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Indecomposable nonorientable PD_3–complexes

JONATHAN A HILLMAN

We show that the orientable double covering space of an indecomposable, nonorientable PD_3–complex has torsion-free fundamental group.

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One of the foundational results of Wall [12] on Poincaré duality complexes was the fact that there is a well-defined notion of connected sum for such complexes. In dimensions $n > 2$ the fundamental group of a connected sum of two PD_n–complexes is the free product of the groups of the summands. This notion is of particular interest when $n = 3$ for, by the well-known work of Kneser and Milnor, every closed orientable 3–manifold has an essentially unique factorization into indecomposable 3–manifolds. (The corresponding assertion for closed nonorientable 3–manifolds is slightly more complicated.) Moreover, such a 3–manifold is indecomposable with respect to connected sum if and only if its fundamental group is indecomposable with respect to free product. It is perhaps less widely known that Turaev [11] has shown that each of these results extends to the context of PD_3–complexes.

Indecomposable, orientable 3–manifolds are either aspherical, have finite fundamental group or have fundamental group $\mathbb{Z}$. This is no longer true for PD_3–complexes, although Crisp [3] has shown that (in the orientable case) the indecomposables are either aspherical or have virtually free fundamental group. There are examples of the latter kind with fundamental group neither finite nor $\mathbb{Z}$; see Hillman [9].

Let X be an indecomposable PD_3–complex, with fundamental group π and orientation character w. In [9] we showed that if $w \neq 1$ and π is virtually free then X is homotopy equivalent to $S^2 \tilde{\times} S^1$ or $\mathbb{RP}^2 \times S^1$, so $\pi \cong \mathbb{Z}$ or $\pi \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. In particular, $\pi^+ = \mathrm{Ker}(w)$ is torsion-free. We shall show that this remains true if $w \neq 1$ and π is *not* virtually free. This result is surely well-known for 3–manifolds. We give a short proof for this case in Section 2, which uses the "projective plane theorem" of Epstein [6] and a result from Hillman [9]. (The fact that $\mathbb{RP}^2$ does not bound provides a further restriction here which is not yet known in general.) Our main result is Theorem 6 in Section 3:

Theorem *Let X be an indecomposable, nonorientable* PD_3*–complex such that π has infinitely many ends. Then $\pi \cong \pi^+ \rtimes \mathbb{Z}/2\mathbb{Z}^-$ and π^+ is torsion-free, but not free.*

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By passing to Sylow subgroups of the torsion in π, we may reduce potential counter-examples to special cases, which are eliminated by Lemmas 3, 4 and 5. The arguments are similar to those of [9].

1 Notation and major cited results

In order that this paper be reasonably self-contained we shall give here some of the notation and results used in [9].

Let X be a PD_3–complex, with fundamental group π and orientation character w, and let X^+ be the orientable covering space, with fundamental group $\pi^+ = \mathrm{Ker}(w)$. If $H \leq \pi$ then we shall write $H^+ = H \cap \pi^+$. It is convenient to say that such a subgroup H is *orientable* if $H = H^+$. (This usage depends upon the orientation character w.) Let $\mathbb{Z}/2\mathbb{Z}^-$ denote a subgroup of order two on which $w \neq 1$.

If G is a group, $|G|$, G' and ζG shall denote the order, commutator subgroup and centre of G, while if $H \leq G$ then $C_G(H)$ and $N_G(H)$ are the centralizer and normalizer, respectively. Let $F(r)$ be the free group of rank r.

If R is a ring, two finitely presentable left R–modules M and N are *stably isomorphic* if $M_1 \oplus R^a \cong N \oplus R^b$ for some a, $b \geq 0$. Let $[M]$ denote the stable isomorphism class of M.

A homomorphism $w\colon G \to \{\pm 1\}$ defines an anti-involution of $\mathbb{Z}[G]$ by $\overline{g} = w(g)g^{-1}$ for all $g \in G$. Tietze move considerations show that if A is any finite presentation matrix for the augmentation ideal I_G then the stable isomorphism class of the left $\mathbb{Z}[G]$–module J_G with presentation matrix the conjugate transpose $\bar{A}^{\mathrm{tr}}$ is well-defined [11].

A *graph of groups* $(\mathcal{G}, \Gamma)$ consists of a graph Γ with origin and target functions o and t from the set of edges $E(\Gamma)$ to the set of vertices $V(\Gamma)$, and a family $\mathcal{G}$ of groups G_v for each vertex v and subgroups $G_e \leq G_{o(e)}$ for each edge e, with monomorphisms $\phi_e\colon G_e \to G_{t(e)}$. (We shall usually suppress the maps ϕ_e from our notation.) In considering paths in Γ we shall not require that the edges be compatibly oriented.

