v\} \in \lambda_\square(T)$ and $\{t_v\} \in \chi_\blacktriangle(T)$. For any $f \in \Omega\,\mathrm{Operad}(\square; O_c)$, the map ξ_c^f is defined on each polytope $\mathcal{B}\square(T)$ by induction on the number of vertices of T using the operadic structure of O_c:

$$\xi_{T;c}^f\colon \mathcal{B}\square(T) \to O(n;c)$$
$$\{x_v; t_v\} \mapsto f_{|r|}(x_r; t_r)\big(\xi_{T_1;c}^f(\{x^1; t^1\}), \ldots, \xi_{T_{|r|};c}^f(\{x^{|r|}; t^{|r|}\})\big),$$

with T_i the subtree of T whose trunk coincides with the ith input edge of the root of T.

In our case, a cofibrant replacement of $\mathcal{A}ct_{>0}$ in the model category $\mathrm{Bimod}_{\mathcal{A}ct_{>0}}$ is the $\{o;c\}$–sequence

$$\mathcal{B}\square_{\{o;c\}}(n;c) = \mathcal{B}\square_{\{o;c\}}(n;o) = \mathcal{B}\square(n) \quad \text{for } n > 0,$$

and the empty set otherwise, with the obvious $\mathcal{A}ct_{>0}$–bimodule structure. The space $\mathcal{B}\square_{\{o;c\}}(n;o)$ has a polytope subdivision $\{\mathcal{B}\square_{\{o;c\}}(T)\}_T$ indexed by tree_n^o. The space

$\mathcal{B}\diamond_{\{o;c\}}(T)$ is the product of the following two spaces:

$$\lambda_{\diamond}(T) = \prod_{v \in V(T)} \mathcal{WA}\big(n; f(e_0(v))\big),$$

$$\chi_{\blacktriangle}(T) = \{\{t_v\}_{v \in V(T)} \mid t_v \in [0,1] \text{ and } t_{v_1} < t_{v_2} \text{ if } v_1 < v_2\} \subset [0,1]^{|V(T)|}.$$

A point in $\mathcal{B}\diamond_{\{o;c\}}(T)$ is denoted by $\{x_v; t_v\}$. For any $f \in \Omega\operatorname{Operad}(\mathcal{WA}; O)$, the map ξ^f is defined by $\xi^f_{n;c}$ as before, and $\xi^f_{n;o}$ by induction on the number of vertices of T using the operadic structure of O:

$$\xi^f_{T;o} \colon \mathcal{B}\diamond_{\{o;c\}}(T) \to O(n;o)$$

$$\{x_v; t_v\} \mapsto f_{|r|;o}(x_r; t_r)\big(\xi^f_{T_1;c}(\{x^1; t^1\}), \dots, \xi^f_{T_{|r|};o}(\{x^{|r|}; t^{|r|}\})\big),$$

with T_i the subtree of T whose trunk coincides with the ith input edge of the root of T. It defines a map from $\Omega\operatorname{Operad}_{\{o;c\}}(\mathcal{WA}; O)$ to $\operatorname{Bimod}_{Act_{>0}}(\mathcal{B}\diamond_{\{o;c\}}; O)$, which is a weak equivalence using the same arguments as Turchin in [19]. $\qquad\square$

Theorem 6.2 *Assume O is an $\{o;c\}$–operad such that $O(0;c) \simeq O(1;c) \simeq *$ and $O(1;o) \simeq *$. If $\eta \colon Act \to O$ is a map of $\{o;c\}$–operads, the pair $(\operatorname{sTot}(O_c); \operatorname{sTot}(O_o))$ is weakly equivalent to the $\mathcal{SC}_2$–space*

$$\big(\Omega^2\operatorname{Operad}^h(As_{>0}; O_c); \Omega^2\big(\operatorname{Operad}^h(As_{>0}; O_c); \operatorname{Operad}^h_{\{o;c\}}(Act_{>0}; O)\big)\big).$$

Proof By [19, Theorem 7.2], we know that the space $\operatorname{sTot}(O_c)$ is weakly equivalent to $\Omega^2\operatorname{Operad}(\diamond; O_c)$. Proposition 6.1 implies that the projection of ξ onto the closed part gives rise to the commutative diagram

$$
\begin{array}{ccc}
\Omega\operatorname{Operad}_{\{o;c\}}(\mathcal{WA}; O) & \xrightarrow{\ \xi\ } & \operatorname{Bimod}_{Act_{>0}}(\square; O) \\
{\scriptstyle \Omega(p_2)}\big\downarrow & & \big\downarrow{\scriptstyle p_1} \\
\Omega\operatorname{Operad}(\diamond; O_c) & \xrightarrow{\ \xi_c\ } & \operatorname{Bimod}_{As_{>0}}(\square_c; O_c),
\end{array}
$$

where p_1 and p_2 are respectively the maps (8) and (9). Since the homotopy fibers commute with the homotopy limits, we have the following:

$$\Omega(\operatorname{Bimod}_{As_{>0}}(\square_c; O_c); \operatorname{Bimod}_{Act_{>0}}(\square; O))$$

$$\simeq \operatorname{hofib}\big(\operatorname{Bimod}_{Act_{>0}}(\square; O) \xrightarrow{p_1} \operatorname{Bimod}_{As_{>0}}(\square_c; O_c)\big)$$

$$\simeq \operatorname{hofib}\big(\Omega\operatorname{Operad}_{\{o;c\}}(\mathcal{WA}; O) \xrightarrow{\Omega(p_2)} \Omega\operatorname{Operad}(\diamond; O_c)\big)$$

$$\simeq \Omega\operatorname{hofib}\big(\operatorname{Operad}_{\{o;c\}}(\mathcal{WA}; O) \xrightarrow{p_2} \operatorname{Operad}(\diamond; O_c)\big)$$

$$\simeq \Omega^2\big(\operatorname{Operad}(\diamond; O_c); \operatorname{Operad}_{\{o;c\}}(\mathcal{WA}; O)\big). \qquad\square$$

Acknowledgments

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