msp

Algebraic & Geometric Topology 16 (2016) 3005–3027

The η–inverted $\mathbb{R}$–motivic sphere

BERTRAND J GUILLOU
DANIEL C ISAKSEN

We use an Adams spectral sequence to calculate the $\mathbb{R}$–motivic stable homotopy groups after inverting η. The first step is to apply a Bockstein spectral sequence in order to obtain h_1–inverted $\mathbb{R}$–motivic Ext groups, which serve as the input to the η–inverted $\mathbb{R}$–motivic Adams spectral sequence. The second step is to analyze Adams differentials. The final answer is that the Milnor–Witt $(4k-1)$–stem has order 2^{u+1}, where u is the 2–adic valuation of $4k$. This answer is reminiscent of the classical image of J. We also explore some of the Toda bracket structure of the η–inverted $\mathbb{R}$–motivic stable homotopy groups.

14F42; 55T15, 55Q45

1 Introduction

The first exotic property of motivic stable homotopy groups is that the Hopf map η is not nilpotent. This means that inverting η can be useful for understanding the global structure of motivic stable homotopy groups.

In Andrews and Miller [3] and Guillou and Isaksen [5], the η–inverted $\mathbb{C}$–motivic 2–completed stable homotopy groups $\hat{\pi}^{\mathbb{C}}_{*,*}[\eta^{-1}]$ were explicitly computed to be

$$\mathbb{F}_2[\eta^{\pm 1}][\mu, \varepsilon]/\varepsilon^2.$$

This result naturally suggests that one should study the structure of η–inverted motivic stable homotopy groups over other fields.

In the present article, we consider the η–inverted $\mathbb{R}$–motivic 2–completed stable homotopy groups $\hat{\pi}^{\mathbb{R}}_{*,*}[\eta^{-1}]$. Our main tool is the motivic Adams spectral sequence, which takes the form

$$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[h_1^{-1}] \implies \hat{\pi}^{\mathbb{R}}_{*,*}[\eta^{-1}].$$

Here $\mathcal{A}^{\mathbb{R}}$ is the $\mathbb{R}$–motivic Steenrod algebra, and $\mathbb{M}_2^{\mathbb{R}}$ is the motivic $\mathbb{F}_2$–cohomology of $\mathbb{R}$. We will exhaustively compute this spectral sequence.

Published: 7 November 2016 DOI: 10.2140/agt.2016.16.3005

We begin with computing the Adams E_2–page $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[h_1^{-1}]$ using the ρ–Bockstein spectral sequence; see Hill [6] and Dugger and Isaksen [4]. This spectral sequence takes the form

$$\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})[\rho][h_1^{-1}] \implies \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[h_1^{-1}],$$

where $\mathcal{A}^{\mathbb{C}}$ is the $\mathbb{C}$–motivic Steenrod algebra and $\mathbb{M}_2^{\mathbb{C}}$ is the motivic $\mathbb{F}_2$–cohomology of $\mathbb{C}$.

The input to the ρ–Bockstein spectral sequence is completely known from Guillou and Isaksen [5]. In order to deduce differentials, one first observes, as in Dugger and Isaksen [4], that the groups

$$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[\rho^{-1}, h_1^{-1}]$$

with ρ and h_1 both inverted are easy to describe. Then there is only one pattern of ρ–Bockstein differentials that is consistent with this ρ–inverted calculation.

Having obtained the Adams E_2–page $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[h_1^{-1}]$, the next step is to compute Adams differentials. The extension of scalars functor from $\mathbb{R}$–motivic homotopy theory to $\mathbb{C}$–motivic homotopy theory induces a map

$$\begin{array}{ccc} \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[h_1^{-1}] & \Longrightarrow & \hat{\pi}^{\mathbb{R}}_{*,*}[\eta^{-1}] \\ \downarrow & & \downarrow \\ \mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})[h_1^{-1}] & \Longrightarrow & \hat{\pi}^{\mathbb{C}}_{*,*}[\eta^{-1}] \end{array}$$

of Adams spectral sequences. The bottom Adams spectral sequence is completely understood; see Andrews and Miller [3] and Guillou and Isaksen [5]. The Adams d_2 differentials in the top spectral sequence can then be deduced by the comparison map.

This leads to a complete description of the h_1–inverted $\mathbb{R}$–motivic Adams E_3–page. Over $\mathbb{C}$, it turns out that the h_1–inverted Adams spectral sequence collapses at this point. However, over $\mathbb{R}$, there are higher differentials that we deduce from manipulations with Massey products and Toda brackets.

In the end, we obtain an explicit description of the h_1–inverted $\mathbb{R}$–motivic Adams E_∞–page, from which we can read off the η–inverted stable motivic homotopy groups over $\mathbb{R}$.

In order to state the result, we need a bit of terminology. Because η belongs to $\hat{\pi}^{\mathbb{R}}_{1,1}$, it makes sense to use a grading that is invariant under multiplication by η. The Milnor–Witt n–stem is the direct sum $\Pi_n = \bigoplus_p \hat{\pi}^{\mathbb{R}}_{p+n,p}$. Then multiplication by η is an endomorphism of the Milnor–Witt n–stem.

Theorem 1.1 (1) *The η–inverted Milnor–Witt 0–stem $\Pi_0[\eta^{-1}]$ is $\mathbb{Z}_2[\eta^{\pm 1}]$, where $\mathbb{Z}_2$ is the ring of 2–adic integers.*

(2) *If $k > 1$, then the η–inverted Milnor–Witt $(4k-1)$–stem $\Pi_{4k-1}[\eta^{-1}]$ is isomorphic to $\mathbb{Z}/2^{u+1}[\eta^{\pm 1}]$ as a module over $\mathbb{Z}_2[\eta^{\pm 1}]$, where u is the 2–adic valuation of $4k$.*

(3) *The η–inverted Milnor–Witt n–stem $\Pi_n[\eta^{-1}]$ is zero otherwise.*

For degree reasons, the product structure on $\hat{\pi}^{\mathbb{R}}_{*,*}[\eta^{-1}]$ is very simple. However, there are many interesting Toda brackets. We explore much of the 3–fold Toda bracket structure in this article. In particular, we will show that all of $\hat{\pi}^{\mathbb{R}}_{*,*}[\eta^{-1}]$ can be constructed inductively via Toda brackets, starting from just 2 and the generator of the Milnor–Witt 3–stem.

Theorem 1.1 gives a familiar answer. These groups have the same order as the classical image of J. For example, Π_3 consists of elements of order 8, which is the same as the order of the image of J in the classical 3–stem. Similarly, Π_7 consists of elements of order 16, which is the same as the order of the image of J in the classical 7–stem. One might expect a geometric proof that directly compares the classical image of J spectrum with the η–inverted $\mathbb{R}$–motivic sphere. However, higher structure in the form of Toda brackets suggests that such a direct proof is not possible.

We also observe that our calculations are reminiscent of the classical Adams spectral sequence for v_1–periodic homotopy at odd primes, as carried out in Andrews [2]. We are not aware of a structural reason why the calculations are so similar.

The calculation of the η–inverted $\mathbb{R}$–motivic homotopy groups leads to questions about η–inverted motivic homotopy groups over other fields. We leave it to the reader to speculate on the behavior of these η–inverted groups over other fields.

Acknowledgements Guillou was supported by Simons Collaboration Grant 282316. Isaksen was supported by NSF grant DMS-1202213.

2 Preliminaries

2.1 Notation

We continue with notation from [4] as follows:

(1) $\mathbb{M}_2^{\mathbb{C}} = \mathbb{F}_2[\tau]$ is the motivic cohomology of $\mathbb{C}$ with $\mathbb{F}_2$ coefficients, where τ has bidegree $(0,1)$.

(2) $\mathbb{M}_2^{\mathbb{R}} = \mathbb{F}_2[\tau,\rho]$ is the motivic cohomology of $\mathbb{R}$ with $\mathbb{F}_2$ coefficients, where τ and ρ have bidegrees $(0,1)$ and $(1,1)$, respectively.

(3) $\mathcal{A}^{\mathrm{cl}}$ is the classical mod 2 Steenrod algebra.

(4) $\mathcal{A}^{\mathbb{C}}$ is the mod 2 motivic Steenrod algebra over $\mathbb{C}$.

(5) $\mathcal{A}^{\mathbb{R}}$ is the mod 2 motivic Steenrod algebra over $\mathbb{R}$.

(6) $\mathrm{Ext}_{\mathrm{cl}}$ is the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$.

(7) $\mathrm{Ext}_{\mathbb{C}}$ is the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$.

(8) $\mathrm{Ext}_{\mathbb{R}}$ is the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$.