The *fundamental group* of $(\mathcal{G}, \Gamma)$ is the group $\pi\mathcal{G}$ with presentation

$$\langle G_v, t_e \mid t_e g t_e^{-1} = \phi_e(g)\ \forall g \in G_e,\ t_e = 1\ \forall e \in E(T)\rangle,$$

where T is some maximal tree for Γ. Different choices of maximal tree give isomorphic groups. We may assume that $(\mathcal{G}, \Gamma)$ is *reduced*: if an edge joins distinct vertices then the edge group is isomorphic to a proper subgroup of each of these vertex groups. The corresponding π–tree T is incompressible in the terminology of [5], so T and $\mathcal{G}$ are essentially unique, by [5, Proposition IV.7.4]. An edge e is a *loop isomorphism* at v if $o(e) = t(e) = v$ and the inclusions induce isomorphisms $G_e \cong G_v$.

Since fundamental groups of PD_n–complexes are $\mathbb{F}\mathrm{P}_2$ [12], π is the fundamental group of a finite graph of groups $(\mathcal{G},\Gamma)$, where all vertex groups are finite or have one end and all edge groups are finite. (See [5, Theorem VI.6.3].) We may assume that π is indecomposable as a proper free product, by the splitting theorem, so $(\mathcal{G},\Gamma)$ is *indecomposable*: all edge groups are nontrivial. A graph of groups $(\mathcal{G},\Gamma)$ is *admissible* if it is reduced, all vertex groups are finite or one-ended groups and all edge groups are nontrivial finite groups.

Turaev gave the following characterization of the group pairs (π,w) which may be realized by finite PD_3–complexes [11]:

Theorem *Let π be a finitely presentable group and $w\colon \pi\to\{\pm1\}$ a homomorphism. Then there is a finite PD_3–complex K with $\pi_1(K)\cong\pi$ and $w_1(K)=w$ if and only if $[I_\pi]=[J_\pi]$.*

We wish to adapt the results from [9, Section 7] to the cases when π has infinitely many ends and $w\neq 1$. In particular, we use the following two results to control the possible edge groups:

(1) **Crisp's theorem** [3, Theorem 17] *If X is a PD_3–complex and $g\in\pi=\pi_1(X)$ has prime order p and infinite centralizer $C_\pi(g)$ then $p=2$, g is orientation-reversing and $C_\pi(g)$ has two ends.*

(2) **The normalizer condition** [10, Proposition 5.4.2] *A proper subgroup of a nilpotent group is properly contained in its normalizer.*

Note also that if G is a finite subgroup of π then the centralizer $C_\pi(G)$ has finite index in the normalizer $N_\pi(G)$.

The main result (Theorem 6 below) involves consideration of the finite groups with periodic cohomology, of period dividing 4. A finite group has cohomological period 2 if and only if it is cyclic, and has cohomological period 4 if and only if it is a product $B\times\mathbb{Z}/d\mathbb{Z}$ with $(|B|,d)=1$, where B is a generalized quaternionic group $\mathbb{Z}/a\mathbb{Z}\rtimes Q(2^i)$ (with a odd), an extended binary polyhedral group T_k^* (of order $2^3\cdot 3^k$), O_k^* (of order $2^4\cdot 3^k$) or $I^*=\mathrm{SL}(2,5)$ (of order $2^3\cdot 3\cdot 5$) or a metacyclic group $\mathbb{Z}/a\mathbb{Z}\rtimes_{-1}\mathbb{Z}/2^e\mathbb{Z}$ (for some odd a and $e\geq 1$).

There seems to be no one reference with a complete proof of the above assertion. The six families of finite groups with periodic cohomology are determined in [1, pages 142–150]:

(1) $\mathbb{Z}/a\mathbb{Z}\rtimes\mathbb{Z}/b\mathbb{Z}$;

(2) $\mathbb{Z}/a\mathbb{Z}\rtimes(\mathbb{Z}/b\mathbb{Z}\times Q(2^i))$ for $i\geq 3$;

(3) $\mathbb{Z}/a\mathbb{Z} \rtimes (\mathbb{Z}/b\mathbb{Z} \times T_k^*)$ for $k \geq 1$;
(4) $\mathbb{Z}/a\mathbb{Z} \rtimes (\mathbb{Z}/b\mathbb{Z} \times O_k^*)$ for $k \geq 1$;
(5) $(\mathbb{Z}/a\mathbb{Z} \rtimes \mathbb{Z}/b\mathbb{Z}) \times \mathrm{SL}(2, p)$ for $p \geq 5$ prime;
(6) $\mathbb{Z}/a\mathbb{Z} \rtimes (\mathbb{Z}/b\mathbb{Z} \times \mathrm{TL}(2, p))$ for $p \geq 5$ prime.

