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Thickness, relative hyperbolicity, and randomness in Coxeter groups

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APPENDIX WRITTEN JOINTLY WITH PIERRE-EMMANUEL CAPRACE

For right-angled Coxeter groups W_Γ, we obtain a condition on Γ that is necessary and sufficient to ensure that W_Γ is *thick* and thus not relatively hyperbolic. We show that Coxeter groups which are not thick all admit canonical minimal relatively hyperbolic structures; further, we show that in such a structure, the peripheral subgroups are both parabolic (in the Coxeter group-theoretic sense) and strongly algebraically thick. We exhibit a polynomial-time algorithm that decides whether a right-angled Coxeter group is thick or relatively hyperbolic. We analyze random graphs in the Erdős–Rényi model and establish the asymptotic probability that a random right-angled Coxeter group is thick.

In the joint appendix, we study Coxeter groups in full generality, and we also obtain a dichotomy whereby any such group is either strongly algebraically thick or admits a minimal relatively hyperbolic structure. In this study, we also introduce a notion we call *intrinsic horosphericity*, which provides a dynamical obstruction to relative hyperbolicity which generalizes thickness.

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Introduction

The notion of relative hyperbolicity was introduced by Gromov [38], then developed by Farb [35]. This notion is both sufficiently general to include many important classes of groups, including all (uniform and nonuniform) lattices in rank-one semisimple Lie groups, yet is sufficiently restrictive that it allows for powerful geometric, algebraic and algorithmic results to be proven; see Arzhantseva, Minasyan and Osin [1], Druțu [27], Druțu and Sapir [30] and Farb [35]. Further, relatively hyperbolicity admits numerous geometric, topological and dynamical formulations which are all equivalent; see eg Bowditch [12], Dahmani [21], Druțu and Sapir [29], Osin [44], Sisto [45; 46] and Yaman [48].

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Let G be a finitely generated group and $\mathcal{P}$ a finite collection of proper subgroups of G. The group G is *hyperbolic relative to the subgroups* $\mathcal{P}$ if collapsing the left cosets of $\mathcal{P}$ to finite-diameter sets, in any (hence every) word metric on G, yields a δ–hyperbolic space, and if the collection $\mathcal{P}$ satisfies the *bounded coset property* which, roughly speaking, requires that in the δ–hyperbolic metric space obtained as above, any pair of quasigeodesics with the same endpoints travels through the collapsed cosets in approximately the same manner. The subgroups in $\mathcal{P}$ are called *peripheral subgroups*. We say a group is *relatively hyperbolic* when there is some collection of subgroups for which this holds. A collection $\mathcal{P}$ of peripheral subgroups of the relatively hyperbolic group G is *minimal* if, for any other relatively hyperbolic structure $(G, \mathcal{Q})$ on G, each $P \in \mathcal{P}$ is conjugate into some $Q \in \mathcal{Q}$. Relatively hyperbolic groups do not always admit minimal structures; see Behrstock, Druțu and Mosher [5, Theorem 6.3]. We will follow the convention of requiring the subgroups to be proper, which rules out the trivial case of G being hyperbolic relative to itself. Note also that a group G is hyperbolic relative to hyperbolic subgroups if and only if G is hyperbolic.

We will also be interested in the notion of *thickness* which was introduced by Behrstock, Druțu and Mosher [5] as a powerful geometric obstruction to relative hyperbolicity which holds in many interesting cases, including most mapping class groups, right-angled Artin groups, lattices in higher-rank semisimple Lie groups, and elsewhere. Thickness is defined inductively: At the base level, *thick of order* 0, it is characterized by linear divergence. Roughly, a group is *thick of order n* if it is a "network of left cosets of subgroups" which are thick of lower orders. This essentially means that the union of these cosets is the entire space, and any two points in the space can be connected by a sequence of these cosets which successively intersect along infinite-diameter subsets; the precise definition appears in Section 1.2. Thickness has proven to be an important invariant for obtaining upper bounds on divergence, and we shall utilize this below; cf Behrstock and Charney [3], Behrstock and Druțu [4], Behrstock and Hagen [7], Brock and Masur [13] and Sultan [47]. In a relatively hyperbolic group, any thick subgroup must be contained inside a peripheral subgroup; see [5, Corollary 7.9, Theorem 4.1]. This fact yields the useful application that any relatively hyperbolic structure in which the peripheral subgroups are thick is a minimal relatively hyperbolic structure; see [29, Theorem 1.8] and [5, Corollary 4.7].

In this paper, we study thickness and relative hyperbolicity in the setting of Coxeter groups. One reason to do so is that Coxeter groups have many interesting properties, making them a standard testing ground in geometric group theory. For example, these groups are known to act properly on CAT(0) cube complexes (see Niblo and Reeves [43]), which allows them to be studied using the tools of CAT(0) geometry. In particular, this connects them to the study of thickness of cubulated groups initiated in [7].

We first specialize to the case of right-angled Coxeter groups, the class of which is diverse; for instance, each right-angled Artin group is a finite-index subgroup of a right-angled Coxeter group; see Davis and Januszkiewicz [24]. The right-angled Coxeter group W_Γ is generated by involutions indexed by vertices of the finite simplicial graph Γ; the relations are commutation relations corresponding to edges. We prove that, for every right-angled Coxeter group, either it is thick or it admits a canonical relatively hyperbolic structure in which the peripheral subgroups are thick:

Theorem I (right-angled Coxeter groups are thick or relatively hyperbolic) *Let $\mathcal{T}$ be the class consisting of the finite simplicial graphs Λ such that W_Λ is strongly algebraically thick. Then for any finite simplicial graph Γ, either $\Gamma \in \mathcal{T}$ or there exists a collection $\mathbb{J}$ of induced subgraphs of Γ such that $\mathbb{J} \subset \mathcal{T}$, W_Γ is hyperbolic relative to the collection $\{W_J : J \in \mathbb{J}\}$, and this relatively hyperbolic structure is minimal.*

One application of this theorem is to the quasi-isometric classification of Coxeter groups. As thickness is a quasi-isometric invariant, this provides a way to distinguish the thick Coxeter groups from many other groups. A more refined classification also follows from this result using the theorem which states that the quasi-isometric image of a group which is hyperbolic relative to thick peripheral subgroups is also hyperbolic relative to thick peripheral subgroups, each of which is quasi-isometric to one of the peripherals in the source; see [5, Corollary 4.8] and [27]. Prior to this application of Theorem I, major methods of classifying right-angled Coxeter groups included using classification theorems in right-angled Artin groups (ie Behrstock and Neumann [9], Behrstock, Januszkiewicz and Neumann [8] and Bestvina, Kleiner and Sageev [10]) in conjunction with results about commensurability between right-angled Artin and Coxeter groups (for instance, results in Davis and Januszkiewicz [24]) and, for some hyperbolic right-angled Coxeter groups, applying a result in Crisp and Paoluzzi [20].

Additionally, Theorem I provides an effective classification theorem because $\mathcal{T}$ can be characterized combinatorially as follows:

Theorem II (combinatorial characterization of thick right-angled Coxeter groups) *Let $\mathcal{T}$ be the class of finite simplicial graphs whose corresponding right-angled Coxeter groups are strongly algebraically thick. Then $\mathcal{T}$ is the smallest class of graphs satisfying the following conditions:*

(1) *$K_{2,2} \in \mathcal{T}$, where $K_{2,2}$ is the complete bipartite graph on two sets of two elements, ie a 4–cycle.*

(2) *Let $\Gamma \in \mathcal{T}$ and let $\Lambda \subset \Gamma$ be an induced subgraph which is not a clique. Then the graph obtained from Γ by coning off Λ is in $\mathcal{T}$.*

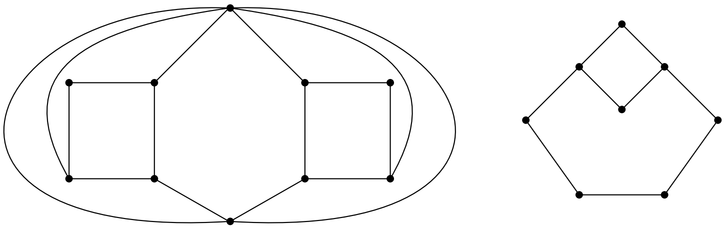

Figure 1: A graph in $\mathcal{T}$ (left) and a graph not in $\mathcal{T}$ (right)

(3) *Let* $\Gamma_1, \Gamma_2 \in \mathcal{T}$*, and suppose there exists a graph* Γ *which is not a clique and which arises as a subgraph of each of the* Γ_i*. Then the union* Λ *of* Γ_1 *and* Γ_2 *along* Γ *is in* $\mathcal{T}$*, and so is any graph obtained from* Λ *by adding any collection of edges joining vertices in* $\Gamma_1 - \Gamma$ *to vertices of* $\Gamma_2 - \Gamma$*.*

Theorems I and II together imply that any thick right-angled Coxeter group is strongly algebraically thick. A special case of this is that W_Γ is thick of order 0 if and only if it is the product of two infinite right-angled Coxeter groups; see Proposition 2.11, which generalizes a result of Dani and Thomas [22, Theorem 4.1].

Figure 1 illustrates examples of graphs in and not in $\mathcal{T}$. See also Remark 2.8. The right-angled Coxeter groups with polynomial divergence constructed by Dani and Thomas [22] are strongly algebraically thick; this was shown in [loc. cit.] and can also be verified either by observing that the corresponding graphs are in $\mathcal{T}$, or by combining the fact that they have subexponential divergence with Theorem I and the exponential divergence of any relatively hyperbolic group.

An important consequence of the above characterization of the class $\mathcal{T}$ is that it allows thickness/relative hyperbolicity to be detected algorithmically:

Theorem III (polynomial algorithm for relative hyperbolicity; Theorem 4.1) *There exists a polynomial-time algorithm to decide if a given graph is in* $\mathcal{T}$*, and hence whether a given right-angled Coxeter group is* (*strongly algebraically*) *thick or relatively hyperbolic.*

Random graphs

We consider right-angled Coxeter groups on random graphs in the Erdős–Rényi model [31]: $G(n, p(n))$ is the class of graphs on n vertices with the probability measure corresponding to independently declaring each pair of vertices to be adjacent with probability $p(n)$. The results of this section are summarized in Figure 2.

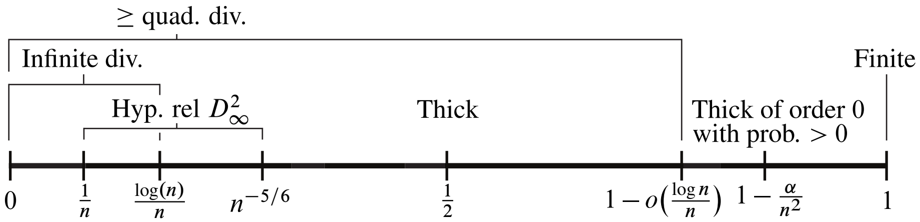

Figure 2: The results of Section 3 illustrated on the same spectrum of densities as addressed conjecturally in Figure 4. Each listed property occurs aas at the given density, unless the specific asymptotic probability is stated.

An important result of Erdős and Rényi states that a random graph is asymptotically almost surely (aas) connected when $p(n)$ grows more quickly that $(\log n)/n$, and is aas disconnected when $p(n) = o((\log n)/n)$. This implies that for slowly growing $p(n)$, when $\Gamma \in G(n, p(n))$, the right-angled Coxeter group W_Γ is aas a nontrivial free product, and hence relatively hyperbolic. In light of Theorem I, it is natural to wonder if there are densities at which a random right-angled Coxeter group is relatively hyperbolic but not a free product. The following gives a positive answer to this question; the technical terms in this theorem will be defined in Section 3.