(9) $\hat{\pi}_{*,*}^{\mathbb{C}}$ is the motivic stable homotopy ring of the 2–completed motivic sphere spectrum over $\mathbb{C}$.

(10) $\hat{\pi}_{*,*}^{\mathbb{R}}$ is the motivic stable homotopy ring of the 2–completed motivic sphere spectrum over $\mathbb{R}$.

(11) Π_n is the Milnor–Witt n–stem $\bigoplus_p \hat{\pi}_{p+n,p}^{\mathbb{R}}$.

(12) $\mathcal{R} = \mathbb{F}_2[\rho, h_1^{\pm 1}]$.

(13) The symbols v_1^4 and P are used interchangeably for the Adams periodicity operator.

2.2 Grading conventions

We follow [7] in grading Ext according to (s, f, w), where:

(1) f is the Adams filtration, ie the homological degree.

(2) $s+f$ is the internal degree, ie that corresponding to the first coordinate in the bidegree of the Steenrod algebra.

(3) s is the stem, ie the internal degree minus the Adams filtration.

(4) w is the weight.

Following this grading convention, the elements τ and ρ, as elements of $\mathrm{Ext}_{\mathbb{R}}$, have degrees $(0,0,-1)$ and $(-1,0,-1)$ respectively.

We will consider the groups $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ in which h_1 has been inverted. The degree of h_1 is $(1,1,1)$. As in [5], for this purpose it is convenient to introduce the following gradings whose values are zero for h_1:

(5) $mw = s - w$ is the Milnor–Witt degree.

(6) $c = s + f - 2w$ is the Chow degree.

In order to avoid notational clutter, we will often drop h_1 from the notation. Since h_1 is a unit, no information is lost by doing this. The correct powers of h_1 can always be recovered by checking degrees.

For example, in Lemma 3.1 below, we claim that there is a differential $d_3^\rho(v_1^4) = \rho^3 v_2$ in the ρ–Bockstein spectral sequence. Strictly speaking, this formula is nonsensical because $d_3^\rho(v_1^4)$ has Adams filtration 5 while v_2 has Adams filtration 1. The correct full formula is $d_3^\rho(v_1^4) = \rho^3 h_1^4 v_2$.

If we are to ignore multiples of h_1, we must rely on gradings that take value 0 on h_1. This explains our preference for Milnor–Witt degree mw and Chow degree c.

3 The ρ–Bockstein spectral sequence

Recall [6; 4] that the ρ–Bockstein spectral sequence takes the form

$$\mathrm{Ext}_{\mathbb{C}}[\rho] \Longrightarrow \mathrm{Ext}_{\mathbb{R}}.$$

After inverting h_1, by [5, Theorem 1.1] this takes the form

$$\mathcal{R}[v_1^4, v_2, v_3, \dots] \Longrightarrow \mathrm{Ext}_{\mathbb{R}}[h_1^{-1}],$$

where $\mathcal{R} = \mathbb{F}_2[\rho, h_1^{\pm 1}]$. Table 1 lists the generators of the Bockstein E_1–page.

(mw, c)	generator
$(0, 1)$	ρ
$(4, 4)$	v_1^4
$(3, 1)$	v_2
$(7, 1)$	v_3
$(15, 1)$	v_4
$(2^n - 1, 1)$	v_n

Table 1: Bockstein E_1–page generators

Lemma 3.1 *In the ρ–Bockstein spectral sequence, there are differentials*

$$d^\rho_{2^n-1}(v_1^{2^n}) = \rho^{2^n-1} v_n \quad \text{for } n \geq 2.$$

All other nonzero differentials follow from the Leibniz rule.

The first few examples of these differentials are $d_3(v_1^4) = \rho^3 v_2$, $d_7(v_1^8) = \rho^7 v_3$ and $d_{15}(v_1^{16}) = \rho^{15} v_4$.

Proof Inverting ρ induces a map

$$\begin{array}{ccc} \mathrm{Ext}_{\mathbb{C}}[h_1^{-1}][\rho] & \overset{\rho\text{–Bss}}{\Longrightarrow} & \mathrm{Ext}_{\mathbb{R}}[h_1^{-1}] \\ {\scriptstyle \rho\text{–inv}}\downarrow & & \downarrow{\scriptstyle \rho\text{–inv}} \\ \mathrm{Ext}_{\mathbb{C}}[h_1^{-1}][\rho^{\pm 1}] & \overset{\rho\text{–Bss}}{\Longrightarrow} & \mathrm{Ext}_{\mathbb{R}}[h_1^{-1}, \rho^{-1}] \end{array}$$

of ρ–Bockstein spectral sequences. We will establish differentials in the ρ–inverted spectral sequence. The map of spectral sequences then implies that the same differentials occur when ρ is not inverted.

Recall [4, Theorem 4.1] there is an isomorphism $\mathrm{Ext}_{\mathrm{cl}}[\rho^{\pm 1}] \cong \mathrm{Ext}_{\mathbb{R}}[\rho^{-1}]$ sending the classical element h_0 to the motivic element h_1. Using also that $\mathrm{Ext}_{\mathrm{cl}}[h_0^{-1}] = \mathbb{F}_2[h_0^{\pm 1}]$, it follows $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}, \rho^{-1}]$ is isomorphic to $\mathcal{R}[\rho^{-1}]$. Then the ρ–inverted ρ–Bockstein spectral sequence takes the form

$$\mathcal{R}[\rho^{-1}][v_1^4, v_2, v_3, \dots] \overset{\rho\text{–Bss}}{\Longrightarrow} \mathcal{R}[\rho^{-1}].$$

Because the target of the ρ–inverted spectral sequence is very small, essentially everything must either support a differential or be hit by a differential.

The ρ–Bockstein differentials have degree $(-1, 0)$ with respect to the grading (mw, c) used in Table 1. The elements $\rho^k v_2$ cannot support differentials because there are no elements in the Milnor–Witt 2–stem. The only possibility is that after inverting ρ, there is a ρ–Bockstein differential $d_3(v_1^4) = \rho^3 v_2$.

Then the ρ–inverted E_4–page is $\mathcal{R}[v_1^8, v_3, v_4, \dots]$. The elements $\rho^k v_3$ cannot support differentials because the ρ–inverted E_4–page has no elements in the Milnor–Witt 6–stem. The only possibility is that after inverting ρ, there is a ρ–Bockstein differential $d_7(v_1^8) = \rho^7 v_3$.

In general, the ρ–inverted E_{2^n-1}–page is $\mathcal{R}[v_1^{2^n}, v_n, v_{n+1}, \dots]$. The elements $\rho^k v_n$ cannot support differentials because the ρ–inverted E_{2^n-1}–page has no elements in the Milnor–Witt (2^n-2)–stem. The only possibility is that after inverting ρ, there is a ρ–Bockstein differential $d_{2^n-1}(v_1^{2^n}) = \rho^{2^n-1} v_n$. □

The ρ–Bockstein E_∞–page can be directly computed from the Leibniz rule and the differentials in Lemma 3.1. For example, $d_3(v_1^4) = \rho^3 v_2$, so $d_3(v_1^{4+8k}) = \rho^3 v_1^{8k} v_2$. This establishes the relation $\rho^3 v_1^{8k} v_2 = 0$.

To ease the notation in Proposition 3.2, we write P rather than v_1^4.

Proposition 3.2 *The ρ–Bockstein E_∞–page is the $\mathcal{R}$–algebra on the generators $P^{2^{n-1}k}v_n$ for $n \geq 2$ and $k \geq 0$ (see Table 2), subject to the relations*

$$\rho^{2^n-1}P^{2^{n-1}k}v_n = 0$$

for $n \geq 2$ and $k \geq 0$, and

$$P^{2^{n-1}k}v_n \cdot P^{2^{m-1}j}v_m + P^{2^{n-1}(k+2^{m-n}j)}v_n \cdot v_m = 0$$

for $m \geq n \geq 2$, $k \geq 0$ and $j \geq 0$.