Here a, b and the order of the quotient by the metacyclic subgroup $\mathbb{Z}/a\mathbb{Z} \rtimes \mathbb{Z}/b\mathbb{Z}$ are relatively prime. See [1, pages 142–150] for further details on the groups $\mathrm{TL}(2, p)$ (with $\mathrm{TL}(2, p)' \cong \mathrm{SL}(2, p)$, of index 2) and the actions in the semidirect products. If such a group G contains a semidirect product $\mathbb{Z}/m\mathbb{Z} \rtimes_\theta \mathbb{Z}/n\mathbb{Z}$, where θ has image of order k, then the cohomological period of G is a multiple of $2k$. (See [2, Exercise 6, page 159].) The class of groups of period dividing 4 follows on applying this criterion to the groups of the above list.

2 3–manifolds

The result is relatively easy (and no doubt well-known) in the case of irreducible 3–manifolds, as we may use the sphere theorem, as strengthened by Epstein [6].

Theorem 1 *Let M be an indecomposable, nonorientable 3–manifold with fundamental group π. If π has infinitely many ends then $\pi \cong \pi^+ \rtimes \mathbb{Z}/2\mathbb{Z}^-$ and π^+ is torsion-free, but not free.*

Proof Let $\mathcal{P}$ be a maximal set of pairwise nonparallel 2–sided projective planes in M. Then $\mathcal{P}$ is nonempty, since M is indecomposable and π has infinitely many ends. In particular, $\pi \cong \pi^+ \rtimes \mathbb{Z}/2\mathbb{Z}^-$, since the inclusion of a member of $\mathcal{P}$ splits $w = w_1(M)\colon \pi \to \mathbb{Z}/2\mathbb{Z}$. Let $\mathcal{P}^+$ be the preimage of $\mathcal{P}$ in M^+. Then $\mathcal{P}^+$ is a set of disjoint 2–spheres in M^+, and the components of $M^+ \setminus \mathcal{P}^+$ each double cover a component of $M \setminus \mathcal{P}$. Each such component of $M \setminus \mathcal{P}$ is indecomposable [6].

Suppose that $M \setminus \mathcal{P}$ has a component Y with virtually free fundamental group. Then the double DY is indecomposable (see [9, Lemma 2.4]), nonorientable and $\pi_1(DY)$ is virtually free. Moreover, $\pi_1(DY) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}^-$, since the inclusion of a boundary component of Y splits w. (See [9, Theorems 7.1 and 7.4].) But then $DY \cong \mathbb{RP}^2 \times S^1$, so $Y \cong \mathbb{RP}^2 \times [0,1]$. This is contrary to the hypothesis that the members of $\mathcal{P}$ are nonparallel. Thus the components of $M \setminus \mathcal{P}$ are punctured aspherical 3–manifolds.

Let Γ be the graph with vertex set $\pi_0(M \setminus \mathcal{P})$ and edge set $\mathcal{P}$, with an edge joining contiguous components. Then $\pi^+ \cong G * F(s)$, where G is a free product of PD_3–groups (corresponding to the fundamental groups of the components of $M \setminus \mathcal{P}$), and $s = \beta_1(\Gamma)$. Hence π^+ is torsion-free. □

We remark also that each component Y of $M \setminus \mathcal{P}$ has an even number of boundary components, since $\chi(\partial Y)$ is even (for any odd-dimensional manifold Y), by Poincaré duality. Thus the vertices of the graph Γ have even valence.

Example The canonical involution ι of the topological group $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ has 8 isolated fixed points (the points of order 2). Let X be the complement of an equivariant open regular neighbourhood of the fixed point set, and let $M = D(X/\langle\iota\rangle)$. Then M is indecomposable and nonorientable, and $\pi \cong (\mathbb{Z}^3 * \mathbb{Z}^3 * F(7)) \rtimes \mathbb{Z}/2\mathbb{Z}^-$.

3 PD_3–complexes

Suppose now that X is an indecomposable PD_3–complex, with fundamental group π and orientation character w. Then π is finitely presentable, so $\pi \cong \pi\mathcal{G}$, where $(\mathcal{G}, \Gamma)$ is an admissible graph of groups.

