

Veech groups of infinite-genus surfaces

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We show that every countable subgroup $G < GL_+(2, \mathbb{R})$ without contracting elements is the Veech group of a tame translation surface S of infinite genus for infinitely many different topological types of S . Moreover, we prove that as long as every end has genus, there are no restrictions on the topological type of S to realize all possible *uncountable* Veech groups.

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Introduction

A surface S endowed with an atlas whose transition functions are translations is called a *translation surface*. To each such surface, we can associate the group $\text{Aff}_+(S)$ formed by all orientation-preserving homeomorphisms of S which are affine in local coordinates. It is easy to see that the derivative of any affine homeomorphism of S is constant; hence we have a well-defined map

$$(1) \quad \text{Aff}_+(S) \xrightarrow{D} GL_+(2, \mathbb{R}).$$

The main purpose of this article is the study of the image $\Gamma(S)$ of this map. When S is a compact translation surface,¹ $\Gamma(S)$ lies in $SL(2, \mathbb{R})$, is Fuchsian, and hence acts on the hyperbolic plane by Möbius transformations. In a milestone paper [12], W A Veech proved the geodesic flow on a compact translation surface S for which $\Gamma(S)$ is a lattice behaves, roughly speaking, like the geodesic flow on a flat torus. For this reason, it is standard to call $\Gamma(S)$ *the Veech group* of the surface S . Aside from questions regarding the dynamical properties of the geodesic flow, it is natural to investigate which subgroups of $GL_+(2, \mathbb{R})$ can be realized as Veech groups. For compact translation surfaces, this is a difficult question. Moreover, simpler instances of this problem, such as the existence of hyperbolic cyclic Veech groups, are still open.

¹That is, when the metric completion of S with respect to the natural flat metric is a compact surface of genus $g \geq 1$.

In this paper, we address the aforementioned realization problem from the perspective of *infinite-type* tame translation surfaces.² This kind of translation surface appears naturally when studying irrational polygonal billiards (see [11] by the second author) or infinite coverings of compact translation surfaces (see Hooper and Weiss [4]). Given that translation surfaces are orientable, the topological type of an infinite-type translation surface is determined by its genus $g = g(S) \in \mathbb{N} \cup \{\infty\}$ and a couple of nested compact, metrizable and totally disconnected spaces $\text{Ends}_\infty(S) \subset \text{Ends}(S)$. Here $\text{Ends}(S)$ is the space of ends of S and $\text{Ends}_\infty(S)$ is formed by those ends that carry (infinite) genus. Reciprocally, every couple of nested closed subspaces of the Cantor set $X_\infty \subset X \subset 2^\omega$ can be realized as the space of ends and the space of ends with genus of some surface. The following result describes in general terms what is to be expected from the Veech group of a tame translation surface.

Theorem 0.1 (Przytycki, Schmithüsen and Valdez [8]) *The Veech group $\Gamma(S)$ of a tame translation surface is*

- (1) *countable and does not contain contracting elements, or*
- (2) *conjugated to $P := \left\{ \begin{pmatrix} 1 & t \\ 0 & s \end{pmatrix} : t \in \mathbb{R} \text{ and } s \in \mathbb{R}^+ \right\}$, or*
- (3) *conjugated to $P' < \text{GL}_+(2, \mathbb{R})$, the subgroup generated by P and $-\text{Id}$, or*
- (4) *equal to $\text{GL}_+(2, \mathbb{R})$.*

It is not difficult to see that condition (4) above implies that S is isometric to either the plane or a ramified covering of the plane. Hence it is natural to ask if all other expected groups can be realized within *the same topological class* of an infinite-type tame translation surface:

Question 0.2 *Let $X_\infty \subset X \subset 2^\omega$ be a nested couple of closed subspaces of the Cantor set. Is it possible to realize any subgroup of $\text{GL}_+(2, \mathbb{R})$ satisfying (1), (2) or (3) in Theorem 0.1 as the Veech group of some tame translation surface S satisfying $X_\infty = \text{Ends}_\infty(S)$ and $X = \text{Ends}(S)$?*

For the simplest instance of this question $X_\infty = X = \{*\}$, the answer is positive:

Theorem 0.3 [ibid] *Any subgroup of $\text{GL}_+(2, \mathbb{R})$ satisfying (1), (2) or (3) in Theorem 0.1 can be realized as the Veech group of an infinite-genus tame translation with only one end.*

²These are surfaces whose fundamental group is not finitely generated and whose natural flat metric has only singularities of (possibly infinite) conic type. Compact translation surfaces are tame, but not all translation surfaces of infinite type are tame. We refer the reader to Bowman and the second author [1] for a general discussion on singularities of translation surfaces.

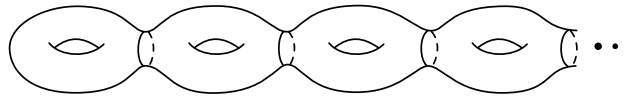


Figure 1: The Loch Ness monster

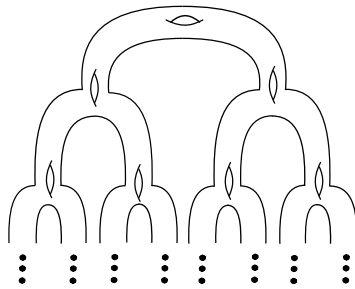


Figure 2: The blooming Cantor tree

The main contribution of this article is to show that Question 0.2 has a positive answer for a large topological class of infinite-type tame translation surfaces. For instance, in Section 2, we show that within the topological class of infinite-type tame translation surfaces S for which $\text{Ends}_\infty(S) = \text{Ends}(S)$, there are no restrictions to realize *uncountable* Veech groups:

Theorem 0.4 *Let X be any closed subset of the Cantor set and $P, P' < \text{GL}_+(2, \mathbb{R})$ be as in Theorem 0.1. Then there exist tame translation surfaces S and S' for which*

- $\text{Ends}(S) = \text{Ends}_\infty(S) = \text{Ends}(S') = \text{Ends}_\infty(S') = X$, and
- *the Veech groups of S and S' are conjugated to P and P' , respectively.*

The rest of the results we present deal with realizing *countable subgroups* of $\text{GL}_+(2, \mathbb{R})$ *without contracting elements* as Veech groups of infinite-type tame translation surfaces satisfying $\text{Ends}_\infty(S) = \text{Ends}(S)$. We present them according to the following heuristic picture: if one was to order infinite-genus surfaces satisfying $\text{Ends}_\infty(S) = \text{Ends}(S)$ according to their topological complexity, the simplest surface would be the one for which the space of ends is just a singleton and the most sophisticated would be the one for which the space of ends is homeomorphic to the Cantor set. These surfaces are called the *Loch Ness monster* and the *blooming Cantor tree*, respectively; see Figures 1 and 2. The nomenclature is due to Phillips and Sullivan [7] and Ghys [3].

Theorem 0.5 *Let $G < \text{GL}_+(2, \mathbb{R})$ be any countable subgroup without contracting elements. Then there exists a tame translation surface S homeomorphic to the blooming Cantor tree whose Veech group is G .*

In other words, Question 0.2 has a positive answer if we move from the simplest topological type of an infinite-genus surface without planar ends to the most sophisticated one. After this, we show that for infinitely many cases “in between”, Question 0.2 also has a positive answer.

Theorem 0.6 *Let X be a countable closed subspace of the Cantor set with characteristic system $(k, 1)$, where $k \in \mathbb{N} \cup \{0\}$, and $G < \mathrm{GL}_+(2, \mathbb{R})$ any countable subgroup without contracting elements. Then there exists a tame translation surface S whose spaces of ends $\mathrm{Ends}_\infty(S) = \mathrm{Ends}(S)$ are homeomorphic to X and whose Veech group is G .*

Recall that a countable compact Hausdorff space X has *characteristic system* (k, n) if its k^{th} Cantor–Bendixson derivative, which we denote by X^k , is a finite set of n points [6]. We remark that for $k = 0$, the statement of Theorem 0.6 is the statement of Theorem 0.3 for countable subgroups of $\mathrm{GL}_+(2, \mathbb{R})$ without contracting elements.

On the other hand, we know from the Cantor–Bendixson theorem that every uncountable closed subset of the Cantor set 2^ω is homeomorphic to a subset of the form $B \sqcup U \subset 2^\omega$, where B is homeomorphic to 2^ω and U is countable and discrete.

Theorem 0.7 *Let $B \sqcup U$ be an uncountable closed subset of the Cantor set, where B is homeomorphic to the Cantor set and U is countable, discrete and its boundary ∂U is just one point. Then for any countable subgroup $G < \mathrm{GL}_+(2, \mathbb{R})$ without contracting elements, there exists a tame translation surface S whose spaces of ends $\mathrm{Ends}_\infty(S) = \mathrm{Ends}(S)$ are homeomorphic to $B \sqcup U$ and whose Veech group is G .*

The proof of Theorem 0.4 is largely inspired by the proof of Theorem 0.3 found in [8] and a convenient characterization of spaces of ends using binary trees that we introduce in the next section. On the other hand, Theorems 0.5, 0.6 and 0.7 are a consequence of an abstract and general construction that we call a *puzzle* and define in Section 3. As a matter of fact, some of the main results found in [8] can be deduced using this construction. Sadly, puzzles do not work to give an answer to all possible instances of Question 0.2. For example, it is still unknown if every countable subgroup $G < \mathrm{GL}_+(2, \mathbb{R})$ can be realized as the Veech group of an infinite-type translation surface S whose spaces of ends $\mathrm{Ends}_\infty(S) = \mathrm{Ends}(S)$ contain only $n \geq 2$ elements. However, given the evidence provided by the results shown in this article we conjecture that Question 0.2 *always has a positive answer*.

Remark 0.8 All the translation surfaces constructed in this article will have infinite area. We believe that this hypothesis is crucial to answer Question 0.2 positively. For finite area infinite-type translation surfaces, little is known about the Veech group; see, for example, [1].

Reader’s guide In Section 1, we discuss the basic concepts needed to develop the proofs of our main results. Namely, we recall the topological classification of all orientable surfaces, the Cantor–Bendixson derivative and the notion of a characteristic system. We also show that every closed subset X of the Cantor set is the space of ends of a subgraph T_X of an infinite (binary) tree. In particular, we introduce a decomposition of T_X into a countable union of infinite paths which is crucial to the constructions we present. We finish this section by explaining the construction of the “building block” used to define puzzles. In Section 2, we prove Theorem 0.4. Finally, in Section 3, we introduce the so called puzzles and show how Theorems 0.5, 0.6 and 0.7 follow from the same general construction.

1 Preliminaries

We begin this section recalling the topological classification of orientable surfaces, the Cantor–Bendixson derivative and the notion of a characteristic system. We then introduce a model for spaces of ends of surfaces based on subgraphs of an infinite (binary) tree. We finish by explaining the construction of the *elementary piece* that will be used in the next section to define puzzles and give the proofs of Theorems 0.5, 0.6 and 0.7.

1.1 Topological classification of surfaces

Through this text we will only work with *orientable* surfaces S without boundary. It is a well-known fact that any such surface with finitely generated fundamental group is determined up to homeomorphism by its genus and number of punctures. When the fundamental group of the surface is not finitely generated, new topological invariants, namely *the spaces of ends of S* , are needed to determine the surface up to homeomorphism. In what follows, we review these invariants in detail, for they will be used in the proof of our main results. For more details, we refer the reader to [9].

A *pre-end* of a connected surface S is a nested sequence $U_1 \supset U_2 \supset \dots$ of connected open subsets of S such that the boundary of U_n in S is compact for every $n \in \mathbb{N}$, and for any compact subset K of S there exists $l \in \mathbb{N}$ such that $U_l \cap K = \emptyset$. We shall denote the pre-end $U_1 \supset U_2 \supset \dots$ as $(U_n)_{n \in \mathbb{N}}$. Two such sequences $(U_n)_{n \in \mathbb{N}}$ and $(U'_n)_{n \in \mathbb{N}}$ are said to be equivalent if for any $i \in \mathbb{N}$, there exists $j \in \mathbb{N}$ such that $U'_j \subset U_i$, and vice versa. We denote by $\text{Ends}(S)$ the corresponding set of equivalence classes and call each equivalence class $[U_n]_{n \in \mathbb{N}} \in \text{Ends}(S)$ an *end* of S . We endow $\text{Ends}(S)$ with a topology by specifying a prebasis as follows: for any open subset $W \subset S$ whose boundary is compact, we define $W^* := \{[U_n]_{n \in \mathbb{N}} \in \text{Ends}(S) : W \supset U_l \text{ for } l \text{ sufficiently large}\}$. We call the corresponding topological space *the space of ends of S* .