(mw, c)	generator	ρ–torsion
$(0, 1)$	ρ	∞
$(0, 0)$	h_1	∞
$(3, 1) + k(8, 8)$	$P^{2k}v_2$	3
$(7, 1) + k(16, 16)$	$P^{4k}v_3$	7
$(15, 1) + k(32, 32)$	$P^{8k}v_4$	15
$(2^n - 1, 1) + k(2^{n+1}, 2^{n+1})$	$P^{2^{n-1}k}v_n$	$2^n - 1$

Table 2: Bockstein E_∞–page generators

Remark 3.3 In practice, the relations mean that every P can be shifted onto the v_n with minimal n in any monomial. Thus an $\mathcal{R}$–module basis is given by monomials of the form $P^{2^{n-1}k}v_n \cdot v_{m_1} \cdots v_{m_a}$, where $n \leq m_1 \leq \cdots \leq m_a$. For example,

$$P^2v_2 \cdot P^4v_2 = P^6v_2 \cdot v_2, \quad P^4v_2 \cdot P^8v_3 = P^{12}v_2 \cdot v_3, \quad P^4v_3 \cdot P^{48}v_5 = P^{52}v_3 \cdot v_5.$$

4 The Adams E_2–page

Having obtained the ρ–Bockstein E_∞–page in Section 3, our next task is to consider hidden extensions in $\mathrm{Ext}_\mathbb{R}[h_1^{-1}]$. We will show that there are no hidden relations. This will require some careful analysis of degrees, as well as some manipulations with Massey products.

The ρ–Bockstein E_∞–page is an associated graded object of $\mathrm{Ext}_\mathbb{R}[h_1^{-1}]$. Elements of the E_∞–page only determine elements of $\mathrm{Ext}_\mathbb{R}[h_1^{-1}]$ up to higher filtration. Therefore, we must be careful about choosing specific generators of $\mathrm{Ext}_\mathbb{R}[h_1^{-1}]$.

We will show in Lemma 4.1 that $P^{2^{n-1}k}v_n$ detects a unique element of $\mathrm{Ext}_\mathbb{R}[h_1^{-1}]$. Therefore, we may unambiguously use the same notation $P^{2^{n-1}k}v_n$ for an element of $\mathrm{Ext}_\mathbb{R}[h_1^{-1}]$.

In general, the ρ–Bockstein spectral sequence does not allow for hidden extensions by ρ. More precisely, if x is an element of the ρ–Bockstein E_∞–page such that $\rho^k x = 0$, then x detects an element of $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ that is also annihilated by ρ^k. Beware that x might detect more than one element of $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$, and some such elements might not be annihilated by ρ^k. Nevertheless, there is always at least one element that is annihilated by ρ^k.

For example, the relation $\rho^{2^{n-1}} P^{2^{n-1}k} v_n = 0$ in the ρ–Bockstein E_∞–page lifts to give the same relation in $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$.

Lemma 4.1 *For each $n \geq 2$ and $k \geq 0$, the element $P^{2^{n-1}k} v_n$ of the Bockstein E_∞–page detects a unique element of* $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$.

Proof We need to show that in the ρ–Bockstein E_∞–page, $P^{2^{n-1}k} v_n$ does not share bidegree with an element of higher filtration.

First suppose that $P^{2^{n-1}k} v_n$ has the same bidegree as $\rho^b P^{2^{m-1}j} v_m$. Then

$$(2^n - 1, 1) + k(2^{n+1}, 2^{n+1}) = (2^m - 1, 1) + j(2^{m+1}, 2^{m+1}) + b(0, 1).$$

Considering only the Milnor–Witt degree, we have

$$2^n(2k+1) = 2^m(2j+1).$$

Therefore, $n = m$ and $k = j$, so $b = 0$.

Suppose that $P^{2^{n-1}k} v_n$ shares bidegree with some element x. By Remark 3.3, we may assume that x is of the form $\rho^b P^{2^{m_1-1}j} v_{m_1} \cdot v_{m_2} \cdots v_{m_a}$, where $m_1 \leq m_2 \leq \cdots \leq m_a$. Since $\rho^{2^{m_1}-1} P^{2^{m-1}j} v_{m_1} = 0$, we may also assume that $b \leq 2^{m_1} - 2$. Because of the previous paragraph, we may assume that $a \geq 2$. We wish to show that $b = 0$.

We first show that $n \geq m_a$. Let $u(x)$ be the difference $mw - c$. We have that $u(P^{2^{m_1-1}j} v_{m_1}) = 2^{m_1} - 2$ and $u(\rho) = -1$. Since $b \leq 2^{m_1} - 2$, it follows that $u(\rho^b P^{2^{m_1-1}j} v_{m_1}) \geq 0$. Thus

$$2^n - 2 = u(P^{2^{n-1}k} v_n) = u(\rho^b P^{2^{m_1-1}j} v_{m_1}) + u(v_{m_2} \cdots v_{m_a}) \geq u(v_{m_a}) = 2^{m_a} - 2,$$

so that $n \geq m_a$.

Now consider the Milnor–Witt and Chow degrees modulo 4. We have

$$(-1, 1) \equiv (-a, a+b) \pmod 4,$$

so $a \equiv 1 \pmod 4$ and $b \equiv 0 \pmod 4$. Thus either $b = 0$, which was what we wanted to show, or $b \geq 4$.

We may now assume that $b \geq 4$. Since $\rho^4 P^{2j} v_2 = 0$, we must have $m_1 \geq 3$, so that all m_i, and also n, are at least 3.

Next, consider degrees modulo 8. Comparing degrees gives

$$(-1, 1) \equiv (-a, a+b) \pmod 8.$$

Thus $b \equiv 0 \pmod 8$, so that $b \geq 8$. Since $\rho^8 P^{4j} v_3 = 0$, we must have $j_1 \geq 4$, and therefore n and all other j_i are also at least 4. This argument can be continued to establish that b and n must be arbitrarily large under the assumption that $b > 0$. □

Lemma 4.2 *For each $n \geq 2$ and $k \geq 0$, the element $P^{2^{n-1}k} v_n \cdot v_n$ of the Bockstein E_∞–page detects a unique element of* $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$.

Proof The Milnor–Witt degree of $P^{2^{n-1}k} v_n \cdot v_n$ is even, while the Milnor–Witt degree of $\rho^b P^{2^{m-1}j} v_m$ is odd. Therefore, these elements cannot share bidegree.

Now suppose that the element $P^{2^{n-1}k} v_n \cdot v_n$ has the same bidegree as the element $\rho^b P^{2^{m_1-1}j} v_{m_1} \cdot v_{m_2} \cdots v_{m_a}$, with $m_1 \leq m_2 \leq \cdots \leq m_a$, $b \leq 2^{m_1} - 2$ and $a \geq 2$. The rest of the proof is essentially the same as the proof of Lemma 4.1. Consider $u = mw - c$ to get that $n \geq m_a$. Then consider congruences $(-2, 2) \equiv (-a, a+b)$ modulo higher and higher powers of 2 to obtain that $b = 0$. □

Remark 4.3 The obvious generalization of Lemma 4.2 to elements of the form $P^{2^{n-1}k} v_n \cdot v_m$ is false. For example, $P^2 v_2 \cdot v_5$ has the same degree as $\rho^4 v_3^6$.

Remark 4.4 Lemmas 4.1 and 4.2 are equivalent to the claim that there are no ρ multiples in the ρ–Bockstein E_∞–page in the same bidegrees as either $P^{2^{n-1}k} v_n$ or $P^{2^{n-1}k} v_n \cdot v_n$. This implies that there are also no ρ multiples in $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ that share bidegree with these elements; we will need this fact later.

Lemma 4.5 $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ *is zero when the Milnor–Witt stem mw and the Chow degree c are both equal to $2i$ with $i \geq 1$.*

Proof Under the condition $mw = c = 2i$, inspection of Table 1 shows the ρ–Bockstein E_1–page consists of products of elements of the form v_1^4 or $\rho^{2^n + 2^m - 4} v_n v_m$. In the E_∞–page, $\rho^{2^n + 2^m - 4} v_n v_m = 0$ since $\rho^{2^n - 1} v_n = 0$. Also, v_1^{4k} supports a differential for all $k \geq 0$. □

Lemma 4.6 *For each $n \geq 2$, $k \geq 0$ and $m > n$, we have a Massey product*

$$P^{2^{n-1}k + 2^{m-2}} v_n = \langle \rho^{2^m - 2^n} v_m, \rho^{2^n - 1}, P^{2^{n-1}k} v_n \rangle$$

in $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ *with no indeterminacy.*

Proof The Bockstein differential $d^{\rho}_{2^m-1}(P^{2^{m-2}}) = \rho^{2^m-1} v_m$ and May's convergence theorem [8, Theorem 4.1] imply that the Massey product is detected by $P^{2^{n-1}k+2^{m-2}} v_n$ in the ρ–Bockstein E_∞–page. There are no crossing Bockstein differentials as all classes are in nonnegative ρ–filtration. Lemma 4.1 says that this ρ–Bockstein E_∞–page element detects a unique element of $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$.