Theorem IV (low density, Theorem 3.4) *Suppose $p(n)n \to \infty$ and $p(n)^6 n^5 \to 0$. For $\Gamma \in G(n, p(n))$, the group W_Γ is aas hyperbolic relative to a nonempty collection of $D_\infty \times D_\infty$ subgroups; the same holds for $W_{\Gamma'}$, where $\Gamma' \subseteq \Gamma$ is the giant component of Γ.*

Intuitively, the probability of thickness should increase with the growth rate of $p(n)$, up to the point where Γ is aas sufficiently dense that W_Γ is either finite or virtually cyclic. The following result confirms this intuition.

Theorem V (high density, Theorem 3.9) *Suppose that $(1-p(n))n^2 \to \alpha \in [0,\infty)$. Then for $\Gamma \in G(n, p(n))$, the group W_Γ is*

(1) *finite with probability tending to $\beta = e^{-\alpha/2}$,*

(2) *virtually $\mathbb{Z}$ with probability tending to $\gamma = \frac{1}{2}\alpha e^{-\alpha/2}$,*

(3) *virtually $\mathbb{Z}^k$ for $k \geq 2$, and thus thick of order 0, with probability tending to $1-(\beta+\gamma)$.*

The following describes the situation at a natural choice of "intermediate" $p(n)$:

Theorem VI (intermediate density) *For $\Gamma \in G\left(n, \frac{1}{2}\right)$, the group W_Γ is aas thick.*

We conjecture that for all $p \in (0,1)$, the group W_Γ is aas thick for $\Gamma \in G(n,p)$.[1] This conjecture is strongly supported by computer experiments; for example, for $n = 200$ and for each of several values of p, we tested 50 random graphs and found *all* to correspond to thick right-angled Coxeter groups. For any given $p \in (0,1)$, we expect the strategy used in the proof of Theorem VI will work. However, there are two serious complications to implementing this strategy for any particular p: first, combinatorially, the requisite set-up may be more intricate, and second, establishing the base case of the induction is likely to be computationally prohibitive for some values of p, since it involves checking all graphs of a size depending on p for membership in $\mathcal{T}$.

One of our motivations for our study of random Coxeter groups was the results of Charney and Farber [18] on hyperbolicity of random right-angled Coxeter groups. More recently, results have been obtained about cohomological properties of such random groups by Davis and Kahle [25]. Together with our results, this represents the beginning of a systematic study of random Coxeter groups.

General Coxeter groups

In the appendix, we generalize Theorems I and II to all Coxeter groups. Accordingly, we recommend reading the first part of the appendix, Section A.1, concurrently with Section 2 in order to see how the results on thickness versus relative hyperbolicity for right-angled Coxeter groups generalize to arbitrary Coxeter groups, as well as the limitations of the generalization. In the latter vein, as shown by the example in Remark 2.9, there is no characterization of strongly algebraically thick Coxeter groups that are not right-angled purely in terms of the underlying graph of the free Coxeter diagram.

Theorem I generalizes as follows:

Theorem VII (minimal relatively hyperbolic structures for Coxeter groups) *Let (W,S) be a Coxeter system. Then there is a (possibly empty) collection $\mathcal{J}$ of subsets of S enjoying the following properties:*

(i) *The parabolic subgroup W_J is strongly algebraically thick for every $J \in \mathcal{J}$.*

(ii) *W is relatively hyperbolic with respect to $\mathcal{P} = \{W_J \mid J \in \mathcal{J}\}$.*

In particular, $\mathcal{P}$ is a minimal relatively hyperbolic structure for W.

Theorem II takes the following form for general Coxeter groups. Note that thickness is now described using a class of labeled graphs instead of a class of graphs.

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[1] While this paper was circulating as a preprint, a resolution of a strong form of this conjecture was obtained by Behrstock, Falgas-Ravry, Hagen and Susse [6].

Theorem VIII (classification of thick Coxeter groups) *The class* $\mathbb{T}$ *of Coxeter systems* (W,S) *for which* W *is strongly algebraically thick is the smallest class satisfying:*

1. $\mathbb{T}$ *contains the class* $\mathbb{T}_0$ *of all irreducible* affine *Coxeter systems* (W,S) *with* S *of cardinality at least* 3*, as well as all Coxeter systems of the form* $(W, S_1 \cup S_2)$ *with* W_{S_1} *and* W_{S_2} *irreducible nonspherical and* $[W_{S_1}, W_{S_2}] = 1$*.*
2. *Suppose* $(W, S \cup s)$ *has the properties that* $s^{\perp}$ *is nonspherical and* (W_S, S) *belongs to* $\mathbb{T}$*. Then* $(W, S \cup s)$ *belongs to* $\mathbb{T}$*.*
3. *Suppose* (W,S) *has the property that there exist* $S_1, S_2 \subseteq S$ *with* $S_1 \cup S_2 = S$*,* $(W_{S_1}, S_1), (W_{S_2}, S_2) \in \mathbb{T}$ *and* $W_{S_1 \cap S_2}$ *nonspherical. Then* $(W,S) \in \mathbb{T}$*.*

We also introduce the notion, which we feel will be of independent interest, of an *intrinsically horospherical* group, ie one for which every proper isometric action of Γ on a proper hyperbolic geodesic metric space fixes a unique point at infinity. Any group G admits a collection of maximal intrinsically horospherical subgroups, and any relatively hyperbolic structure on G has the property that every maximal intrinsically horospherical subgroup is conjugate into a peripheral subgroup. We show that any thick group is intrinsically horospherical. In the case of Coxeter groups, we say more:

Corollary IX *Let* (W,S) *be a Coxeter system. Then the following conditions are equivalent:*

- (I) (W,S) *is in* $\mathbb{T}$*.*
- (II) W *is strongly algebraically thick.*
- (III) W *is intrinsically horospherical.*
- (IV) W *is not relatively hyperbolic with respect to any family of proper subgroups.*
- (V) W *is not relatively hyperbolic with respect to any family of proper Coxeter-parabolic subgroups.*

Outline

In Section 1, we discuss background on Coxeter groups, thickness and divergence. Sections 2, 3 and 4 are devoted to right-angled Coxeter groups: In the second section, we treat Theorems I and II. In the third section, we study right-angled Coxeter groups presented by random graphs, dealing in particular with Theorems IV, V and VI. In the fourth section, we produce an algorithm for testing whether a given graph is in $\mathcal{T}$. We also include source code containing an implementation of a refined version of this algorithm; this program is needed for a computation in the proof of Theorem VI. (This source code is available from the authors' web pages and on the arXiv.) In the appendix, we study arbitrary Coxeter groups and introduce the notion of intrinsic horosphericity; in particular, we prove Theorems VII and VIII and Corollary IX.

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1 Preliminaries

In this section, we review definitions and facts related to Coxeter groups, divergence and thick metric spaces. A comprehensive discussion of Coxeter groups can be found in [23]. The notion of divergence used here is due to Gersten [36]. Our consideration of divergence in the setting of Coxeter groups was motivated largely by the discussion in [22] and, to some extent, by questions about divergence in cubulated groups (of which Coxeter groups are examples) raised in [7]. Thick spaces and groups were introduced in [5], and we also refer to results of [4].

1.1 Background on Coxeter groups

Throughout this paper, we confine our discussion to finitely generated Coxeter groups. A *Coxeter group* is a group of the form

$$\langle S \mid (st)^{m_{st}} : s,t \in S\rangle,$$

where each $m_{ss}=1$, and for $s \neq t$, either $m_{st} \geq 2$ or there is no relation between s and t of this form. Also, $m_{st}=m_{ts}$ for each $s,t\in S$. The pair (W,S) is a *Coxeter system*.

The Coxeter group W is *reducible* if there are nonempty sets $S_1, S_2 \subset S$ such that $S = S_1 \sqcup S_2$, and for all $s_1 \in S_2, s_2 \in S_2$, we have $m_{s_1s_2}=2$. If W is not reducible, then it is *irreducible*. The Coxeter system (W,S) is said to be *(ir-)reducible* if W has the corresponding property.

To the Coxeter system (W,S), we associate a bilinear form $\langle -,-\rangle$ on $\mathbb{R}[S]$ defined by $\langle s,t\rangle = -\cos(\pi/m_{st})$ when there is a relation $(st)^{m_{st}}$, and $\langle s,t\rangle=-1$ otherwise. It is well known that this bilinear form is positive definite if and only if W is finite, in which case the Coxeter system (W,S) is *spherical*. Otherwise, (W,S) is nonspherical (or *aspherical*). If the bilinear form is positive semidefinite and (W,S) is irreducible, then there is a short exact sequence $\mathbb{Z}^n \to W \to W_0$, where $n+1=|S|$ and W_0 is a finite Coxeter group. In this case, the Coxeter system (W,S) is *(irreducible) affine*.

For any $J \subset S$, the subgroup $W_J := \langle J \rangle \subset W$ is a *parabolic* subgroup. Evidently, W_J is again a Coxeter group and (W_J, J) a Coxeter system. The subset J is *spherical, irreducible, affine, etc.* if the Coxeter system (W_J, J) has the same property.

Right-angled Coxeter groups If each relation in the above presentation has the form $(st)^2$, then W is a *right-angled Coxeter group*. In this case, let Γ be the graph with vertex set S and with an edge joining $s, t \in S$ if and only if $(st)^2 = 1$, ie if and only if the involutions s and t commute. Then W decomposes as a graph product: the underlying graph is Γ, and the vertex groups are the subgroups $\langle s \rangle \cong \mathbb{Z}_2$ and $s \in S$.

Conversely, given a finite simplicial graph Γ with vertex set S and edge set $\mathcal{E}$, there is a right-angled Coxeter group

$$W_\Gamma := \langle S \mid s^2, (st)^2 : s, t \in S, (s,t) \in \mathcal{E} \rangle.$$

For example, if Γ is disconnected, then W_Γ is isomorphic to the free product of the parabolic subgroups generated by the vertex sets of the various components, while if Γ decomposes as a nontrivial join, then W_Γ is isomorphic to the product of the parabolic subgroups generated by the factors of the join. For $J \subset S$, the parabolic subgroup $W_J \leq W_\Gamma$ is isomorphic to the right-angled Coxeter group W_Λ, where Λ is the subgraph of Γ induced by J.

Finally, we remark that if W_Γ is a right-angled Coxeter group, then there exists a CAT(0) cube complex $\widetilde{X}_\Gamma$ on which W_Γ acts properly discontinuously and cocompactly. This CAT(0) cube complex is the *Davis complex* X_Γ, which is obtained from the universal cover of the presentation complex of W_Γ by collapsing bigons to edges, noting that each remaining 2–cell is a 2–cube, and then iteratively attaching a k–cube whenever its vertex set is contained in the $(k-1)$–skeleton, for $k \geq 3$; see [23] for details. We will make use of the existence of such a CAT(0) cube complex in the proof of Proposition 2.11.

1.2 Background on divergence and thickness

Given functions $f, g\colon \mathbb{R}_+ \to \mathbb{R}_+$, we write $f \preceq g$ if for some $K \geq 1$, we have $f(s) \leq Kg(Ks + K) + Ks + K$ for all $s \in \mathbb{R}_+$, and $f \asymp g$ if $f \preceq g$ and $g \preceq f$.