Lemma 2 *Let X be an indecomposable, nonorientable* PD_3*–complex with $\pi = \pi_1(X) \cong \pi\mathcal{G}$, where $(\mathcal{G}, \Gamma)$ is an admissible graph of groups.*

(1) *If e is an edge with $G_{o(e)}$ or $G_{t(e)}$ infinite, then $G_e = \mathbb{Z}/2\mathbb{Z}^-$.*

(2) *If $X \not\simeq S^2 \tilde{\times} S^1$ then $\pi \cong \pi^+ \rtimes \mathbb{Z}/2\mathbb{Z}^-$.*

(3) *If all finite vertex groups are 2–groups then they are nonorientable and all edge groups are $\mathbb{Z}/2\mathbb{Z}^-$.*

Proof Suppose first that the vertex groups are all finite. Then $X \simeq S^2 \tilde{\times} S^1$ (if all the vertex groups are orientation-preserving) or $\mathbb{R}P^2 \times S^1$ (otherwise), by Theorems 7.1 and 7.4 of [9], respectively, so the lemma holds. Hence we may assume that $(\mathcal{G}, \Gamma)$ has at least one infinite vertex group G_v and at least one edge e with $o(e) = v$ or $t(e) = v$. If $w(g) = 1$ for some $g \in G_e$ of prime order then both $G^+_{o(e)}$ and $G^+_{t(e)}$ would be finite, by [3, Theorem 14]. But then G_v would be finite, contrary to hypothesis. Thus $G_e = \mathbb{Z}/2\mathbb{Z}^-$, and the inclusion of G_e into π splits w, so $\pi \cong \pi^+ \rtimes \mathbb{Z}/2\mathbb{Z}^-$.

Suppose that all finite subgroups are 2–groups. Let f be an edge such that the vertex groups $G_{o(f)}$ and $G_{t(f)}$ are finite. If $G_f = G_{o(f)}$ (or $G_{t(f)}$) then f must be a loop isomorphism, since $(\mathcal{G}, \Gamma)$ is reduced. But then $C_\pi(G_f)$ is infinite, so $G_f = \mathbb{Z}/2\mathbb{Z}^-$, by Crisp's theorem. Since $(\mathcal{G}, \Gamma)$ is reduced, f must be the only edge, contrary to the assumption that there is an infinite vertex group. Thus we may assume that $G_{o(f)}$ and $G_{t(f)}$ each properly contain G_f. Since $G_{o(f)}$ and $G_{t(f)}$ are 2–groups and hence nilpotent, $N_\pi(G_f)$ is infinite, by the normalizer condition. Since $C_\pi(G_f)$ has finite index in $N_\pi(G_e)$ we must have $G_f = \mathbb{Z}/2\mathbb{Z}^-$, by Crisp's theorem. Since Γ is connected it follows easily that every finite vertex group is nonorientable and every edge group is $\mathbb{Z}/2\mathbb{Z}^-$. □

The next two lemmas consider two parallel special cases, involving a prime p, which is odd or 2, respectively.

Lemma 3 *Let X be an indecomposable* PD_3*–complex with $\pi = \pi_1(X) \cong \kappa \rtimes W$, where κ is orientable and torsion-free, and W has order $2p$ for some odd prime p. Then X is orientable.*

Proof Suppose that X is not orientable. Then π and κ are infinite. Since π has a subgroup W of finite order > 2, we may assume that $\pi \cong \pi\mathcal{G}$, where $(\mathcal{G}, \Gamma)$ is an admissible graph of groups with $r \geq 1$ finite vertex groups and at least one edge. Let $s = \beta_1(\Gamma)$.

Each finite vertex group is mapped injectively by any projection from π onto W with kernel κ. If a vertex group G_v has prime order then every edge e with one vertex at v is a loop isomorphism, since $(\mathcal{G}, \Gamma)$ is reduced. But then Γ has just one vertex and $\pi \cong G_v \rtimes F$, which contradicts the hypothesis. Hence all finite vertex groups are isomorphic to W. If an edge e is a loop isomorphism then $G_e^+ \cong \mathbb{Z}/p\mathbb{Z}$ has infinite normalizer, contradicting Crisp's theorem. If there is an edge e with G_e of order p then both of the vertex groups $G_{o(e)}$ and $G_{t(e)}$ are finite, by Lemma 2. But then $[G_{o(e)} : G_e] = [G_{t(e)} : G_e] = 2$, so $N_\pi(G_e)$ is infinite, which again contradicts Crisp's theorem. Since the orientation character w factors through W it follows that every edge group is $\mathbb{Z}/2\mathbb{Z}^-$ and w is nontrivial on every vertex group.

Since each edge group is $\mathbb{Z}/2\mathbb{Z}^-$, w is nontrivial on each vertex group, so $\pi^+ = \pi\mathcal{G}^+$ is the fundamental group of a graph of groups $(\mathcal{G}^+, \Gamma)$ with the same underlying graph Γ, trivial edge groups and vertex groups G_v^+ for all $v \in V(\Gamma)$. Hence $\pi^+ \cong G * F(s) * P$, where G is a free product of orientable PD_3–groups and P is a free product of r copies of $\mathbb{Z}/p\mathbb{Z}$. We have $P \cong F(t) \rtimes \mathbb{Z}/p\mathbb{Z}$ for some $t \geq 0$. (In fact, $t = (p-1)(r-1)$, by a simple virtual Euler characteristic argument.)