The indeterminacy of the bracket is generated by products of the form $\rho^{2^m-2^n} v_m \cdot x$ and $y \cdot P^{2^{n-1}k} v_n$, where x and y have appropriate bidegrees. We showed in Lemma 4.5 that 0 is the only possibility for x or y. □

Remark 4.7 Lemma 4.6 gives many different Massey products for the same element. For example,

$$P^8 v_2 = \langle \rho^4 v_3, \rho^3, P^6 v_2 \rangle = \langle \rho^{12} v_4, \rho^3, P^4 v_2 \rangle = \langle \rho^{28} v_5, \rho^3, v_2 \rangle.$$

Lemma 4.8 *For $m > n \geq 2$, there is a Massey product*

$$P^{2^{n-1}k+2^{m-2}} v_n = \langle P^{2^{n-1}k} v_n, \rho^{2^m-2} v_m, \rho \rangle$$

in $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ with no indeterminacy.

Proof The Massey product formula follows from the Bockstein differential

$$d^{\rho}_{2^m-1}(P^{2^{m-2}}) = \rho^{2^m-1} v_m$$

and May's convergence theorem [8, Theorem 4.1]. There are no crossing Bockstein differentials as all classes are in nonnegative ρ–filtration. As in the proof of Lemma 4.6, we need Lemma 4.1 to tell us that the element $P^{2^{n-1}k+2^{m-2}} v_n$ of the ρ–Bockstein E_∞–page detects a unique element of $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$.

The indeterminacy of the bracket is generated by products of the form $P^{2^{n-1}k} v_n \cdot x$ and $y \cdot \rho$. We showed in Lemma 4.5 that 0 is the only possibility for x. We observed in Remark 4.4 that $y \cdot \rho$ must be zero because there are no multiples of ρ in the appropriate bidegree. □

The relations in the Bockstein E_∞–page given in Proposition 3.2 may lift to $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ with additional terms that are multiples of ρ. In other words, there may be hidden relations in the Bockstein spectral sequence. For example, for degree reasons it is possible that $P^2 v_2 \cdot P^{16} v_5 + P^{18} v_2 \cdot v_5$ equals $\rho^4 P^{16} v_3 \cdot v_3^5$. Proposition 4.9 shows that there are no such hidden terms in the relations in $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$.

Proposition 4.9 *There are no hidden relations in the Bockstein spectral sequence.*

Proof The relation $\rho^{2^{n-1}} P^{2^{n-1}k} v_n = 0$ in the ρ–Bockstein E_∞–page lifts to give the same relation in $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$, as we observed in the discussion preceding Lemma 4.1. Therefore, we need only compute the products $P^{2^{n-1}k} v_n \cdot P^{2^{m-1}j} v_m$ in $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ for $m \geq n$.

Lemma 4.6 implies that $P^{2^{n-1}k} v_n \cdot P^{2^{m-1}j} v_m$ equals

$$P^{2^{n-1}k} v_n \langle \rho^{2^m} v_{m+1}, \rho^{2^m-1}, P^{2^{m-1}(j-1)} v_m \rangle.$$

Shuffle to obtain

$$\langle P^{2^{n-1}k} v_n, \rho^{2^m} v_{m+1}, \rho^{2^m-1} \rangle P^{2^{m-1}(j-1)} v_m.$$

This expression is contained in

$$\langle P^{2^{n-1}k} v_n, \rho^{2^{m+1}-2} v_{m+1}, \rho \rangle P^{2^{m-1}(j-1)} v_m,$$

which equals $P^{2^{n-1}k+2^{m-1}} v_n \cdot P^{2^{m-1}(j-1)} v_m$ by Lemma 4.8.

By induction, $P^{2^{n-1}k} v_n \cdot P^{2^{m-1}j} v_m$ equals $P^{2^{n-1}(k+2^{m-n}j)} v_n \cdot v_m$. □

Theorem 4.10 $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ *is the $\mathcal{R}$–algebra on the generators $P^{2^{n-1}k} v_n$ for $n \geq 2$ and $k \geq 0$ (see Table 2), subject to the relations*

$$\rho^{2^n-1} P^{2^{n-1}k} v_n = 0$$

for $n \geq 2$ and $k \geq 0$, and

$$P^{2^{n-1}k} v_n \cdot P^{2^{m-1}j} v_m + P^{2^{n-1}(k+2^{m-n}j)} v_n \cdot v_m = 0$$

for $m \geq n \geq 2$, $k \geq 0$ and $j \geq 0$.

Proof This follows immediately from Propositions 3.2 and 4.9. □

Remark 4.11 Analogously to Remark 3.3, an $\mathcal{R}$–module basis for $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ is given by monomials of the form $P^{2^{n-1}k} v_n \cdot v_{m_1} \cdots v_{m_a}$, where $n \leq m_1 \leq \cdots \leq m_a$.

5 Adams differentials

Before computing with the h_1–inverted $\mathbb{R}$–motivic Adams spectral sequence, we will consider convergence. A priori, there could be an infinite family of homotopy classes linked together by infinitely many hidden η multiplications. These classes would not be detected in $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$. Lemma 5.1 implies that this cannot occur for degree reasons.

Lemma 5.1 *Let $m > 0$ be a fixed Milnor–Witt stem. There exists a constant A such that $\mathrm{Ext}_{\mathbb{R}}^{(s,f,w)}$ vanishes when $s - w = m$, s is nonzero, $f > A$ and $f > s + 1$.*

Lemma 5.1 can be restated in the following more casual form: within a fixed Milnor–Witt stem, there exists a horizontal line and a line of slope 1 such that $\mathrm{Ext}_{\mathbb{R}}$ vanishes in the region above both lines, except in the 0–stem. Figure 1 depicts the shape of the vanishing region.

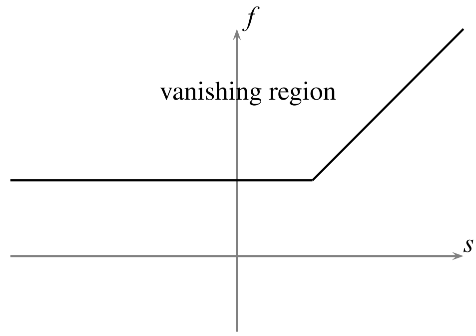

Figure 1: The vanishing region in a Milnor–Witt stem

Proof This argument occurs in $\mathrm{Ext}_{\mathbb{R}}$, where h_1 has not been inverted.

As explained in [4, Theorem 4.1], the elements in the m–stem of the classical Ext groups $\mathrm{Ext}_{\mathrm{cl}}$ correspond to elements of $\mathrm{Ext}_{\mathbb{R}}$ in the Milnor–Witt m–stem that remain nonzero after ρ is inverted, ie that support infinitely many multiplications by ρ. Each stem of $\mathrm{Ext}_{\mathrm{cl}}$ is finite except for the 0–stem. For $m > 0$, choose A to be larger than the Adams filtrations of all of the elements in the m–stem of $\mathrm{Ext}_{\mathrm{cl}}$. Then A is larger than the Adams filtrations of every element of $\mathrm{Ext}_{\mathbb{R}}$ in the Milnor–Witt m–stem that remain nonzero after ρ is inverted.

Let x be a nonzero element of $\mathrm{Ext}_{\mathbb{R}}^{(s,f,w)}$ such that $s - w = m$, $f > A$ and $f > s + 1$. We will show that s must equal zero.

The choice of A guarantees that x is annihilated by some positive power of ρ. Suppose that $\rho^k x = 0$ but $\rho^{k-1} x$ is nonzero, for some $k > 0$. Then there must be a differential in the ρ–Bockstein spectral sequence of the form $d_k(y) = \rho^k x$, where y is an element of $\mathrm{Ext}_{\mathbb{C}}$ in degree $(s - k + 1, f - 1, w - k)$.

The argument from [1] establishes a vanishing line of slope 1 in the nonzero stems of $\mathrm{Ext}_{\mathbb{C}}$. The conditions $f > s + 1$ and $k > 0$ imply that the element y lies strictly above this vanishing line, so it must be of the form $\tau^a h_0^b$ with $b \geq 1$. The only ρ–Bockstein differentials on such classes are $d_1(\tau^{2c+1} h_0^b) = \rho \tau^{2c} h_0^{b+1}$, which implies that x must be of the form $\tau^{2c} h_0^b$. This shows that $s = 0$. □

The h_1–inverted motivic Adams spectral sequence over $\mathbb{C}$ was studied in [5; 3]. It takes the form

$$\mathbb{F}_2[h_1^{\pm 1}, P, v_2, v_3, \dots] \implies \hat{\pi}^{\mathbb{C}}_{*,*}[\eta^{-1}],$$

where $\hat{\pi}^{\mathbb{C}}_{*,*}[\eta^{-1}]$ is the η–inverted motivic stable homotopy ring of the 2–completed motivic sphere spectrum over $\mathbb{C}$. This spectral sequence has differentials

$$d_2(P^k v_n) = P^k v_{n-1}^2$$

for all $k \geq 0$ and all $n \geq 3$. As usual, we omit any powers of h_1.

