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On RO(G)–graded equivariant "ordinary" cohomology where G is a power of $\mathbb{Z}/2$

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We compute the complete RO(G)–graded coefficients of "ordinary" cohomology with coefficients in $\mathbb{Z}/2$ for $G = (\mathbb{Z}/2)^n$. As an important intermediate step, we identify the ring of coefficients of the corresponding geometric fixed point spectrum, revealing some interesting algebra. This is a first computation of its kind for groups which are not cyclic p–groups.

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1 Introduction

The notion of a cohomology theory graded by elements of the real representation ring (RO(G)–graded cohomology) is a key concept of equivariant stable homotopy theory of a finite or compact Lie group G. Like much of stable homotopy theory, perhaps one of the first known examples was K-theory. Atiyah and Singer [4] introduced equivariant K-theory of a compact Lie group G and proved that it is naturally RO(G)–graded. In fact, Bott periodicity identifies many of the "dimensions" in RO(G), and relates others to "twistings" (see Karoubi [7] and, for a more recent treatment, Freed, Hopkins and Teleman [9]). Pioneered by Adams and Greenlees [10], the general RO(G)–graded stable homotopy theory found firm foundations in the fundamental book of Lewis, May and Steinberger [22].

Despite the clear importance of the concept, beyond K-theory, calculations of RO(G)–graded cohomology are few and far in between. Perhaps the most striking case is "ordinary" RO(G)–graded cohomology. Bredon [5] discovered $\mathbb{Z}$–graded G–equivariant cohomology associated with a *coefficient system* which is "ordinary" in the sense that the cohomology of a point is concentrated in a single dimension. It was later discovered (Lewis, May and McClure [20]) that such a theory becomes RO(G)–graded when the coefficient system enjoys the structure of a *Mackey functor* (see Dress [8]), which means that it allows building in an appropriate concept of *transfer*. Strikingly, the RO(G)–graded coefficients were not known in any single nontrivial case.

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Complete calculations of $\mathrm{RO}(\mathbb{Z}/2)$–graded coefficients, however, are important in Real-oriented stable homotopy theory, because they exhibit the analogy with the complex-oriented case. Real orientation was, once again, discovered first by Atiyah [3] in the case of K-theory, and was subsequently extended to cobordism by Landweber [19]. $\mathrm{RO}(\mathbb{Z}/2)$–graded cohomology with coefficients in the Burnside ring Mackey functor was calculated by Stong [21]. A systematic pursuit of real-oriented homotopy theory was started by Araki [2], and developed further by Hu and Kriz [13] with many calculations, including a complete calculation of the $\mathrm{RO}(G)$–graded coefficients of Landweber's Real cobordism spectrum. In the process, Hu and Kriz [13] also calculated the $\mathrm{RO}(\mathbb{Z}/2)$–graded ordinary cohomology of the "constant" Mackey functors $\mathbb{Z}$ and $\mathbb{Z}/2$ (ie the Mackey functors uniquely extending the constant coefficient systems). A major development was the work of Hill, Hopkins and Ravenel [12], who partially extended the calculations of [13] to $\mathbb{Z}/(2^k)$ (with special interest in $k=3$), and applied this to solving the Kervaire–Milnor problem by showing the nonexistence of manifolds of Kervaire invariant 1 in dimensions >126. A still more complete calculation of $\mathrm{RO}(G)$–graded ordinary equivariant cohomology of the constant Mackey functors for $G=\mathbb{Z}/(2^k)$ was more recently given in Hu and Kriz [14].

Still, no calculations of $\mathrm{RO}(G)$–graded cohomology beyond K-theory were known for groups other than where G is a primary cyclic group. In a spin-off of their joint solution with Ormsby [16] of Thomason's homotopy limit problem for Hermitian K-theory, Hu and Kriz [15] computed the $\mathrm{RO}(G)$–graded coefficients of topological Hermitian cobordism, which has $G=\mathbb{Z}/2\times\mathbb{Z}/2$. However, this is a rather special case, where many periodicities occur.

The purpose of the present paper is to calculate the $\mathrm{RO}(G)$–graded coefficients of the ordinary equivariant cohomology of the "constant" $\mathbb{Z}/2$ Mackey functor for $G=(\mathbb{Z}/2)^n$. There are several reasons to focus on this case. The group $(\mathbb{Z}/2)^n$ has an exceptionally simply described real representation ring, thus eliminating the need to handle representation-theoretical exceptions such as distinguishing between real and complex (let alone, quaternionic) representations. The coefficients $\mathbb{Z}/2$ are more convenient than $\mathbb{Z}$, since they eliminate the need to consider extensions. Despite all this, the complete answer is complicated, however, and in general, we are only able to present it in the form of the cohomology of an n–stage chain complex.