Proposition 1.1 [9, Proposition 3] *The space of ends of a connected surface S is totally disconnected, compact and Hausdorff. In particular, $\text{Ends}(S)$ is homeomorphic to a closed subspace of the Cantor set.*

A surface is said to be *planar* if all of its compact subsurfaces are of genus zero. An end $[U_n]_{n \in \mathbb{N}}$ is called *planar* if there exist $l \in \mathbb{N}$ such that U_l is planar. The *genus* of a surface S is the maximum of the genera of its compact subsurfaces. We remark that if a surface S has *infinite genus*, there exists no finite set \mathcal{C} of mutually nonintersecting simple closed curves with the property that $S \setminus \mathcal{C}$ is *connected and planar*. We define $\text{Ends}_\infty(S) \subset \text{Ends}(S)$ as the set of all ends of S which are not planar. It follows from the definitions that $\text{Ends}_\infty(S)$ forms a closed subspace of $\text{Ends}(S)$.

Theorem 1.2 (classification of orientable surfaces [13, Chapter 5]) *Let S and S' be two orientable surfaces of the same genus. Then S and S' are homeomorphic if and only if both $\text{Ends}_\infty(S) \subset \text{Ends}(S)$ and $\text{Ends}_\infty(S') \subset \text{Ends}(S')$ are homeomorphic as nested topological spaces.*

Of special interest in this paper is the surface of infinite genus and only one end, and the surface without planar ends whose ends space is homeomorphic to the Cantor set. The former is known as *the Loch Ness monster* (Figure 1) and the latter as the *blooming Cantor tree* (Figure 2). The nomenclature is due to [7] and [3], respectively. We remark that a surface S has only one end if and only if for every compact subset $K \subset S$ there exists a compact $K' \subset S$ such that $K \subset K'$ and $S \setminus K'$ is connected; see [10].

1.2 Cantor–Bendixson derivative

We recall briefly the Cantor–Bendixson theorem and how the Cantor–Bendixson rank classifies countable compact Polish spaces up to homeomorphism.

Theorem 1.3 (Cantor–Bendixson [5]) *Let X be a Polish space. Then X can be uniquely written as $X = B \sqcup U$, where B is perfect and U is countable and open.*

Corollary 1.4 *Any uncountable Polish space contains a closed subset homeomorphic to the Cantor set.*

Given a topological space X , the *Cantor–Bendixson derivative* of X is the set $X' := \{x \in X : x \text{ is a limit point of } X\}$. For every $k \in \mathbb{N}$, the k^{th} *Cantor–Bendixson derivative* of X is defined as $X^k := (X^{k-1})'$, where $X^0 = X$. A countable Polish space is said to have *characteristic system* $(k, 1)$ if $X^k \neq \emptyset$, $X^{k+1} = \emptyset$ and the cardinality of X^k is exactly one.

Remark 1.5 Any countable Polish space with characteristic system $(k, 1)$ is homeomorphic to the ordinal number³ $\omega^k + 1$, where ω is the least infinite ordinal. In particular, every two countable Polish spaces with characteristic system $(k, 1)$ are homeomorphic. For further details, see [6].

The following results will be used in the proof of Theorem 0.7.

Theorem 1.6 Let $X_1 = B_1 \sqcup U_1$ and $X_2 = B_2 \sqcup U_2$ be two uncountable closed subsets of the Cantor set. Suppose that both boundaries ∂U_1 and ∂U_2 are just singletons. Then X_1 and X_2 are homeomorphic.

Proof The Cantor set 2^ω is identified with the topological group $\prod_{i \in \mathbb{N}} \mathbb{Z}_2^i$; see [5, page 50]. Recall that this group acts by homeomorphisms (freely and transitively) on itself. Let us denote this action by

$$(2) \quad \alpha: \prod_{i \in \mathbb{N}} \mathbb{Z}_2^i \times \prod_{i \in \mathbb{N}} \mathbb{Z}_2^i \rightarrow \prod_{i \in \mathbb{N}} \mathbb{Z}_2^i, \quad ([x_n]_{n \in \mathbb{N}}, [y_n]_{n \in \mathbb{N}}) \mapsto [x_n + y_n]_{n \in \mathbb{N}}.$$

By hypothesis, there are homeomorphisms $f_j: B_j \rightarrow \prod_{i \in \mathbb{N}} \mathbb{Z}_2^i$, $j = 1, 2$ and an element $[z_n]_{n \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} \mathbb{Z}_2^i$ such that the homeomorphism $f_2^{-1} \circ \alpha|_{[z_n]_{n \in \mathbb{N}}} \circ f_1$ sends ∂U_1 to ∂U_2 . On the other hand, if we let \overline{U}_j denote the closure of U_j in X_j for $j = 1, 2$, there exists a homeomorphism $h: \overline{U}_1 \rightarrow \overline{U}_2$. In particular, $h(\partial U_1) = \partial U_2$. Hence

$$F: X_1 \rightarrow X_2, \quad x \mapsto \begin{cases} f(x) & \text{if } x \in B_1, \\ h(x) & \text{if } x \in \overline{U}_1, \end{cases}$$

is a well-defined bijection. We claim that F is the desired homeomorphism. Let V be an open set in X_2 containing $F(\partial U_1) = \partial U_2$. There are open sets W_1 and W_2 in X_1 containing the point ∂U_1 such that $f(W_1 \cap B_1) \subset V \cap B_2$ and $h(W_2 \cap \overline{U}_1) \subset V \cap \overline{U}_2$. Then $F(W_1 \cap W_2) \subset V$. This proves that the function F is *continuous* at ∂U_1 . Since F is defined by gluing homeomorphisms at this point, this is sufficient to prove that F is continuous. On the other hand, the function F is *closed* because every continuous function from a compact Hausdorff space onto a Hausdorff space is closed; see [2, page 226]. Therefore, F is a homeomorphism. □

1.3 The Cantor binary tree

One of the fundamental objects we use in the construction of infinite translation surfaces with prescribed Veech group is the infinite 3-regular tree. This graph plays a

³Recall that an ordinal number can be made into a topological space by endowing it with the order topology.

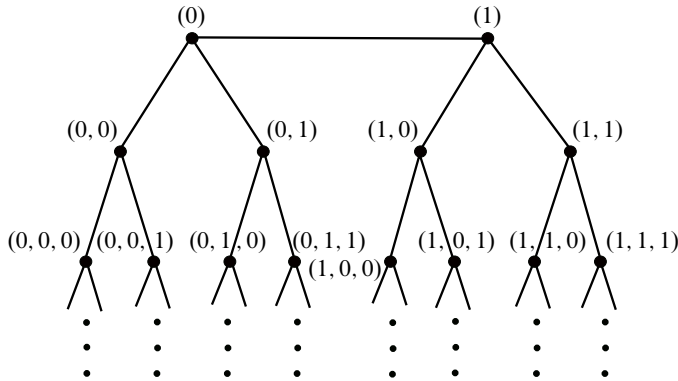


Figure 3: Cantor binary tree $T2^\omega$

distinguished role since the space of ends of any surface is homeomorphic to a subspace of its space of ends. We give binary coordinates to the vertex set of the infinite 3–regular tree, for we use these in a systematic way during the proofs of our main results.

For every $n \in \mathbb{N}$, let $2^n := \prod_{i=1}^n \{0, 1\}_i$ and $\pi_i: 2^n \rightarrow \{0, 1\}$ be the projection onto the i^{th} coordinate. We define $V := \{D_s : D_s \in 2^n \text{ for some } n \in \mathbb{N}\}$ and E as the union of $((0), (1))$ with the set

$$\{(D_s, D_t) : D_s \in 2^n \text{ and } D_t \in 2^{n+1} \text{ for some } n \in \mathbb{N}, \text{ and } \pi_i(D_s) = \pi_i(D_t) \text{ for every } i \in \{1, \dots, n\}\}.$$

The infinite 3–regular tree with vertex set V and edge set E will be called the *Cantor binary tree* and denoted by $T2^\omega$; see Figure 3.

Remark 1.7 Let $(v_n)_{n \in \mathbb{N}}$, where $v_n \in 2^\omega$, be an infinite simple path in $T2^\omega$. If we define V_n as the connected component of $T2^\omega \setminus \{v_n\}$ such that $v_{n+1} \in V_n$, then $[V_n] \in \text{Ends}(T2^\omega)$ is completely determined by $(v_n)_{n \in \mathbb{N}}$. Hence, if we endow $\{0, 1\}$ and $2^\omega := \prod_{i \in \mathbb{N}} \{0, 1\}_i$ with the discrete and product topologies, respectively, the map

$$(3) \quad f: \prod_{i \in \mathbb{N}} \{0, 1\}_i \rightarrow \text{Ends}(T2^\omega), \quad (x_n)_{n \in \mathbb{N}} \mapsto (v_n := (x_1, \dots, x_n)),$$

is a homeomorphism between the standard binary Cantor set and the space of ends of $T2^\omega$.

Remark 1.8 Sometimes we will abuse notation and denote by $f((x_n)_{n \in \mathbb{N}})$ both the end defined by the infinite path $(v_n := (x_1, \dots, x_n))_{n \in \mathbb{N}}$ and the infinite path itself in $T2^\omega$.

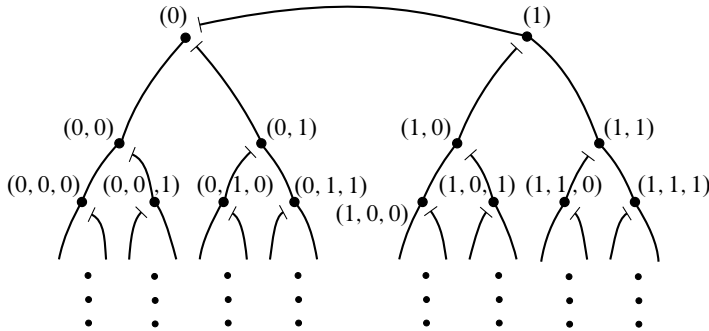


Figure 4: Countable family of infinite paths \mathfrak{T}

Lemma 1.9 *Let X be a closed subset of the Cantor set. Then there exists a connected subgraph $T_X \subset T2^\omega$ such that its ends space $\text{Ends}(T_X)$ is homeomorphic to X .*

Proof Without loss of generality we suppose that $X \subset \prod_{i \in \mathbb{N}} \{0, 1\}_i$. Define

$$(4) \quad T_X := \left(\bigcup_{(x_n)_{n \in \mathbb{N}} \in X} f((x_n)_{n \in \mathbb{N}}) \right) \cup ((0), (1)) \subset T2^\omega,$$

where f is the map defined in (3). □

The constructions of translation surfaces that we present follow at most a countable number of steps. In order to integrate the graphs T_X into these constructions, it is important to be able to express them as a *countable* union of infinite paths which intersect in at most a vertex.

Example 1.10 Define \mathfrak{T} as the union of

$$((0), (1), (1, 1), (1, 1, 1), \dots) \quad \text{and} \quad ((0), (0, 0), (0, 0, 0), \dots)$$

with the countable set of infinite paths,

$$\{(D_s), (D_s, x), (D_s, x, x), (D_s, x, x, x), \dots\} \in T2^\omega : \\ D_s \in 2^n, x \in \{0, 1\}, \text{ and } \pi_n(D_s) \neq x\}_{n \in \mathbb{N}}.$$

Then it is easy to see that $T2^\omega = \bigcup_{\gamma \in \mathfrak{T}} \gamma$. Moreover, any two infinite paths in \mathfrak{T} are either disjoint or intersect in just one vertex (see Figure 4).