Lemma 5.2 *In the h_1–inverted $\mathbb{R}$–motivic Adams spectral sequence, there are differentials*

$$d_2(P^{2^{n-1}k} v_n) = P^{2^{n-1}k} v_{n-1}^2$$

for all $k \geq 0$ and all $n \geq 3$.

Proof There is an extension of scalars functor from $\mathbb{R}$–motivic homotopy theory to $\mathbb{C}$–motivic homotopy theory. This functor induces a map

$$\begin{array}{ccc} \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[h_1^{-1}] & \Longrightarrow & \hat{\pi}^{\mathbb{R}}_{*,*}[\eta^{-1}] \\ \downarrow & & \downarrow \\ \mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})[h_1^{-1}] & \Longrightarrow & \hat{\pi}^{\mathbb{C}}_{*,*}[\eta^{-1}] \end{array}$$

from the $\mathbb{R}$–motivic Adams spectral sequence to the $\mathbb{C}$–motivic Adams spectral sequence. This map takes ρ to zero.

The above map of spectral sequences implies that the $\mathbb{R}$–motivic Adams differential $d_2(P^{2^{n-1}k} v_n)$ equals $P^{2^{n-1}k} v_{n-1}^2$ plus terms that are divisible by ρ. Lemma 4.2 implies that there are no possible additional terms in the relevant bidegree. □

Our next task is to completely describe the Adams E_3–page. First, we explore some elements that survive to the E_3–page. We will consider these elements more carefully in Proposition 5.4.

Despite the differential $d_2(P^{4k} v_3) = P^{4k} v_2^2$, the element $\rho^3 P^{4k} v_3$ survives to the E_3–page because $\rho^3 P^{4k} v_2^2$ is zero. Similarly, $\rho^{2^{n-1}-1} P^{2^{n-1}k} v_n$ survives to the E_3–page.

The element $P^2 v_2^2$ looks like it should be hit by an Adams d_2 differential on $P^2 v_3$. However, $P^2 v_3$ did not survive the ρ–Bockstein spectral sequence. Therefore, there is nothing to hit $P^2 v_2^2$ and it survives to the Adams E_3–page. The same observation applies to the elements $P^{2^{n-1}(2j+1)} v_n^2$.

We record the following simple computation, as we will employ it several times.

Lemma 5.3 *Let S be an $\mathbb{F}_2$–algebra. Let $B = S[w_1, w_2, \dots]$ be a polynomial ring in infinitely many variables, and define a differential on B by $\partial(w_n) = w_{n-1}^2$ for $n \geq 2$. Then $\mathrm{H}^*(B, \partial) \cong S[w_1]/w_1^2$.*

In fact, we will use a slight generalization of Lemma 5.3 in which $\partial(w_n)$ is equal to $u_n w_{n-1}^2$, where u_n is a unit in S. This generalization implies, for example, that the h_1–inverted $\mathbb{C}$–motivic Adams E_3–page is $\mathbb{F}_2[h_1^{\pm 1}, P, v_2]/v_2^2$.

Proposition 5.4 *The h_1–inverted $\mathbb{R}$–motivic Adams E_3–page is free as an $\mathcal{R}$–module on the generators listed in Table 3 for $n \geq 2$, $k \geq 0$ and $j \geq 0$. Almost all products of these generators are zero, except that*

$$P^{4k} v_2 \cdot P^{4j+2} v_2 = P^{4k+4j+2} v_2^2$$

and for $n \geq 3$,

$$\rho^{2^{n-1}-1} P^{2^{n-1} \cdot 2k} v_n \cdot \rho^{2^{n-1}-1} P^{2^{n-1}(2j+1)} v_n = \rho^{2^n-2} P^{2^{n-1}(2k+2j+1)} v_n^2.$$

(mw, c)	generator	ρ–torsion
$(0,0)$	1	∞
$(3,1) + k(8,8)$	$P^{2k} v_2$	3
$(7,4) + k(16,16)$	$\rho^3 P^{4k} v_3$	4
$(15,8) + k(32,32)$	$\rho^7 P^{8k} v_4$	8
$(2^n - 1, 2^{n-1}) + k(2^{n+1}, 2^{n+1})$	$\rho^{2^{n-1}-1} P^{2^{n-1}k} v_n$	2^{n-1}
$(6,2) + (2j+1)(8,8)$	$P^{2(2j+1)} v_2^2$	3
$(14,2) + (2j+1)(16,16)$	$P^{4(2j+1)} v_3^2$	7
$(30,2) + (2j+1)(32,32)$	$P^{8(2j+1)} v_4^2$	15
$(2^{n+1} - 2, 2) + (2j+1)(2^{n+1}, 2^{n+1})$	$P^{2^{n-1}(2j+1)} v_n^2$	$2^n - 1$

Table 3: $\mathcal{R}$–module generators for the Adams E_3–page

Remark 5.5 The relations in Proposition 5.4 are just the ones that are obvious from the notation. For example,

$$v_2 \cdot P^2 v_2 = P^2 v_2^2, \qquad \rho^3 P^4 v_3 \cdot \rho^3 P^8 v_3 = \rho^6 P^{12} v_3^2.$$

Proof of Proposition 5.4 Let $\mathrm{Ext}\langle k, b\rangle$ be the $\mathbb{F}_2[h_1^{\pm 1}]$–submodule of the h_1–inverted $\mathbb{R}$–motivic Adams E_2–page on generators of the form $\rho^b P^k v_{m_1} v_{m_2} \cdots v_{m_a}$ such that $m_1 \leq m_2 \leq \cdots \leq m_a$. Note that $b \leq 2^{m_1} - 2$ in this situation, since $\rho^{2^{m_1}-1} P^k v_{m_1} = 0$.

Also, k must be a multiple of 2^{m_1-1}. By Lemma 5.2 and the fact that ρ is a permanent cycle, each $\mathrm{Ext}\langle k,b\rangle$ is a differential graded submodule. Thus it suffices to compute the cohomology of each $\mathrm{Ext}\langle k,b\rangle$.

We start with $\mathrm{Ext}\langle 0,b\rangle$, which is equal to $\rho^b\cdot\mathbb{F}_2[h_1^{\pm 1},v_m,v_{m+1},\dots]$ as a differential graded $\mathbb{F}_2[h_1^{\pm 1}]$–module, where m is the smallest integer such that $b\le 2^m-2$. Now Lemma 5.3 implies that $H^*(\mathrm{Ext}\langle 0,b\rangle,d_2)$ is a free $\mathbb{F}_2[h_1^{\pm 1}]$–module on two generators ρ^b and $\rho^b v_m$.

So far, we have demonstrated that the powers of ρ and the elements

$$v_2,\quad \rho v_2,\quad \rho^2 v_2,\quad \rho^3 v_3,\dots,\rho^6 v_3,\quad \rho^7 v_4,\dots$$

are present in the h_1–inverted ℝ–motivic Adams E_3–page.

The module $\mathrm{Ext}\langle k,b\rangle$ is zero when k is odd.

Now assume that k is equal to 2 modulo 4. If $b\le 2$, then $\mathrm{Ext}\langle k,b\rangle$ is equal to $\rho^b P^k v_2\cdot\mathbb{F}_2[h_1^{\pm 1},v_2,v_3,\dots]$ as a differential graded $\mathbb{F}_2[h_1^{\pm 1}]$–module. Lemma 5.3 implies that $H^*(\mathrm{Ext}\langle k,b\rangle,d_2)$ is a free $\mathbb{F}_2[h_1^{\pm 1}]$–module on two generators $\rho^b P^k v_2$ and $\rho^b P^k v_2^2$. If $b\ge 3$, then $\mathrm{Ext}\langle k,b\rangle$ is zero because $\rho^3 P^k v_2=0$.

We have now shown that the elements

$$P^k v_2,\quad \rho P^k v_2,\quad \rho^2 P^k v_2,\quad P^k v_2^2,\quad \rho P^k v_2^2,\quad \rho^2 P^k v_2^2$$

are present in the h_1–inverted ℝ–motivic Adams E_3–page for all k congruent to 2 modulo 4.