Our method is based on *isotropy separation*, a term coined by Greenlees and May [11], to mean considering separately the contributions of subgroups of G. An isotropy separation spectral sequence was developed in Abram and Kriz [1], but we use a different spectral sequence here. The reason is that in [1], we are not concerned with $\mathrm{RO}(G)$–graded coefficients, but rather with computing the complete $\mathbb{Z}$–graded coefficients of equivariant complex cobordism of a finite abelian group G as a ring.

Based on generalizing the method of Kriz [17] in the case of $G = \mathbb{Z}/p$, in the case of $\mathbb{Z}$–graded equivariant complex cobordism, one can set up a spectral sequence of rings which collapses to E^2 in a single filtration degree. This means that the complete ring structure can be recovered, which is a special property of complex cobordism. It is worth mentioning that the spectral sequence of [1] contains many "completed" (for example, uncountable) terms.

The case of ordinary RO(G)–graded equivariant cohomology is quite different, however, in that the spectral sequence fails to collapse to a single degree. Even for $G = \mathbb{Z}/p$, we observe that a part of the coefficients are in filtration degree 0 and a part in filtration degree -1 (graded homologically). This caused us to give up, at least for now, calculating the complete ring structure, and use a spectral sequence which is more amenable to calculations instead.

Another key ingredient in our computation is the concept of *geometric fixed points* of an RO(G)–graded equivariant cohomology theory. This concept was introduced (using a different terminology) by tom Dieck [6], who calculated the geometric fixed points of equivariant complex cobordism. As far as we know, the term geometric fixed points was coined by Greenlees and May, and is recorded in Lewis, May, Steinberger and McClure [22]. Unlike actual fixed points, the geometric fixed point coefficients are *periodic* with respect to all nontrivial irreducible real representations of G. Thus, instead of RO(G)–graded, the geometric fixed points are, again, only $\mathbb{Z}$–graded. This is a big advantage in expressing the answer. Note that the ring RO($(\mathbb{Z}/2)^n$) is huge: it is the free abelian group on 2^n generators! On the downside, the term "geometric" fails to carry the expected implications in the case of ordinary equivariant cohomology: we know of no geometry that would help calculating them. Still, in the case $G = (\mathbb{Z}/2)^n$, a complete calculation of the geometric fixed point ring of $H\mathbb{Z}/2$ is possible using spectral sequence methods. This is our Theorem 2.

The main method of this paper is, basically, setting up another spectral sequence which enables the calculation of the coefficients of $H\mathbb{Z}/2_{(\mathbb{Z}/2)^n}$ by investigating how they differ from the coefficients of the geometric fixed points. There results a spectral sequence, which, in a fairly substantial range of RO(G)–graded dimensions, collapses to E^2 in degree 0. More precisely, the range is, graded homologically, suspensions by elements of RO(G) where summands of nontrivial irreducible representations occur with *nonpositive* coefficients. Alternately, graded cohomologically, this is the range of suspensions by actual representations, possibly minus a trivial representation. (As it turns out, however, in this case, when the trivial representation has a negative coefficient, the cohomology group is 0.) In this case, we can both recover the complete ring structure, since the ring embeds into the ring of geometric fixed points tensored with RO(G). We also have a nice concise formula for the Poincaré series in this case (Theorem 5).

In the case of completely general $\mathrm{RO}(G)$–dimension with $G = (\mathbb{Z}/2)^n$, we are only able to give a spectral sequence in n filtration degrees, which collapses to E^2 and calculates the $\mathrm{RO}(G)$–graded coefficient group of $H\mathbb{Z}/2_G$. Thus, this gives an algebraically defined chain complex whose homology are the desired groups (Theorem 7). We give an example of a complete calculation of the Poincaré series of the $\mathrm{RO}(G)$–graded coefficients of $H\mathbb{Z}/2_{\mathbb{Z}/2\times\mathbb{Z}/2}$ (the case $n = 2$), which clearly shows that the answer gets complicated, and additional complications arise for $n \geq 3$.

The present paper is organized as follows: In Section 2, we introduce the necessary conventions and notation. In Section 3, we compute the geometric fixed points. In Section 4, we compute the coefficients in dimensions involving elements of $\mathrm{RO}(G)$ where nontrivial irreducible representations have nonpositive coefficients (graded homologically). In Section 5, we calculate the chain complex computing the complete $\mathrm{RO}(G)$–graded coefficients of $H\mathbb{Z}/2_G$ for $G = (\mathbb{Z}/2)^n$. In Section 6, we treat the example of $n = 2$. The authors apologize to the readers for not stating their theorems in the introduction. Even in the prettiest cases, the theorems involve quite a lot of notation and technical prerequisites. We prefer to state them properly in the text.