Let $a \in \pi$ be such that $a^2 = 1$ and $w(a) = -1$, and let $\lambda \cong \kappa \rtimes \mathbb{Z}/2\mathbb{Z}^-$ be the subgroup generated by κ and a. Then λ is also the group of a PD_3–complex, since it has finite index in π. The involution of π^+ induced by conjugation by a maps each indecomposable factor which is not infinite cyclic to a conjugate of an isomorphic factor [7]. However, its behaviour on the free factor $F(s)$ may be more complicated.

Let $w\colon \mathbb{Z}[\pi] \to R = \mathbb{Z}[\langle a \rangle] = \mathbb{Z}[a]/(a^2 - 1)$ be the linear extension of the orientation character. Then $I_{\langle a \rangle} \cong \widetilde{\mathbb{Z}} = R/(a+1)$. We may factor out the action of π^+ on a $\mathbb{Z}[\pi]$–module by tensoring with R. The derived sequence of the functor $R \otimes_w -$ applied to the augmentation sequence

$$0 \to I_\pi \to \mathbb{Z}[\pi] \to \mathbb{Z}$$

gives an exact sequence

$$0 \to H_1(\pi; R) = \kappa/\kappa' \to R \otimes_w I_\pi \to R \to \mathbb{Z} \to 0.$$

The inclusion of $\langle a \rangle$ into π splits the epimorphism from $R \otimes_w I_\pi$ onto $I_{\langle a \rangle}$, so $R \otimes_w I_\pi \cong \kappa/\kappa' \oplus \widetilde{\mathbb{Z}}$.

Let γ be the normal subgroup of π generated by $G \cup F(s)$ and let H be the image of γ in κ/κ'. Then similar arguments show that

$$R \otimes_w I_\pi = H \oplus (R \otimes_w I_{\pi/\gamma}),$$
$$R \otimes_w I_\lambda = H \oplus (R \otimes_w I_{\lambda/\gamma}).$$

The groups P and its normal subgroup $F(t)$ have presentations

$$P = \langle b_i,\ 1 \le i \le r \mid b_i^p = 1\ \forall i \rangle$$

and

$$F(t) = \langle x_{i,j},\ 1 \le i \le r-1,\ 1 \le j \le p-1 \mid \ \rangle,$$

where $x_{i,j}$ has image $b_1^j b_{i+1}^{-j}$ in P for $1 \le i \le r-1$ and $1 \le j \le p-1$. (If $p = 2$ we shall write x_i instead of $x_{i,1}$ for $1 \le i \le r-1$.)

The quotient $\pi/\langle\langle G \rangle\rangle$ is the fundamental group of the (possibly *un*reduced) graph of groups $(\bar{\mathcal{G}}, \Gamma)$ with vertex groups W (or $\mathbb{Z}/2\mathbb{Z}^-$) and edge groups $\mathbb{Z}/2\mathbb{Z}^-$, obtained by replacing each infinite vertex group G_v of $(\mathcal{G}, \Gamma)$ by $G_v/G_v^+ = \mathbb{Z}/2\mathbb{Z}^-$. Thus if W is abelian (so has an unique element of order 2) then $\pi/\langle\langle G \rangle\rangle \cong (F(s) * P) \times \mathbb{Z}/2\mathbb{Z}^-$. Hence $\pi/\gamma \cong P \times \mathbb{Z}/2\mathbb{Z}^-$ and $\lambda/\gamma \cong F(t) \times \mathbb{Z}/2\mathbb{Z}^-$, so

$$R \otimes_w I_{\pi/\gamma} \cong (R/(p, a-1))^r \oplus \widetilde{\mathbb{Z}},$$
$$R \otimes_w I_{\lambda/\gamma} \cong (R/(a-1))^t \oplus \widetilde{\mathbb{Z}} = \mathbb{Z}^t \oplus \widetilde{\mathbb{Z}}.$$

The quotient ring $R/pR = \mathbb{F}_p[a]/(a^2-1)$ is semisimple, so p–torsion R–modules have unique factorizations as sums of simple modules. Since $I_\pi \otimes_w R$ and $I_\lambda \otimes_w R$ satisfy Turaev's criterion (and projective R–modules are $\mathbb{Z}$–torsion-free), the p–torsion submodule of $R \otimes_w I_\pi$ has the same numbers of summands of types $R/(p, a-1)$ and $R/(p, a+1)$, and similarly for $R \otimes_w I_\lambda$. Since $R \otimes_w I_{\lambda/\gamma}$ is p–torsion-free, the number of summands of types $R/(p, a-1)$ and $R/(p, a+1)$ in H must also be equal. On the other hand, $R \otimes_w I_{\pi/\gamma}$ has $r > 0$ summands of type $R/(p, a-1)$ and none of type $R/(p, a+1)$. These conditions are inconsistent, so π is not the group of a nonorientable PD_3–complex.