Lemma 1.11 *Let $X \subset 2^\omega$ be a closed subset of the Cantor set, and let T_X be the subgraph given by Lemma 1.9. Then there exists a countable set of infinite paths \mathfrak{T}_X such that $T_X = \bigcup_{\gamma \in \mathfrak{T}_X} \gamma$. Moreover, every two infinite paths $\gamma, \gamma' \in \mathfrak{T}_X$ intersect in at most one vertex.*

Proof Let us first suppose that X is uncountable. By Theorem 1.3, $\text{Ends}(T_X) = B \sqcup U$, where B is homeomorphic to the Cantor set and U is countable, discrete and open. If $U = \emptyset$, then without loss of generality, we can suppose that $T_X = T2^\omega$ and treat this case as in Example 1.10. Let us suppose then that $U = \{u_n\}_{n \in \mathbb{N}}$. By Lemma 1.9, we have

$$T_X = \left(\bigcup_{x \in B \sqcup U} f(x) \right) \cup ((0), (1)) = \left(\bigcup_{x \in B} f(x) \right) \cup \left(\bigcup_{k, n \in \mathbb{N}} f((v_k^n)) \right) \cup ((0), (1)) \subset T2^\omega,$$

where $f((v_k^n)_{k \in \mathbb{N}}) = u_n$. From Example 1.10, we deduce that there exists a countable family \mathfrak{T}_B of infinite paths of the subgraph $T_B := (\bigcup_{x \in B} f(x)) \cup ((0), (1)) \subset T_X$ such that $T_B = \bigcup_{\gamma \in \mathfrak{T}_B} \gamma$, and any two different infinite paths belonging to \mathfrak{T}_B intersect in at most one vertex. Now we inductively construct the rest of the desired family of infinite paths. For the infinite path $f((v_k^1)_{k \in \mathbb{N}}) := ((v_1^1), (v_1^1, v_2^1), \dots) = u_1$ in T_X , there exists $j(1) \in \mathbb{N}$ such that $(v_1^1, \dots, v_{j(1)}^1) \in T_B$; however, for each $i > j(1)$, we have that $(v_1^1, \dots, v_i^1) \notin T_B$. We define $\gamma_1 := (v_{j(1)}^1, v_{j(1)+1}^1, \dots)$ and the subgraph

$$(5) \quad T_X^1 := T_B \cup \{\gamma_1\} \subset T_X.$$

We remark that $\mathfrak{T}_X^1 := \mathfrak{T}_B \cup \{\gamma_1\}$ is a countable family of infinite paths in T_X^1 such that $T_X^1 = \bigcup_{\gamma \in \mathfrak{T}_X^1} \gamma$, and any two different paths in \mathfrak{T}_X^1 intersect in at most one vertex. Now suppose we have found the desired countable family \mathfrak{T}_X^{n-1} of infinite paths for the subgraph $T_X^{n-1} \subset T_X$, and let $f((v_k^n)_{k \in \mathbb{N}}) := ((v_1^n), (v_1^n, v_2^n), \dots) = u_n$ in T_X . Then there exists $j(n) \in \mathbb{N}$ such that $(v_1^n, \dots, v_{j(n)}^n) \in T_X^{n-1}$, but for each $i > j(n)$, $(v_1^n, \dots, v_i^n) \notin T_X^{n-1}$. We define $\gamma_n := (v_{j(n)}^n, v_{j(n)+1}^n, \dots)$ and the subgraph

$$(6) \quad T_X^n := T_X^{n-1} \cup \{\gamma_n\} \subset T_X.$$

The desired countable family of infinite paths is given by $\mathfrak{T}_X := \mathfrak{T}_B \cup \{\gamma_n : n \in \mathbb{N}\}$. We finish the proof by remarking that if X is countable, the desired family of countable infinite paths is obtained from the preceding recursive construction taking $T_X^1 = \{\gamma_1\}$ as base case. □

Corollary 1.12 *Let X be a countable closed subset of the Cantor set. Then the sets $\text{Ends}(T_X)$ and \mathfrak{T}_X are in bijection.*

By the way we constructed the family \mathfrak{T}_X (see Remark 1.7) we obtain the following:

Corollary 1.13 *Let $X = B \sqcup U$ be the decomposition of an uncountable subset of the Cantor set given by Theorem 1.3. Suppose that the boundary $\partial U \subset B$ is just one point. Then there exists an infinite path $\tilde{\gamma} \in \mathfrak{T}_X$ such that the end $[V_n]_{n \in \mathbb{N}}$ of T_X defined by $\tilde{\gamma}$ is precisely ∂U .*

1.4 Translation surfaces and the Veech group

We recall some general aspects of translation surfaces. For more details, we refer the reader to [8] and the references within. A translation surface is a real surface S whose transition functions are translations. Every translation surface inherits a natural flat metric from the plane via pull back. We denote by \widehat{S} the *metric completion* of S with respect to its natural flat metric.

Definition 1.14 [8] A translation surface S is called *tame* if for every point $x \in \widehat{S}$, there exists a neighborhood $U_x \subset \widehat{S}$ which is either isometric to some neighborhood of the Euclidean plane or to the neighborhood of the branching point of a cyclic branched covering of the unit disk in the Euclidean plane. In the latter case, we call x a *cone angle singularity of angle $2n\pi$* if the cyclic covering is of (finite) order $n \in \mathbb{N}$, and an *infinite cone angle singularity* when the cyclic covering is infinite.

We denote by $\text{Sing}(S) \subset \widehat{S}$ the set of cone angle singularities of S . A geodesic in S is called *singular* if it has one endpoint in $\text{Sing}(S)$ and no singularities in its interior. A singular geodesic having both endpoints in $\text{Sing}(S)$ is called a *saddle connection*. To every saddle connection γ we can associate two *holonomy vectors* $\{v, -v\} \subset \mathbb{R}^2$ by developing the translation surface structure along γ . Analogously, one can associate two unit vectors to every singular geodesic, which we will also call holonomy vectors. Two saddle connections or singular geodesics are said to be *parallel* if their corresponding holonomy vectors are parallel.

An *affine diffeomorphism* is a map $f: S \rightarrow S$ which is affine on charts. We denote by $\text{Aff}_+(S)$ the group of all orientation-preserving affine diffeomorphisms. Given that S is a translation surface, the differential of every element in $\text{Aff}_+(S)$ is constant. Hence we have a well-defined group morphism

$$D: \text{Aff}_+(S) \rightarrow \text{GL}_+(2, \mathbb{R}),$$

where $D(f)$ is the differential matrix of f . The image of D , that we denote by $\Gamma(S) := D(\text{Aff}_+(S))$, is called *the Veech group* of S ; see [12].

We finish this section by recalling that $\text{GL}_+(2, \mathbb{R})$ acts on the set of all translation surfaces by postcomposition on charts. For every $g \in \text{GL}_+(2, \mathbb{R})$, we let $S_g := g \cdot S$ and let $\bar{g}: S_{\text{Id}} \rightarrow S_g$ be the corresponding affine diffeomorphism.

1.5 Auxiliary constructions

In the following paragraphs we introduce some elementary constructions needed to prove our main results. All these constructions are based on the same principle: *to*

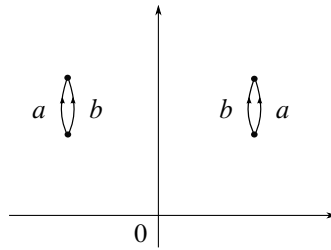


Figure 5: Gluing marks

glue translation surfaces along parallel marks. A mark m on a translation surface S is a finite length geodesic having no singular points in its interior. As with saddle connections, we can associate to each mark two holonomy vectors by developing the translation structure along them. Two marks on S are parallel if their respective holonomy vectors are parallel.

Definition 1.15 (gluing marks) Let m and m' be two disjoint parallel marks of the same length on a translation surface S . We cut S along m and m' , which turns S into a surface with boundary consisting of four straight segments. We glue these segments back using translations to obtain a tame translation surface S' different from the one we started from. We say that S' is obtained from S by *regluing* along m and m' .

We denote by $m \sim_{\text{glue}} m'$ the operation of gluing the marks m and m' , and $S' = S/m \sim_{\text{glue}} m'$. In Figure 5, we depict the gluing of two marks on the plane. We remark that the operation of gluing marks can also be performed for marks on different surfaces. In any case, $\text{Sing } S' \setminus \text{Sing } S$ is formed by two 4π cone angle singularities; that is, S tame implies S' tame.

Lemma 1.16 Let S_1 and S_2 be two translation surfaces homeomorphic to the Loch Ness monster and $M^j := \{m_i^j : i \in \mathbb{N}\}$ a discrete⁴ family of marks on S_j , $j = 1, 2$ such that m_i^1 and m_i^2 are parallel of the same length for every $i \in \mathbb{N}$. Then

$$S := \left(\bigcup_{j \in \{1,2\}} S_j \right) / (m_i^1 \sim_{\text{glue}} m_i^2)_{i \in \mathbb{N}}$$

is a tame translation surface homeomorphic to the Loch Ness monster (see Figure 6).

⁴By discrete we mean that M^j , as a set of marks, does not accumulate in the metric completion of the surface.

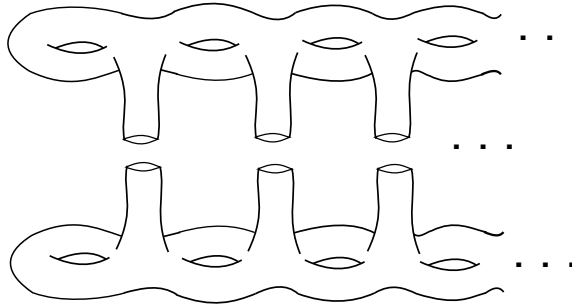


Figure 6: Gluing marks on the Loch Ness monsters

Proof Given that the family of marks M^j does not accumulate in \widehat{S}_j , the quotient S is a tame translation surface. Let K be a compact subset of S . We will show that there exists a compact subset $K \subset K' \subset S$ such that $S \setminus K'$ is connected. By the remark after Theorem 1.2, this implies that S has only one end. Let $f: S_1 \rightarrow S_2$ be a homeomorphism such that $f(m_i^1) = m_i^2$ for each $i \in \mathbb{N}$, and define the projections

$$\pi_1: \bigcup_{j \in \{1,2\}} (S_j \setminus M^j) \rightarrow (S_1 \setminus M^1), \quad x \mapsto \begin{cases} x, & \text{if } x \in S_1, \\ f^{-1}(x), & \text{if } x \in S_2, \end{cases}$$

$$\pi_2: \bigcup_{j \in \{1,2\}} (S_j \setminus M^j) \rightarrow (S_2 \setminus M^2), \quad x \mapsto \begin{cases} x & \text{if } x \in S_2, \\ f(x) & \text{if } x \in S_1. \end{cases}$$

We denote by $\pi: S_1 \cup S_2 \rightarrow S$ the standard projection. For each $j \in \{1, 2\}$, the closure of $\pi_j(\pi^{-1}(K) \setminus (M^1 \cup M^2))$ in S_j is compact. We denote it by K_j . Since S_j has only one end, there exists a compact subset $K'_j \subset S_j$ such that $K_j \subset K'_j$ and $S_j \setminus K'_j$ is connected. Define K' as the closure of

$$\pi \left(\bigcup_{j \in \{1,2\}} \pi_j^{-1}(K'_j \cap (S_j \setminus M^j)) \right)$$

in S . By construction, $K \subset K'$ and $S \setminus K'$ is connected. To conclude the proof, we remark that, since cutting along marks does not destroy genus, for each $j \in \{1, 2\}$, the open surface $S_j \setminus M^j$ with boundary of S_j has infinite genus and is naturally embedded in S . □

Construction 1.17 (elementary piece) Let X be a closed subset of the Cantor set, T_X the subgraph of $T2^\omega$ given by Lemma 1.9, and \mathfrak{T}_X a countable family of infinite paths as in Lemma 1.11. Suppose that for each path $\gamma \in \mathfrak{T}_X$, we have a tame translation surface $S(\gamma)$ homeomorphic to the Loch Ness monster and a countable family of marks $M := \{m_k : k \in \mathbb{N}\} \subset S(\gamma)$ which do not accumulate on the metric completion $\widehat{S}(\gamma)$.

Suppose that the vertex set of γ is labeled by $(v_k = (x_1, \dots, x_k))$ as in (3). For each $k \in \mathbb{N}$, we label the k^{th} mark m_k of the family M with $v_k \in \gamma$ and define the elementary piece associated to X and the family of translation surfaces $\{S(\gamma)\}_{\gamma \in \mathfrak{T}_X}$ as

$$(7) \quad S_{\text{elem}} := \left(\bigcup_{\gamma \in \mathfrak{T}_X} S(\gamma) \right) / \sim,$$

where \sim is the equivalence relation given by gluing marks with the same labels.

Lemma 1.18 *For every closed subset X of the Cantor set, $S_{\text{elem}} = S_{\text{elem}}(X)$ is a tame translation surface with $\text{Ends}(S_{\text{elem}}) = \text{Ends}_\infty(S_{\text{elem}})$ homeomorphic to X .*

Proof Tameness follows from two facts: any two infinite paths in \mathfrak{T}_X intersect in at most one vertex, and marks never accumulate on the metric completion $\widehat{S(\gamma)}$. The rest of the proof goes as follows. First we prove that $\text{Ends}(S_{\text{elem}})$ is homeomorphic to X . The idea is to define an embedding $i: T_X \hookrightarrow S_{\text{elem}}$ and show that it induces a homeomorphism between $\text{Ends}(T_X)$ and $\text{Ends}(S_{\text{elem}})$. Finally, we prove that S_{elem} has no planar ends.

