Algebraic & Geometric Topology 17 (2017) 111–130

Notes on the knot concordance invariant Upsilon

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Ozsváth, Stipsicz and Szabó have defined a knot concordance invariant Υ_K taking values in the group of piecewise linear functions on the closed interval $[0,2]$. This paper presents a description of one approach to defining Υ_K and proving its basic properties.

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1 Introduction

Ozsváth, Stipsicz and Szabó [5] used the Heegaard Floer knot complex $\mathrm{CFK}^-(K)$ of a knot $K \subset S^3$ to define a piecewise linear function $\Upsilon_K(t)$ with domain $[0,2]$. The function $K \to \Upsilon_K$ induces a homomorphism from the smooth knot concordance group to the group of functions on the interval $[0,2]$. Among its properties, $\Upsilon_K(t)$ provides bounds on the four-genus, $g_4(K)$, the three-genus, $g_3(K)$, and, consequently, the concordance genus, $g_c(K)$. This note describes a simple approach to defining $\Upsilon_K(t)$ using $\mathrm{CFK}^\infty(K)$ and proving its basic properties.

Acknowledgments Thanks go to Jen Hom, Slaven Jabuka, Swatee Naik, Peter Ozsváth, Shida Wang and C M Michael Wong for their comments. Matt Hedden pointed out the structure theorem for filtered knot complexes presented in the appendix and its usefulness in simplifying a key proof. Suggestions from the referee led to valuable improvements in the exposition. The author was supported by a Simons Foundation grant and by NSF-DMS-1505586.

2 Knot complexes

We begin by describing the algebraic structure of the Heegaard Floer complex of a knot K, denoted $\mathrm{CFK}^\infty(K)$, first defined in Ozsváth and Szabó [9]. This is a vector space over the field $\mathbb{F}$ with two elements. To simplify notation, we write $\mathrm{CF}(K)$ for $\mathrm{CFK}^\infty(K)$. Here we summarize its basic properties:

Published: 26 January 2017 DOI: 10.2140/agt.2017.17.111

- The chain complex $\mathrm{CF}(K)$ has an integer valued grading and the boundary map ∂ is of degree -1. The grading is called the *Maslov grading*. The grading of a homogeneous element is denoted $\mathrm{gr}(x)$.
- The complex $\mathrm{CF}(K)$ has an *Alexander filtration* consisting of an increasing sequence of subcomplexes. The filtration level of an element $x \in \mathrm{CF}(K)$ is denoted $\mathrm{Alex}(x)$.
- There is a similar filtration, called the *algebraic filtration*, and filtration levels of elements are denoted $\mathrm{Alg}(x)$.
- There is an action of the Laurent polynomial ring $\mathbb{F}[U, U^{-1}]$ on $\mathrm{CF}(K)$. The action of U commutes with ∂, lowers gradings by 2, and lowers Alexander and algebraic filtration levels by 1.
- Let Λ denote $\mathbb{F}[U, U^{-1}]$. As a Λ–module, $\mathrm{CF}(K)$ is free on a finite set of generators, $\{x_i\}_{1 \le i \le r}$. To simplify notation, we suppress the indexing set. The set of elements $\{U^k x_i\}_{k \in \mathbb{Z}}$ forms a bifiltered graded basis for $\mathrm{CF}(K)$: for any triple of integers, (g, m, n), the subspace of $\mathrm{CF}(K)$ spanned by elements of grading g, Alexander filtration level less than or equal to m, and algebraic filtration level less than or equal to n, has as basis a subset of $\{U^k x_i\}$.
- The singly filtered complex $(\mathrm{CF}(K), \mathrm{Alg})$ with Λ–structure is chain homotopy equivalent to complex $\mathcal{T} \cong \Lambda$ where $1 \in \Lambda$ has grading 0 and filtration level 0, and the boundary map is trivial. (The same statement holds for the Alexander grading, but we do not use this fact.)

The construction of $\mathrm{CF}(K)$ depends on a series of choices. However, there is a natural definition of chain homotopy equivalence for graded, bifiltered chain complexes with Λ–action. A key result of [9] is that in this sense, the chain homotopy equivalence class of $\mathrm{CF}(K)$ is a well-defined knot invariant.

As an example, Figure 1 presents a schematic diagram of the complex for the torus knot $T(3,7)$. As a Λ–module it has nine filtered generators, with algebraic and Alexander filtration levels indicated by the first and second coordinate, respectively. Five of the generators, indicated with black dots, have grading 0; the four white dots represent generators of grading one. The boundary map is indicated by the arrows. The rest of $\mathrm{CF}(K)$ is the direct sum of the U^k translates for $k \in \mathbb{Z}$ of this finite complex; for instance, applying U shifts the diagram one down and to the left.

3 Filtrations

We now discuss more general filtrations on vector spaces. In our applications, the vector space will be $\mathrm{CF}(K)$.

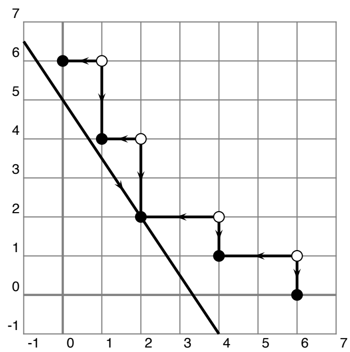

Figure 1: $\mathrm{CFK}^\infty(T(3,7))$

Definition 3.1 A *real-valued (discrete) filtration* on a vector space C is a collection of subspaces $\mathcal{F} = \{C_s\}$ indexed by the real numbers. This collection must satisfy the following properties:

(1) $C_{s_1} \subseteq C_{s_2}$ if $s_1 \le s_2$.

(2) $C = \bigcup_{s \in \mathbb{R}} C_s$.

(3) $\bigcap_{s \in \mathbb{R}} C_s = \{0\}$.

(4) **discreteness** C_{s_2}/C_{s_1} is finite-dimensional when $s_1 \le s_2$.

Given a discrete filtration $\mathcal{F} = \{C_s\}$ on C, we can define an associated function on C, which we temporarily also denote by $\mathcal{F}$, given by $\mathcal{F}(x) = \min\{s \in \mathbb{R} \mid x \in C_s\}$. Notice that $\mathcal{F}^{-1}((-\infty, s]) = C_s$.

Given an arbitrary real-valued function f on C, one can define an associated filtration with $C_s = \mathrm{Span}(f^{-1}((-\infty, s]))$. The resulting filtration need not be discrete.

Notation In cases in which more than one filtration might be under consideration, we will write $(C, \mathcal{F})_s$ rather than C_s.

Definition 3.2 A set of vectors $\{z_i\}$ in the real filtered vector space C is called a *filtered basis* if it is linearly independent and every C_s has some subset of $\{z_i\}$ as a basis. If C is also graded, $C = \bigoplus_{i=-\infty}^{\infty} G_i$, then we say the basis is a filtered graded basis if each $C_s \cap G_k$ has a subset of $\{z_i\}$ as a basis.

4 The definition of the filtration $\mathcal{F}_t$ on $\mathrm{CF}(K)$

For any $t \in [0,2]$, the convex combination of Alexander and algebraic filtrations, $\frac{t}{2}\,\mathrm{Alex} + \bigl(1-\frac{t}{2}\bigr)\,\mathrm{Alg}$, defines a real-valued function on $\mathrm{CF}(K)$, to which we associate a filtration denoted $\mathcal{F}_t$. That is, for all $s \in \mathbb{R}$, $(\mathrm{CF}(K), \mathcal{F}_t)_s$ is spanned by all vectors $x \in \mathrm{CF}(K)$ such that $\frac{t}{2}\,\mathrm{Alex}(x) + \bigl(1-\frac{t}{2}\bigr)\,\mathrm{Alg}(x) \le s$.