Recent developments: odd primes, and hyperplane arrangements While this paper was under review, several developments took place. A generalization of the present result to $(\mathbb{Z}/p)^n$ for p an odd prime was found by Holler. The authors also found out that the ring described in Theorem 2 is a previously known object in algebraic geometry, related to a certain compactification of complements of hyperplane arrangements referred to as *the reciprocal plane*.

More concretely, for a set $S = \{z_\alpha\}$ of equations of hyperplanes through 0 in an affine space $\mathrm{Spec}(F[u_1, \dots, u_n])$ of a field F, one considers the subring R_S of

$$\left(\prod_{\alpha\in S} z_\alpha\right)^{-1} F[u_1, \dots, u_n] \tag{1}$$

generated by the elements z_α^{-1} (which correspond to our elements x_α). The ring was first described by Terao [24], and a particularly nice presentation was found by Proudfoot and Speyer [23]. In the case of an odd prime p, one deals analogously with the subring Ξ_S of

$$\left(\prod_{\alpha\in S} z_\alpha\right)^{-1} F[u_1, \dots, u_n] \otimes_F \Lambda(du_1, \dots, du_n) \tag{2}$$

generated by z_α^{-1} and $d\log(z_\alpha)$, which are topologically in dimensions 2 and 1, respectively. The analogues of the constructions of [23; 24] in this graded-commutative case, and the reciprocal plane compactification, were recently worked out by S Kriz [18].

Our emphasis is quite different form the authors of [23; 24], who, doing classical algebraic geometry, were mostly interested in characteristic 0. Their arguments, however, work in general. The ring described in Theorem 2 (and its $\mathbb{Z}/p$ analogue discovered by Holler, ie the geometric fixed point ring of $H\mathbb{Z}/p_G$ where $G = (\mathbb{Z}/p)^n$) is related to the hyperplane arrangement of *all* hyperplanes through 0 in the n–dimensional affine space over $\mathbb{Z}/p$. It follows, however, from the descriptions of [23; 24; 18] that for a subset S' of a hyperplane arrangement S, the ring $R_{S'}$ (resp. $\Xi_{S'}$) is a subring of R_S (resp. Ξ_S). It follows in turn that for *every* hyperplane arrangement in $G = (\mathbb{Z}/p)^n$, the $\mathbb{Z}$–graded part of the coefficient ring of the spectrum

$$\bigwedge_{\alpha \in S} S^{\infty\alpha} \wedge H\mathbb{Z}/p_G$$

is R_S for $p = 2$, and Ξ_S for any odd prime p.

2 Conventions and notation

Throughout this paper, let $G = (\mathbb{Z}/2)^n$. Then the real representation ring of G is canonically identified as

$$\mathrm{RO}(G) = \mathbb{Z}[G^*],$$

where $G^* = \mathrm{Hom}(G, \mathbb{Z}/2)$. Recall [22] that for $H \subseteq G$, we have the family $\mathcal{F}[H]$ consisting of all subgroups $K \subset G$ with $H \not\subseteq K$. (In the case of $H = G$, we see that $\mathcal{F}[G]$ is simply the family $\mathcal{P}$ of proper subgroups of G.) Recall further that for any family $\mathcal{F}$ (a set of subgroups of G closed under subconjugation, which is the same as closed under subgroups, as G is commutative), we have a cofibration sequence

$$E\mathcal{F}_+ \to S^0 \to \widetilde{E\mathcal{F}},$$

where $E\mathcal{F}$ is a G–CW-complex whose K–fixed point set is contractible when $K \in \mathcal{F}$ and empty otherwise. For our choice of G, we may then choose a model

$$\widetilde{E\mathcal{F}[H]} = \bigwedge_{\substack{\alpha \in G^* \\ \alpha|H \neq 0}} S^{\infty\alpha}. \tag{3}$$

Here $S^{\infty\alpha}$ is the direct limit of $S^{n\alpha}$ with respect to the inclusions

$$S^0 \to S^\alpha \tag{4}$$

given by sending the non-basepoint to 0. The other construction we use is the family $\mathcal{F}(H)$ of all subgroups of a subgroup $H \subseteq G$. We will write simply

$$EG/H = E\mathcal{F}(H).$$

The cardinality of a finite set S will be denoted by $|S|$. We will also adopt a convention from [13] where, for an $\mathrm{RO}(G)$–graded spectrum E, the $\mathbb{Z}$–indexed coefficients (= homotopy groups) of E are denoted by E_*, while the $\mathrm{RO}(G)$–indexed coefficients will be denoted by $E_\star$. As is customary, we will also denote by $S(V)$ the unit sphere of a representation V, while by S^V we denote the 1–point compactification of V. The $\mathrm{RO}(G)$–graded dimension of a homogeneous element $x \in E_\star$ will be denoted by $|x|$.