The embedding Let $[U(\gamma)_n]_{n \in \mathbb{N}}$ be the end of $S(\gamma)$. Up to making an isotopy on $S(\gamma)$, for every $k \in \mathbb{N}$, we can suppose that the marks $\{v_1, \dots, v_k\} \subset S(\gamma)$ are contained in $S(\gamma) \setminus U(\gamma)_k$. It is then easy to see that there exists an infinite simple embedded path $\delta: [0, \infty) \rightarrow S(\gamma)$ such that $\delta(0)$ is an extremity of the mark labeled with the vertex v_1 , all marks M are contained in $\delta([0, \infty))$, and $\delta^{-1}(U(\gamma)_k) = (k, \infty)$. In other words, the image of δ goes only once over each mark and runs into the only end of the surface. On the vertex set $V(\gamma) = (v_k)_{k \in \mathbb{N}}$, we define $i_\gamma(v_k)$ as the only endpoint in the mark labeled with v_k satisfying the following property: if $\overline{i_\gamma(v_k)i_\gamma(v_{k+1})}$ is the segment in the image of δ going from $i_\gamma(v_k)$ to $i_\gamma(v_{k+1})$ and p_{k+1} is the endpoint on the mark v_{k+1} different from $i_\gamma(v_{k+1})$, then $p_{k+1} \in \overline{i_\gamma(v_k)i_\gamma(v_{k+1})}$. We extend the map i_γ to all edges by declaring the image of the edge (v_k, v_{k+1}) to be $\overline{i_\gamma(v_k)i_\gamma(v_{k+1})}$. This defines an embedding

$$i_\gamma: \gamma \hookrightarrow S(\gamma)$$

by declaring that the image of the edge (v_k, v_{k+1}) is precisely the subarc of δ joining $i_\gamma(v_k)$ to $i_\gamma(v_{k+1})$. Now consider two infinite paths $\gamma, \gamma' \in \mathfrak{T}_X$ sharing a vertex v_k . This means that $S(\gamma)$ and $S(\gamma')$ are glued along the mark labeled with v_k when constructing S_{elem} . By the way we defined i_γ and $i_{\gamma'}$, we have that their images in S_{elem} only intersect in $i_\gamma(v_k) = i_{\gamma'}(v_k)$. Therefore, we can glue the family of maps $\{i_\gamma\}_{\gamma \in \mathfrak{T}_X}$ to define an embedding

$$(8) \quad i: T_X \hookrightarrow S_{\text{elem}}.$$

We remark that in general, we cannot use the set of paths in \mathfrak{T}_X to characterize all points in $\text{Ends}(T_X)$ because \mathfrak{T}_X is always countable, but the cardinality of $\text{Ends}(\mathfrak{T}_X)$ is the same as that of X , which may be uncountable. To overcome this difficulty, let us define the family \mathfrak{D}_X of *descending paths* of T_X as the set of all infinite paths $(D_n)_{n \in \mathbb{N}}$ of T_X such that $D_n \in 2^n$; see Section 1.3. Every descendent path $(D_n)_{n \in \mathbb{N}}$ defines an element $[\mathcal{U}_n]_{n \in \mathbb{N}} \in \text{Ends}(T_X)$ as follows: for every vertex D_n , let $\mathcal{U}_n \subset T_X$ be the connected component of $T_X \setminus D_n$ containing D_{n+1} . This defines a bijection between $\text{Ends}(T_X)$ and \mathfrak{D}_X . We use the family of descending paths of T_X to define a homeomorphism between $\text{Ends}(T_X)$ and $\text{Ends}(S_{\text{elem}})$.

Let $(D_n)_{n \in \mathbb{N}} \in \mathfrak{D}_X$ be the descending path determining $[\mathcal{U}_n]_{n \in \mathbb{N}} \in \text{Ends}(T_X)$. For each vertex D_n , we consider two cases:

- (1) Suppose that there exist two paths $\gamma_\alpha, \gamma_\beta \in \mathfrak{T}_X$ such that $D_n \in \gamma_\beta \cap \gamma_\alpha$. Let $k_0, k'_0 \in \mathbb{N}$ be the smallest positive integers such that $i_{\gamma_\alpha}(D_n) \notin U(\gamma_\alpha)_{k_0} \in [U(\gamma_\alpha)_k]_{k \in \mathbb{N}}$ and $i_{\gamma_\beta}(D_n) \notin U(\gamma_\beta)_{k'_0} \in [U(\gamma_\beta)_k]_{k \in \mathbb{N}}$. Then we define W_n as the connected component of $S_{\text{elem}} \setminus \partial U(\gamma_\alpha)_{k_0} \cup \partial U(\gamma_\beta)_{k'_0}$ containing $i(D_{n+1})$.
- (2) Suppose $D_n \in \gamma_\alpha \in \mathfrak{T}_X$ but is not contained in any other infinite path of \mathfrak{T}_X . Let k_0 be the least positive integer such that $i_{\gamma_\alpha}(D_n) \notin U(\gamma_\alpha)_{k_0} \in [U(\gamma_\alpha)_k]_{k \in \mathbb{N}}$. Then we define W_n as the connected component of $S_{\text{elem}} \setminus \partial U(\gamma_\alpha)_{k_0}$ containing $i(D_{n+1})$.

It is easy to check that the map

$$i_*: \text{Ends}(T_X) \rightarrow \text{Ends}(S_{\text{elem}})$$

given by $i_*([\mathcal{U}_n]_{n \in \mathbb{N}}) := [W_n]_{n \in \mathbb{N}}$ is well defined. We now prove that it is a closed continuous bijection, hence a homeomorphism.

Injectivity Consider two different infinite paths $(D_n)_{n \in \mathbb{N}}$ and $(D'_n)_{n \in \mathbb{N}}$ in \mathfrak{D}_X defining ends $[\mathcal{U}_n]_{n \in \mathbb{N}}$ and $[\mathcal{U}'_n]_{n \in \mathbb{N}}$ of T_X . Then there exists $N \in \mathbb{N}$ such that for all $m > N$, we have that $D_m \neq D'_m$. This implies that $W_m \cap W'_m = \emptyset$; hence $i_*([\mathcal{U}_n]_{n \in \mathbb{N}}) = [W_n]_{n \in \mathbb{N}} \neq [W'_n]_{n \in \mathbb{N}} = i_*([\mathcal{U}'_n]_{n \in \mathbb{N}})$.

Surjectivity Consider an end $[W_n]_{n \in \mathbb{N}}$ of S_{elem} . By the way the embedding (8) was defined, $[i^{-1}(W_n \cap i(T_X))]$ defines an end in T_X . Let $(D_n)_{n \in \mathbb{N}} \in \mathfrak{D}_X$ be the descending path defining $[i^{-1}(W_n \cap i(T_X))]$ and $[\mathcal{U}_n]_{n \in \mathbb{N}} = [i^{-1}(W_n \cap i(T_X))]$. Then $i_*([\mathcal{U}_n]_{n \in \mathbb{N}}) = [W_n]_{n \in \mathbb{N}}$.

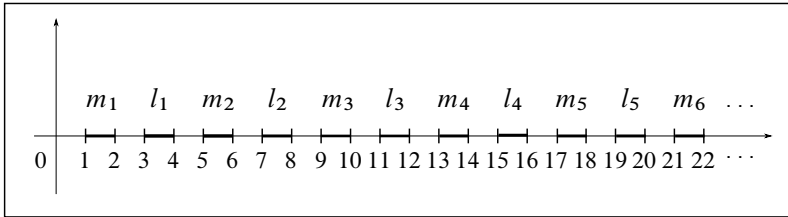


Figure 7: Tame Loch Ness monster S_P

The map i_* is continuous Let $W \subset S_{\text{elem}}$ be an open set with compact boundary and $W^* \subset \text{Ends}(S_{\text{elem}})$ the basic open set it defines in the space of ends. Since (8) is an embedding, we have that $\mathcal{U} := i^{-1}(W \cap T_X)$ is an open set with compact boundary, and $i_*(\mathcal{U}^*) \subset W^*$.

The map i_* is closed, for it is a bijection from a compact Hausdorff space into a Hausdorff space. Finally, we remark that S_{elem} has no planar ends, for each piece $S(\gamma)$ used in its construction has infinite genus. \square

2 Proof of Theorem 0.4

The proof that we present relies on Construction 1.17 and Lemma 1.18. First we define a tame translation surface S_P homeomorphic to the Loch Ness monster whose Veech group is exactly P . This surface comes with an infinite family of marks which do not accumulate on the boundary; hence we are in shape to perform Construction 1.17 taking all $S(\gamma)$ to be equal to S_P and X an arbitrary closed subset of the Cantor set. We then check that the Veech group of the resulting elementary piece S_{elem} is P . The case for P' is treated analogously.

Construction 2.1 Consider a copy \mathbb{E} of the Euclidean plane equipped with a fixed origin $\bar{0}$ and an orthogonal basis $\beta = \{e_1, e_2\}$. On \mathbb{E} , we define⁵ two infinite families of marks,

$$L := \{l_i = ((4i-1)e_1, 4ie_1) : i \in \mathbb{N}\} \quad \text{and} \quad M := \{m_i = ((4i-3)e_1, (4i-2)e_1) : i \in \mathbb{N}\},$$

and the tame Loch Ness monster (see Figure 7)

$$(9) \quad S_P := \mathbb{E} / (l_{2i-1} \sim_{\text{glue}} l_{2i})_{i \in \mathbb{N}}.$$

Given that $\{\pm e_1\}$ is the set of all holonomy vectors of S_P and there are no saddle connections on the half plane $x < 0$, the Veech group of S_P is P . We remark that S_P

⁵Marks are given by their ends points.

comes with the infinite family of marks $M = \{m_k\}_{k \in \mathbb{N}}$, none of which has been glued to another mark.

Let X be a closed subset of the Cantor set and

$$(10) \quad S_{\text{elem}} := \left(\bigcup_{\gamma \in \mathfrak{I}_X} S(\gamma) \right) / \sim$$

the tame translation surface obtained by performing Construction 1.17 with initial data X and all $S(\gamma)$ equal to S_P . By Lemma 1.18, S_{elem} has no planar ends, and its ends space is homeomorphic to X . Since all marks M in S_P are parallel, all saddle connections in S_{elem} are parallel as well. Hence we can define for each $g \in P$ an affine diffeomorphism f on S_P whose differential is exactly g and which fixes all points in the marks belonging to L and M . Since S_{elem} is obtained by gluing copies of S_P along saddle connections, these affine diffeomorphisms can be glued together to define a global affine diffeomorphism $f \in \text{Aff}_+(S_{\text{elem}})$ whose differential is precisely g . In other words, P is a subgroup of $\Gamma(S_{\text{elem}})$. To see that $\Gamma(S_{\text{elem}})$ is a subgroup of P , it is sufficient to remark that there is only one horizontal direction defining infinite singular geodesics in S_{elem} , and this direction has to be fixed by the (linear) action of $\Gamma(S_{\text{elem}})$ on the plane. The construction of a tame translation surface having ends space homeomorphic to X and Veech group equal to P' is done exactly as we did for P , except that instead of taking all copies of $S(\gamma)$ equal to S_P , we take them equal to the surface $S_{P'}$ defined in the following paragraph.

Construction 2.2 Consider a copy \mathbb{E} of the Euclidean plane equipped with a fixed origin $\bar{0}$ and an orthogonal basis $\beta = \{e_1, e_2\}$. On \mathbb{E} we define two infinite families of marks,

$$M^+ := \{m_i^+ = ((4i - 3)e_1, (4i - 2)e_1) : i \in \mathbb{N}\},$$

$$M^- := \{m_i^- = ((3 - 4i)e_1, (2 - 4i)e_1) : i \in \mathbb{N}\},$$

and

$$L^+ := \{l_i^+ = ((4i - 1)e_1, 4ie_1) : i \in \mathbb{N}\},$$

$$L^- := \{l_i^- = ((1 - 4i)e_1, -4ie_1) : i \in \mathbb{N}\},$$

and the tame Loch Ness monster (see Figure 8)

$$(11) \quad S_{P'} := \mathbb{E} / (l_{2i-1}^+ \sim_{\text{glue}} l_{2i}^+, l_{2i-1}^- \sim_{\text{glue}} l_{2i}^-)_{i \in \mathbb{N}}.$$

Corollary 2.3 *There exists a tame translation surface S of genus zero with Veech group P such that its space of ends is homeomorphic to X .*

Indeed, consider the copy \mathbb{E} of the Euclidean plane described by Construction 2.1 with only the family of marks M , and proceed verbatim.