Theorem 4.1 *If $0 \le t \le 2$, the filtration $\mathcal{F}_t$ on $\mathrm{CF}(K)$ is a filtration by subcomplexes and is discrete. The action of U lowers filtration levels by 1.*

Proof To see that these are subcomplexes, suppose that $x \in (\mathrm{CF}(K), \mathcal{F}_t)_s$. Write $x = \sum x_i$, where $\frac{t}{2}\,\mathrm{Alex}(x_i) + \bigl(1-\frac{t}{2}\bigr)\,\mathrm{Alg}(x_i) \le s$ for all i. Since $\partial x = \sum \partial x_i$, we only need to check that $\partial x_i \in (\mathrm{CF}(K), \mathcal{F}_t)_s$ for each i. Let x_i have $\mathrm{Alex}(x_i) = a$ and $\mathrm{Alg}(x_i) = b$. Then $\mathrm{Alex}(\partial x_i) = a' \le a$ and $\mathrm{Alg}(\partial x_i) = b' \le b$. Since both $\frac{t}{2}$ and $1-\frac{t}{2}$ are nonnegative, $\frac{t}{2}a' + \bigl(1-\frac{t}{2}\bigr)b' \le \frac{t}{2}a + \bigl(1-\frac{t}{2}\bigr)b \le s$, as desired.

The discreteness of the filtration depends on two properties of $\mathrm{CF}(K)$. First, letting g denote the three-genus, $g_3(K)$, according to [8] one has $-g \le \mathrm{Alex}(x) - \mathrm{Alg}(x) \le g$ for all x. From this it follows that for given $s_1 < s_2$, there are k_1 and k_2 in $\mathbb{R}$ such that

$$(\mathrm{CF}(K), \mathrm{Alex})_{k_1} \subseteq (\mathrm{CF}(K), \mathcal{F}_t)_{s_1} \subseteq (\mathrm{CF}(K), \mathcal{F}_t)_{s_2} \subseteq (\mathrm{CF}(K), \mathrm{Alex})_{k_2}.$$

(The values of k_1 and k_2 can be chosen to be $s_1 - \bigl(1-\frac{t}{2}\bigr)g$ and $s_2 + \bigl(1-\frac{t}{2}\bigr)g$, respectively, but we do not need this level of detail.) Second, the Alexander filtration is discrete, so the quotient $(\mathrm{CF}(K), \mathrm{Alex})_{k_2}/(\mathrm{CF}(K), \mathrm{Alex})_{k_1}$ is finite-dimensional.

Finally, that U lowers filtration levels by one is immediate. □

5 The definition of $\Upsilon_K(t)$

For each $t \in [0,2]$ and for all $s \in \mathbb{R}$, the set $(\mathrm{CF}(K), \mathcal{F}_t)_s \subset \mathrm{CF}(K)$ is a subcomplex. Thus, we can make the following definition:

Definition 5.1 Let

$$\nu(\mathrm{CF}(K), \mathcal{F}_t) = \min\bigl\{s \mid H_0((\mathrm{CF}(K), \mathcal{F}_t)_s) \to H_0(\mathrm{CF}(K)) \text{ is surjective}\bigr\}.$$

Definition 5.2 $\Upsilon_K(t) = -2\nu(\mathrm{CF}(K), \mathcal{F}_t)$.

5.1 Example

Consider the knot $K = T(3,7)$ with $\mathrm{CF}(K)$ as illustrated in Figure 1. The portion of the complex shown has homology $\mathbb{F}$ at grading 0.

The subcomplex $(\mathrm{CF}(K), \mathcal{F}_t)_s$ is generated by the bifiltered generators with Alexander and algebraic filtration levels satisfying

$$\mathrm{Alex} \leq \frac{2}{t}s + \Big(1 - \frac{2}{t}\Big)\,\mathrm{Alg}\,. \tag{5-1}$$

Observation The lattice points which contain a filtered generator at filtration level t all lie on a line of slope

$$m = 1 - \frac{2}{t},$$

with lattice points parametrized by the pair $(\mathrm{Alg}, \mathrm{Alex})$. Alternatively, if a line of slope m contains distinct lattice points representing bifiltration levels of generators at the same $\mathcal{F}_t$ filtration level, then

$$t = \frac{2}{1-m}.$$

In the diagram for $T(3,7)$ shown in Figure 1, the illustrated line in the plane corresponds to $t = \frac{4}{5}$ and $s = 2$. Since the lower half-plane bounded by this line contains a generator of $H_0(\mathrm{CF}(K))$, while no half-plane bounded by a parallel line with smaller value of s contains such a generator, we have $\Upsilon_K\big(\frac{4}{5}\big) = -2(2) = -4$.

Continuing with $K = T(3,7)$, it is now clear that for $m < -2$ — that is, for $t < \frac{2}{3}$ — the least s for which $(\mathrm{CF}(K), \mathcal{F}_t)_s$ contains a generator of $H_0(\mathrm{CF}(K))$ corresponds to the line through $(0,6)$, which has filtration level $\frac{t}{2}6 + \big(1 - \frac{t}{2}\big)0 = 3t$.

For $-2 < m < -1$ — that is, for $\frac{2}{3} < t < 1$ — the least s for which $(\mathrm{CF}(K), \mathcal{F}_t)_s$ contains a generator of $H_0(\mathrm{CF}(K))$ corresponds to the line through $(2,2)$, which has filtration level $\frac{t}{2}2 + \big(1 - \frac{t}{2}\big)2 = 2$. Multiplying by -2 and checking the value $t = \frac{2}{3}$ yields

$$\Upsilon_{T(3,7)}(t) = \begin{cases} -6t & \text{if } 0 \leq t \leq \frac{2}{3}, \\ -4 & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases}$$

6 An alternative definition of ν and Υ

In the appendix we prove Theorem A.1, which has as an immediate consequence the following result:

Theorem 6.1 *The filtered graded chain complex* $(\mathrm{CF}(K), \mathcal{F}_t)$ *is isomorphic to a filtered graded complex of the form*

$$\mathcal{T} \oplus \mathcal{A},$$

where $\mathcal{T} \oplus \mathcal{A}$ *has the structure of a* Λ*–module and the isomorphism is a* Λ*–module isomorphism. The summand* $\mathcal{T}$ *has the properties that*

(1) *it is isomorphic to* Λ *as a* Λ*–module;*

(2) *the element* $1 \in \Lambda \cong \mathcal{T}$ *has grading* 0.

Furthermore, $\mathcal{A}$ *is acyclic as an unfiltered complex.*

Notice that since all gradings in $\mathcal{T}$ are even, the boundary operator restricted to $\mathcal{T}$ is trivial.

When placed in this simple form, the computation of $\nu((\mathrm{CF}(K), \mathcal{F}_t))$ is simple: it is the $\mathcal{F}_t$ filtration level of $1 \in \Lambda \cong \mathcal{T}$. Hence, we have the following result:

Corollary 6.2 $\Upsilon_K(t)$ *equals* -2 *times the* $\mathcal{F}_t$*–filtration level of* $1 \in \Lambda \cong \mathcal{T}$ *for the decomposition* $(\mathrm{CF}(K), \mathcal{F}_t) \cong \mathcal{T} \oplus \mathcal{A}$.