Definition 1.1 (divergence) Let (M, d) be a geodesic metric space, let $\delta \in (0,1)$ and $\gamma \geq 0$, and let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be given by $f(r) = \delta r - \gamma$. Given $a, b, c \in M$ with $d(c, \{a, b\}) = r > 0$, let $\operatorname{div}_f(a, b; c) = \inf\{|P|\}$, where P varies over all paths in M joining a to b and avoiding the ball of radius $f(r)$ about c. If no such path exists, $\operatorname{div}_f(a,b;c)=\infty$. The *divergence function* $\operatorname{Div}_f^M\colon \mathbb{R}_+\to\mathbb{R}_+$ of M is then defined by

$$\operatorname{Div}_f^M(s)=\sup\{\operatorname{div}_f(a,b;c) : d(a,b)\leq s\}.$$

Note that M has finite divergence if and only if M has one end.

Given a function $g\colon \mathbb{R}_+\to\mathbb{R}_+$, we say that M has *divergence of order at most* g if for some f as above, $\operatorname{Div}_f^M(s)\preccurlyeq g(s)$. Much of the interest in divergence comes from the fact that the divergence function of M is a quasi-isometry invariant in the following sense: if M_1 and M_2 are quasi-isometric geodesic metric spaces and $\operatorname{Div}_f^{M_1}\asymp g$, then $\operatorname{Div}_{f'}^{M_2}\asymp g$ for some f'. In particular, the divergence of a finitely generated group is well defined up to the relation $\asymp$. A group has linear divergence if and only if it does not have cut-points in any asymptotic cone. Such spaces are called *wide*; see [2; 28].

One family of metric spaces which are particularly amenable to divergence computations is the family of *thick* spaces, as introduced in [5]. Thickness is a quasi-isometrically invariant notion, and this family of spaces is partitioned into quasi-isometrically invariant subclasses by their *order of thickness*, which is a nonnegative integer. In the present paper, we work with a refinement of the notion of thickness which is tuned for the study of finitely generated groups:

Definition 1.2 (strongly algebraically thick [4]) A finitely generated group G is said to be *strongly algebraically thick of order* 0 if it is *wide*. For $n\geq 1$, the finitely generated group G is *strongly algebraically thick of order at most* n if there exists a finite collection $\mathcal{H}$ of subgroups such that:

(1) Each $H\in\mathcal{H}$ is strongly algebraically thick of order at most $n-1$.

(2) $\langle\bigcup_{H\in\mathcal{H}} H\rangle$ has finite index in G.

(3) There exists $C\geq 0$ such that for all $H,H'\in\mathcal{H}$, there is a sequence $H=H_1,\dots,H_k=H'$ with each $H_i\in\mathcal{H}$ such that for all $i\leq k$, the intersection $H_i\cap H_{i+1}$ is infinite and the C–neighborhood of $H_i\cap H_{i+1}$ (with respect to some fixed word metric on G) is path-connected.

(4) For all $H\in\mathcal{H}$, any two points in H can be connected in the C–neighborhood of H by a (C,C)–quasigeodesic.

G is *strongly algebraically thick of order* n if G is strongly algebraically thick of order at most n but is not strongly algebraically thick of order at most $n-1$.

As shown in [4], if G is strongly algebraically thick of order n, then G, with any word metric, is a (strongly) thick metric space. In the present paper, we are particularly interested in the following consequences of strong algebraic thickness:

Proposition 1.3 (upper bound on divergence [4, Corollary 4.17]) *Let G be a finitely generated group that is strongly algebraically thick of order n. Then the divergence function of G is of order at most s^{n+1}.*

Proposition 1.4 (nonrelative hyperbolicity [5, Corollary 7.9]) *Let G be strongly algebraically thick. Then G is not hyperbolic relative to any collection of proper subgroups.*

Note that the above establishes that the divergence function of thick groups is qualitatively different from that of relatively hyperbolic groups, as the latter class has divergence functions which are at least exponential; cf [45, Theorem 1.3].

2 Hyperbolicity relative to thick subgroups: the right-angled case

In this section, Γ will denote a finite simplicial graph and W_Γ will denote the associated right-angled Coxeter group. We will postpone proofs of most of the results of this section to the appendix, where we will consider them in the context of arbitrary Coxeter groups. We focus on the right-angled case here, both for the benefit of readers specifically interested in the right-angled case and because these groups are cocompactly cubulated, which allow for more refined results, such as those in Proposition 2.11 and in Section 3.

We will adopt the following:

Convention 2.1 When we say *graph*, we will always mean a finite simplicial graph (ie no multiedges or monogons). Graphs will often be denoted by Greek letters. When we say Λ is a subgraph of Γ, or when we write $\Lambda \subset \Gamma$, we will mean the *full induced subgraph*; ie a pair of vertices of Λ spans an edge in Λ if and only if they span one in Γ.

We begin by defining the class of graphs $\mathcal{T}$ that we discussed briefly in the introduction.

Definition 2.2 (new graphs from old) If Γ is a graph and $\Lambda \subset \Gamma$, then we say that the graph Γ' is obtained by *coning off* Λ if the graph Γ' can be obtained from Γ by adding one new vertex along with edges between that vertex and each vertex of Λ. Given two graphs Γ_1 and Γ_2 with isomorphic subgraphs Γ, we say the *union of* Γ_1 *and* Γ_2 *along* Γ is the graph obtained by taking the disjoint union of the graphs Γ_1 and Γ_2 and identifying the corresponding Γ subgraphs of Γ_i by the given isomorphism taking one of the Γ subgraphs to the other. Given two graphs Γ_1 and Γ_2 with isomorphic subgraphs Γ, we say that a graph Γ' is a *generalized union of* Γ_1 *and* Γ_2 *along* Γ if Γ' can be obtained from the associated union by adding a collection of edges between vertices of $\Gamma_1 \setminus \Gamma$ and vertices of $\Gamma_2 \setminus \Gamma$.

Definition 2.3 (thick graphs) The set of *thick graphs*, $\mathcal{T}$, is the smallest set of graphs satisfying the following conditions:

(1) $K_{2,2} \in \mathcal{T}$.

(2) If $\Gamma \in \mathcal{T}$ and $\Lambda \subset \Gamma$ is any induced subgraph of diameter greater than one, then the graph obtained by *coning off* Λ is in $\mathcal{T}$.

(3) Let $\Gamma_1, \Gamma_2 \in \mathcal{T}$ with both Γ_i containing an isomorphic subgraph, Γ, which is not a clique. Then any generalized union of the Γ_i along Γ is in $\mathcal{T}$.

When W is a right-angled Coxeter group, there are no irreducible affine Coxeter systems (W, S) with S of cardinality at least 3. In particular, it is straightforward to check that a right-angled Coxeter group is defined by a graph in $\mathcal{T}$ if and only if the group is in the class of right-angled Coxeter groups $\mathbb{T}$ which is defined at the beginning of Section A.1. The next result is thus a consequence of Proposition A.2.

Theorem 2.4 *For each $\Gamma \in \mathcal{T}$, the right-angled Coxeter group W_Γ is strongly algebraically thick.*

The main result of this section is the following, which provides an effective classification theorem with our explicit description of $\mathcal{T}$.

Theorem 2.5 *Let Γ be a graph. The right-angled Coxeter group W_Γ satisfies exactly one of the following:*

- *it is strongly algebraically thick and $\Gamma \in \mathcal{T}$, or*
- *it is hyperbolic relative to a (possibly empty) minimal collection $\mathbb{A}$ of parabolic subgroups for which each $W_\Lambda \in \mathbb{A}$ is strongly algebraically thick and with each such $\Lambda \in \mathcal{T}$.*

If a group is hyperbolic relative to the empty collection of subgroups, then it is hyperbolic; hence if $\mathbb{A}$ is empty, then W_Γ is hyperbolic.

Theorem 2.5 can now be proven considering the collection of all maximal subgraphs of Γ that belong to $\mathcal{T}$ and checking that conditions (RH1)–(RH3) of [15, Theorem A$'$] hold. We postpone the proof of this to the appendix.

Remark 2.6 An alternative way to prove Theorem 2.5 is to define $\mathcal{T}$ to be the set of finite graphs whose corresponding right-angled Coxeter groups are thick. It would then suffice to establish the following statements about induced subgraphs J_1, J_2 of Γ belonging to $\mathcal{T}$:

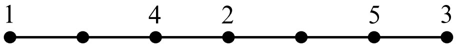

Figure 3: A length-6 geodesic in Γ shows that $\Gamma \in \mathcal{F}$.

(1) If $J_1 \cap J_2$ is aspherical, then the subgraph induced by $J_1 \cup J_2$ belongs to $\mathcal{T}$.

(2) If $v \in \Gamma - J_1$ and the link of v in J_1 is nonempty and aspherical, then $J_1 \cup \{v\} \in \mathcal{T}$.

(3) Joins of aspherical subgraphs belong to $\mathcal{T}$.

Our explicit definition of $\mathcal{T}$ allows us to characterize thick right-angled Coxeter groups, as we do now.

Corollary 2.7 *W_Γ is strongly algebraically thick if and only if $\Gamma \in \mathcal{T}$.*

Proof If W_Γ is strongly algebraically thick, then Γ is not relatively hyperbolic by [5, Corollary 7.9]. Thus by Theorem 2.5, we must have $W_\Gamma \in \mathcal{T}$. In the other direction: by Theorem 2.4, if $\Gamma \in \mathcal{T}$, then W_Γ is strongly algebraically thick. □

Remark 2.8 From Corollary 2.7, we know that all right-angled Coxeter groups which are wide have corresponding graphs in $\mathcal{T}$. As we shall see in Proposition 2.11, these graphs all decompose as nontrivial joins, and thus in particular, the number of squares in these graphs is linear in the number of vertices. In the case of right-angled Coxeter groups which are thick of order 1, it was proven in [22] that each vertex in the corresponding graph is contained in a square; hence in that case as well, the number of squares is at least linear in the number of vertices.

Accordingly, it is natural to expect that a graph in $\mathcal{T}$ contains "many" squares relative to the number of vertices it contains. However, this is not the case in general. Indeed, for all sufficiently large $N \in \mathbb{N}$, the set of graphs in $\mathcal{T}$ containing at most N squares is infinite. We define a class of graphs $\mathcal{F}$ consisting of graphs Γ such that $\Gamma \in \mathcal{T}$ and Γ contains vertices $v_1, \dots, v_5$ for which $d(v_i, v_{i+1}) \geq 3$ for each i. If $\Gamma \in \mathcal{F}$, then the graph obtained by joining v_i and v_{i+1} by a path of length 2 is also in $\mathcal{F}$, and it has the same number of squares as Γ and strictly more vertices. Any element of $\mathcal{T}$ of diameter at least 6 is in $\mathcal{F}$, since it has an induced subgraph which is in $\mathcal{F}$, namely, the path of length 6 (as shown in Figure 3).

The claim now follows for some N since $\mathcal{T}$ contains graphs of arbitrarily large diameter, as we shall now show. Begin with a graph $\Gamma_0 \in \mathcal{T}$ of diameter $d \geq 3$ with the additional property that some vertex v_0 of Γ_0 lies at distance d from nonadjacent vertices u_0 and w_0 (for example, the graph in Figure 1 (left)). Form Γ_1 from Γ_0 by adding two new vertices u_1 and w_1, each joined by an edge to u_0 and w_0. By Theorem 2.4, $\Gamma_1 \in \mathcal{T}$. By construction, the distance in Γ_1 from each of u_1 and w_1 to v_0 is $d+1$, so the diameter has increased. Finally, the triple v_0, u_1, w_1 shows that Γ_1 has the property needed to repeat this procedure. Hence, the existence of graphs in $\mathcal{T}$ of arbitrarily large diameter follows by induction.