If W is not abelian then it has an unique conjugacy class of elements of order 2, and $\pi/\gamma \cong P \rtimes \mathbb{Z}/2\mathbb{Z}^-$ and $\lambda/\gamma \cong F(t) \rtimes \mathbb{Z}/2\mathbb{Z}^-$ have presentations

$$\langle a,\, b_i,\, 1 \le i \le r \mid a^2 = 1,\, b_i^p = 1,\, ab_ia = b_i^{-1}\ \forall i\rangle$$

and

$$\langle a,\, x_{i,j},\, 1 \le i \le r-1,\, 1 \le j \le p-1 \mid a^2 = 1,\, ax_{ij}a = x_{i,p-j}\ \forall i, j\rangle,$$

respectively. (In particular, $\lambda/\gamma \cong F(t/2) * \mathbb{Z}/2\mathbb{Z}^-$.) In this case,

$$R \otimes_w I_{\pi/\gamma} \cong (R/(p, a+1))^r \oplus \widetilde{\mathbb{Z}},$$
$$R \otimes_w I_{\lambda/\gamma} \cong R^{t/2} \oplus \widetilde{\mathbb{Z}}.$$

Consideration of the p–torsion submodules again shows that $R \otimes_w I_\pi$ and $R \otimes_w I_\lambda$ cannot both satisfy Turaev's criterion, and hence that π is not the group of a non-orientable PD_3–complex. Thus X must be orientable. □

The case $p = 2$ involves slightly different calculations.

Lemma 4 *Let X be an indecomposable* PD_3*–complex with $\pi = \pi_1(X) \cong \kappa \rtimes W$, where κ is orientable and torsion-free, and W has order 4. Then X is orientable.*

Proof As in Lemma 3, we suppose that X is not orientable, so π and κ are infinite, and may assume that $\pi \cong \pi\mathcal{G}$, where $(\mathcal{G}, \Gamma)$ is an admissible graph of groups with $r \ge 1$ finite vertex groups and at least one edge. We continue with the notation P, γ, a and R from Lemma 3.

The inclusions of the edge groups split w, by Lemma 2. In this case, $W \cong (\mathbb{Z}/2\mathbb{Z})^2 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}^-$ and has two orientation-reversing elements. Note that P is now a free product of r copies of $\mathbb{Z}/2\mathbb{Z}$.

The quotient π/γ is the group of a finite graph of groups with all vertex groups W and edge groups $\mathbb{Z}/2\mathbb{Z}^-$. Since P is a free product of cyclic groups, π/γ has a presentation

$$\langle a,\, b_i,\, 1 \le i \le r \mid a^2 = 1,\, b_i^2 = (aw_i)^2 = (aw_ib_i)^2 = 1\ \forall i\rangle,$$

where $w_i = 1$ and $w_i \in F(t)$ for $2 \le i \le r$. The free subgroup $F(t)$ has basis $\{x_i \mid 1 \le i \le r-1\}$, where x_i has image b_1b_{i+1} in P, and λ/γ has a presentation

$$\langle a,\, x_i,\, 1 \le i \le r-1 \mid a^2 = 1,\, ax_ia = x_ib_{i+1}w_{i+1}b_{i+1}w_{i+1}^{-1}\ \forall i\rangle.$$

In this case,

$$R \otimes_w I_{\pi/\gamma} \cong (R/(2, a-1))^r \oplus \widetilde{\mathbb{Z}},$$
$$R \otimes_w I_{\lambda/\gamma} \cong \mathbb{Z}^{r-1} \oplus \widetilde{\mathbb{Z}}.$$

Since $R/(2,a+1) = R/(2,a-1)$, torsion considerations do not appear to help. If $r>1$ we may instead compare the quotients by the $\mathbb{Z}$–torsion submodules, as in [9, Lemma 7.3], since finitely generated torsion-free R–modules are direct sums of copies of R, $\mathbb{Z}$ and $\widetilde{\mathbb{Z}}$, by [4, Theorem 74.3]. We again conclude that π is not the group of a nonorientable PD_3–complex.