7 Products and additivity

According to [9], there is a (graded) chain homotopy equivalence of complexes

$$\mathrm{CF}(K_1) \otimes_\Lambda \mathrm{CF}(K_2) \simeq \mathrm{CF}(K_1 \# K_2)$$

that preserves the Λ–structure.

Each of $\mathrm{CF}(K_1)$, $\mathrm{CF}(K_2)$ and $\mathrm{CF}(K_1 \# K_2)$ has an algebraic filtration. To distinguish these, we write Alg^1, Alg^2 and $\mathrm{Alg}^{1,2}$. Similarly, the Alexander and $\mathcal{F}_t$ filtrations will be distinguished with superscripts.

Momentarily we write $\mathrm{CF}_1 = \mathrm{CF}(K_1)$ and $\mathrm{CF}_2 = \mathrm{CF}(K_2)$. For each $t \in [0,2]$ the filtrations $\mathcal{F}_t^1$ and $\mathcal{F}_t^2$ on CF_1 and CF_2 induce a filtration $\mathcal{F}_t^1 \otimes \mathcal{F}_t^2$ on $\mathrm{CF}_1 \otimes_\Lambda \mathrm{CF}_2$, defined via

$$(\mathrm{CF}_1 \otimes_\Lambda \mathrm{CF}_2, \mathcal{F}_t^1 \otimes \mathcal{F}_t^2)_s = \mathrm{Image}\Big(\bigoplus_{s_1+s_2=s} (\mathrm{CF}_1, \mathcal{F}_t^1)_{s_1} \otimes_{\mathbb{F}} (\mathrm{CF}_2, \mathcal{F}_t^2)_{s_2} \to (\mathrm{CF}_1, \mathcal{F}_t^1) \otimes_\Lambda (\mathrm{CF}_2, \mathcal{F}_t^2) \Big).$$

Notice that the direct sum is infinite and each summand is infinitely generated. Again, according to [9], for the connected sum of knots, the equivalence

$$\mathrm{CF}(K_1) \otimes_\Lambda \mathrm{CF}(K_2) \simeq \mathrm{CF}(K_1 \# K_2)$$

is a filtered equivalence for both the Alexander and algebraic filtrations. To state this explicitly,

$$(\mathrm{CF}(K_1), \mathrm{Alex}^1) \otimes_\Lambda (\mathrm{CF}(K_2), \mathrm{Alex}^2) \simeq (\mathrm{CF}(K_1 \# K_2), \mathrm{Alex}^{1,2})$$

and

$$(\mathrm{CF}(K_1), \mathrm{Alg}^1) \otimes_\Lambda (\mathrm{CF}(K_2), \mathrm{Alg}^2) \simeq (\mathrm{CF}(K_1 \# K_2), \mathrm{Alg}^{1,2}).$$

Theorem 7.1 *For all* $t \in [0,1]$,

$$(\mathrm{CF}(K_1), \mathcal{F}^1_t) \otimes_\Lambda (\mathrm{CF}(K_2), \mathcal{F}^2_t) \simeq (\mathrm{CF}(K_1 \# K_2), \mathcal{F}^{1,2}_t).$$

Proof Fix bases $\{x_i\}$ and $\{y_i\}$ for the free Λ–modules $\mathrm{CF}(K_1)$ and $\mathrm{CF}(K_2)$ such that the sets of all translates $\{U^k x_i\}$ and $\{U^k y_i\}$ for $k \in \mathbb{Z}$ form graded bifiltered bases for $\mathrm{CF}(K_1)$ and $\mathrm{CF}(K_2)$ (as $\mathbb{F}$–vector spaces). The $\mathbb{F}$–vector space $\mathrm{CF}(K_1) \otimes_\Lambda \mathrm{CF}(K_2)$ is generated by the set of all tensor products, $\{U^k x_i \otimes U^j x_l\}$, but note that these do not form a basis; for instance, $Ux \otimes y = x \otimes Uy$.

When selecting elements from $\{U^k x_i\}$, we will sometimes refer to them as x, and similarly for y. Note that in particular, for such basis elements, $\mathrm{Alg}^{1,2}(x \otimes y) = \mathrm{Alg}^1(x) + \mathrm{Alg}^2(y)$ and $\mathrm{Alex}^{1,2}(x \otimes y) = \mathrm{Alex}^1(x) + \mathrm{Alex}^2(y)$.

The proof of the theorem consists of showing that the filtrations $\mathcal{F}^1_t \otimes \mathcal{F}^2_t$ and $\mathcal{F}^{1,2}_t$ on $\mathrm{CF}(K_1) \otimes_\Lambda \mathrm{CF}(K_2)$ are the same.

If an element $z \in \mathrm{CF}(K_1) \otimes_\Lambda \mathrm{CF}(K_2)$ has $\mathcal{F}^{1,2}_t$ filtration level s, then it can be written as the sum of elements $x \otimes y$ with

$$\frac{t}{2}\,\mathrm{Alex}(x \otimes y) + \left(1 - \frac{t}{2}\right)\mathrm{Alg}(x \otimes y) \le s.$$

This is the same as

$$\frac{t}{2}\,\mathrm{Alex}(x) + \left(1 - \frac{t}{2}\right)\mathrm{Alg}(x) + \frac{t}{2}\,\mathrm{Alex}(y) + \left(1 - \frac{t}{2}\right)\mathrm{Alg}(y) \le s.$$

This implies that $\mathcal{F}^1_t(x) + \mathcal{F}^2_t(y) \le s$. This in turn implies that $(\mathcal{F}^1_t \otimes \mathcal{F}^2_t)(x \otimes y) \le s$. Thus, $(\mathcal{F}^1_t \otimes \mathcal{F}^2_t)(z) \le \mathcal{F}^{1,2}_t(z)$ for all $z \in \mathrm{CF}(K_1) \otimes_\Lambda \mathrm{CF}(K_2)$.

Similarly, suppose that $z \in \mathrm{CF}(K_1) \otimes_\Lambda \mathrm{CF}(K_2)$ has $\mathcal{F}^1_t \otimes \mathcal{F}^2_t$ filtration level s. Then it is the sum of elements $x \otimes y$, each of which satisfies $\mathcal{F}^t_1(x) + \mathcal{F}^t_2(y) \le s$. This can be expanded and rewritten as

$$\frac{t}{2}(\mathrm{Alex}(x) + \mathrm{Alex}(y)) + \left(1 - \frac{t}{2}\right)(\mathrm{Alg}(x) + \mathrm{Alg}(y)) \le s.$$

In other words, z is the sum of elements $x \otimes y$ with $\mathcal{F}_t^{1,2}(x \otimes y) \le s$. Hence, $\mathcal{F}_t^{1,2}(x \otimes y) \le s$. □

Theorem 7.1, along with Theorem 6.1, offers a fast proof of the additivity of Υ:

Theorem 7.2 $\Upsilon_{K_1 \# K_2}(t) = \Upsilon_{K_1}(t) + \Upsilon_{K_2}(t)$ *for each* $t \in [0,2]$.

Proof One only needs to check this for complexes of the form $\mathcal{T} \oplus \mathcal{A}$, as given in Theorem 6.1. Acyclic summands do not affect the value of $\Upsilon_K(t)$. Thus, we only need consider the case of complexes $\mathcal{T}(K_1) \otimes_\Lambda \mathcal{T}(K_2)$, for which the statement is clear. □

Similarly, Theorem 6.1 offers a fast proof of the following:

Theorem 7.3 $\Upsilon_{-K}(t) = -\Upsilon_K(t)$ *for an arbitrary knot* K.