Remark 2.9 (Theorem 2.4 does not hold for general Coxeter groups) Given a (not necessarily right-angled) Coxeter system (W, S), there is a naturally associated labeled graph Γ, the *free Coxeter diagram*, with vertex set S and an edge labeled $n \geq 2$ joining vertices s and t that satisfy a relation $(st)^n = 1$. Note that since $m_{ss} = 1$ for all $s \in S$, this graph is simplicial. Furthermore, if (W, S) is right angled, then all labels are 2, and Γ is the graph considered above.

If the Coxeter group W is not right-angled, the thickness of W cannot be characterized by a purely graph-theoretic property of the free Coxeter diagram. Indeed, there exists a hyperbolic Coxeter group W whose free Coxeter diagram is a 4–cycle: Consider the Coxeter system determined by the presentation

$$W = \langle s, t, u, v \mid s^2, t^2, u^2, v^2, (st)^n, (su)^2, (uv)^2, (tv)^2 \rangle,$$

with $n \geq 3$. The labeled graph Γ is a 4–cycle, with the edge joining s, t labeled $n \geq 3$ and all other edges labeled 2. However, the group W is a Fuchsian group, being generated by reflections in the sides of a 4–gon in $\mathbb{H}^2$ with angles $\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{\pi}{2}$ and $\frac{\pi}{n}$. Being hyperbolic, W cannot be thick.

Combining the upper bound on divergence of strongly thick spaces given in [4, Corollary 4.17], the fact that relatively hyperbolic groups have exponential divergence (see eg [45, Theorem 1.3]) and Theorem 2.5, we obtain:

Corollary 2.10 *Let Γ be a connected graph. Then the divergence function of W_Γ is either exponential or bounded above by a polynomial.*

2.1 Characterizing thickness of order 0

As it turns out, the class $\mathcal{T}_0$ of graphs Γ for which W_Γ is wide admits a simple description as we shall see below. The triangle-free case of this result was previously established using different techniques in [22, Theorem 4.1]. We note that since there exist wide Coxeter groups which are not products (for instance the 3–3–3 triangle group), the following result does not generalize beyond the right-angled case.

Proposition 2.11 *$\mathcal{T}_0$ is the set of graphs of the form $(\Gamma_1 \star \Gamma_2) \star K$, where Γ_1 and Γ_2 are aspherical and K is a (possibly empty) clique.*

Proof If Γ decomposes as in the statement of the proposition, then W_Γ decomposes as the product of infinite subgroups $(W_{\Gamma_1} \times W_{\Gamma_2}) \times \mathbb{Z}_2^{|K|}$, whence W_Γ has linear divergence and is therefore wide, ie $\Gamma \in \mathcal{T}_0$. Conversely, suppose that W_Γ has linear divergence, and let $\widetilde{X}_\Gamma$ be the Davis complex (see [23]). Then $\widetilde{X}_\Gamma$ is a CAT(0) cube complex on which W_Γ acts properly and cocompactly by isometries. Each hyperplane H of $\widetilde{X}_\Gamma$ is regarded as being labeled by a pair $(v, g) \in \Gamma^{(0)} \times W_\Gamma$, where gvg^{-1} acts as an inversion in the hyperplane H.

Recall that W_Γ acts *essentially*, in the sense of [17], on $\widetilde{X}_\Gamma$ if for each hyperplane H, the two components of $\widetilde{X}_\Gamma - H$ each contain points in some W_Γ–orbit which are arbitrarily far from H. A hyperplane without this property is called *inessential.*

Suppose that the action of W_Γ on $\widetilde{X}_\Gamma$ is *essential.* Then since W_Γ is wide, it contains no rank-one isometry of $\widetilde{X}_\Gamma$, and hence the rank-rigidity theorem of [17] implies that there exist unbounded convex subcomplexes $\widetilde{Y}$ and $\widetilde{Y}'$ such that $\widetilde{X}_\Gamma = \widetilde{Y} \times \widetilde{Y}'$. It follows that the link of the vertex in $\widetilde{X}_\Gamma$ decomposes as the join of aspherical subgraphs. But this link is exactly Γ, and hence Γ has the desired form.

Now we may assume W_Γ is not acting essentially on $\widetilde{X}_\Gamma$. Thus, by definition, there exists an inessential hyperplane $H_{(v,1)}$, and it is easy to see that every generator must commute with v. Indeed, if $H_{(w,1)}$ and $H_{(v,1)}$ are disjoint hyperplanes, then $\langle v, w\rangle\{H_{(w,1)}\}$ contains hyperplanes arbitrarily far from $H_{(v,1)}$ in each of its half-spaces. Let K be the clique in Γ whose vertices label such inessential hyperplanes. Then $\Gamma = \Gamma' \star K$, where Γ' is an aspherical set whose vertices label essential hyperplanes of $\widetilde{X}_\Gamma$. This provides the desired decomposition of Γ' as the join of aspherical subsets. □

3 Random right-angled Coxeter groups

We now consider the right-angled Coxeter group W_Γ, where Γ is a random graph in the following sense. Let $p\colon \mathbb{N} \to [0, 1]$ be a function such that $p(n)\binom{n}{2}$ has a limit in $\mathbb{R} \cup \{\infty\}$ as $n \to \infty$. A random graph on n vertices is formed by declaring each pair of vertices to span an edge, independently of other pairs, with probability $p = p(n)$. In other words, we define $G(n, p)$ to be the probability space consisting of simplicial graphs with n vertices where, for each graph Γ on n vertices, $\mathbb{P}(\Gamma) = p^E(1-p)^{\binom{n}{2}-E}$, where E is the number of edges in Γ. This model of random graphs was introduced by Gilbert in [37], and is both contemporaneous with and very similar to the Erdős–Rényi model of random graphs first studied in [31; 32]. For a survey of more recent results on random graphs, see [19].

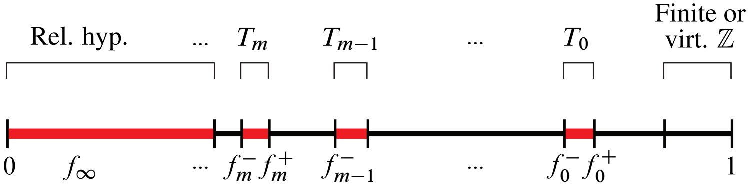

Figure 4: Prevalence of thickness along the "spectrum" of densities $p(n)$, if the answer to Question is positive; bold intervals are where, conjecturally, W_Γ is aas thick of a specified order.

Since the assignment $\Gamma \mapsto W_\Gamma$ of a finite simplicial graph to the corresponding right-angled Coxeter group is bijective [42], it is sensible to define "generic" properties of right-angled Coxeter groups with reference to the above model of random graphs. More precisely, if $\mathcal{P}$ is some property of right-angled Coxeter groups and $\mathcal{G}$ is a class of finite simplicial graphs such that W_Γ has the property $\mathcal{P}$ if and only if $\Gamma \in \mathcal{G}$, then we say that W_Γ satisfies $\mathcal{P}$ *asymptotically almost surely (aas)* if $\mathbb{P}(\Gamma \in \mathcal{G} \cap G(n,p)) \to 1$ as $n \to \infty$. We emphasize that the notion of asymptotically almost surely depends on the choice of probability function p even though it is customary to not explicitly mention p in the notation.

The following question describes the authors' best guess regarding the behavior of thickness and relative hyperbolicity for random right-angled Coxeter groups. In this section, we will provide both theorems and computations that motivate this picture, but we lead with it to contextualize the theorems that follow.

Question Let T_m be the set of graphs Γ for which W_Γ is thick of order $m \geq 0$, and denote by T_∞ the set of graphs for which W_Γ is hyperbolic relative to proper subgroups. Do there exist functions $f_m^-, f_m^+\colon \mathbb{N} \to [0,1]$, for $m \geq 0$, such that $f_m^- = o(f_m^+)$, $f_m^+ = O(f_{m-1}^-)$ and

$$\lim_{n\to\infty} \mathbb{P}\big(\Gamma \in T_m \mid \Gamma \in G(n,p(n))\big) = \begin{cases} 0 & \text{if } p(n)/f_m^-(n) \to 0, \\ 1 & \text{if } p(n)/f_m^-(n) \to \infty \text{ and } p(n)/f_m^+(n) \to 0, \end{cases}$$

for all $m \geq 0$? Similarly, does there exist f_∞ such that W_Γ is asymptotically almost surely relatively hyperbolic when $\Gamma \in G(n,p(n))$ and $p = o(f_\infty)$?

The situation that would occur in the event of a positive answer to Question is illustrated heuristically in Figure 4. Given $p_1, p_2\colon \mathbb{N} \to [0,1]$, we place p_1 to the left of p_2 in the picture of $[0,1]$ if and only if $p_1 = o(p_2)$. Compare also Figure 2, which summarizes the results of this section.

In the interval where W_Γ is aas relatively hyperbolic, it is interesting to speculate whether the order of thickness of the peripheral subgroups might be determined by $p(n)$, especially in view of Theorem 3.4, which we will see below. In other words, one could

a	n	Prop. thick
1.95	2000	0.53
1.95	2100	0.515
1.95	4000	0
2	2000	0.8
2	2500	0.46
2	3000	0.19
2	4000	0.025
2.5	2500	1
2.5	3000	0.53
2.5	4000	0
3	4000	0.5
3	5000	0
4	4000	1
4	10000	1
5	4000	1
5	10000	1
10	4000	1
10	10000	1

Table 1: Experimental proportion of $\Gamma \in G(n,(a\log n)/n)$ that are thick. For each a, this proportion tends to 0 as $n \to \infty$ by Theorem 3.4 but, as illustrated, may do so quite slowly.

sensibly ask if there are functions $g_m^{\pm}$ such that W_Γ is aas hyperbolic relative to groups that are thick of order n for p between g_m^- and g_m^+, and if there is a function g_∞ such that W_Γ is aas hyperbolic — ie hyperbolic relative to hyperbolic subgroups — when $p = o(g_\infty)$. In fact, Charney and Farber have established that we can take $g_\infty(n) = n^{-1}$: when $np(n) \to 0$, the group W_Γ is aas hyperbolic, and if $p(n) \to 0$ and $p(n)n \to \infty$, then aas W_Γ is not hyperbolic [18]. However, identifying the functions g_m appears to be an open question.

The results in this section are summarized in Figure 2. These results are consistent with a positive answer to Question, but there are significant "gaps" in the spectrum about which nothing is presently known.

Remark 3.1 (thickness and connectivity) If Γ is disconnected, then W_Γ splits as a nontrivial free product and is therefore not thick. Hence the function f_∞ from Question, if it exists, must satisfy $\log n/(nf_\infty) \to 0$, by Theorem 3.4 (as shown in Figure 2), since $(\log^6 n)/n \to 0$. In other words, there are densities at which Γ is aas connected but W_Γ is not aas thick. However, the convergence to 0 of the proportion of random graphs at density $O((\log n)/n)$ is quite slow. This is illustrated in Table 1, which shows data selected from the output of many computer experiments;[2] for correctly chosen $a > 0$, even at $n = 10000$ it is not yet clear that W_Γ is not aas thick at density $(a\log n)/n$.

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[2] Source code available from the authors and at arXiv.

3.1 Behavior at low densities

In the next theorem, we collect a few facts about random right-angled Coxeter groups. Recall from [23, Theorem 8.7.4] that W_Γ is one-ended provided Γ has no separating clique.