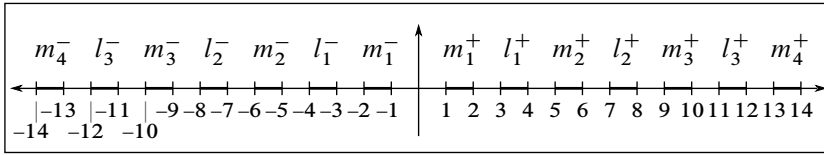


Figure 8: Tame Loch Ness monster S_P

3 Proofs of main results

In this section, we present the proofs of Theorems 0.5, 0.6 and 0.7. These follow from a general construction that we introduce in the first section of this section.

3.1 Puzzles

Let (X, G, H) be a triple where X is a closed subset of the Cantor set, and G is a countable subgroup of $GL_+(2, \mathbb{R})$, without contracting elements, generated⁶ by H . To the triple (X, G, H) we will associate a family of tame translation surfaces $\mathfrak{P}(X, G, H) = \{S_g : g \in G\}$. This set will be called a *puzzle* and each of its elements a *piece*. As notation suggests, each piece of the puzzle is an affine copy of a fixed tame translation surface, which will be called the *elementary piece of the puzzle*, for it is obtained through Construction 1.17. The pieces S_g of the puzzle are endowed with families of marks that do not accumulate in the metric completion; hence we can glue them together to form a *tame* translation surface $S_{\mathfrak{P}}$ that we will call the *assembled surface*. The Veech group of the surface $S_{\mathfrak{P}}$ is G . Moreover, this surface satisfies $\text{Ends}(S_{\mathfrak{P}}) = \text{Ends}_{\infty}(S_{\mathfrak{P}})$, and we will give an explicit description of this space of ends that will allow us to prove, for different instances of X , Theorems 0.5, 0.6 and 0.7.

The elementary piece of a puzzle As its name suggests, this piece is obtained using Construction 1.17. Recall that this construction has as initial data a closed subset X of the Cantor set and, for each path γ in the countable family of paths \mathfrak{T}_X given by Lemma 1.11, a tame Loch Ness monster $S(\gamma)$ endowed with an infinite family of marks that do not accumulate. We build the surfaces $S(\gamma)$ gluing together a series of tame translation surfaces that we define in what follows. With this purpose in mind, we choose an enumeration⁷ $G := \{g_1, \dots, g_{|G|}\}$ and $H := \{h_1, \dots, h_{|H|}\}$ for elements in G and H respectively.

⁶We think of H as generator subset closed under inverse elements.

⁷By enumeration of these groups when $|G|$ or $|H|$ is infinite, we mean to write them as an infinite sequence.

Buffer Loch Ness monster For every $h_j \in H$ and $g \in G$, we construct a tame translation surface $S(g, gh_j)$ homeomorphic to the Loch Ness monster. The purpose of these surfaces is to separate singularities during the construction of $S_{\mathfrak{N}}$, thus guaranteeing tameness. This idea already appears in Construction 4.4 in [8].

For each element $h_j \in H$, consider two copies $\mathbb{E}(j, 1)$ and $\mathbb{E}(j, 2)$ of the Euclidean plane equipped with origins $\bar{0}$ and an orthogonal basis $\beta = \{e_1, e_2\}$ for each. On $\mathbb{E}(j, 1)$, we draw the two families of marks

$$\begin{aligned} \check{M}^j &:= \{\check{m}_i^j = (4ie_1, (4i + 1)e_1) : i \in \mathbb{N}\}, \\ L_1 &:= \{l_i^1 = ((4i + 2)e_1, (4i + 3)e_1) : i \in \mathbb{N}\}, \end{aligned}$$

and similarly, on $\mathbb{E}(j, 2)$,

$$\begin{aligned} h_j \check{M}^{-j} &:= \{h_j \check{m}_i^{-j} = (2ie_2, e_1 + 2ie_2) : i \in \mathbb{N}\}, \\ L_2 &:= \{l_i^2 = ((2i + 1)e_2, e_1 + (2i + 1)e_2) : i \in \mathbb{N}\}. \end{aligned}$$

We define (see Figure 9) *the buffer surface*

$$(12) \quad S(\text{Id}, h_j) := \left(\bigcup_{k=1}^2 \mathbb{E}(j, k) \right) / (l_i^1 \sim_{\text{gluc}} l_i^2)_{i \in \mathbb{N}}.$$

By Lemma 1.16, the surface $S(\text{Id}, h_j)$ is a tame translation surface with infinitely many conic singularities of angle 4π homeomorphic to the Loch Ness monster. For each $g \in G$, we define

$$(13) \quad S(g, gh_j) := g \cdot S(\text{Id}, h_j).$$

That is, $S(g, gh_j)$ is the affine copy of $S(\text{Id}, h_j)$ obtained by postcomposing every chart by the affine transformation associated to the matrix g . The family of marks \check{M}^j and $h_j \check{M}^{-j}$ on $S(g, gh_j)$ are relabeled as $g\check{M}^j$ and $gh_j \check{M}^{-j}$, respectively. These marks will be used later to construct $S_{\mathfrak{N}}$. As said before, the purpose of the surfaces $S(g, gh_j)$ is to separate singularities during the construction of $S_{\mathfrak{N}}$, thus guaranteeing tameness. The following lemma is essential to assure this property.

Lemma 3.1 [8, Lemma 4.5] *For every $g \in G$ and $h_j \in H$, the distance between $g\check{M}^j$ and $gh_j \check{M}^{-j}$ is at least $1/\sqrt{2}$.*

Decorated Loch Ness monster Using $S(\text{Id}, h_j)$ defined above, we construct a tame translation surface S homeomorphic to the Loch Ness monster and having only one conic singularity of angle 6π . This surface will be called the decorated Loch Ness monster, and its purpose is to force the Veech group of $S_{\mathfrak{N}}$ to be exactly G . This idea also appears in Construction 4.6 in [8].

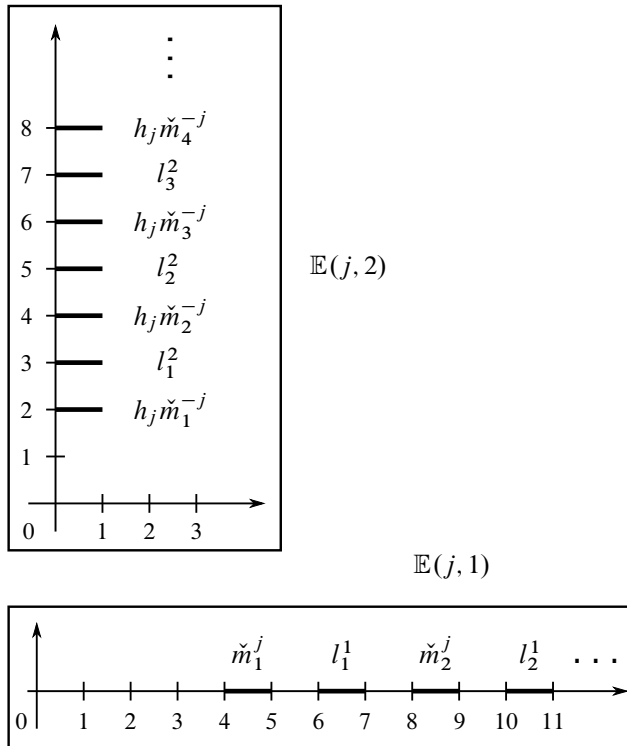


Figure 9: Buffer Loch Ness monster $S(\text{Id}, h_j)$

Consider a copy \mathbb{E} of the Euclidean plane equipped with an origin $\bar{0}$ and the same orthogonal basis $\beta = \{e_1, e_2\}$ as before. On \mathbb{E} we draw the following families of marks:

$$M := \{m_i = ((4i-1)e_1, 4ie_1) : i \in \mathbb{N}\},$$

$$M^j := \{m_i^j = ((2i-1)e_1 + (j+1)e_2, 2ie_1 + (j+1)e_2) : i \in \mathbb{N}, j \in \{0, \dots, |H|\}\}.$$

Now, we shall recursively define new families of marks on \mathbb{E} .

For $j = 1$, we can choose a point $(x_1, y_1) \in \mathbb{E}$ where $x_1 > 0$ and $y_1 < 0$ such that the family of marks

$$M^{-1} := \{m_i^{-1} = (ix_1e_1 + y_1e_2, ix_1e_1 + h_1^{-1}e_1 + y_1e_2) : i \in \mathbb{N}\} \subset \mathbb{E}$$

is disjoint from the families M and M^j defined in the previous paragraph. For $|H| \geq j > 1$, we can choose a point $(x_j, y_j) \in \mathbb{E}$ where $x_j > 0$ and $y_j < 0$ such that the family of marks

$$M^{-j} := \{m_i^{-j} = (ix_je_1 + y_je_2, ix_je_1 + h_j^{-1}e_1 + y_je_2) : i \in \mathbb{N}\} \subset \mathbb{E}$$

is disjoint from the families of marks defined in M and M^j above, and in step $j - 1$.

On the other hand, let $\pi: \tilde{\mathbb{E}} \rightarrow \mathbb{E}$ be the threefold cyclic covering of \mathbb{E} branched over the origin, and

$$\tilde{M}^0 := \{\tilde{m}_i^0 : i \in \mathbb{N}\}$$

one of the three (disjoint) families of marks on $\tilde{\mathbb{E}}$ defined by $\pi^{-1}(M)$. In addition, consider preimages \tilde{t}_1 and \tilde{t}_2 of $t_1 := (e_2, 2e_2)$ and $t_2 := (-e_2, -2e_2)$, respectively, which are in the same fold of $\tilde{\mathbb{E}}$ as \tilde{M}^0 . We define the *decorated surface* as

$$(14) \quad S := \left(\mathbb{E} \cup \tilde{\mathbb{E}} \cup \bigcup_{h_j \in H} S(\text{Id}, h_j) \right) / \sim,$$

where \sim is the equivalent relation given by gluing marks as follows (see Figure 10):

- (1) $\tilde{t}_1 \sim_{\text{glue}} \tilde{t}_2$ on $\tilde{\mathbb{E}}$.
- (2) $m_i^0 \sim_{\text{glue}} \tilde{m}_i^0$ on \mathbb{E} and $\tilde{\mathbb{E}}$, respectively.
- (3) $m_i^j \sim_{\text{glue}} \check{m}_i^j$ for each $i \in \mathbb{N}$ and for each $j \in \{1, \dots, |H|\}$, on \mathbb{E} and $S(\text{Id}, h_j)$, respectively.

By Lemma 1.16, the decorated surface S is a tame translation surface homeomorphic to the Loch Ness monster with infinitely many cone angle singularities of angle 4π and only one singular point of angle 6π .

Remark 3.2 The decorated surface S has $2|H| + 1$ families of marks left without gluing. These are $h_j \check{M}^{-j}$, M^{-j} and M , where $j \in \{1, \dots, |H|\}$. These marks will be used in what follows to define the elementary piece of the puzzle and to glue the surfaces S_g forming the puzzle to form the surface $S_{\mathfrak{P}}$.

Construction 3.3 (elementary piece associated to puzzle $\mathfrak{P}(X, G, H)$) Consider a closed subset X of the Cantor set, the graph T_X with ends space homeomorphic to X given by Lemma 1.9 and the countable family \mathfrak{T}_X of paths given by Lemma 1.11 and decomposing T_X . Pick an arbitrary path $\tilde{\gamma} \in \mathfrak{T}_X$. We define $S(\tilde{\gamma})$ as a copy of the decorated Loch Ness monster described above. We endow this tame translation surface with the family of marks M (see preceding remark). Clearly, the family M does not accumulate in the metric completion of $S(\tilde{\gamma})$. Moreover, for each $k \in \mathbb{N}$, we label the k^{th} mark m_k of the family M in $S(\tilde{\gamma})$ with the vertex $v_k \in \tilde{\gamma}$.