Proof According to [9], the complexes $\mathrm{CF}(K)$ and $\mathrm{CF}(-K)$ are duals: $\mathrm{CF}(-K) \cong \mathrm{CF}(K)^*$. More precisely, $\mathrm{CF}(-K)$ is isomorphic to the complex $\mathrm{Hom}_{\mathbb{F}}(\mathrm{CF}(K), \mathbb{F})$, having underlying vector space the space of $\mathbb{F}$–homomorphisms with finite-dimensional (that is, finite) support.

If we fix a basis $\{x_i\}$ of $\mathrm{CF}(K)$ as a Λ–module such that the set $\{U^k x_i\}$ forms a graded bifiltered basis of $\mathrm{CF}(K)$, then we can denote the elements of the dual basis by $(U^k x_i)^*$. The dual complex is readily understood in terms of these bases:

(1) An easy exercise shows that the action of U on the dual basis is of the form $U(U^k x_i)^* = (U^{k-1} x_i)^*$. In particular, the set $\{x_i^*\}$ forms a basis for the Λ–module $\mathrm{CF}(K)^*$.

(2) For any filtration $\mathcal{F}$ on $\mathrm{CF}(K)$, we can define a filtration $\mathcal{F}^*$ on the dual space as follows:

$$(\mathrm{CF}(K)^*, \mathcal{F}^*)_s = \big\{\phi \in \mathrm{CF}(K)^* \mid \phi\big((\mathrm{CF}(K), \mathcal{F})_{-s'}\big) = 0 \text{ for all } s' > s\big\}.$$

The choice of signs ensures that the dual filtration is increasing. Thus, $\mathcal{F}^*(x_i^*) = -\mathcal{F}(x_i)$.

(3) The boundary operator for the dual space acts in the expected way with respect to basis elements: if x is a component of ∂y, then y^* is a component of ∂x^*.

These three observations are easily summarized in terms of diagrams such as in Figure 1: the diagram for $\mathrm{CF}(-K)$ is obtained from that for $\mathrm{CF}(K)$ by rotating the figure by 180 degrees around the origin and reversing all the arrows.

There are two filtrations on $\mathrm{CF}(-K)$ of interest. The first is $\frac{t}{2}\,\mathrm{Alex}^* + \big(1-\frac{t}{2}\big)\,\mathrm{Alg}^*$; the second is $\mathcal{F}_t^* = \big(\frac{t}{2}\,\mathrm{Alex} + \big(1-\frac{t}{2}\big)\,\mathrm{Alg}\big)^*$. By using the chosen basis and its dual basis, it is possible to see that these two filtrations are the same, as follows. We use coordinates (i,j) for the plane. For a basis vector x, its dual vector x^* is in $\mathcal{F}_t^*$ if and only if it lies on or above the line $\frac{t}{2}j + \big(1-\frac{t}{2}\big)i = -t$. If this is the case, then when rotated 180 degrees about the origin it lies on or below the line $\frac{t}{2}j + \big(1-\frac{t}{2}\big)i = t$. These are precisely the dual vectors for which $\frac{t}{2}\,\mathrm{Alex}^* + \big(1-\frac{t}{2}\big)\,\mathrm{Alg}^* \le t$.

The proof of the theorem is now reduced to an elementary calculation for the simple complex $\mathcal{T}(K)$ and its dual $\mathcal{T}(K)^*$. □

8 Basic properties of $\Upsilon_K(t)$ and $\Upsilon_K'(t)$

We now present some basic results concerning $\Upsilon_K(t)$ and its derivative. An initial observation is that $\Upsilon_K(0) = 0$ and, since $\mathrm{CF}(K)$ is finitely generated, $\Upsilon_K(t)$ is continuous at 0. Thus, we focus on $t > 0$.

Theorem 8.1

(1) *$\Upsilon(K)$ is a continuous piecewise linear function for every knot K.*

(2) *At a nonsingular point of $\Upsilon_K'(t)$, the value of $|\Upsilon_K'(t)|$ is $|i-j|$, where (i,j) is the bifiltration level of some filtered generator of $\mathrm{CF}(K)$ with homological grading 0.*

(3) *Singularities in $\Upsilon_K'(t)$ can occur only at values of t such that some line of slope $1-\frac{2}{t}$ contains at least two lattice points, (i,j) and (i',j'), each of which represents the algebraic and Alexander gradings of filtered generators of $\mathrm{CF}(K)$ of homological grading 0.*

(4) *If $\Upsilon_K'(t)$ has a singularity at t, then the jump in $\Upsilon_K'(t)$ at t, denoted $\Delta\Upsilon_K'(t)$, satisfies $|\Delta\Upsilon_K'(t)| = \frac{2}{t}|i-i'|$ for some pair (i,i') for which there are lattice points (i,j) and (i',j') as in the previous item.*

Proof The proof is discussed in terms of the diagram of the complex, as illustrated for the knot $T(3,7)$ in the previous section.

Suppose $\Upsilon_K(t) = -2s$ and there is exactly one lattice point (i,j) with $\frac{t}{2}j + \big(1-\frac{t}{2}\big)i = s$ which represents the bifiltration level of a filtered generator of $\mathrm{CF}(K)$. (This will be the case for all but a finite number of values of t.) For a nearby t, say t', the value of $\Upsilon_K(t') = -2s'$ will be such that the same vertex (at (i,j)) lies on the line

$\frac{t'}{2}j + \left(1 - \frac{t'}{2}\right)i = s'$. That is, for all nearby values of t, the value of s is given by $\frac{t}{2}j + \left(1 - \frac{t}{2}\right)i$. Written differently,

$$\Upsilon_K(t) = -2i + (i - j)t.$$

In particular, we see that $\Upsilon_K(t)$ is piecewise linear off a finite set.

Now consider a singular value of t, at which $\Upsilon_K(t) = -2s$ and there are two or more pairs (i, j) for which $\frac{t}{2}j + \left(1 - \frac{t}{2}\right)i = s$. Notice that this line in the (i, j)–plane has slope $m = 1 - \frac{2}{t}$. For t' close to t and $t' < t$, we have

$$\Upsilon_K(t') = -2i + (i - j)t'$$

for one of those pairs (i, j). If t' is near t and $t' > t$, then

$$\Upsilon_K(t') = -2i' + (i' - j')t'$$

for another of these pairs, (i', j'), which may be the same. Notice that these are equal at t, giving the continuity of $\Upsilon_K(t)$.

We now see that a singularity of $\Upsilon_K(t)$ occurs if $j - i \neq j' - i'$. With these observations, the proofs of (1), (2) and (3) are complete.

For (4), our computations have shown that the change in $\Upsilon'_K(t)$, denoted $\Delta\Upsilon'_K(t)$, is given by $\Delta\Upsilon'_K(t) = (j - j') - (i - i')$ for some appropriate (i, j) and (i', j'). Since both are assumed to lie on a line of slope $1 - \frac{2}{t}$, we have $j - j' = \left(1 - \frac{2}{t}\right)(i - i')$, so

$$\Delta\Upsilon'_K(t) = \left(1 - \frac{2}{t}\right)(i - i') - (i - i') = -\frac{2}{t}(i - i').$$

This completes the proof of the theorem. □

Corollary 8.2 *For any knot K and for $t = \frac{p}{q}$ with $\gcd(p, q) = 1$,*

$$\frac{t}{2}\Delta\Upsilon'_K(t) = kp,$$

where k is some integer if p is odd, or half-integer if p even.