The case when $p=2$ and $r=1$ requires a little more work. Let N be the R–module presented by the transposed conjugate of $\binom{2}{a-1}$. If $\{e, f\}$ is the standard basis for R^2 then $N = R^2/R(2e+(a+1)f)$. The $\mathbb{Z}$–torsion submodule of N is generated by the image of $(a-1)e$ and has order 2, but is not a direct summand. The quotient of N by its $\mathbb{Z}$–torsion submodule is generated by the images of e and $f-e$, and is a direct sum $\mathbb{Z}\oplus\widetilde{\mathbb{Z}}$. In particular, it has no free summand. It now follows easily that $H\oplus\widetilde{\mathbb{Z}}\oplus R/(2,a-1)$ is not stably isomorphic to $H\oplus\widetilde{\mathbb{Z}}\oplus N$. Therefore I_π and I_λ cannot both satisfy Turaev's criterion, so π is not the group of a nonorientable PD_3–complex. Thus X must be orientable. □

Our final lemma is needed to cope with three exceptional cases.

Lemma 5 *Let $G = H \rtimes \mathbb{Z}/2\mathbb{Z}$, where $H = T_1^*$, O_1^* or I^*. Suppose that every element of G divisible by 4 is in H. Then G has a subgroup W of order 6 such that $[W : W\cap H] = 2$.*

Proof Let g be an element of order 2 whose image generates G/H.

Suppose first that $H = T_1^*$, with presentation

$$\langle x,\, y,\, z \mid x^2 = (xy)^2 = y^2,\, z^3 = 1,\, zxz^{-1} = y,\, zyz^{-1} = xy\rangle.$$

Then $\zeta T_1^* = \langle x^2\rangle$ has order 2. The outer automorphism group $\mathrm{Out}(T_1^*)$ is generated by the class of the involution ρ which sends x, y and z to y^{-1}, x^{-1} and z^2, respectively. (See [8, page 221].) Hence ρ preserves the subgroup S of order 3 generated by z.

If conjugation by g induces an inner automorphism of T_1^*, there is an $h\in T_1^*$ such that $gxg^{-1} = hxh^{-1}$ for all $x\in T_1^*$. Then $gh = hg$ and h^2 is central in T_1^*, so $(h^{-1}g)^2 = h^2$ has order dividing 4. Therefore $h^{-1}g$ has order 2, by hypothesis.

Otherwise we may assume that there is an $h\in G^+$ such that $gxg^{-1} = h\rho(x)h^{-1}$ for all $x\in T_1^*$, so ρ is conjugation by $h^{-1}g$. Since ρ is an involution, $(h^{-1}g)^2$ is central in T_1^*. We again see that $h^{-1}g$ has order 2. In each case, $h^{-1}g$ normalizes S, so the subgroup W generated by S and $h^{-1}g$ has order 6, while $h^{-1}g\notin H$, so $[W : W\cap H] = 2$.

The commutator subgroup of O_1^* is T_1^*. Since this is a characteristic subgroup, it is preserved by g. The group T_1^* is a nonnormal subgroup of I^*, of index 5. Since g acts as an involution on the set of conjugates of T_1^*, we may assume that it preserves T_1^*. In each case the lemma follows easily from its validity for $H = T_1^*$. □

We may now give our main result.

Theorem 6 *Let X be an indecomposable, nonorientable* PD_3*–complex such that $\pi = \pi_1(X)$ has infinitely many ends. Then:*

(1) $\pi \cong \pi\mathcal{G}$*, where $(\mathcal{G}, \Gamma)$ is an admissible graph of groups with all vertex groups one-ended and all edge groups* $\mathbb{Z}/2\mathbb{Z}^-$.

(2) $\pi \cong \pi^+ \rtimes \mathbb{Z}/2\mathbb{Z}^-$.

(3) $\pi^+ \cong G * H$*, where G is a nontrivial free product of* PD_3*–groups and H is free. In particular, π^+ is torsion-free.*

Proof Let $\pi \cong \pi\mathcal{G}$, where $(\mathcal{G}, \Gamma)$ is an admissible graph of groups. At least one vertex group is infinite, for otherwise π has two ends, by [9, Theorems 7.1 and 7.4]. Hence $\pi^+ \cong G * H$, where G is a nontrivial free product of PD_3–groups and H is virtually free. Therefore π^+ is virtually torsion-free. Let κ be the intersection of the conjugates in π of a torsion-free subgroup of finite index in π^+, and let $\phi\colon \pi \to \pi/\kappa$ be the canonical projection. Then κ is orientable, torsion-free and of finite index, and w factors through π/κ.

If F is a finite subgroup then $\phi|_F$ is injective, and $\phi^{-1}(\phi(F))$ has finite index in π. Hence $\phi^{-1}(\phi(F))$ has a graph of groups structure in which all finite vertex groups are isomorphic to subgroups of F. In particular, if F is a nonorientable 2–group then at least one of these vertex groups is a nonorientable 2–group, so there is a $g \in F$ such that $g^2 = 1$ and $w(g) = -1$, by Lemma 2(3). Hence, if, moreover, F is cyclic, then it has order 2.