Theorem 3.2 *W_Γ asymptotically almost surely decomposes as a nontrivial free product if and only if there exists $\epsilon > 0$ such that $p(n) < ((1-\epsilon)\log n)/n$. Hence, if $p(n) < ((1-\epsilon)\log n)/n$, then the divergence of W_Γ is aas infinite.*

If there exists $\epsilon > 0$ such that $p(n) > ((1+\epsilon)\log n)/n$, and there exists $k \in \mathbb{N}$ such that $n^k p(n)^{k^2} \to 0$, then aas Γ has no separating clique, and hence W_Γ is aas one-ended and has finite-divergence function.

Proof W_Γ admits a nontrivial free product decomposition if and only if Γ is disconnected, and $\log n/n$ is the threshold for $p(n)$ above which connectedness occurs aas and below which disconnectedness occurs aas; see [32].

Let $K_n = K_n(\Gamma)$ equal 1 or 0 according to whether Γ is disconnected. For $0 \le j \le n$, let $K_n^j(\Gamma) = \sum_\Lambda K_{n-j}(\Gamma - \Lambda)$, where Λ varies over the size-j subgraphs of Γ. Then $\mathbb{E}(K_n^j) = \binom{n}{j}\mathbb{E}(K_{n-j})p^{\binom{j}{2}}$ is an upper bound for the expected number of separating j–simplices, and the expected number of separating simplices in Γ is therefore bounded by

$$\sum_{j=0}^{n-2}\binom{n}{j}\mathbb{E}(K_{n-j})p^{\binom{j}{2}}.$$

Now, for $p(n) > (1+\epsilon)(\log(n))/n$ and $p = o(1)$, Theorem 1 of [31] implies that $\sum_{j\le k}\binom{n}{j}\mathbb{E}(K_{n-j})p^{\binom{j}{2}}$ tends to 0 for any fixed k. If $p(n)$ is sufficiently small to ensure that aas all cliques in Γ have size $O(1)$, ie if there exists k such that $\binom{n}{k}p^{\binom{k}{2}} \to 0$, then the preceding sum bounds the limiting expected number of separating cliques of any size, and the proof is complete. $\square$

Because of the hypothesis that $n^k p(n)^{k^2} \to 0$ for some $k \in \mathbb{N}$, the second assertion of Theorem 3.2 says nothing about how many ends W_Γ aas has when $\Gamma \in G(n,p)$ and $p \ne o(1)$. This should be expected in light of Theorem 3.9 below, which shows that if $p(n) \to 1$ sufficiently quickly, the random right-angled Coxeter group W_Γ will have 2 or 0 ends with positive probability. However, it is likely possible to improve the second assertion to show that W_Γ is aas one-ended for a wider range of p, provided we still have $p \not\to 1$ as $n \to \infty$, using the fact that aas all cliques in Γ have size in $O(\log n)$ provided $p \not\to 1$, by an application of Markov's inequality.

Indeed, under the assumptions that $p(n) \geq 5(\log(n))/n$ and $p \not\to 1$, it is proven in [34, Lemma 4.1] that linearly many edges must be removed to disconnect Γ; thus the bound on the size of cliques, as noted above, implies that there are no separating cliques. It would be interesting to know if this last comment can be improved to hold when $p(n) \geq (1+\epsilon)(\log(n))/n$ and $p \not\to 1$.

Theorem 3.3 *If for some $\epsilon > 0$, we have $1 - p(n) \geq (1+\epsilon)(\log n)/n$, then W_Γ is not thick of order 0, and hence has at least quadratic divergence, aas.*

Proof Let Γ' be the *complement* of Γ, ie the graph with the same vertex set as Γ, but with each pair of vertices adjacent if and only if they are nonadjacent in Γ. Observe that Γ decomposes as a nontrivial join if and only if Γ' is disconnected. Moreover, note that if $\Gamma \in \mathcal{G}(n,p)$, then $\Gamma' \in \mathcal{G}(n, 1-p)$. Hence if $1 - p(n) \geq (1+\epsilon)(\log(n))/n$ for some $\epsilon > 0$, then Γ' is asymptotically almost surely connected; ie Γ is asymptotically almost surely not a nontrivial join for such $p(n)$. In this case, we thus have that W_Γ is not thick of order 0 and hence has superlinear divergence. By [17, Corollary B], since W_Γ acts cocompactly on its Davis complex, it contains a periodic rank-one geodesic, and thus by [40, Proposition 3.3], the divergence of W_Γ is at least quadratic. □

Theorem 3.4 *If $p(n)n \to \infty$ and $p(n)^6 n^5 \to 0$, then the following holds asymptotically almost surely: Γ has a component Γ' such that $W_{\Gamma'}$ is hyperbolic relative to a nonempty collection of proper subgroups each isomorphic to $D_\infty \times D_\infty$. Hence W_Γ is aas hyperbolic relative to a nonempty collection of proper $D_\infty \times D_\infty$ subgroups, at least one of which is not a proper free factor of W_Γ.*

Remark 3.5 Of greatest interest are densities $p(n)$ growing faster than $(\log n)/n$ but slower than $n^{-1/6}$. At such densities, Theorem 3.2 and Theorem 3.4 together ensure that W_Γ is asymptotically almost surely one-ended and hyperbolic relative to $D_\infty \times D_\infty$ subgroups.

Proof of Theorem 3.4 Since $pn \to \infty$, [33] together with [11, Theorem 2.2(ii)] implies that aas Γ has a *giant component* Γ' containing a positive proportion $\alpha \in (0,1)$ of the vertices, and every other component Γ_i has no more than $O(\log n)$ vertices. It suffices to show that, a.a.s, Γ' contains $K_{2,2}$ as an induced proper subgraph and Γ does not contain $K_{2,3}$. Indeed, the second assertion together with Lemma 3.8 implies that every element of $\mathcal{T}$ arising as an induced subgraph of Γ' is isomorphic to $K_{2,2}$. The first assertion, together with Theorem 2.5, will then complete the proof.

$K_{2,3}$ is aas absent Since $p(n)^6 n^5 \to 0$ as $n \to \infty$ by hypothesis, Corollary 5 of [32] implies that, aas, Γ, and therefore Γ', does not contain $K_{2,3}$.

An induced $K_{2,2}$ aas appears in Γ' Let $v_1, \dots, v_4$ be distinct vertices in the random size-n graph Γ, and let the random variable $I(v_1, \dots, v_4)$ take the value 1 or 0 according to whether or not $\{v_1, \dots, v_4\}$ is the vertex set of an induced $K_{2,2}$ in Γ. The random variable $S_n = \sum_{v_1,v_2,v_3,v_4} I(v_1, \dots, v_4)$ counts each induced $K_{2,2}$ in Γ 24 times, reflecting the eight automorphisms of $K_{2,2}$ and the three ways of choosing which pairs of vertices in $K_{2,2}$ will be nonadjacent. Since there are $\binom{n}{4}$ such quadruples, and each forms an induced copy of $K_{2,2}$ exactly when there is some permutation $\sigma\colon \{1,2,3,4\} \to \{1,2,3,4\}$ such that $v_{\sigma(i)}$ is adjacent to $v_{\sigma(i)+1}$ for each i, and the remaining two possible edges are absent, we have $\mathbb{E}(S_4) = 24\binom{n}{4}p^4(1-p)^2$.

Let $N \in \mathbb{N}$ and let $\epsilon \in (0,1)$. The preceding discussion shows that since $p(n)n \to \infty$, there exists $N_1 \in \mathbb{N}$ such that $\mathbb{E}(S_n) \geq N/\epsilon$ for all $n \geq N_1$. The proof of Theorem 4.1 of [18] shows that since $pn \to \infty$ and $(1-p)n^2 \to \infty$,

$$\frac{\mathbb{E}(S_n)^2}{\mathbb{E}(S_n^2)} \to 1,$$

so there exists $N_2 \in \mathbb{N}$ such that

$$\frac{\mathbb{E}(S_n)^2}{\mathbb{E}(S_n^2)} > 1-\epsilon$$

for $n \geq N_2$. The Paley–Zygmund inequality implies that for all $n \geq \max\{N_1, N_2\}$,

$$\begin{aligned} \mathbb{P}(S_n \geq N) &\geq \mathbb{P}(S_n \geq \epsilon\,\mathbb{E}(S_n)) \\ &\geq (1-\epsilon)^2 \frac{\mathbb{E}(S_n)^2}{\mathbb{E}(S_n^2)} > (1-\epsilon)^3. \end{aligned}$$

This implies that for each $N \in \mathbb{N}$, we have $\lim_n \mathbb{P}(S_n < N) = 0$. Lemma 3.7 below states that aas, every component of Γ is either a tree or equal to Γ', so it suffices to find squares in Γ. We have shown that $\mathbb{P}(S_n < 48) \to 0$ as $n \to \infty$, so Γ' aas contains at least two induced copies of $K_{2,2}$. □

Remark 3.6 The fact that W_Γ is hyperbolic relative to $D_\infty \times D_\infty$ subgroups that are not free factors can be seen slightly more easily as follows. First we produce induced $K_{2,2}$ subgraphs in Γ and verify that Γ aas does not contain $K_{2,3}$, as in the proof of Theorem 3.4. Then we observe that by Theorem 5.16 of [11], Γ aas has no component which is a 4–cycle. Theorem 3.4 is, of course, a stronger conclusion since it rules out the possibility that $W_{\Gamma'}$ is hyperbolic and every 4–cycle lies in a unicyclic component that is not a 4–cycle.

Lemma 3.7 *Let $\Gamma \in G(n, p(n))$, with $p(n)$ satisfying the hypotheses of Theorem 3.4. Asymptotically almost surely, each component of Γ is either the giant component or a tree.*

Proof of Lemma 3.7 This follows immediately from [11, Theorem 6.10(iii)] and [11, Theorem 2.2(ii)]. □

Lemma 3.8 *If $\Lambda \in \mathcal{T}$, then either $\Lambda \cong K_{2,2}$ or Λ contains $K_{2,3}$.*

Proof Since Λ must contain the join of two subgraphs of diameter at least 2, we have that $|\Lambda^0| \geq 4$ and either $\Lambda \cong K_{2,2}$ or $|\Lambda| \geq 5$. In the latter case, suppose that each maximal join in Λ is isomorphic to $K_{2,2}$ and let $\Lambda_0 \subset \Lambda$ be such a join. Then no two nonadjacent vertices in Λ_0 have a common adjacent vertex, since otherwise Λ_0 would extend to a copy of $K_{2,3}$. Hence $\Lambda \cong K_{2,2}$, a contradiction. □

3.2 Behavior at high densities

Charney–Farber showed in [18] that a random right-angled Coxeter group on n vertices is aas finite when $(1-p(n))n^2 \to 0$ as $n \to \infty$. The following description of random right-angled Coxeter groups for rapidly growing $p(n)$ generalizes this result.