Proof By Theorem 8.1(4), $\left|\frac{t}{2}\Delta\Upsilon'_K(t)\right| = |i - i'|$ for some pair of integers i and i', where there are two lattice points on a line of slope $m = 1 - \frac{2}{t}$. Thus, we want to constrain the possible differences between the first coordinates of such lattice points.

For $t = \frac{p}{q}$, we have $m = -(2q - p)/p$. Since $\gcd(p, q) = 1$, in reduced terms this is either $m = -(2q - p)/p$ or $m = -\left(q - \frac{p}{2}\right)/\frac{p}{2}$ if p is odd or even, respectively. Two lattice points on such a line have first coordinates differing by a multiple of p or of $\frac{p}{2}$ if p is odd or even, respectively. The completes the proof. □

9 The three-genus, $g_3(K)$

Theorem 9.1 $|\Upsilon'_K(t)| \leq g_3(K)$ *for nonsingular points of* $\Upsilon'_K(t)$.

Proof According to [8], if K is of genus g, then all elements of $\mathrm{CF}(K)$ have filtration level (i,j), where

$$-g \leq i - j \leq g.$$

It follows immediately from Theorem 8.1(2) that $|\Upsilon'_K(t)| \leq g_3(K)$. □

We also observe that the genus of K constrains the possible points of singularity of $\Upsilon'_K(t)$.

Theorem 9.2 *Suppose that* $\Upsilon'_K(t)$ *has a singularity at* $t = \frac{p}{q}$, *with* $\gcd(p,q) = 1$. *Then:*

- *If* p *is odd,* $q \leq g_3(K)$.
- *If* p *is even,* $q \leq 2g_3(K)$.

Proof Suppose that a line of slope $m = -\frac{a}{b}$, where $0 < b < a$, contains two distinct points of the form (i,j) with $|i-j| \leq g_3(K)$. It follows quickly that the genus bound implies

$$a \leq 2g_3(K) - b.$$

To express this in terms of t, suppose $t = \frac{p}{q}$ with $\gcd(p,q) = 1$. Then

$$m = 1 - \frac{2}{t} = -\frac{2q-p}{p}.$$

If p is odd, then $\gcd(2q-p,p) = 1$. If p is even, say $p = 2k$, then $\gcd(2q-p,p) = \gcd(2q,p) = 2$ and $m = -(q-k)/k$, with q and k relatively prime.

In the first case, with p odd, we have $2q - p \leq 2g_3(K) - p$, so $q \leq g_3(K)$.

In the second case, with p even, we have $q - k \leq 2g_3(K) - k$, so $q \leq 2g_3(K)$. □

10 $\Upsilon_K(t)$ as a knot concordance invariant

If knots K_1 and K_2 are concordant, then there is an equality among d–invariants: $d(S^3_N(K_1), \mathfrak{s}_m) = d(S^3_N(K_2), \mathfrak{s}_m)$ for all $N \in \mathbb{Z}$ and $m \in \mathbb{Z}$ with $-\frac{N-1}{2} \leq m \leq \frac{N-1}{2}$. Here $S^3_N(K)$ denotes N surgery on K, d is the Heegaard Floer correction term, and $\mathfrak{s}_m$ is a Spin^c structure, with m given by a specific enumeration of Spin^c structures;

all are described in [6]. (In the case that N is odd, this range of m includes all possible Spin^c structures.)

If N is large, then $d(S^3_N(K_1), \mathfrak{s}_0) = D(K) + S(N)$, where $D(K)$ is the largest grading of a class z in the homology of $\mathrm{CF}(K)_{\{i \le 0, j \le 0\}}$ for which $U^k z$ is nontrivial for all $k > 0$, and $S(N)$ is some rational function defined on the integers, independent of K.

In the case that K is slice, we see that the maximal grading $D(K) = D(u)$, where u is the unknot. This implies that $D(K) = 0$ for a slice knot K. We have a nesting of complexes

$$\mathrm{CF}(K)_{\{i \le 0,\, j \le 0\}} \subset (\mathrm{CF}(K), \mathcal{F}_t)_0.$$

Since $(0,0)$ is at $\mathcal{F}_t$ filtration level 0, it follows that $\nu(\mathrm{CF}(K), \mathcal{F}_t) \le 0$; thus $\Upsilon_K(t) \ge 0$.

However, $-K$ is also slice, so $-\Upsilon_K(t) \ge 0$. It follows that $\Upsilon_K(t) = 0$. An additive invariant of knots that vanishes on slice knots is a concordance invariant.

11 The concordance-genus

The concordance-genus $g_c(K)$ of a knot K, defined in [4], is the minimal genus among all knots concordant to K. Since $\Upsilon_K(t)$ is a concordance invariant, the genus bounds in Section 9 apply to the concordance genus.

Theorem 11.1 *For all nonsingular points of $\Upsilon_K(t)$, $|\Upsilon'_K(t)| \le g_c(K)$. The jumps in $\Upsilon'_K(t)$ occur at rational numbers $\frac{p}{q}$. For p odd, $q \le g_c(K)$. If p is even, $\frac{q}{2} \le g_c(K)$.*

12 Bounds on the four-genus, $g_4(K)$

Let $\mathrm{CF}(K)_{0,m}$ denote the bifiltered subcomplex $\mathrm{CF}(K)_{\{i \le 0,\, j \le m\}}$. We let $\nu^-(K)$ denote the minimum value of m such that the homology of $\mathrm{CF}(K)_{0,m}$ contains a nontrivial grading 0 element of the homology of $\mathrm{CF}(K)$, which we recall is isomorphic to Λ with 1 at grading 0. There is the following result of Hom and Wu [1], built from work of Rasmussen [10]. (In [1] the invariant ν^+ is described; the equivalence with ν^- is presented in [5].)

Proposition 12.1 [1, Proposition 2.4] $\nu^- \le g_4(K)$.

Based on this, we show that $\Upsilon_K(t)$ provides a bound on $g_4(K)$.

Theorem 12.2 $|\Upsilon_K(t)| \leq t g_4(K)$ *for all* $t \in [0,2]$.

Proof Since $(0,m)$ is at $\mathcal{F}_t$ filtration level $\frac{tm}{2}$, we have the containment

$$\mathrm{CF}(K)_{0,m} \subset (\mathrm{CF}(K), \mathcal{F}_t)_{tm/2}.$$

Since $\mathrm{CF}(K)_{0,\nu^-}$ contains an element of grading 0 in the homology of $\mathrm{CF}(K)$, so does the subcomplex $(\mathrm{CF}(K), \mathcal{F}_t)_{t\nu^-/2}$. Thus, $\nu(\mathrm{CF}(K), \mathcal{F}_t) \leq \frac{1}{2} t\nu^-$. By the previous proposition, $\nu(\mathrm{CF}(K), \mathcal{F}_t) \leq \frac{1}{2} t g_4(K)$.

Considering $-K$, we have $\nu(\mathrm{CF}(-K), \mathcal{F}_t) \leq \frac{1}{2} t g_4(-K)$; it follows that

$$-\nu(\mathrm{CF}(K), \mathcal{F}_t) \leq \tfrac{1}{2} t g_4(K).$$

Combining these yields

$$|\nu(\mathrm{CF}(K), \mathcal{F}_t)| \leq \tfrac{1}{2} t g_4(K).$$

Multiplying by -2 yields the desired conclusion. □

13 Crossing change bounds

Here we sketch a proof of [5, Proposition 1.10]. The argument is essentially the same as used in [3] to prove the corresponding fact about $\tau(K)$.