For each infinite path γ in $\mathfrak{T}_X \setminus \{\tilde{\gamma}\}$, we define $S(\gamma)$ as a copy of the Loch Ness monster described in Construction 2.1. By construction, each $S(\gamma)$ is endowed with a countable family of marks which, abusing notation, we also denote by M . These marks

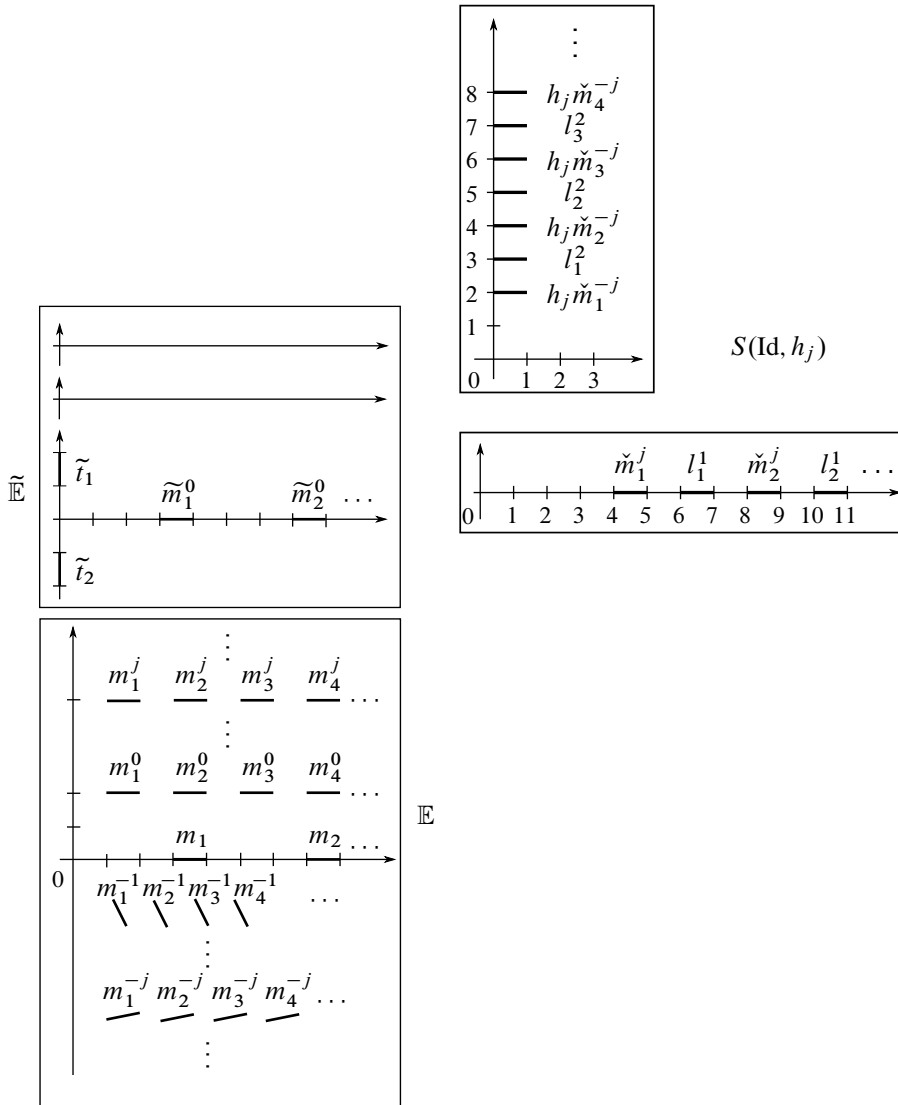


Figure 10: Decorated Loch Ness monster S

do not accumulate on the metric completion. For each $k \in \mathbb{N}$, we label the k^{th} mark m_k of the family M in $S(\gamma)$ with the vertex $v_k \in \gamma$. The elementary piece associated to the puzzle $\mathfrak{P}(X, G, H)$, which we denote by S_{elem} , is defined as the tame translation surface obtained by performing Construction 1.17 on the initial data X and the marked translations surfaces

$$S(\tilde{\gamma}) \cup \left(\bigcup_{\gamma \in \mathfrak{I}_X \setminus \tilde{\gamma}} S(\gamma) \right).$$

Remark 3.4 The elementary piece S_{elem} has the following properties:

- (1) From Lemma 1.18, we can deduce that the surface S_{elem} is a tame translation surface with space of ends homeomorphic to X and no planar ends.
- (2) The families of marks $h_j \check{M}^{-j}$ and M^{-j} , for each $j \in \{1, \dots, |H|\}$, have (still) not been glued to other marks.
- (3) By construction, there is *only one* end $[U_n^{\text{elem}}]_{n \in \mathbb{N}}$ of S_{elem} having the following property: for every n , there exists U_n^{elem} such that $U_n^{\text{elem}} \cap (h_j \check{M}^{-j} \cup M^{-j})$ are infinitely many marks. This end is defined by taking complements of large balls centered at the origin in the Euclidean plane \mathbb{E} defining the decorated Loch Ness monster $S(\tilde{\gamma})$. We will call this end the *distinguished end* of the elementary piece S_{elem} .
- (4) On the other hand, every other end $[U_n]_{n \in \mathbb{N}} \in \text{Ends}(S_{\text{elem}}) \setminus \{[U_n^{\text{elem}}]_{n \in \mathbb{N}}\}$ satisfies that there exists an n such that the intersections of U_n with the families of marks $h_j \check{M}^{-j} \cup M^{-j}$ in $S(\tilde{\gamma})$ are empty, for $j \in \{1, \dots, |H|\}$.

Definition 3.5 (puzzle and assembled surface) Let X be a closed subset of the Cantor set, G a countable subgroup of $\text{GL}_+(2, \mathbb{R})$, without contracting elements, generated by H . For each $g \in G$, let $S_g := g \cdot S_{\text{elem}}$ be the affine copy of S_{elem} obtained by postcomposing its translation atlas with the linear transformation defined by g . For each $j \in \{1, \dots, |H|\}$, we denote by $gh_j \check{M}^{-j}$ and gM^{-j} the families of marks on S_g given by the image of the families of marks $h_j \check{M}^{-j}$ and M^{-j} via the affine diffeomorphism $\bar{g}: S_{\text{elem}} \rightarrow S_g$; see Section 1.4. We define the *puzzle* associated to the triplet (X, G, H) as the set of marked surfaces,⁸

$$(15) \quad \mathfrak{P}(X, G, H) := \{S_g : g \in G\}.$$

The *assembled surface* associated to the puzzle $\mathfrak{P}(X, G, H)$ is defined as

$$(16) \quad S_{\mathfrak{P}} := \left(\bigcup_{g \in G} S_g \right) / \sim,$$

where \sim is the equivalent relation given by gluing marks as follows. Given an edge (g, gh_j) of the Cayley graph $\text{Cay}(G, H)$, we glue⁹ for each $i \in \mathbb{N}$, the mark $gh_j \check{m}_i^{-j} \in gh_j \check{M}^{-j} \subset S_g$ to the mark $gh_j m_i^{-j} \in gh_j M^{-j} \subset S_{gh_j}$.

⁸That is, on S_g we consider the families of marks $gh_j \check{M}^{-j}$ and gM^{-j} .

⁹We remark that, by construction, marks we glue are indeed parallel.

Remark 3.6 The pieces S_g of the puzzle inherit all affine invariant properties from S_{elem} , namely:

- (1) S_g is a tame translation surface without planar ends whose space of ends is homeomorphic to X .
- (2) For every $g \in G$, there is *only one* end $[U_n^g]$ of S_g having the following property: for every n , there exists U_n^g such that there are infinitely many marks of the form $U_n^g \cap (gh_j \check{M}^{-j} \cup gM^{-j})$. We will call this end the *distinguished end* of S_g .
- (3) Every other end $[U_n]_{n \in \mathbb{N}} \in \text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\}$ satisfies the following property: there exists an n such that the intersection $U_n \cap (gh_j \check{M}^{-j} \cup gM^{-j}) = \emptyset$ for each $j \in \{1, \dots, |H|\}$. We will call these kind of ends *common ends*.

Property (3) above tells us that every common end $[U_n]_{n \in \mathbb{N}} \in \text{Ends}(S_g)$ has a representative U_n that avoids the families of marks used to assemble the surface $S_{\mathfrak{P}}$. Therefore, we can *embed* U_n into $S_{\mathfrak{P}}$ (using the identity) and induce an embedding for common ends,

$$(17) \quad i_g: \text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\} \hookrightarrow \text{Ends}(S_{\mathfrak{P}}), \quad [U_n]_{n \in \mathbb{N}} \mapsto [U_n]_{n \in \mathbb{N}}.$$

We remark that the end $i_g([U_n]_{n \in \mathbb{N}})$ of $S_{\mathfrak{P}}$ is not planar. The following proposition describes the space of ends of the assembled surface *as a set*.

Theorem 3.7 *The assembled surface $S_{\mathfrak{P}}$ is a tame translation surface without planar ends, and its Veech group is G . Moreover,*

$$(18) \quad \text{Ends}(S_{\mathfrak{P}}) = \{[\tilde{U}_n]_{n \in \mathbb{N}}\} \cup \left(\bigsqcup_{g \in G} i_g(\text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\}) \right).$$

We insist that the preceding is an equality in the category of sets, not in the category of topological spaces. In particular, this result says that all distinguished ends in the puzzle $\mathfrak{P}(X, G, H)$ *merge into a single end* $\{[\tilde{U}_n]_{n \in \mathbb{N}}\}$ when constructing $S_{\mathfrak{P}}$. We will call this end the *secret end* of $S_{\mathfrak{P}}$.

Proof We begin by showing that the assembled surface $S_{\mathfrak{P}}$ is tame. This follows from the following two facts:

- *The surface $S_{\mathfrak{P}}$ is a complete metric space.*¹⁰ Indeed, let $(x_n)_{n \in \mathbb{N}} \subset S_{\mathfrak{P}}$ be a Cauchy sequence. Lemma 3.1 implies that the cost (in distance inside $S_{\mathfrak{P}}$) to

¹⁰This is with respect to the distance induced by the natural flat metric on $S_{\mathfrak{P}}$.

escape from a piece S_g of the puzzle is at least $1/\sqrt{2}$. Hence the sequence (x_n) is eventually contained in the closure (in $S_{\mathfrak{P}}$) of the open subset

$$U(g) := S_g \setminus \bigcup_{j=1}^{|H|} (gh_j\check{M}^{-j} \cup gM^{-j}),$$

and this closure is, by construction, a complete metric subspace of $S_{\mathfrak{P}}$.

- *The set of singularities of $S_{\mathfrak{P}}$ is discrete in $S_{\mathfrak{P}}$.* This follows from Lemma 3.1 and the fact that the set of singularities at each piece S_g of the puzzle is discrete.

We now prove that the Veech group of $S_{\mathfrak{P}}$ is G . For every $g, g' \in G$, there is a natural affine diffeomorphism $f_{gg'}: S_g \rightarrow S_{g'g}$ whose differential is precisely g' . These transformations send parallel marks to parallel marks; therefore, one can glue all the $f_{gg'}$ together to induce an affine diffeomorphism in the quotient $F_{g'}: S_{\mathfrak{P}} \rightarrow S_{\mathfrak{P}}$ whose differential is precisely g' . Since g' was arbitrary, we have that $G < \Gamma(S_{\mathfrak{P}})$. When constructing the elementary piece S_{elem} , we added in purpose a decorated Loch Ness monster so that S_{elem} has only one 6π singularity $x(\text{Id})$ and only three saddle connections γ_1, γ_2 and γ_3 issuing from it. Moreover, the holonomy vectors of these saddle connections are $\{\pm e_1, \pm e_2\}$. This implies that every piece S_g in the puzzle $\mathfrak{P}(X, G, H)$ has *only one* singularity $x(g)$ of total angle 6π and only three saddle connections γ_1, γ_2 and γ_3 issuing from it. The holonomy vectors of these are $\{\pm g \cdot e_1, \pm g \cdot e_2\}$. On the other hand, suppose that an affine diffeomorphism $f \in \text{Aff}_+(S_{\mathfrak{P}})$ sends $x(\text{Id}) \in S_{\text{elem}} = S_{\text{Id}}$ to $x(g)$. Its derivative Df must then send $\{\pm e_1, \pm e_2\}$ to $\{\pm g \cdot e_1, \pm g \cdot e_2\}$ and have positive determinant. The only possibility is $Df = g$; therefore, $\Gamma(S) < G$.