Next assume that k is equal to 4 modulo 8. If $b\le 2$, then $\mathrm{Ext}\langle k,b\rangle$ is the free $\mathbb{F}_2[h_1^{\pm 1}]$–module on generators $\rho^b P^k v_{m_1}\cdots v_{m_a}$ such that m_1 equals 2 or 3, and $m_1\le\cdots\le m_a$. There is a short exact sequence

$$0\to \mathrm{Ext}\langle k,b\rangle\to \rho^b P^k\cdot\mathbb{F}_2[h_1^{\pm 1},v_2,v_3,\dots]\to \rho^b P^k\cdot\mathbb{F}_2[h_1^{\pm 1},v_4,v_5,\dots]\to 0,$$

where the differential is defined on the second and third terms in the obvious way. By Lemma 5.3, the homology of the middle term has two generators $\rho^b P^k$ and $\rho^b P^k v_2$, while the homology of the right term has two generators $\rho^b P^k$ and $\rho^b P^k v_4$. Analysis of the long exact sequence in homology shows that $H^*(\mathrm{Ext}\langle k,b\rangle,d_2)$ has two generators $\rho^b P^k v_2$ and $\rho^b P^k v_3^2$.

Now assume that $3\le b\le 6$. Since $\rho^b P^k v_2=0$, we get that $\mathrm{Ext}\langle k,b\rangle$ is equal to $\rho^b P^k v_3\cdot\mathbb{F}_2[h_1^{\pm 1},v_3,v_4,\dots]$. Lemma 5.3 implies that $H^*(\mathrm{Ext}\langle k,b\rangle,d_2)$ is a free $\mathbb{F}_2[h_1^{\pm 1}]$–module on two generators $\rho^b P^k v_3$ and $\rho^b P^k v_3^2$.

Finally, if $b \geq 7$, then $\mathrm{Ext}\langle k, b\rangle$ is zero because $\rho^7 P^k v_2 = 0$ and $\rho^7 P^k v_3 = 0$. This finishes the argument when k is equal to 4 modulo 8, and we have shown that $\mathrm{Ext}_{\mathbb{R}}[h_1^{\pm 1}]$ contains the elements

$$P^k v_2,\ \rho P^k v_2,\ \rho^2 P^k v_2,$$
$$\rho^3 P^k v_3, \dots, \rho^6 P^k v_3,$$
$$P^k v_3^2,\ \rho P^k v_3^2, \dots, \rho^6 P^k v_3^2.$$

Analysis of the other cases is the same as the argument for $k \equiv 4$ modulo 8. The details depend on the value of k modulo 2^i and inequalities of the form $2^j - 1 \leq b \leq 2^{j+1} - 2$. In each case there is a short exact sequence of differential graded modules whose first term is $\mathrm{Ext}\langle k, b\rangle$ and whose other two terms have homology that is computed by Lemma 5.3. □

We have now calculated the h_1–inverted $\mathbb{R}$–motivic E_3–page. This E_3–page is displayed in Figure 2. Beware that the grading on this chart is not the same as in a standard Adams chart. The Milnor–Witt stem $mw = s - w$ is plotted on the horizontal axis, while the Chow degree $c = s + f - 2w$ is plotted on the vertical axis. As a result, an Adams d_r differential has slope $-r + 1$, rather than slope $-r$. Vertical lines in Figure 2 represent multiplications by ρ.

Our next goal is to establish the Adams d_3 differentials. Inspection of Figure 2 reveals that the only possible nonzero d_3 differentials might be supported on elements of the form $\rho^b P^{2^{n-1}k} v_n$ for $n \geq 4$. In fact, these differentials all occur, as indicated in Figure 2 by lines that go left one unit and up two units. We will establish these d_3 differentials by first proving a homotopy relation in Lemma 5.6.

Lemma 5.6 *For each $n \geq 2$ and $j \geq 0$, the element $P^{2^{n-1}(2j+1)} v_n^2$ is a permanent cycle that detects a ρ–divisible element of the η–inverted $\mathbb{R}$–motivic homotopy groups.*

Proof Inspection of Figure 2 shows that $P^{2^{n-1}(2j+1)} v_n^2$ cannot support a differential.

Lemma 4.8 implies that

$$P^{2^{n-1}(2j+1)} v_n^2 \in \langle \rho, \rho^{2^{n+1}-2} v_{n+1}, P^{2^n j} v_n^2 \rangle \quad \text{in } \mathrm{Ext}_{\mathbb{R}}[h_1^{-1}].$$

In fact, the Massey product has no indeterminacy because of Remark 4.4 and Lemma 4.5.

We will now apply Moss's convergence theorem [10, Theorem 1.2] to this Massey product. There is an Adams differential $d_2(P^{2^n j} v_{n+1}) = P^{2^n j} v_n^2$, so $P^{2^n j} v_n^2$ detects the homotopy element 0. By inspection of Figure 2, $\rho^{2^{n+1}-2} v_{n+1}$ is a permanent

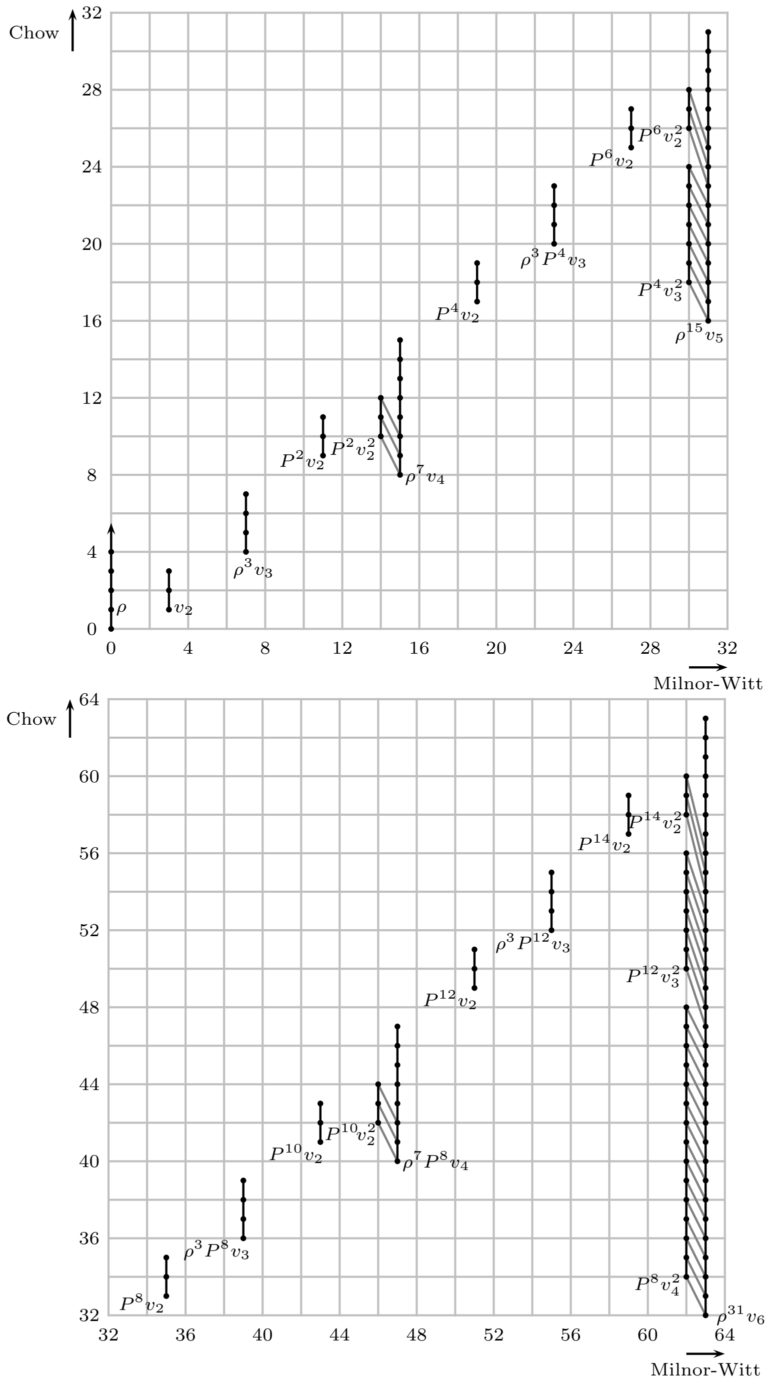

Figure 2: The η–inverted $\mathbb{R}$–motivic Adams E_3–page

cycle; let α be a homotopy element detected by it. Moreover, $\rho\alpha$ is zero in homotopy because there are no classes in higher filtration that could detect it.