Theorem 13.1 *Let K_- and K_+ be knots with identical diagrams, except at one crossing which is either negative or positive, respectively. Then, for $t \in [0,1]$,*

$$\Upsilon_{K_+}(t) \leq \Upsilon_{K_-}(t) \leq \Upsilon_{K_+}(t) + t.$$

Proof First note that $K_- \# -K_+$ can be changed into the slice knot $K_+ \# -K_+$ by changing a negative crossing to positive. Thus, $g_4(K_- \# -K_+) \leq 1$. It follows that

$$-t \leq \Upsilon_{K_-}(t) - \Upsilon_{K_+}(t) \leq t. \tag{13-1}$$

Next, note that $K_- \# -K_+ \# \, T(2,3)$ can be changed into the slice knot $K_+ \# -K_+$ by changing one negative crossing to positive and one positive crossing to negative. Thus, it too has four-genus at most 1: it bounds a singular disk with two singularities of opposite sign, and these can be tubed together. A simple computation for $T(2,3)$ yields $\Upsilon_{T(2,3)}(t) = -t$ for $0 \leq t \leq 1$. Thus,

$$-t \leq \Upsilon_{K_-}(t) - \Upsilon_{K_+}(t) - t \leq t,$$

which we rewrite as

$$0 \leq \Upsilon_{K_-}(t) - \Upsilon_{K_+}(t) \leq 2t. \tag{13-2}$$

Combining (13-1) and (13-2),

$$0 \le \Upsilon_{K_-}(t) - \Upsilon_{K_+}(t) \le t.$$

Adding $\Upsilon_{K_+}(t)$ to all terms yields the desired conclusion,

$$\Upsilon_{K_+}(t) \le \Upsilon_{K_-}(t) \le \Upsilon_{K_+}(t) + t. \qquad \square$$

Note This argument can be easily modified to show that if there is a singular concordance from K to J with a single positive double point, then $\Upsilon_K(t) \le \Upsilon_J(t) \le \Upsilon_K(t) + t$.

14 The Ozsváth–Szabó τ–invariant and $\Upsilon_K(t)$ for small t

For small t, $\Upsilon_K(t)$ is determined by the τ invariant defined in [7]. We review the definition below. Here is the statement of the result:

Theorem 14.1 *For t small, $\Upsilon_K(t) = -\tau(K)t$.*

The subquotient complex $\mathrm{CF}(K)_{\{i\le 0\}}/\mathrm{CF}(K)_{\{i<0\}}$ will be denoted $\widehat{\mathrm{CF}}(K)$. (Usually, $\widehat{\mathrm{CF}}$ is written $\widehat{\mathrm{CFK}}$.) It is filtered by the Alexander filtration and has homology $\mathbb{F}$, supported in grading 0. The invariant $\tau(K)$ is defined to be the least integer τ such that the map on homology $H_0(\widehat{\mathrm{CF}}(K)_{\{j\le\tau\}}) \to H_0(\widehat{\mathrm{CF}}(K)) \cong \mathbb{F}$ is surjective.

We wish to relate $\tau(K) = \tau$ to an invariant of $\mathrm{CF}(K)$. The needed technical result is the following:

Lemma 14.2 *If $\tau(K) = \tau$, then there is a cycle $w \in \mathrm{CF}(K)_{\{i\le 0,\, j\le\tau\}\cup\{i<0\}}$ representing a nontrivial element in $H_0(\mathrm{CF}(K))$.*

Proof From the definition of τ we see that there is a chain $x \in \mathrm{CF}(K)_{\{i\le 0,\, j\le\tau\}\cup\{i<0\}}$ that in the quotient $\widehat{\mathrm{CF}}(K)$ is a cycle that represents a generator of the homology group $H_0(\widehat{\mathrm{CF}}(K))$.

Since the chain x represents a cycle in $\widehat{\mathrm{CF}}(K)$, it has the property that $\partial x = y$, where $y \in \mathrm{CF}(K)_{i<0}$. Note that y is a cycle and $\mathrm{gr}(y) = -1$. Since $H_{-1}(\mathrm{CF}(K)_{i<0}) = 0$, there is a chain $z \in \mathrm{CF}(K)_{i<0}$ with $\partial z = y$. Thus, $x + z$ is a cycle in the complex $\mathrm{CF}(K)_{\{i\le 0,\, j\le\tau\}\cup\{i<0\}}$. The map $H_0(\mathrm{CF}(K)_{i\le 0}) \to H_0(\widehat{\mathrm{CF}}(K))$ is an isomorphism; both groups are isomorphic to $\mathbb{F}$. Thus, $x+z$ represents a generator of $H_0(\mathrm{CF}(K)_{i\le 0})$. The map $H_0(\mathrm{CF}(K)_{i\le 0}) \to H_0(\mathrm{CF}(K))$ is an isomorphism, completing the proof. $\square$

Proof of Theorem 14.1 For t small, we consider the filtration $\mathcal{F}_t$ and the filtration level $s = \frac{t}{2}\tau$. Then one has $\mathrm{CF}(K)_s = \mathrm{CF}(K)_{\{i \leq 0,\, j \leq \tau\} \cup \{i<0\}}$. By Lemma 14.2, this subcomplex contains a cycle that represents an element of grading 0 in $H(\mathrm{CF}(K))$. Thus, for this $\mathcal{F}_t$ filtration, $\nu \leq \frac{t}{2}\tau$.

On the other hand, suppose that $\nu < \frac{t}{2}\tau$. Then there would exist a cycle

$$z \in \mathrm{CF}(K)_{\{i \leq 0,\, j \leq \tau - 1\} \cup \{i<0\}}$$

representing a generator of $H(\mathrm{CF}(K))$ of grading 0. However, the image of z in $\widehat{\mathrm{CF}}(K)$ would be an element in $\widehat{\mathrm{CF}}(K)_{\tau-1}$ that represents a generator of $H_0(\widehat{\mathrm{CF}}(K))$. But τ is by definition the lowest level at which this can occur. Thus, we see that $\nu = \frac{t}{2}\tau$.

To conclude, recall that $\Upsilon_K(t) = -2\nu$, so $\Upsilon_K(t) = -\tau(K)t$, as desired. □

Note With care, one can check that in this argument, the condition that t be small can be made precise by requiring that $t < 1/g_3(K)$. Of course, once the result is established for some set of small t, then Theorem 9.2 provides the bound $t < 1/g_3(K)$.

15 Equivalence of definitions of $\Upsilon_K(t)$

In this section we explain why $\Upsilon_K(t)$ as defined here agrees with that of [5].

Beginning with $\mathrm{CF}(K)$, a new complex $t\mathrm{CF}(K)$ can be constructed as follows. As an $\mathbb{F}$–vector space,

$$t\mathrm{CF}(K) = \mathrm{CF}(K) \otimes_\Lambda \mathbb{F}[v^{1/n}, v^{-1/n}],$$

where U acts on $\mathbb{F}[v^{1/n}]$ via multiplication by v^2. This has the structure of an $\mathbb{F}[v^{1/n}, v^{-1/n}]$–module. To simplify notation, we write $\Lambda' = \mathbb{F}[v^{1/n}, v^{-1/n}]$.