Theorem 3.9 *Suppose $(1-p(n))n^2 \to \alpha$ as $n \to \infty$ for some $\alpha \in [0,\infty)$, and let the random variable M_n count the number of "missing edges" in $\Gamma \in \mathcal{G}(n,p)$, ie the number of pairs of distinct vertices that are not joined by an edge. Then $M_n = O(1)$ aas, and the following hold:*

1. *With probability tending to $e^{-\alpha/2}$, $M_n = 0$ and the group W_Γ is finite.*
2. *With probability tending to $\frac{1}{2}\alpha e^{-\alpha/2}$, $M_n = 1$ and the group W_Γ is virtually $\mathbb{Z}$ and thus hyperbolic.*
3. *With probability tending to $1-(1+\frac{1}{2}\alpha)e^{-\alpha/2}$, $M_n \geq 2$ and the group W_Γ is virtually $\mathbb{Z}^{M_n}$, and is thus thick of order 0 and has linear divergence.*

Proof **Finite and virtually $\mathbb{Z}$** If $M_n = 0$, then Γ is a complete graph, so $W_\Gamma \cong \mathbb{Z}_2^n$ is finite. Conversely, if W_Γ is finite, then since any two nonadjacent vertices together generate a subgroup isomorphic to D_∞, we see that $M_n = 0$. Similarly, W_Γ is virtually $\mathbb{Z}$ if and only if $M_n = 1$.

For $k \geq 0$, we have

$$\mathbb{P}(M_n = k) = \binom{\binom{n}{2}}{k}(1-p(n))^k p^{\binom{n}{2}-k},$$

and

$$p(n)^{\binom{n}{2}-k} \sim e^{-\alpha/2}.$$

Hence $\mathbb{P}(M_n = 0) \to e^{-\alpha/2}$, while $\mathbb{P}(M_n = 1) \sim \binom{n}{2}(\alpha/n^2)e^{-\alpha/2} \to \frac{1}{2} - \alpha e^{-\alpha/2}$. This establishes the first two assertions.

Thick of order 0 For each vertex $v \in \Gamma$, let I_v be 1 or 0 according to whether or not v belongs to exactly one missing edge, so that $\mathbb{P}(I_v = 1) = \mathbb{E}(I_v) = n(1-p(n))p(n)^{n-2}$. Let $E_n = \sum_v I_v$ count the number of vertices belonging to exactly one missing edge, and observe that $\mathbb{E}(E_n) = n^2(1-p(n))p(n)^{n-2} \sim \alpha$.

Similarly, let J_v be 1 or 0 according to whether or not v belongs to at least one missing edge, and let $F_n = \sum_v J_v$ count the vertices appearing in at least one missing edge. Note that $\mathbb{P}(J_v = 1) = \mathbb{E}(J_v) = 1 - p(n)^{n-1}$. Hence

$$\begin{aligned}\mathbb{E}(F_n) &= n(1-p(n)^{n-1}) \\ &= n\left[1 - \left(1 - \frac{\alpha}{n^2}\right)^{n-1}\right] \\ &= \frac{\alpha n(n-1)}{n^2} + o(1) \sim \alpha.\end{aligned}$$

Since $F_n \geq E_n$, and $\mathbb{E}(F_n - E_n) \to 0$, aas $F_n = E_n$. In other words, aas every vertex occurs in at most one missing edge. Therefore, aas there are pairwise-distinct vertices $v_1, \dots, v_k, w_1, \dots, w_k$ such that v_i and w_i are not adjacent for all i, and every other pair of vertices spans an edge. This implies that W_Γ is virtually the product of k copies of D_∞.

The above argument shows that aas $M_n = \frac{1}{2}E_n$. For distinct vertices v and w, we have

$$\mathbb{P}(I_v I_w = 1) = (n-1)^2 p^{2n-5}(1-p)^2 + p^{2n-4}(1-p),$$

from which a computation shows that $\mathbb{E}(M_n) \to \frac{1}{8}\alpha(\alpha+1)$. It follows from Markov's inequality that $M_n = O(1)$ aas. □

3.3 Constant-density behavior

In this section, we prove:

Theorem 3.10 *For $\Gamma \in G\left(n, \frac{1}{2}\right)$, the group W_Γ is aas thick.*

The following lemma isolates the most crucial estimates we will use in the proof of the theorem.

Lemma 3.11 *Let $\pi_n = \mathbb{P}\left(\Gamma \notin \mathcal{T} \mid \Gamma \in G\left(n, \frac{1}{2}\right)\right)$. Then the following hold:*

1. $\pi_{2n} \leq \pi_n^2 + f(n)$, *where* $f(n) = 2n \sum_{i=0}^{n} \binom{n}{i} 2^{-n-\binom{i}{2}}$.
2. $\pi_{2n} \leq \pi_n^2 + 2\pi_n(1-\pi_n)(nc(n)/2^n t(n)) + (1-\pi_n)^2$, *where $c(n)$ is the number of cliques in the disjoint union of all $\mathcal{T}$–graphs on n vertices (with the 0–clique counted once), and $t(n)$ is the total number of $\mathcal{T}$–graphs on n vertices.*
3. $\pi_{n+1} \leq \pi_n + f(n)$.

Proof Let $\Gamma \in G\big(2n, \frac{1}{2}\big)$ and let $A \sqcup B$ be a partition of $\Gamma^{(0)}$ into sets of size n. For $v \in B$, we denote by $\mathrm{Link}_A(v)$ the set of vertices in A adjacent to v. Note that if $\Gamma \notin \mathcal{T}$, then one of the following holds:

(i) The subgraphs generated by A and B are not in $\mathcal{T}$.

(ii) There exists $v \in B$ [or $v \in A$] such that $\mathrm{Link}_A(v)$ [or $\mathrm{Link}_B(v)$] is a (possibly empty) clique.

To establish this dichotomy, first we assume (i) does not hold, and hence without loss of generality, we may assume the subgraph generated by A is in $\mathcal{T}$. If additionally, (ii) does not hold, we show this yields $\Gamma \in \mathcal{T}$, which is a contradiction. Condition (ii) implies that for each vertex v of B, the set $\mathrm{Link}_A(v)$ is nonempty and has diameter exceeding 1. Now, for each $v \in B$ we have that the subgraph Γ_v of Γ generated by $A \cup \{v\}$ is in $\mathcal{T}$ since it is obtained by coning off a set of diameter at least 2 and applying Definition 2.3(2). Also, for each $v, v' \in B$, since the graphs Γ_v and $\Gamma_{v'}$ are both thick and their intersection is the thick graph generated by A, we see that the graph generated by $A \cup \{v, v'\}$, which is the generalized union of Γ_v and $\Gamma_{v'}$, is thus thick by Definition 2.3(3). Thus, by adding one vertex from B at a time in the above way we see that $\Gamma \in \mathcal{T}$.

Next, we claim that $\mathbb{P}((\mathrm{i})) = \pi_n^2$. Indeed, since in the construction of Γ, edges joining pairs of vertices in A are added independently of those joining vertices in B, the events "A generates a subgraph in $\mathcal{T}$" and "B generates a subgraph in $\mathcal{T}$" are independent. Moreover, the subgraphs of Γ generated by A and B are in $G\big(n, \frac{1}{2}\big)$. It follows that (i) occurs with probability π_n^2, whence

$$\pi_{2n} \leq \pi_n^2 + \mathbb{P}((\mathrm{ii})).$$

We finally show that $\mathbb{P}((\mathrm{ii})) \leq f(n)$. To this end, let $\mathcal{V}$ be the number of vertices of B whose links in A are (possibly empty) cliques. Then $\mathbb{P}((\mathrm{ii})) \leq 2\,\mathbb{P}(\mathcal{V} > 0)$ and $\mathbb{P}(\mathcal{V} > 0) \leq \mathbb{E}(\mathcal{V})$. The initial factor of 2 reflects the fact that we are assuming that $A \in \mathcal{T}$ and counting vertices in B whose links in A are cliques; (ii) could just as easily occur with the roles of A and B reversed.

For each $v \in B$, if $\mathrm{Link}_A(v)$ has k vertices, then it is generated by one of $\binom{n}{k}$ subsets of A. Each such subset is a clique with probability $2^{-\binom{k}{2}}$, and such a subset generates $\mathrm{Link}_A(v)$ with probability $2^{-k}2^{k-n} = 2^{-n}$, reflecting the fact that the k vertices of the putative link must be adjacent to v, and the $n-k$ remaining vertices of A must not. Summing over k yields the probability that $\mathrm{Link}_A(v)$ is a clique, so $\mathbb{E}(\mathcal{V}) = n\sum_{k=0}^{n}\binom{n}{k}2^{-n-\binom{k}{2}}$, and (1) follows.

To establish (2), write $\Gamma^{(0)} = A \sqcup B$ as above. If $\Gamma \notin \mathcal{T}$, then one of the following holds:

(a) The subgraphs generated by A and B are both not in $\mathcal{T}$. This event occurs with probability π_n^2.

(b) Exactly one of the subgraphs generated by A and B belongs to $\mathcal{T}$. In this case, suppose that A generates a subgraph in $\mathcal{T}$. This subgraph is among the $t(n)$ graphs of its size in $\mathcal{T}$, and as above, B must contain a vertex v whose link in A generates one of the $c(n)$ possible cliques. There are n choices for this vertex, and each has a given clique as its link with probability at most 2^{-n}. Hence this situation occurs with probability at most $2\pi_n(1-\pi_n)nc(n)2^{-n}t(n)^{-1}$.

(c) The subgraphs generated by A and B both belong to $\mathcal{T}$. In this case, it must be true that some vertex in A has link in B a clique (or vice versa), but we do not use this fact; we just note that the probability of this event is certainly at most $(1-\pi_n)^2$.

Finally, to establish (3), regard the size-$(n+1)$ graph Γ as the subgraph of Γ generated by $A \sqcup \{v\}$, with v a vertex. If $\Gamma \notin \mathcal{T}$, then either $A \notin \mathcal{T}$ or the link of v is a clique. The claim now follows by arguing as in the proof of (1). Note that in this case, since the two parts are not symmetric and we are looking at the link of only one point rather than n, this removes a factor of $2n$ from the second term in the sum, and actually establishes the stronger fact that $\pi_{n+1} \le \pi_n + f(n)/2n$. □

Remark 3.12 The relation between the first two parts of the above lemma are as follows. In the language of conditional probability, to prove Lemma 3.11(1), we use the fact that

$$\pi_{2n} \le \mathbb{P}[A, B \notin \mathcal{T}] + \mathbb{P}[\text{(ii)}].$$

Whereas, for Lemma 3.11(2) we exploited the following:

$$\pi_{2n} \le \mathbb{P}[A, B \notin \mathcal{T}] + 2\,\mathbb{P}[A \in \mathcal{T}, B \notin \mathcal{T}] \cdot \mathbb{P}[\text{(ii)}_B \mid A \in \mathcal{T}, B \notin \mathcal{T}] + \mathbb{P}[A, B \in \mathcal{T}],$$

where (ii)_B is the same as (ii) except that we require only the condition on links of vertices of B. We then sum over these probabilities to yield Lemma 3.11(2).

We will make use of the following estimate:

Lemma 3.13 *Let X_n be a binomial random variable with mean $\frac{1}{2} \cdot n$ and variance $\frac{1}{4} \cdot n$. Then for all $M \le \frac{1}{2}n$, we have*

$$\mathbb{P}(X_n \le M) \le \exp\Big(-\frac{n}{2} + 2M - \frac{2M^2}{n}\Big).$$

Proof Viewing X_n as the sum of n Bernoulli trials, this follows from Hoeffding's inequality [39]. □

Lemma 3.14 *The function f of Lemma 3.11 has the following properties:*

(1) $f(n) \xrightarrow{n} 0$ *exponentially, and in particular,* $\sum_{n\geq 0} f(n) < \infty$.

(2) $f(n) < 0.03760$ *for all* $n \geq 18$.