Moss's convergence theorem says that the Toda bracket $\langle \rho, \alpha, 0 \rangle$ contains an element that is detected by $P^{2^{n-1}(2j+1)} v_n^2$. This Toda bracket consists entirely of multiples of ρ. □

Lemma 5.7 *$d_3(\rho^{2^{n-1}-1} P^{2^{n-1}k} v_n) = P^{2^{n-3}+2^{n-1}k} v_{n-2}^2$ for $n \geq 4$.*

Proof Lemma 5.6 shows that $P^{2^{n-3}+2^{n-1}k} v_{n-2}^2$ detects a class that is divisible by ρ. By inspection of Figure 2, there are no classes in lower filtration. Therefore, $P^{2^{n-3}+2^{n-1}k} v_{n-2}^2$ must detect zero, ie must be hit by a differential. It is apparent from Figure 2 that there is only one possible differential. □

Lemma 5.8 describes the higher Adams differentials.

Lemma 5.8 *For $n \geq r+1$ and $r \geq 3$,*

$$d_r(\rho^{2^n - 2^{n-r+2} - r + 2} P^{2^{n-1}k} v_n) = P^{2^{n-1}k + 2^{n-2} - 2^{n-r}} v_{n-r+1}^2.$$

Proof The proof is essentially the same as the proof of Lemma 5.7. In the Milnor–Witt stem congruent to 2 modulo 4, Lemma 5.6 implies that every homotopy element is divisible by ρ. This implies that they must all be hit by differentials. Figure 2 indicates that there is just one possible pattern of differentials. □

From Lemma 5.8, it is straightforward to derive the h_1–inverted Adams E_∞–page, as shown in Figure 3.

Proposition 5.9 *The h_1–inverted Adams E_∞–page is the $\mathcal{R}$–module on generators given in Table 4 for $n \geq 2$.*

(mw, c)	generator	ρ–torsion
$(0, 0)$	1	∞
$(3, 1) + k(8, 8)$	$P^{2k} v_2$	3
$(7, 4) + k(16, 16)$	$\rho^3 P^{4k} v_3$	4
$(15, 11) + k(32, 32)$	$\rho^{10} P^{8k} v_4$	5
$(2^n - 1, 2^n - n - 1) + k(2^{n+1}, 2^{n+1})$	$\rho^{2^n - n - 2} P^{2^{n-1}k} v_n$	$n+1$

Table 4: $\mathcal{R}$–module generators for the Adams E_∞–page

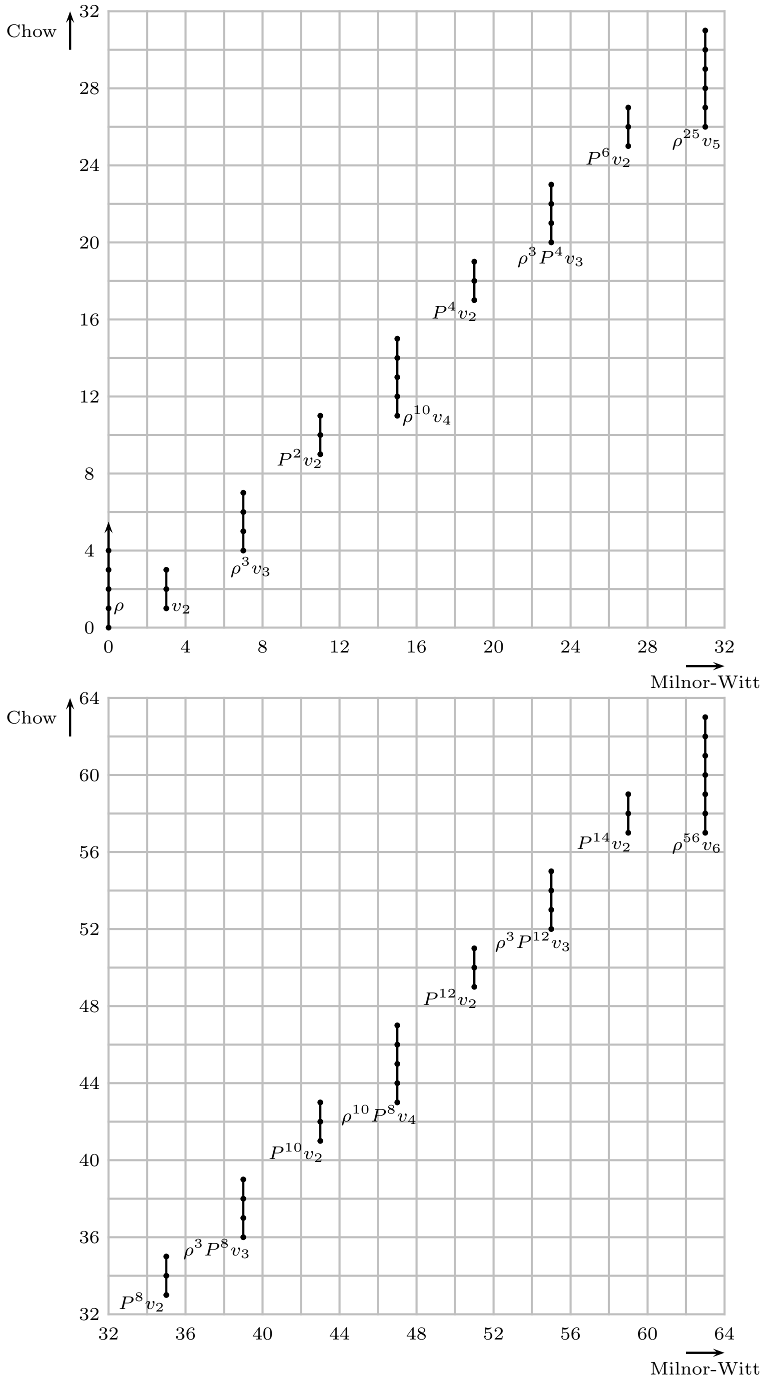

Figure 3: The η–inverted $\mathbb{R}$–motivic Adams E_∞–page

6 η–inverted homotopy groups

From the h_1–inverted Adams E_∞–page, it is a short step to the η–inverted stable homotopy ring. First we must choose generators. Recall that Π_n is the Milnor–Witt n–stem $\bigoplus_p \hat{\pi}^{\mathbb{R}}_{p+n,p}$.

Definition 6.1 For k nonnegative and n at least 2, let $P^{2^{n-1}k}\lambda_n$ be an element of $\Pi_{2^{n+1}k+2^n-1}[\eta^{-1}]$ that is detected by $\rho^{2^n-n-2}P^{2^{n-1}k}v_n$.

There are choices in these definitions, which are measured by Adams E_∞–page elements in higher filtration. For example, there are four possible choices for λ_2 because of the presence of ρv_2 and $\rho^2 v_2$ in higher filtration.

Theorem 6.2 *The η–inverted $\mathbb{R}$–motivic stable homotopy ring, as a $\mathbb{Z}_2[\eta^{\pm 1}]$–module, is generated by 1 and $P^{2^{n-1}k}\lambda_n$ for $n \geq 2$ and $k \geq 0$. The generator $P^{2^{n-1}k}\lambda_n$ lies in $\Pi_{2^{n+1}k+2^n-1}[\eta^{-1}]$ and is annihilated by 2^{n+1}. All products are zero, except for those involving 2 or η.*

Proof In the η–inverted stable homotopy ring, ρ and 2 differ by a unit because $\rho\eta^2 = -2\eta$; see [9]. Therefore, the ρ–torsion information given in Proposition 5.9 translates to 2–torsion information in homotopy.

Except for 1, all $\mathbb{Z}_2[\eta^{\pm 1}]$–module generators lie in Milnor–Witt stems that are congruent to 3 modulo 4. Therefore, such generators must multiply to zero. □

Table 5 lists all generators through the Milnor–Witt 63–stem. The table also identifies Toda brackets that contain each generator. These Toda brackets are computed in Section 7.

Table 5 also reveals a pattern that matches the classical image of J.