Assume that there is a nonorientable finite vertex group G_v. Then G_v has a nonorientable Sylow 2–subgroup $S(2)$, so there is a $g \in S(2)$ such that $g^2 = 1$ and $w(g) = -1$. The orientable subgroup G_v^+ has periodic cohomology, with period dividing 4, by [9, Theorems 4.3 and 4.6]. Moreover, every element of G_v divisible by 4 is in G_v^+, by the argument of the previous paragraph.

Let g be an element of order 2 whose image generates G_v/G_v^+. We may assume that $G_v^+ \cong B \times \mathbb{Z}/d\mathbb{Z}$, where B is either $\mathbb{Z}/a\mathbb{Z} \rtimes Q(2^i)$ (with a odd and $i \geq 3$), T_k^* or O_k^* (for some $k \geq 1$), I^* or $\mathbb{Z}/a\mathbb{Z} \rtimes_{-1} \mathbb{Z}/2^e\mathbb{Z}$ (with a odd and $e \geq 1$), as in the penultimate paragraph of Section 1 above. Suppose first that G_v^+ is not a 2–group.

Then it has a nontrivial subgroup S of order p for some odd prime p. If $d > 1$ we may assume that p divides d, and then S is characteristic in G_v^+. This is also the case if $G_v^+ \cong \mathbb{Z}/a\mathbb{Z} \rtimes Q(8)$ or $\mathbb{Z}/a\mathbb{Z} \rtimes_{-1} \mathbb{Z}/2^e\mathbb{Z}$ with a odd (so p divides a), or $G_v^+ \cong T_k^*$ or O_k^* with $k > 1$ (so $p = 3$). In these cases, S is normalized by g and the subgroup H generated by S and g has order $2p$. The remaining possibilities are that $G_v^+ \cong T_1^* \times \mathbb{Z}/d\mathbb{Z}$, $O_1^* \times \mathbb{Z}/d\mathbb{Z}$ or $I^* \times \mathbb{Z}/d\mathbb{Z}$. For these cases we appeal to Lemma 5 to see that G_v has a nonorientable subgroup W of order $2p$.

Since $\phi^{-1}\phi(W)$ has finite index in π, it is again the group of a nonorientable PD_3–complex. This complex has an indecomposable factor whose group has W as one of its finite vertex groups, so has fundamental group $\kappa \rtimes W$. But this factor is nonorientable, so contradicts Lemma 3.

Therefore we may assume that G_v^+ is a 2–group. If $S(2)^+ \neq 1$ (ie if G_v^+ is a nontrivial 2–group) it is cyclic or generalized quaternionic, so has an unique central element of order 2 (see [9, Lemma 2.1]). Hence G_v has a finite index subgroup $W \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}^-$. As before, passage to $\phi^{-1}\phi(W)$ leads to a contradiction, by Lemma 4.

Therefore all finite vertex groups are orientable. But the graph Γ is connected, and any edge connecting a finite vertex group to an infinite vertex group must be nonorientable, as in Lemma 2. Since there is at least one infinite vertex group there can be no finite vertex groups.

The second assertion follows from part (2) of Lemma 2, and $\pi^+ = \pi\mathcal{G}^+$ is the fundamental group of a graph of groups $(\mathcal{G}^+, \Gamma)$ with the same underlying graph Γ, trivial edge groups and vertex groups G_v^+ all PD_3–groups. Hence π^+ is torsion-free, but not free. □

As observed at the end of Section 2, when X is a 3–manifold and $(\mathcal{G}, \Gamma)$ is an admissible graph of groups such that $\pi = \pi\mathcal{G}$, all vertices of Γ have even valence. Can this observation be extended to the case of PD_3–complexes? Although there are indecomposable PD_3–complexes which are not homotopy equivalent to 3–manifolds [9; 12], it remains possible that every indecomposable, nonorientable PD_3–complex is homotopy equivalent to a 3–manifold.

Corollary 7.5 of [9] follows immediately from Crisp's theorem and Theorem 6. (The argument in [9] assumed that π is virtually free.) We restate it here:

Corollary 7 *Let X be a PD_3–complex and $g \in \pi = \pi_1(X)$ a nontrivial element of finite order. If $C_\pi(g)$ is infinite then g has order 2 and is orientation-reversing, and $C_\pi(g) = \langle g \rangle \times \mathbb{Z}$.* □

Question Are there any examples other than $\mathbb{R}P^2 \times S^1$ of indecomposable PD_3–complexes whose groups have a central element of order 2 with infinite centralizer?

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School of Mathematics and Statistics F07, University of Sydney
Sydney NSW 2006, Australia

jonathan.hillman@sydney.edu.au

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