Proof Let $M = \lfloor n^{a/b} \rfloor$ for natural numbers $a < b$, and define (I) and (II) by writing

$$f(n) = 2n\Bigg[\underbrace{\sum_{i=0}^{M}\binom{n}{i}2^{-n-\binom{i}{2}}}_{\text{(I)}} + \underbrace{\sum_{i=M+1}^{n}\binom{n}{i}2^{-n-\binom{i}{2}}}_{\text{(II)}}\Bigg].$$

For each n,

$$\text{(I)} \leq 2^{-n}\sum_{i=0}^{M}\binom{n}{i} = \mathbb{P}(X_n \leq M),$$

where X_n is a binomial random variable with mean $n\cdot\frac{1}{2}$. From Lemma 3.13, we have, for $M \leq n/2$,

$$\begin{aligned}\text{(I)} &\leq \exp\Big[-\frac{n}{2} + 2M - \frac{2M^2}{n}\Big]\\ &\leq e^{-n/2}e^{2\lfloor n^{a/b}\rfloor}e^{-2\lfloor n^{a/b}\rfloor^2/n} := g(n,M).\end{aligned}$$

We also have

$$\begin{aligned}\text{(II)} &\leq 2^{-n-\binom{M}{2}}\sum_{i=M+1}^{n}\binom{n}{i}\\ &\leq 2^{-n-\binom{M}{2}}\Big(2^n - \sum_{i=0}^{M}\binom{n}{i}\Big)\\ &\leq 2^{-\binom{M}{2}} \leq 2^{-n^{a/b}(n^{a/b}-1)/2}.\end{aligned}$$

Suppose now that a and b also satisfy $2a/b > 1$. Then the lemma follows from summing the above estimates: $f(n)$ decays exponentially and is hence summable. This establishes the first assertion.

The second assertion requires a refinement of one of the above bounds. Let $a = 2$ and $b = 3$, and let $M = \lfloor n^{a/b} \rfloor$, X_n and the expressions (I) and (II) be as above. As before, we have

$$\text{(II)} \leq 2^{-n^{2/3}(n^{2/3}-1)/2}.$$

We need to estimate (I) more carefully when $n \geq 18$. We thus write

$$\begin{aligned}(\mathrm{I}) &\leq 2^{-n}\left(\sum_{i=0}^{5}\binom{n}{i}2^{-\binom{i}{2}}\right) + 2^{-\binom{6}{2}}\,\mathbb{P}(X_n \leq \lfloor n^{2/3}\rfloor) \\ &\leq 2^{-n}\left(\sum_{i=0}^{5}\binom{n}{i}2^{-\binom{i}{2}}\right) + 2^{-\binom{6}{2}} g(n, \lfloor n^{2/3}\rfloor) := h(n).\end{aligned}$$

The second inequality is an application of Lemma 3.13, justified by the fact that $n^{2/3} < n/2$ for $n \geq 18$. Hence

$$f(n) \leq 2n \cdot h(n) + 2n \cdot 2^{-n^{2/3}(n^{2/3}-1)/2}.$$

The second term is strictly decreasing for $n \geq 8$, as can be seen by differentiating, and takes a value less than $3.09 \cdot 10^{-5}$ at $n = 18$. Next, a straightforward computation gives

$$g(n, \lfloor n^{2/3}\rfloor) \leq \exp\left(-\frac{n}{2} + 2n^{2/3} - 2n^{1/3} + 4n^{-1/3} - \frac{2}{n}\right),$$

which is decreasing for $n \geq 12$ and, for $n = 18$, yields

$$2n \cdot 2^{-\binom{6}{2}} \cdot g(n, \lfloor n^{2/3}\rfloor) \leq 0.00273.$$

The remaining term can be shown by direct differentiation to decrease for $n \geq 5$, and takes the value 0.3484 at $n = 18$. Combining the above shows that $f(n) \leq 3.09 \cdot 10^{-5} + 0.00273 + 0.03484 = 0.03760$ for $n \geq 18$. □

Remark 3.15 As we will see in the proof of Theorem 3.10, any bound sharper than around $f(18) \leq 0.06045$ is sufficient.

Proof of Theorem 3.10 The idea of the proof is to use Lemma 3.11(1) and the fact that f is small to get convergence to 0 of a subsequence of (π_n). We then use this in order to show that (π_n) converges to 0, and then we apply Lemma 3.11(3) and the summability of f.

Accumulation at 0 implies convergence to 0 For each n and k, Lemma 3.11(3) yields

$$\pi_{n+k} \leq \pi_n + \sum_{i=0}^{k-1} f(i+n) < \pi_n + \sum_{i=n}^{\infty} f(i).$$

Suppose that 0 is an accumulation point of (π_n). Then for each $\epsilon > 0$, we can choose n so that $\pi_n < \frac{1}{2}\epsilon$ and $\sum_{i=n}^{\infty} f(n) < \frac{1}{2}\epsilon$. The latter inequality follows from summability of f, ie from Lemma 3.14(1). Hence for all k, we have $\pi_{n+k} < \epsilon$, ie $\pi_n \xrightarrow{n} 0$.

Nonaccumulation at 0 implies convergence to 1 Suppose now that the subsequence $(\pi_{k\cdot 2^m})_{m\in\mathbb{N}}$ does not have 0 as an accumulation point for some $k\in\mathbb{N}$. Then we claim that $(\pi_{k\cdot 2^m})$ converges to 1. Indeed, consider the smallest accumulation point π of the sequence, and suppose that it is the limit of the subsequence $(\pi_{k\cdot 2^{m_i}})_{i\in\mathbb{N}}$. We have to show $\pi=1$. By Lemma 3.11(1) and the fact that f converges to 0, we get that any accumulation point π' of $(\pi_{k\cdot 2^{m_i+1}})$ satisfies $\pi'\le\pi^2$. As we also have $\pi\le\pi'$, we get $\pi\le\pi^2$, so that $\pi=1$.

A subsequence bounded away from 1 It is thus sufficient to show that the subsequence $(\pi_{k\cdot 2^m})_{m\in\mathbb{N}}$ is bounded away from 1 for some $k\in\mathbb{N}$. In fact, if this is the case, then $(\pi_{k\cdot 2^m})_{m\in\mathbb{N}}$ does not converge to 1, hence it must have 0 as an accumulation point, and hence (π_n) converges to 0 as required. Suppose that for some k, we have $m_0\in\mathbb{N}$ and constants $\alpha,\beta\in[0,1)$ such that $f(k\cdot 2^m)\le\beta$ for all $m\ge m_0$, and $\pi_{k\cdot 2^{m_0}}\le\alpha$. Suppose, moreover, that $\alpha^2+\beta<\alpha$. Then $\pi_{k\cdot 2^{m_0+1}}<\alpha$ by Lemma 3.11(1), and by induction and the same lemma, we have $\pi_{k\cdot 2^m}<\alpha$ for all $m\ge m_0$.

Let $k=9$ and $m_0=1$. The computer program in the online supplement returned the following data:

- $t(9)=14853635863$,
- $c(9)=683846354560$,
- $\pi_9=1-t(9)/2^{\binom{9}{2}}\approx 0.78385$.

Together with Lemma 3.11(2), this implies

$$\pi_{18}\le\alpha:=\left(1-\frac{t(9)}{2^{36}}\right)^2+\left(\frac{t(9)}{2^{36}}\right)^2+2\left(1-\frac{t(9)}{2^{36}}\right)\cdot\frac{t(9)}{2^{36}}\cdot\frac{9\cdot c(9)}{512\cdot t(9)}\approx 0.93537.$$

Lemma 3.14(2) gives $f(n)\le\beta=0.03760$ for all $n\ge 18$. The above discussion, together with the fact that these values satisfy $\alpha^2+\beta<\alpha$, implies that $(\pi_{9\cdot 2^m})$ is bounded away from 1, whence $\pi_n\xrightarrow{n}0$; ie Γ is aas in $\mathcal{T}$. □

4 Detecting thickness algorithmically

In this section, we exhibit a polynomial-time algorithm for deciding whether a finite graph is in $\mathcal{T}$. The construction of the algorithm presented in this section prioritized simplicity over speed. We also provide a C++ implementation of a simple algorithm to compute the constants needed in the proof of Theorem 3.10. The main part of this computer program implements the algorithm for deciding if a given right-angled Coxeter group is thick.

Theorem 4.1 *There exists an algorithm which decides, in polynomial time, whether a graph Γ is in $\mathcal{T}$. Hence the problem of deciding whether a right-angled Coxeter group admits a relatively hyperbolic structure is soluble in polynomial time.*

Proof The second assertion follows from the first by Theorem 2.5. The algorithm takes as input the finite simplicial graph Γ on n vertices and decides whether $\Gamma \in \mathcal{T}$. For ease of exposition, we provide an algorithm which admits an easy description, but we note that there are more efficient algorithms; in particular, the code in the online supplement contains an implementation of a more efficient algorithm for the same task. The steps are:

(1) Make a list $\mathcal{M}$ of all induced $K_{2,2}$ subgraphs of Γ. The running time is in $O(n^4)$ and $|\mathcal{M}|$ is in $O(n^4)$.

(2) Make a list $\mathcal{N}$ of pairs of nonadjacent vertices. The running time is in $O(n^2)$ and $|\mathcal{N}|$ is in $O(n^2)$.

(3) Perform a *union subroutine*; ie for each pair $M, M' \in \mathcal{M}$, determine whether $M \cap M'$ contains some $(v, v') \in \mathcal{N}$. If so, modify $\mathcal{M}$ by removing M and M', and adding the subgraph induced by $M \cup M'$. The running time of a union subroutine is in $O(n^{11})$.

(4) Perform a *coning subroutine*; ie for each $M \in \mathcal{M}$ and each vertex v, determine whether there exists $(w, w') \in \mathcal{N}$ such that $w, w' \in M$ and both are adjacent to v. If so, replace M by the subgraph generated by $M \cup \{v\}$. The running time of a coning subroutine is in $O(n^7)$.

(5) If $\mathcal{M}$ did not change during the coning and union subroutines, then we are finished: the graph is thick if and only if $|\mathcal{M}| = 1$, and the unique element of $\mathcal{M}$ is Γ.

(6) If $\mathcal{M}$ changed, then return to step (2).

The number of union subroutines that modify $\mathcal{M}$ is in $O(n^4)$ since each such union subroutine decreases $|\mathcal{M}|$. The number of coning subroutines that modify $\mathcal{M}$ is in $O(n^5)$ since each such subroutine increases the size of some subgraph in $\mathcal{M}$. Hence the total running time is in $O(n^{15})$. □

4.1 Computing $t(9)$ and $c(9)$

To obtain the values used in the proof of Theorem 3.10, one can use the C++ program in the online supplement, which takes a single command line argument, namely the number n of vertices. We have also checked the computations by hand up to $n = 6$

beyond which they become infeasible. The reader seeking to reproduce our computer computation for $n = 9$ should be aware that the program requires being run for several days with typical 2013 hardware.

The efficiency of the program can be significantly improved. However, we decided to keep the code as simple as possible. Source code for a much more efficient, albeit more complex, version of this program can be obtained from the authors.

Appendix: Generalizing to all Coxeter groups

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All Coxeter groups considered here are assumed finitely generated. In this appendix, we generalize Theorems I and II to Coxeter groups which are not necessarily right angled. Further considerations are contained in Section A.3.

We can summarize the main result in this appendix as follows.

Theorem A.1 (minimal relatively hyperbolic structures) *Let (W, S) be a Coxeter system. Then there is a (possibly empty) collection $\mathcal{J}$ of subsets of S enjoying the following properties:*

(i) *The parabolic subgroup W_J is strongly algebraically thick for every $J \in \mathcal{J}$.*

(ii) *If $J \neq S$ for all $J \in \mathcal{J}$, then W is hyperbolic relative to $\mathcal{P} = \{W_J \mid J \in \mathcal{J}\}$.*

In particular, $\mathcal{P}$ is a minimal relatively hyperbolic structure for W.