Corollary 6.3 *If $k > 1$, then $\Pi_{4k-1}[\eta^{-1}]$ is isomorphic to $\mathbb{Z}/2^{u+1}[\eta^{\pm 1}]$ as a module over $\mathbb{Z}_2[\eta^{\pm 1}]$, where u is the 2–adic valuation of $4k$.*

7 Toda brackets

Even though its primary multiplicative structure is uninteresting, the η–inverted $\mathbb{R}$–motivic stable homotopy ring has rich higher structure in the form of Toda brackets. We will explore some of the 3–fold Toda bracket structure. In particular, we will show that all of the generators can be inductively constructed via Toda brackets, starting

mw	E_∞	$\widehat{\pi}^{\mathbb{R}}_{*,*}[\eta^{-1}]$	2^k–torsion	bracket	indeterminacy
0	1	1	∞		
3	v_2	λ_2	3		
7	$\rho^3 v_3$	λ_3	4	$\langle 2^3, \lambda_2, \lambda_2\rangle$	$2^3\lambda_3$
11	$P^2 v_2$	$P^2\lambda_2$	3	$\langle 2^4, \lambda_3, \lambda_2\rangle$	
15	$\rho^{10} v_4$	λ_4	5	$\langle 2^3, \lambda_2, P^2\lambda_2\rangle$	$2^3\lambda_4$
19	$P^4 v_2$	$P^4\lambda_2$	3	$\langle 2^5, \lambda_4, \lambda_2\rangle$	
23	$\rho^3 P^4 v_3$	$P^4\lambda_3$	4	$\langle 2^5, \lambda_4, \lambda_3\rangle$	
27	$P^6 v_2$	$P^6\lambda_2$	3	$\langle 2^5, \lambda_4, P^2\lambda_2\rangle$	
31	$\rho^{25} v_5$	λ_5	6	$\langle 2^3, \lambda_2, P^6\lambda_2\rangle$	$2^3\lambda_5$
35	$P^8 v_2$	$P^8\lambda_2$	3	$\langle 2^6, \lambda_5, \lambda_2\rangle$	
39	$\rho^3 P^8 v_3$	$P^8\lambda_3$	4	$\langle 2^6, \lambda_5, \lambda_3\rangle$	
43	$P^{10} v_2$	$P^{10}\lambda_2$	3	$\langle 2^6, \lambda_5, P^2\lambda_2\rangle$	
47	$\rho^{10} P^8 v_4$	$P^8\lambda_4$	5	$\langle 2^6, \lambda_5, \lambda_4\rangle$	
51	$P^{12} v_2$	$P^{12}\lambda_2$	3	$\langle 2^6, \lambda_5, P^4\lambda_2\rangle$	
55	$\rho^3 P^{12} v_3$	$P^{12}\lambda_3$	4	$\langle 2^6, \lambda_5, P^4\lambda_3\rangle$	
59	$P^{14} v_2$	$P^{14}\lambda_2$	3	$\langle 2^6, \lambda_5, P^6\lambda_2\rangle$	
63	$\rho^{56} v_6$	λ_6	7	$\langle 2^3, \lambda_2, P^{14}\lambda_2\rangle$	$2^3\lambda_6$

Table 5: $\mathbb{Z}_2[\eta^{\pm 1}]$–module generators for $\widehat{\pi}^{\mathbb{R}}_{*,*}[\eta^{-1}]$

from just 2 and λ_2. Table 5 lists one possible Toda bracket decomposition for each generator of Π_n for all n less than or equal to 63.

We observed in the proof of Theorem 6.2 that the element ρ of the Adams E_∞–page detects the element 2 of the η–inverted stable homotopy ring. We will use this fact frequently in the following results.

Lemma 7.1 *The Toda bracket $\langle 2^3, \lambda_2, \lambda_2\rangle$ contains an element detected by $\rho^3 v_3$ in Π_7, and its indeterminacy is detected by $\rho^6 v_3$.*

Proof Moss's convergence theorem [10, Theorem 1.2] and the differential $d_2(v_3) = v_2^2$ show that $\langle 2^3, \lambda_2, \lambda_2\rangle$ is detected by $\rho^3 v_3$.

The indeterminacy follows from the facts that there are no multiples of λ_2 and that there is a unique multiple of 2^3 in Π_7. □

Remark 7.2 The proof of Lemma 7.1 applies just as well to show that $\langle 2^4, \lambda_3, \lambda_3\rangle$ is detected by $\rho^{10}v_4$ in Π_{15}. In higher stems, the analogous brackets do not produce generators. For example, the Massey product $\langle \rho^5, \rho^{10}v_4, \rho^{10}v_4\rangle$ is already defined in Ext, which implies that the corresponding Toda bracket must be detected in filtration least 27. However, $\rho^{25}v_5$ detects the generator of Π_{31}, and it lies in filtration 26.

Lemma 7.3 *For n at least 2, the Toda bracket $\langle 2^3, \lambda_2, P^{2^{n-1}-2}\lambda_2\rangle$ is detected by the class $\rho^{2^{n+1}-n-3}v_{n+1}$. The indeterminacy in this Toda bracket is generated by $2^3\lambda_{n+1}$.*

Proof This follows from Moss's convergence theorem [10, Theorem 1.2], together with the Adams differential $d_n(\rho^{2^{n+1}-n-6}v_{n+1}) = P^{2^{n-1}-2}v_2^2$. □

Lemma 7.4 *For n at least 2, the Toda bracket $\langle 2^{n+2}, \lambda_{n+1}, P^{2^{n-1}-2}\lambda_2\rangle$ is detected by $P^{2^n-2}v_2$. The Toda bracket has no indeterminacy.*

Proof Lemma 4.8 implies that there is a Massey product

$$P^{2^n-2}v_2 = \langle \rho^{n+2}, \rho^{2^{n+1}-n-3}v_{n+1}, P^{2^{n-1}-2}v_2\rangle,$$

with no indeterminacy. Moss's convergence theorem [10, Theorem 1.2] establishes the desired result. □

Lemma 7.5 *If $m > n \geq 2$, then the Toda bracket $\langle 2^{m+1}, \lambda_m, P^{2^{n-1}k}\lambda_n\rangle$ is detected by $\rho^{2^n-n-2}P^{2^{m-2}+2^{n-1}k}v_n$. The Toda bracket has no indeterminacy.*

Proof Lemma 4.8 implies that there is a Massey product

$$\rho^{2^n-n-2}P^{2^{m-2}+2^{n-1}k}v_n = \langle \rho^{m+1}, \rho^{2^m-m-2}v_m, \rho^{2^n-n-2}P^{2^{n-1}k}v_n\rangle.$$

Moss's convergence theorem [10, Theorem 1.2] establishes the desired result. □

Proposition 7.6 *Every generator $P^{2^{n-1}k}\lambda_n$ of the η–inverted $\mathbb{R}$–motivic stable homotopy ring can be constructed via iterated 3–fold Toda brackets starting from 2 and λ_2.*

Proof Lemmas 7.3 and 7.4 alternately show that the generators λ_n and the generators $P^{2^n-2}\lambda_2$ can be constructed via iterated 3–fold Toda brackets starting from 2 and λ_2. Then Lemma 7.5 shows that any $P^{2^{n-1}k}\lambda_n$ can be constructed. □

Example 7.7 Suppose we wish to find a Toda bracket decomposition for $P^{40}\lambda_3$. Since $40 = 2^{7-2} + 2^{3-1}\cdot 4$, we can apply Lemma 7.5 with $m = 7$, $n = 3$ and $k = 4$ to conclude that $P^{40}\lambda_3$ is detected by the Toda bracket $\langle 2^8, \lambda_7, P^8\lambda_3\rangle$.

References

[1] **J F Adams**, *A finiteness theorem in homological algebra*, Proc. Cambridge Philos. Soc. 57 (1961) 31–36 MR

[2] **M J Andrews**, *The v_1–periodic part of the Adams spectral sequence at an odd prime*, PhD thesis, Massachusetts Institute of Technology (2015) MR Available at `http://search.proquest.com/docview/1720302322`

[3] **M Andrews**, **H Miller**, *Inverting the Hopf map*, preprint (2014) Available at `http://www-math.mit.edu/~hrm/papers/andrews-miller-jul08.pdf`

[4] **D Dugger**, **D C Isaksen**, *Low dimensional Milnor–Witt stems over $\mathbb{R}$*, preprint (2015) arXiv

[5] **B J Guillou**, **D C Isaksen**, *The η–local motivic sphere*, J. Pure Appl. Algebra 219 (2015) 4728–4756 MR

[6] **M A Hill**, *Ext and the motivic Steenrod algebra over $\mathbb{R}$*, J. Pure Appl. Algebra 215 (2011) 715–727 MR

[7] **D C Isaksen**, *Stable stems*, preprint (2014) arXiv

[8] **J P May**, *Matric Massey products*, J. Algebra 12 (1969) 533–568 MR

[9] **F Morel**, *On the motivic π_0 of the sphere spectrum*, from "Axiomatic, enriched and motivic homotopy theory" (J P C Greenlees, editor), NATO Sci. Ser. II Math. Phys. Chem. 131, Kluwer, Dordrecht (2004) 219–260 MR

[10] **R M F Moss**, *Secondary compositions and the Adams spectral sequence*, Math. Z. 115 (1970) 283–310 MR

Department of Mathematics, University of Kentucky
715 Patterson Office Tower, Lexington, KY 40506-0027, United States

Department of Mathematics, Wayne State University
656 W Kirby, Detroit, MI 48202, United States

`bertguillou@uky.edu`, `isaksen@wayne.edu`

Received: 29 October 2015 Revised: 1 March 2016