There are (rational) filtrations Alg and Alex on $t\mathrm{CF}(K)$ which are consistent with those on the Λ–submodule $\mathrm{CF}(K)$. The action of $v^{1/n}$ lowers filtration levels by $1/2n$. Thus, $U = v^2$ lowers filtration levels by 1, as it should. Similarly, the Maslov grading $M(x)$ naturally extends to $t\mathrm{CF}(K)$ so that the action of $v^{1/n}$ lowers this grading by $1/n$, and thus $U = v^2$ continues to lower the Maslov grading by 2.

There is a rational grading on $t\mathrm{CF}(K)$ defined via the Maslov grading, M, along with the algebraic and Alexander filtrations. If x is an element at filtration level (i, j), then:

$$\mathrm{gr}_t(x) = M(x) - t(j - i). \qquad \text{(15-1)}$$

(In [5], only generators at algebraic filtration level 0 are used to define gr_t, so $i = 0$ and the formula $\mathrm{gr}_t(x) = M(x) - t\,\mathrm{Alex}(x)$ is presented.) One checks that U lowers

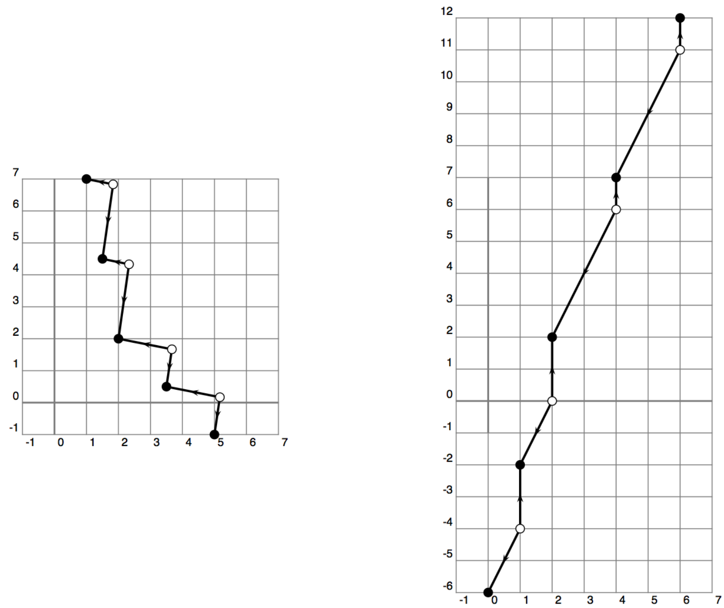

Figure 2: $t\mathrm{CF}_{t=1/3}(T(3,7))$ and $t\mathrm{CF}_{t=2}(T(3,7))$

gr_t–gradings by 2, so, on the extension to $t\mathrm{CF}(K)$, v lowers gradings by 1 and $v^{1/n}$ lowers gradings by $1/n$.

If x is a filtered generator of $\mathrm{CF}(K)$ with $\partial x = \sum y_l$, then the boundary ∂_t is defined so that $\partial_t x = \sum v^{\alpha_l} y_l \in t\mathrm{CF}(K)$, with the values of α_l given explicitly in [5]. This extends naturally to a boundary operator on all of $t\mathrm{CF}(K)$.

Given that the operator ∂_t is well-defined, it is a simple matter to determine its value. Suppose that x is a filtered generator of $\mathrm{CF}(K)$ at filtration level (i,j), Maslov grading g, and suppose also that $\partial x = \sum y_l$. Let y denote one of the terms in this sum, at filtration level (i', j'), necessarily of grading $g-1$. Then x, viewed as an element of $t\mathrm{CF}$, is of grading $g - t(j-i)$, and y has grading $g-1-t(j'-i')$. In $\partial_t x$, the term $v^\alpha y$ appears, and α is such that $\mathrm{gr}_t(v^\alpha y) = \mathrm{gr}_t(x) - 1$. Rewriting this, we have $(g-1) - t(j'-i') - \alpha = g - t(j-i) - 1$. That is,

$$\alpha = t((j-j') - (i-i')). \tag{15-2}$$

As two examples, Figure 2 illustrates the complexes $t\mathrm{CF}(K)$ for $K = T(3,7)$, with $t = \frac{1}{3}$ and $t = 2$. The construction is straightforward using (15-1) and the fact that v shifts along the diagonal a distance of $\frac{1}{2}$ down and to the left. The portion of the complex illustrated was chosen because its homology is $\mathbb{F}$ in grading 0 and represents the generator of the homology of $t\mathrm{CF}$ in grading 0. In the case that $t = \frac{1}{3}$, the full complex consists of the illustrated complex along with all its translates a distance $\frac{k}{6}$ for $k \in \mathbb{Z}$ along the diagonal. In the case of $t = 2$, the translates are those a distance $\frac{k}{2}$ along the diagonal.

It is apparent from these examples that the Alexander filtration is not a filtration of the chain complex, since some arrows increase the Alexander filtration level. However, as is easily verified, the algebraic filtration is a filtration on the chain complex.

Definition 15.1 For $t = \frac{m}{n}$, denote by $t\mathrm{CFK}^-(K)$ the complex $t\mathrm{CF}(K)_{i\leq 0}$.

Note In [5], this complex is denoted $t\mathrm{CFK}(K)$. In fact, it is the complex that is explicitly constructed. Here we first introduced the infinity complex to be consistent with our earlier constructions.

Definition 15.2 For $t = \frac{m}{n}$, let $\Upsilon_K(t)$ be the maximal grading of a class in the homology of $t\mathrm{CFK}^-(K)$ that maps to a nontrivial element in the homology of $t\mathrm{CF}(K)$. Equivalently, it is the maximal grading of a class in the homology of $t\mathrm{CFK}^-(K)$ which is not in the kernel of v^k for all $k > 0$.

Lemma 15.3 *The value of $\Upsilon_K(t)$ as just defined is equal to $-2s$, where s is the least number for which the homology of $t\mathrm{CF}(K)_{i\leq s}$ contains an element of grading 0 that represents a nontrivial element of the homology of $t\mathrm{CF}(K)$.*

Proof This follows from a simple change of coordinates. □

15.1 The two definitions of $\Upsilon_K(t)$ agree

Suppose that, using this definition of $\Upsilon_K(t)$, we have $\Upsilon_K(t) = -2s$. This implies that $t\mathrm{CF}(K)_{i\leq s}$ contains a cycle z representing a nontrivial generator of grading 0 in the homology of $t\mathrm{CF}(K)$. Write $z = \sum x_l$, where the x_l are filtered generators. Some x_l has filtration level (s, j), and none of the x_l has algebraic filtration level greater than s.