We now address (18). Given that this is the most technical part of the proof, we divide our approach in three steps:

- (I) For any two different elements $g \neq g'$ in G , we have that the embeddings defined in (17) have disjoint images

$$(19) \quad i_g(\text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\}) \cap i_{g'}(\text{Ends}(S_{g'}) \setminus \{[U_n^{g'}]_{n \in \mathbb{N}}\}) = \emptyset.$$

Therefore, $\bigsqcup_{g \in G} i_g(\text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\})$ is a subset of $\text{Ends}(S_{\mathfrak{P}})$. Indeed, consider two ends

$$[W_n]_{n \in \mathbb{N}} \in i_g(\text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\}) \quad \text{and} \quad [Z_n]_{n \in \mathbb{N}} \in i_{g'}(\text{Ends}(S_{g'}) \setminus \{[U_n^{g'}]_{n \in \mathbb{N}}\}).$$

Without loss of generality we can suppose that $[W_n]_{n \in \mathbb{N}}$ and $[Z_n]_{n \in \mathbb{N}}$ are ends of the pieces S_g and $S_{g'}$, respectively. Given that $[W_n]_{n \in \mathbb{N}}$ is different from the distinguished $[U_n^g]_{n \in \mathbb{N}}$, there exists a representative W_N which does not intersect any of the buffer surfaces in the decorated Loch Ness monster forming S_g . Since a path from S_g to $S_{g'}$

has to go through one of these buffer surfaces, there exists a representative Z_M such that $W_N \cap Z_M = \emptyset$. Therefore, the ends $[W_n]_{n \in \mathbb{N}}$ and $[Z_n]_{n \in \mathbb{N}}$ are disjoint in $\text{Ends}(S_{\mathfrak{F}})$.

(II) There exists an end $[\tilde{U}_n]_{n \in \mathbb{N}}$, that we will call *the secret end*, in the complement of $\bigsqcup_{g \in G} i_g(\text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\})$ in $\text{Ends}(S_{\mathfrak{F}})$. We construct this end inductively in what follows. First, we choose an enumeration $G := \{g_1, \dots, g_{|G|}\}$ and $H := \{h_1, \dots, h_{|H|}\}$ for elements in G and H , respectively. Since surfaces are σ -compact spaces, for every $g \in G$, there exists an exhaustion of $S_g = \bigcup_{n \in \mathbb{N}} gK_n$ by compact sets whose complements define the ends space of the surface. More precisely, we can write

$$(20) \quad S_g \setminus gK_n := gU_1^n \sqcup \dots \sqcup gU_{k(n)}^n \sqcup \dots \sqcup gU_{i_n}^n,$$

where each $gU_{k(n)}^n$ with $k(n) \in \{1, \dots, i_n\}$ is a connected component whose closure in S_g is noncompact, but has compact boundary, and for every $k(n+1) \in \{1, \dots, i_{n+1}\}$, there exists $k(n) \in \{1, \dots, i_n\}$ such that $gU_{k(n)}^n \supset gU_{k(n+1)}^{n+1}$. In other words, the space $\text{Ends}(S_g)$ are all nested sequences $(gU_{k(n)}^n)_{n \in \mathbb{N}}$. Without loss of generality, we can assume that $[gU_1^n]_{n \in \mathbb{N}}$ defines the distinguished end $[U_n^g]_{n \in \mathbb{N}}$ of S_g for all $g \in G$. Now consider $g_1 \in G$. By taking g_1K_1 big enough, we have the decomposition into connected components,

$$S_{\mathfrak{F}} \setminus g_1K_1 = \tilde{U}_1 \sqcup g_1U_2^1 \sqcup \dots \sqcup g_1U_{i_1}^1,$$

where $g_1U_j^1 \subset S_{g_1}$ and $g_1U_j^1 \cap U_n^{g_1} = \emptyset$ for all $j = 2, \dots, i_1$ and n big enough. By definition, the connected component $\tilde{U}_1 \subset S_{\mathfrak{F}}$ contains $U_1^{g_1} = g_1U_1^1$, has compact boundary, and is not planar, for $U_1^{g_1} \subset \tilde{U}_1$ has infinite genus. To define \tilde{U}_n for $n > 1$, consider the “first” n elements g_1, \dots, g_n in G . By taking g_jK_n big enough, we have the decomposition in connected components

$$S_{\mathfrak{F}} \setminus \bigcup_{k=1}^n g_kK_n = \tilde{U}_n \sqcup \left(\bigsqcup_{k=1}^n (g_kU_2^n \sqcup \dots \sqcup g_kU_{k(n)}^n \sqcup \dots \sqcup g_kU_{i_n}^n) \right),$$

where $g_kU_{k(n)}^n \subset S_{g_k}$ and $g_kU_{k(n)}^n \cap U_m^{g_k} = \emptyset$ for all $k = 1, \dots, n$, $k(n) = 2, \dots, i_n$, and m big enough. By definition, the connected component \tilde{U}_n contains $\bigcup_{k=1}^n U_n^{g_k}$, has compact boundary and is not planar. Moreover, $\tilde{U}_{n-1} \supset \tilde{U}_n$ and, since $\{gK_n\}_{n \in \mathbb{N}}$ is an exhaustion of S_g , for every compact subset K in $S_{\mathfrak{F}}$, there exists N such that $\tilde{U}_N \cap K = \emptyset$. In other words, $[\tilde{U}_n]_{n \in \mathbb{N}}$ is an element of $\text{Ends}(S_{\mathfrak{F}})$ and is not planar. In Figure 11, we depict this secret end when G is an infinite cyclic group.

Now let $[V_n]_{n \in \mathbb{N}} \in \bigsqcup_{g \in G} i_g(\text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\})$. Without loss of generality, we can suppose that $[V_n]_{n \in \mathbb{N}}$ is an element in $\text{Ends}(S_{g_k}) \setminus [U_n^{g_k}]_{n \in \mathbb{N}}$ for some $g_k \in G$. Given that $[V_n]_{n \in \mathbb{N}}$ is not a distinguished end, there exist $m, l \in \mathbb{N}$ such that $V_m \subset S_{g_k} \setminus g_kK_l$, and V_m is disjoint from all buffer surfaces in the decorated

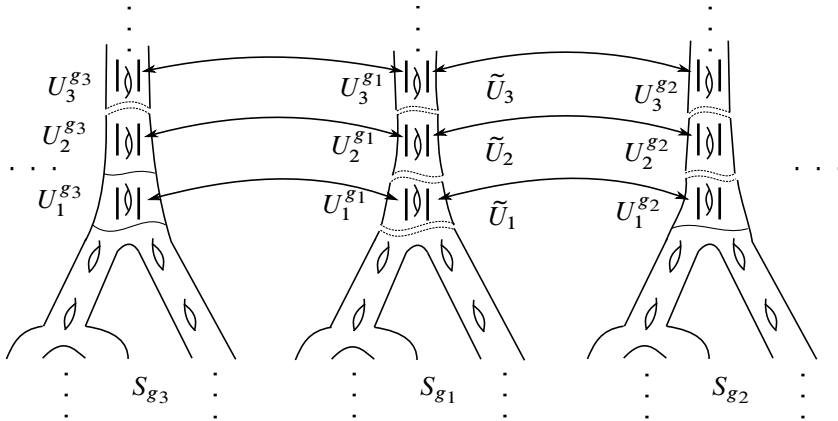


Figure 11: Secret end of $S_{\mathfrak{P}}$ when $G = \mathbb{Z}$

Loch Ness monster in S_{g_k} ; hence $V_m \cap S_{g'} = \emptyset$ for every $g' \neq g$. This implies that $[\tilde{U}_n]_{n \in \mathbb{N}}$ cannot belong to $\bigsqcup_{g \in G} i_g(\text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\})$.

(III) The secret end $[\tilde{U}_n]_{n \in \mathbb{N}}$ and $\bigsqcup_{g \in G} i_g(\text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\})$ is all there is in $\text{Ends}(S_{\mathfrak{P}})$. Consider an end $[W_n]_{n \in \mathbb{N}}$ of $S_{\mathfrak{P}}$. For every $n \in \mathbb{N}$, there exists $l(n) \in \mathbb{N}$ such that

$$W_{l(n)} \subset S_{\mathfrak{P}} \setminus \bigcup_{k=1}^n g_k K_n = \tilde{U}_n \sqcup \left(\bigsqcup_{k=1}^n (g_k U_2^n \sqcup \dots \sqcup g_k U_{i_n}^n) \right).$$

There are two cases to consider. First suppose that there exists $N \in \mathbb{N}$ such that $W_{l(N)} \subset g_k U_j^N$ for some $(k, j) \in \{1, \dots, n\} \times \{1, \dots, i_n\}$. In this situation, we have $[W_n]_{n \in \mathbb{N}} \in i_g(\text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\})$ for some $g \in G$. Suppose now that for every $n \in \mathbb{N}$, there exists $l(n)$ such that $W_{l(n)} \subset \tilde{U}_n$. If we fix n , there exists $k(n)$ such that $\tilde{U}_{k(n)} \cap \partial W_n = \emptyset$. Hence, the open subset $\tilde{U}_{k(n)} \subset S_{\mathfrak{P}}$ is contained in a connected component of $S_{\mathfrak{P}} \setminus \partial W_n$. Given our assumption, there exists $l(k(n)) \in \mathbb{N}$ such that $W_{l(k(n))} \subset \tilde{U}_{k(n)}$. Now, since $\tilde{U}_{k(n)}$ is connected, we have that either $W_n \subset W_{l(k(n))}$ or $W_{l(k(n))} \subset W_n$ implies that the connected component of $S_{\mathfrak{P}} \setminus \partial W_n$ containing $\tilde{U}_{k(n)}$ is precisely W_n . Then we conclude that for every n , there exists $k(n)$ such that $\tilde{U}_{k(n)} \subset W_n$, and hence the ends $[\tilde{U}_n]_{n \in \mathbb{N}}$ and $[W_n]_{n \in \mathbb{N}}$ are the same.

To finish the proof, note that, by construction, all ends in (18) have infinite genus. \square

3.2 Proof of Theorem 0.5

Let 2^ω denote the Cantor set and consider the puzzle $\mathfrak{P}(2^\omega, G, H)$. Following Theorem 3.7, it is sufficient to prove that $\text{Ends}(S_{\mathfrak{P}})$ has no isolated points. Then

let W^* be an open neighborhood of $[U_n]_{n \in \mathbb{N}} \in \text{Ends}(S_{\mathfrak{P}})$. From (18), we have two cases to consider. First, suppose that there exists $g \in G$ such that $[U_n]_{n \in \mathbb{N}} \in i_g(\text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\})$. In this case, there exists an open subset V of $S_{\mathfrak{P}}$ with compact boundary such that $U_l \subset V \subset W \cap S_g$. From Remark 3.4, we know that $\text{Ends}(S_g)$ is homeomorphic to 2^ω . Hence $V^* \setminus [U_n]_{n \in \mathbb{N}} \subset W^* \setminus [U_n]_{n \in \mathbb{N}}$ is not empty. Now suppose that $[U_n]_{n \in \mathbb{N}}$ is equivalent to the secret end $[\tilde{U}_n]_{n \in \mathbb{N}}$. We know that there exists $U_l \subset W$ and, by construction, $U_k \subset U_l$ such that $U_k^* \subset W^*$ contains $\text{Ends}(S_g) \setminus [U_n^g]_{n \in \mathbb{N}}$ for some $g \in G$. Since the latter is homeomorphic to 2^ω , we have that $W^* \setminus [U_n]_{n \in \mathbb{N}}$ is not empty. \square

3.3 Proof of Theorem 0.6

Consider the ordinal number $\omega^k + 1$ for a fixed $k \in \mathbb{N}$. Following Theorem 3.7, it is sufficient to prove that $\text{Ends}(S_{\mathfrak{P}})$ is homeomorphic to $\omega^k + 1$. Roughly speaking, the idea of the proof is the following: First, we properly choose the path $\tilde{\gamma}$ in Construction 3.3 so that the k^{th} iterate of the Cantor–Bendixson derivative on $\text{Ends}(S_g)$ is precisely the distinguished end $[U_n^g]$ for all $g \in G$. Using this and some properties of the secret end for this particular case, we will define a countable topological space Y_k . Finally, we will prove that $\text{Ends}(S_{\mathfrak{P}})$ is homeomorphic to Y_k and that the characteristic system of Y_k is precisely $(k, 1)$.