A.1 Thick Coxeter groups

We consider the class $\mathbb{T}$ of Coxeter systems (W, S) defined as follows.

(1) $\mathbb{T}$ contains the class $\mathbb{T}_0$ of all irreducible *affine* Coxeter systems (W, S) with S of cardinality at least 3, as well as all Coxeter systems of the form $(W, S_1 \cup S_2)$ with W_{S_1} and W_{S_2} irreducible nonspherical and $[W_{S_1}, W_{S_2}] = 1$.

(2) Suppose that $(W, S \cup s)$ is such that $s^\perp$ is nonspherical and (W_S, S) belongs to $\mathbb{T}$. Then $(W, S \cup s)$ belongs to $\mathbb{T}$.

(3) Suppose that (W, S) is such that there exist $S_1, S_2 \subseteq S$ with $S_1 \cup S_2 = S$, $(W_{S_1}, S_1), (W_{S_2}, S_2) \in \mathbb{T}$ and $W_{S_1 \cap S_2}$ nonspherical. Then $(W, S) \in \mathbb{T}$.

Proposition A.2 *For $(W, S) \in \mathbb{T}$, the Coxeter group W is strongly algebraically thick.*

The proof requires the following subsidiary fact.

Lemma A.3 *Let (W,S) be a Coxeter system. Let $s\in S$ and set $K=S\setminus\{s\}$. Then the group $\langle W_K\cup sW_Ks\rangle$ has index at most 2 in W.*

Proof The group $\langle W_K\cup sW_Ks\rangle$ is a reflection subgroup whose fundamental domain for its action on the Cayley graph of (W,S) contains at most two chambers, namely the base vertex 1 and the unique vertex s–adjacent to it, see [26]. □

Proof of Proposition A.2 If (W,S) is in $\mathbb{T}_0$ then the group W is either virtually abelian of rank at least 2 or a direct product of two infinite (Coxeter) groups. In particular, W is wide and, hence, strongly algebraically thick of order 0.

Let $(W,S\cup\{s\})$ be of the form described in item (2) of the definition of $\mathbb{T}$. Lemma A.3 then implies that W contains the group $\langle W_S\cup sW_Ss\rangle$ with index at most 2. Therefore W is strongly algebraically thick, being an algebraic network with respect to the pair of strongly thick groups $\{W_S, sW_Ss\}$.

Finally, let (W,S) be as in item (3) of the definition of $\mathbb{T}$. Then W is strongly algebraically thick, being an algebraic network with respect to the pair of strongly thick groups $\{W_{S_1}, W_{S_2}\}$. □

A.2 Proof of minimal relatively hyperbolic structures theorem

We will use the following criterion for relative hyperbolicity of Coxeter groups, which corrects [14, Theorem A], where a hypothesis on the peripheral subgroups was missing.

Theorem A.4 [15, Theorem A′] *Let (W,S) be a Coxeter system and $\mathcal{J}$ a collection of proper subsets of S. Then W is hyperbolic relative to $\{W_J \mid J\in\mathcal{J}\}$ if and only if the following conditions hold:*

(RH1) *For each irreducible affine subset $K\subseteq S$ of cardinality at least 3, there exists $J\in\mathcal{J}$ such that $K\subseteq J$. Similarly, given any pair of irreducible nonspherical subsets $K_1,K_2\subseteq S$ with $[K_1,K_2]=1$, there exists $J\in\mathcal{J}$ such that $K_1\cup K_2\subseteq J$.*

(RH2) *For all $J_1,J_2\in\mathcal{J}$ with $J_1\neq J_2$, the intersection $J_1\cap J_2$ is spherical.*

(RH3) *For each $J\in\mathcal{J}$ and each irreducible nonspherical $K\subseteq J$, we have $K^\perp\subseteq J$.*

We are now ready to prove Theorem A.1. We will give an explicit description of $\mathcal{J}$:

Theorem A.5 *Let (W,S) be a Coxeter system and let $\mathcal{J}$ be the (possibly empty) collection of all* maximal *subsets $J\subseteq S$ such that $(W_J,J)\in\mathbb{T}$. Then we have:*

(i) *The parabolic subgroup W_J is strongly algebraically thick for every $J\in\mathcal{J}$.*

(ii) *If $\mathcal{J}\neq\{S\}$, then W is hyperbolic relative to $\mathcal{P}=\{W_J \mid J\in\mathcal{J}\}$.*

In particular, $\mathcal{P}$ is a minimal relatively hyperbolic structure for W.

Proof By Moussong's characterization of hyperbolic Coxeter groups [41, Theorem 17.1] (and the fact that S is finite), $\mathcal{J}$ is not empty if and only if W is not hyperbolic, which we assume from now on.

By Proposition A.2, (i) holds.

We are now left to show that $\mathcal{J}$ satisfies the three conditions (RH1)–(RH3) from Theorem A.4.

It is clear that $\mathcal{J}$ satisfies (RH1).

If $J_1, J_2 \in \mathcal{J}$ are distinct, then $W_{J_1 \cap J_2}$ must be spherical. In fact, if it was nonspherical, then we would have $J_1 \cup J_2 \in \mathcal{J}$, contradicting the maximality of either J_1 or J_2. So $\mathcal{J}$ satisfies (RH2).

Let K be a nonspherical subgraph of some $J \in \mathcal{J}$. We have to show that $K^\perp$ is contained in J as well. Indeed, if there was an element $s \in K^\perp \backslash J$, then $J \cup \{s\}$ would be in $\mathbb{T}$, contradicting the maximality of J.

We have now shown the peripherals are in $\mathbb{T}$ and hence thick by Proposition A.2. Thus, as noted in the introduction, minimality now follows from [5, Corollary 4.7]. □

A.3 Intrinsic horosphericity and further corollaries

We say that a discrete group Γ is *(intrinsically) horospherical* if every proper isometric action of Γ on a proper hyperbolic geodesic metric space fixes a unique point at infinity. In particular, the group Γ cannot be virtually cyclic, and every element of infinite order acts as a parabolic isometry in any such Γ–action. As one may expect, thickness and horosphericity are related properties (compare Theorem 4.1 from [5]):

Proposition A.6 *Every strongly algebraically thick group is intrinsically horospherical.*

The proof requires the following result, which follows from the exact same arguments as the proof of Lemma 3.25 in [28].

Lemma A.7 *Let H be a finitely generated group (endowed with its word metric with respect to a finite generating set), (X, d) a metric space and $\mathfrak{q}\colon H \to X$ a map which is Lipschitz up to an additive constant. Given $h \in H$, if the map $\mathbb{Z} \to X, n \mapsto \mathfrak{q}(h^n)$ is a Morse quasigeodesic in X, then h is a Morse element in H.* □

Lemma A.8 *Let H be a group acting properly by isometries on a proper Gromov hyperbolic metric space X. Assume that H has a unique fixed point ξ at infinity of X. Then every infinite subgroup of H has ξ as its unique fixed point at infinity.*

Proof The hypotheses imply that H does not contain any hyperbolic isometry. From Proposition 5.5 in [16], it follows that every subgroup of H either has a bounded orbit or has a unique fixed point at infinity of X. The desired conclusion follows since the H–action on X is proper. □

Proof of Proposition A.6 We argue by induction on the order of thickness. In the base case, let H be a finitely generated group which is wide. Suppose that H acts properly by isometries on a proper Gromov hyperbolic metric space X. H can not contain a hyperbolic isometry since otherwise, Lemma A.7 implies that some asymptotic cone of H has cut-points, which would contradict the assumption that H is wide. Since H is infinite and the H–action on X is proper, it follows from [16, Proposition 5.5] that H fixes a unique point at infinity of X. This proves that strongly algebraically thick groups of order 0 are intrinsically horospherical.

The inductive step is given by the following observation. Let G be an infinite group which is an M–algebraic network with respect to a finite collection $\mathcal{H}$ of subgroups. If each subgroup in $\mathcal{H}$ is intrinsically horospherical, then so is G.

Indeed, let G act properly by isometries on a proper Gromov hyperbolic metric space X. Then each group $H \in \mathcal{H}$ has a unique fixed point ξ_H at infinity of X. Given $H, H' \in \mathcal{H}$, there is a sequence $H = H_1, \dots, H_N = H'$ in $\mathcal{H}$ in which any two consecutive groups have an infinite intersection; see Definition 5.2 in [5]. From Lemma A.8, we deduce that $\xi_H = \xi_{H_1} = \cdots = \xi_{H_n} = \xi_{H'}$. Hence all groups in $\mathcal{H}$ have the same fixed point at infinity, say ξ. By the definition of an algebraic network, this point ξ must be fixed by a finite-index subgroup of G. Thus the G–orbit of ξ is finite.

If this orbit has exactly one point, then G fixes ξ (and no other point at infinity of X), and we are done. If this orbit contains exactly two points, then G is virtually cyclic and hence does not contain any intrinsically horospherical subgroups, which is absurd. If $|G\xi| \geq 3$, then it follows from [38, Proposition-Definition 8.2.L] that G has bounded orbits in X, contradicting the assumption that G is infinite and acts properly. □

Notice that the converse to Proposition A.6 does not hold in general: indeed, horospherical groups include all amenable groups that are not virtually cyclic. In particular, infinite locally finite groups are examples of horospherical groups that are not strongly algebraically thick. By Zorn's lemma, every intrinsically horospherical subgroup of Γ is contained in a maximal one. It is thus a natural question to determine all the maximal intrinsically horospherical subgroups. Theorem A.1 yields the answer to this question when Γ is a Coxeter group.

Corollary A.9 *Let W be a Coxeter group. Then the maximal intrinsically horospherical subgroups of W are parabolic subgroups (in the sense of Coxeter group theory) with respect to any Coxeter generating set. Those parabolic subgroups are precisely the conjugates of the elements of the set $\mathcal{P}$ afforded by Theorem A.1.*

Proof Every strongly algebraically thick group is intrinsically horospherical by Proposition A.6. Moreover, a subgroup of W properly containing a conjugate of an element of $\mathcal{P}$ cannot be intrinsically horospherical by Theorem A.1. Thus the elements of $\mathcal{P}$ are indeed maximal horospherical subgroups. Since W is relatively hyperbolic with respect to $\mathcal{P}$, every intrinsically horospherical subgroup is conjugate to a subgroup of an element of $\mathcal{P}$. □

Corollary A.10 *Let (W, S) be a Coxeter system. Then the following conditions are equivalent:*

- (i) *(W, S) is in $\mathbb{T}$.*
- (ii) *W is strongly algebraically thick.*
- (iii) *W is intrinsically horospherical.*
- (iv) *W is not relatively hyperbolic with respect to any family of proper subgroups.*
- (v) *W is not relatively hyperbolic with respect to any family of proper Coxeter-parabolic subgroups.*
- (vi) *For every collection $\mathcal{J}$ of subsets of S satisfying (RH1)–(RH3), we have $S \in \mathcal{J}$.*

Proof The implication (i) $\Longrightarrow$ (ii) is the content of Proposition A.2. The implication (ii) $\Longrightarrow$ (iii) follows from Proposition A.6. The implication (iii) $\Longrightarrow$ (iv) is straightforward. Property (iv) trivially implies (v). That (v) is equivalent to (vi) follows from Theorem A.4. Applying Theorem A.5, we get that (v) implies (i). □

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