From the regrading formula given in (15-1), $\mathrm{gr}_t(x) = M(x) - t(j-i)$, we see that generators of $\mathrm{CF}(K)$ at filtration level (i, j) and grading 0 yield generators of grading 0 in $t\mathrm{CF}(K)$ at filtration level $\left(i + \frac{t}{2}(j-i), j + \frac{t}{2}(j-i)\right)$. (Recall that shifting down and to the left by t units decreases the grading by $2t$.) We are thus led to consider the transformation

$$(i, j) \mapsto \left(\left(1 - \frac{t}{2}\right)i + \frac{t}{2}j, -\frac{t}{2}i + \left(1 + \frac{t}{2}\right)j\right).$$

Its inverse is given by

$$(i, j) \mapsto \left(\left(1 + \frac{t}{2}\right)i - \frac{t}{2}j, \frac{t}{2}i + \left(1 - \frac{t}{2}\right)j\right).$$

Under this transformation, for a fixed value of s, the vertical line $\{(s,z) \mid z \in \mathbb{R}\}$ is carried to the line (in the CF(K)–plane) $\bigl\{\bigl(\bigl(1+\frac{t}{2}\bigr)s - \frac{t}{2}z, \frac{t}{2}s + \bigl(1-\frac{t}{2}\bigr)z\bigr) \mid z \in \mathbb{R}\bigr\}$. Relabeling the coordinate system (x,y), this is the line

$$y = \Bigl(1 - \frac{2}{t}\Bigr)x + \frac{2}{t}s.$$

Comparing with (5-1), we see that the homology of the filtered complex $(\mathrm{CF}(K), \mathcal{F}_t)_s$ contains a generator of grading 0 that is nontrivial in the homology of CF(K), and that this is not the case for $(\mathrm{CF}(K), \mathcal{F}_t)_{s'}$ for any $s' < s$. Thus, the value of $\Upsilon_K(t)$ as defined in Section 5 is $-2s$, and the definitions agree.

Appendix: A structure theorem for CF(K)

In [2, Chapter 11], vertical and horizontal reductions of CF(K) are discussed. That presentation applies to the filtered complex $(\mathrm{CF}(K), \mathcal{F}_t)$, but adjustments in the details would be required because, for instance, the horizontal and vertical filtrations are integer-valued rather than being real filtrations. Since the argument in the present case is straightforward, we present it in detail.

Viewed as a Λ–module, CF(K) is freely generated by a finite set $\{w_i\}_{1\le i\le m}$. We again simplify notation by suppressing the indexing set and write $\{w_i\}$. This set can be chosen so that the set $\{U^k w_i\}_{k\in\mathbb{Z}}$ forms a bifiltered graded basis for the $\mathbb{F}$–complex CF(K). We will refer to any such set $\{w_i\}$ as a Λ–basis for CF(K). A Λ–module change of basis among the w_i that preserves gradings and filtration levels induces a change of bifiltered graded basis for the $\mathbb{F}$–complex CF(K). We will refer to any such change of basis as a Λ–change of basis of CF(K). Analogous notation will be used when working with the filtered graded complex $(\mathrm{CF}(K), \mathcal{F}_t)$.

Theorem A.1 *Let $t \in [0,2]$. As a Λ–module,* CF(K) *has a basis $\{\alpha, \beta_1, \dots, \beta_k\}$, inducing a splitting of* CF(K) *(as a Λ–module) as the direct sum* $\mathrm{CF}(K) \cong \mathcal{T} \oplus \mathcal{A}$, *where $\mathcal{T}$ is freely generated by α and $\mathcal{A}$ is freely generated by $\{\beta_1, \dots, \beta_k\}$. This splitting has the following properties:*

- $(\mathrm{CF}(K), \mathcal{F}_t) \cong \mathcal{T} \oplus \mathcal{A}$ *as a filtered graded $\mathbb{F}$–complex.*
- *The complex $\mathcal{T}$ has filtered graded basis $\{U^k\alpha\}_{k\in\mathbb{Z}}$, the boundary map is trivial on $\mathcal{T}$, and* $\mathrm{gr}(\alpha) = 0$.
- *The complex $\mathcal{A}$ has filtered graded basis $\{U^k\alpha_i\}_{k\in\mathbb{Z}}$ and has trivial homology:* $H(\mathcal{A}) = 0$.

Proof We begin with the Λ–generating set of CF(K), $\{w_i\}$.

By replacing generators with their U^k translates and renaming the generators, we can decompose this into two subsets: $\{x_i\}$, all of grading 0, and $\{y_i\}$, all of grading 1.

To simplify notation, we abbreviate the filtered graded $\mathbb{F}$–complex $(\mathrm{CF}(K), \mathcal{F}_t)$ by CF_t.

(1) Let A be a cycle in CF_t having the least filtration level among cycles representing nontrivial classes in $H_0(\mathrm{CF}_t)$. After reordering the generators, we can write $A = x_1 + \cdots + x_k$, with the filtration levels nonincreasing. Replacing x_1 with $x_1 + \cdots + x_k$ as the first generating element (over Λ) induces a filtered change of basis for CF_t. Thus, the first element of the Λ–basis, which we now denote A_1, is a cycle of least filtration level representing a nontrivial element of $H_0(\mathrm{CF}_t)$.

(2) Consider the set of all generating elements y_i that have the property that A_1 is a component of ∂y_i. After reordering the basis, we can assume these are $\{y_1, y_2, \dots, y_k\}$ for some k, and that the filtrations are in nondecreasing order. Make the Λ–change of basis that replaces each y_i for $2 \le i \le k$ with $y_i + y_1$. This induces a filtered change of basis of CF_t. Now, the only generator having A_1 as a component of its boundary is y_1, which we relabel B_1.

(3) After perhaps reordering the x_i, we have either

$$\partial B_1 = A_1 \quad \text{or} \quad \partial B_1 = A_1 + x_2 + \cdots + x_k$$

for some $k \ge 2$, with the filtration levels nonincreasing. Since $\partial^2 = 0$, it follows that B_1 is not a component of any element in the image of ∂.

If $\partial B_1 = A_1$, then we see that $\{A_1, B_1\}$ generates an acyclic *summand* of CF_t, and thus A_1 would not represent a nontrivial element in homology.

We have $\partial B_1 = A_1 + x_2 + \cdots x_k$ for some $k \ge 2$. Make the Λ–change of basis that replaces x_2 with $x_2 + \cdots x_k$, now calling this new element A_2. Then $\partial B_1 = A_1 + A_2$. Note that since A_1 is a cycle and $A_1 + A_2 = \partial B_1$ is a cycle, that A_2 is a cycle representing the same homology class as A_1. Hence the filtration level of A_2 is greater than or equal to that of A_1.

(4) We now repeat the previous argument, making a change of basis so that the only basis elements with boundary that include A_2 as a component are B_1 and perhaps a second generator, which we denote B_2.

(5) This step-by-step procedure must eventually stop, at which time there is constructed a summand of the $\mathbb{F}$–complex CF_t,

$$D = A_1 \leftarrow B_1 \rightarrow A_2 \leftarrow B_2 \rightarrow A_3 \leftarrow \cdots \rightarrow B_{k-1} \rightarrow A_k.$$

Note that the process must end with an A_k; if it stopped with a B_k, the resulting complex would be acyclic and thus not contain a nontrivial element in homology. This complex is a summand of the complex CF_t. Note that ΛD is a summand of a direct sum decomposition of CF_t, as a subcomplex and also as a submodule of the Λ–module.

(6) Since A_1 has the lowest filtration level among the A_i, we can replace each A_i with $A_1 + A_i$ to form a new basis. The complex then splits in the following way:

$$A_1 \oplus \big[B_1 \to (A_1{+}A_2) \leftarrow B_2 \to (A_1{+}A_3) \leftarrow \cdots \to B_{k-1} \to (A_1{+}A_k)\big].$$

We let $\mathcal{T} = \Lambda A_1$. It satisfies the required conditions of the theorem. Since, as a Λ–module, $H(\mathcal{T}) \cong H(\mathrm{CF}_t)$, the complementary summand to $\mathcal{T}$ must be acyclic. That complementary summand yields the summand $\mathcal{A}$ in the statement of the theorem. □

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Received: 4 April 2015 Revised: 24 May 2016