Fix a topological embedding $\omega^k + 1 \hookrightarrow \text{Ends}(T2^\omega)$, let $T_{\omega^k + 1}$ be the graph with ends space homeomorphic to ordinal number $\omega^k + 1$ given by Lemma 1.9, and let $\mathfrak{T}_{\omega^k + 1}$ be the countable family of paths given by Lemma 1.11. Given that $\omega^k + 1$ is countable, the sets $\text{Ends}(T_{\omega^k + 1})$ and $\mathfrak{T}_{\omega^k + 1}$ are in bijection; see Corollary 1.12. Let $\tilde{\gamma} \in \mathfrak{T}_{\omega^k + 1}$ be the infinite path corresponding to the only point left in $\text{Ends}(T_{\omega^k + 1})$ after the k^{th} iteration of the Cantor–Bendixson derivative. We perform Construction 3.3 of the elementary piece of the puzzle $\mathfrak{P}(\omega^k + 1, G, H)$ choosing the infinite path $\tilde{\gamma} \in \mathfrak{T}_{\omega^k + 1}$, associated to the decorated Loch Ness monster, as above. With this choice, we assure that the k^{th} iteration of the Cantor–Bendixson derivative on the space of ends of every piece S_g of the puzzle $\mathfrak{P}(\omega^k + 1, G, H)$ is precisely the distinguished end $[U_n^g]_{n \in \mathbb{N}}$. Let $U \subset S_{\mathfrak{P}}$ be a connected open subset with compact boundary defining an open neighborhood U^* of the secret end $[\tilde{U}_n]_{n \in \mathbb{N}}$ in $\text{Ends}(S_{\mathfrak{P}})$. Then there exists a finite subset $G(U^*) \subset G$ such that:

- (1) For every $g \in G(U^*)$, we have $\partial U \cap S_g \neq \emptyset$ and $i_g(\text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\})$ is not properly contained in U^* .
- (2) For every element $g \in G \setminus G(U^*)$, we have $i_g(\text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\}) \subset U^*$ and $\partial U \cap S_g = \emptyset$.

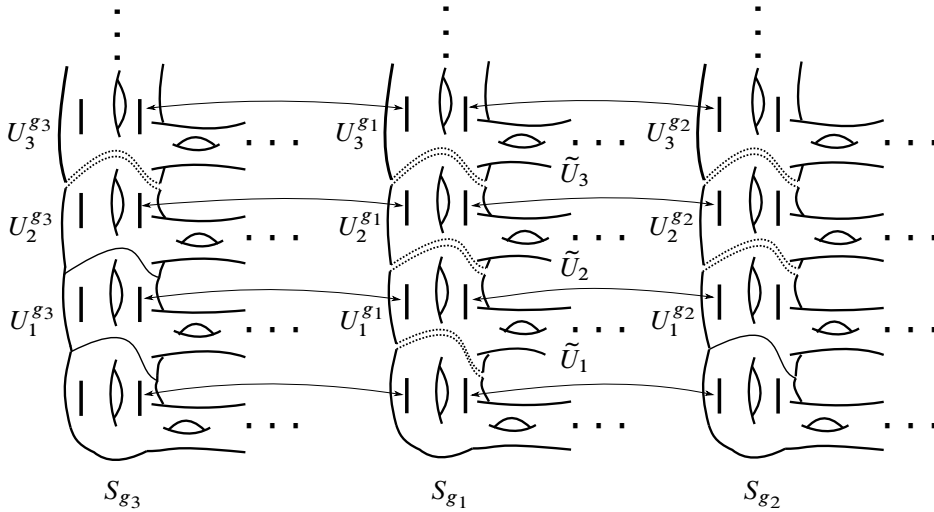


Figure 12: The surface $S_{\mathbb{N}}$ when $G = \mathbb{Z}$ and $\text{Ends}(S_g)$ is homeomorphic to $\omega + 1$

To illustrate these properties, consider the nontrivial example $G = \mathbb{Z}$ and $\text{Ends}(S_g)$ homeomorphic to $\omega + 1$ as depicted in Figure 12.

For every $g \in G$, let ω_g^k be a copy of the ordinal number ω^k , and define

$$(21) \quad Y_k := \{y\} \cup \left(\bigsqcup_{g \in G} \omega_g^k \right),$$

where y is just an abstract point. We endow Y_k with a topology as follows. Let U^* be an open neighborhood of the secret end $[\tilde{U}_n]_{n \in \mathbb{N}}$ of $S_{\mathbb{N}}$, and $G(U^*) \subset G$ a finite subset with $m(U^*)$ elements defined as above. For every $\{\gamma_{g_{j_1}}, \dots, \gamma_{g_{j_{m(U^*)}}}\} \in \prod_{g \in G(U^*)} \omega_g^k$, we define

$$(22) \quad W(G(U^*), \{\gamma_{g_{j_1}}, \dots, \gamma_{g_{j_{m(U^*)}}}\}) \\ := \{y\} \cup \left(\bigsqcup_{n=1}^{m(U^*)} \{\beta \in \omega_{g_{j_n}}^k : \beta \succ \gamma_{g_{j_n}}\} \right) \cup \left(\bigsqcup_{g \in G \setminus G(U^*)} \omega_g^k \right) \subset Y_k.$$

Then

$$\mathcal{B} := \left\{ W(G(U^*), \{\gamma_{g_{j_1}}, \dots, \gamma_{g_{j_{m(U^*)}}}\}) \cup \{W : W \text{ is an open subset of } \omega_g^k \text{ for any } g \in G\} \right\}$$

is the basis for the topology of Y_k . We remark that Y_k is a countable Hausdorff space with respect to this topology.

We now prove that Y_k is homeomorphic to $\text{Ends}(S_{\mathfrak{P}})$. By the way we chose the infinite path $\tilde{\gamma} \in \mathfrak{T}_{\omega^{k+1}}$, there exists a homeomorphism $f_g: i_g(\text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\}) \rightarrow \omega_g^k$ for every $g \in G$. Define $F: \text{Ends}(S_{\mathfrak{P}}) \rightarrow Y_k$ as

$$[V_n]_{n \in \mathbb{N}} \rightarrow \begin{cases} f_g([V_n]_{n \in \mathbb{N}}) & \text{if } [V_n]_{n \in \mathbb{N}} \in i_g(\text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\}) \text{ for some } g \in G, \\ y & \text{if } [V_n]_{n \in \mathbb{N}} \text{ is the secret end } [\tilde{U}_n]_{n \in \mathbb{N}}. \end{cases}$$

To prove that F is a homeomorphism, it is sufficient to prove that it is continuous at the secret end $[\tilde{U}_n]_{n \in \mathbb{N}}$, for every continuous map from a compact and Hausdorff space into a Hausdorff space is a closed map; see [2, page 226]. Consider an open neighborhood $W(G(U^*), \{\gamma_{g_{j_1}}, \dots, \gamma_{g_{j_m(U^*)}}\})$ of y as in (22). Since $G(U^*)$ has $m(U^*) < \infty$ elements, there exists a compact set K in $S_{\mathfrak{P}}$ such that its complement U_K satisfies:

- (1) $U_K^* \cap i_{g_{j_n}}(\text{Ends}(S_{g_{j_n}}) \setminus \{[U_n^{g_{j_n}}]_{n \in \mathbb{N}}\}) \subset F^{-1}(\{\beta \in \omega_{g_{j_n}}^k : \beta \succ \gamma_{g_{j_n}}\})$ for all $n = 1, \dots, m(U^*)$,
- (2) $[\tilde{U}_n]_{n \in \mathbb{N}} \in U_K^*$.

Therefore $F(U_K^*) \subset W(G(U^*), \{\gamma_{g_{j_1}}, \dots, \gamma_{g_{j_m(U^*)}}\})$ as desired, and hence F is continuous.

We claim that the characteristic system of Y_k is $(k, 1)$. To see this, first note that for each $g \in G$, the copy ω_g^k of ω^k figuring in the right-hand side of (21) is (topologically) embedded in Y_k . On the other hand, given that ω^k is not a limit ordinal, y is a limit point of the subset $\bigsqcup_{g \in G} \omega_g^k$. These two facts combined imply that the set of accumulation points of Y_k is precisely Y_{k-1} for every $k \geq 2$, and just the singleton $\{y\}$ when $k = 1$. Hence the $(k-1)^{\text{st}}$ iteration of the Cantor–Bendixson derivative on Y_k yields Y_1 , which has characteristic system $(1, 1)$. This implies that the characteristic system of Y_k is precisely $(k, 1)$, and the proof is complete. \square

3.4 Proof of Theorem 0.7

Following Theorem 3.7, it is sufficient to prove that $\text{Ends}(S_{\mathfrak{P}})$ is homeomorphic to $B \sqcup U$, where by hypothesis, B is homeomorphic to the Cantor set and U is a countable discrete set of points such that $\partial U = u \in B$. As in the proof of Theorem 0.6, we first deal with the elementary piece of $\mathfrak{P}(B \sqcup U, G, H)$. From Corollary 1.13, we know that there exists a path $\tilde{\gamma} \in \mathfrak{T}_{B \sqcup U}$ that defines the end of $T_{B \sqcup U}$ corresponding to u , the boundary of U in $B \sqcup U$. We perform Construction 3.3 of the elementary piece taking the decorated Loch Ness monster as $S(\tilde{\gamma})$, where $\tilde{\gamma}$ is chosen just as mentioned before. This way, for every $g \in G$, we have that $\text{Ends}(S_g) = B_g \sqcup U_g$, with B_g homeomorphic to the Cantor set, U_g is discrete and countable, and ∂U_g is equal to the distinguished end $[U_n^g]_{n \in \mathbb{N}}$. On the other hand, given that the map

$i_g: \text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\} \rightarrow \text{Ends}(S_{\mathfrak{N}})$ defined in (17) is an embedding for every $g \in G$, $\text{Ends}(S_{\mathfrak{N}})$ is uncountable, and

$$(23) \quad i_g(\text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\}) = i_g(B_g \setminus \{[U_n^g]_{n \in \mathbb{N}}\}) \sqcup i_g(U_g).$$

By the Cantor–Bendixson theorem (Theorem 1.3), we can write $\text{Ends}(S_{\mathfrak{N}}) = B_{\mathfrak{N}} \sqcup U_{\mathfrak{N}}$, where $B_{\mathfrak{N}}$ is homeomorphic to the Cantor set and $U_{\mathfrak{N}}$ is discrete and countable. From Theorem 1.6, it is sufficient to show that $\partial U_{\mathfrak{N}}$ is just a point to finish the proof. We achieve this in what follows.

Recall that

$$(24) \quad \text{Ends}(S_{\mathfrak{N}}) = B_{\mathfrak{N}} \sqcup U_{\mathfrak{N}} = \{[\tilde{U}_n]_{n \in \mathbb{N}}\} \cup \left(\bigsqcup_{g \in G} i_g(\text{Ends}(S_g) \setminus \{[U_n^g]_{n \in \mathbb{N}}\}) \right).$$

We claim that $\partial U_{\mathfrak{N}} = \{[\tilde{U}_n]_{n \in \mathbb{N}}\}$. We first remark that, by the choices we made, the secret end of $S_{\mathfrak{N}}$ is not an isolated point of $\text{Ends}(S_{\mathfrak{N}})$; hence $[\tilde{U}_n]_{n \in \mathbb{N}} \in B_{\mathfrak{N}}$. From (23) and (24), we obtain

$$(25) \quad B_{\mathfrak{N}} = \{[\tilde{U}_n]_{n \in \mathbb{N}}\} \cup \left(\bigsqcup_{g \in G} i_g(B_g \setminus \{[U_n^g]_{n \in \mathbb{N}}\}) \right) \quad \text{and} \quad U_{\mathfrak{N}} = \bigsqcup_{g \in G} i_g(U_g).$$

Every neighborhood of the secret end $[\tilde{U}_n]_{n \in \mathbb{N}}$ intersects $i_g(U_g)$, and therefore $[\tilde{U}_n]_{n \in \mathbb{N}} \in \partial U_{\mathfrak{N}}$. Now let’s prove by contradiction that if $[V_n]_{n \in \mathbb{N}} \in \partial U_{\mathfrak{N}}$, then $[V_n]_{n \in \mathbb{N}} = [\tilde{U}_n]_{n \in \mathbb{N}}$. By (25), if $[V_n]_{n \in \mathbb{N}} \neq [\tilde{U}_n]_{n \in \mathbb{N}}$, then we have $[V_n]_{n \in \mathbb{N}} \in i_g(B_g \setminus \{[U_n^g]_{n \in \mathbb{N}}\})$ for some $g \in G$. Without loss of generality, we can suppose that $[V_n]_{n \in \mathbb{N}}$ is actually an end in $B_g \setminus \{[U_n^g]_{n \in \mathbb{N}}\} \subset \text{Ends}(S_g)$. Therefore, there exist $l, n \in \mathbb{N}$ such that $V_l \cap U_n^g = \emptyset$. But then the open set of $S_{\mathfrak{N}}$ given by V_l^* is an open neighborhood of $[V_n]_{n \in \mathbb{N}}$ which lies in the complement of $U_{\mathfrak{N}} = \bigsqcup_{g \in G} i_g(U_g)$, which contradicts $[V_n]_{n \in \mathbb{N}} \in \partial U_{\mathfrak{N}}$. \square

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