3 The geometric fixed points

In this section, we compute the coefficients of the geometric fixed point spectrum $\Phi^G H\mathbb{Z}/2$. We have

$$\Phi^G H\mathbb{Z}/2 = (\widetilde{E\mathcal{F}[G]} \wedge H\mathbb{Z}/2)^G. \tag{5}$$

By (3), suspension of $H\mathbb{Z}/2$ by any nontrivial irreducible real representation of G gives an isomorphism on coefficients, so the coefficients $\left(\Phi^G \Sigma^? H\mathbb{Z}/2\right)_*$ are only $\mathbb{Z}$–graded, not $\mathrm{RO}(G)$–graded. More specifically, we have a cofibration sequence

$$EG/\mathrm{Ker}(\alpha)_+ \to S^0 \to S^{\infty\alpha}, \tag{6}$$

so smashing over all nontrivial 1–dimensional representations α, using (3), we may represent

$$\widetilde{E\mathcal{F}[G]} \wedge H\mathbb{Z}/2$$

as the iterated cofiber of a (2^n-1)–dimensional cube of the form

$$H\mathbb{Z}/2 \wedge \bigwedge_{0\neq\alpha\in G^*} (EG/\mathrm{Ker}(\alpha)_+ \to S^0). \tag{7}$$

Taking coefficients in (7) then gives a spectral sequence converging to $\Phi^G H\mathbb{Z}/2_*$. Now also note that

$$EG/H_1 \times \cdots \times EG/H_k \simeq EG/(H_1 \cap \cdots \cap H_k). \tag{8}$$

From this, we can calculate the spectral sequence associated with the iterated cofiber of the cube (7). Let us grade the spectral sequence homologically, so the term $H\mathbb{Z}/2_*$, which equals $\mathbb{Z}/2$, is in $E^1_{0,0}$. The rest of the E^1–term is then given as

$$E^1_{p,*} = \bigoplus_{S\in\mathcal{S}_p} \mathrm{Sym}_{\mathbb{Z}/2}\left(\left(G/\bigcap\{\mathrm{Ker}(\alpha) \mid \alpha \in S\}\right)^*\right) \cdot y_S, \tag{9}$$

where $\mathcal{S}_p$ is the set of all subsets of $G^* \smallsetminus \{0\}$ of cardinality p. (The last factor y_S of (9) is only a generator written to distinguish the summands.) Now the E^2–term can also be calculated using the following:

Lemma 1 *Consider the differential* ∂ *on*

$$Q_n = \mathbb{Z}/2\{y_S \mid S \subseteq (\mathbb{Z}/2)^n \smallsetminus \{0\}\}$$

given by

$$\partial(y_S) = \sum_{\substack{s\in S \\ \langle S\smallsetminus\{s\}\rangle = \langle S\rangle}} y_{S\smallsetminus\{s\}}. \tag{10}$$

Then the homology is the $\mathbb{Z}/2$*–vector space (freely) generated by a set* F_n *described inductively as follows:*

$$F_1 = \{y_\varnothing, y_{\{(1)\}}\},$$
$$F_n = F_{n-1} \cup \{y_{S\cup\{x\}} \mid S \in F_{n-1}, x \in (\mathbb{Z}/2)^{n-1} \times \{1\}\}.$$

In other words, F_n *consists of the basis elements* y_S *where* S *are all the* $\mathbb{Z}/2$*–linearly independent (in* G^**) subsets in (not necessarily reduced) row echelon form with respect to reversed order of columns (so the first pivot is in the last possible column etc).*

Proof Consider a differential on Q_n given by

$$d(y_S) = \sum_{s\in S} y_{S\smallsetminus\{s\}}. \tag{11}$$

Then the homology is 0 for $n > 0$ and $\mathbb{Z}/2$ for $n = 0$. Now consider an increasing filtration on Q_n by making the filtration degree $\gamma(S)$ of a basis element y_S equal to $\operatorname{rank}\langle S\rangle$, the rank of the $\mathbb{Z}/2$–vector space generated by S. Then the E^1–term is what we are trying to calculate.

On the other hand, in the answer $C = \mathbb{Z}/2(F_n)$ suggested in the statement of the lemma (which, note, consists of elements of E^1), the formula for d^1 is the same as the formula (11) for d. We claim that

$$H_*(C, d) = 0. \tag{12}$$

To see this, note that for any fixed nonempty set S in row echelon form, the subcomplex C_S generated by $y_{S'}$ subsets of $S' \subseteq S$ is just a tensor product of copies of

$$\mathbb{Z}/2 \xrightarrow{\cong} \mathbb{Z}/2, \tag{13}$$

and hence satisfies

$$H_*(C_S, d) = 0.$$

On the other hand, C for $n > 0$ is a sum of the complexes C_S where S ranges over *maximal* linearly independent subsets of $(\mathbb{Z}/2)^n$ in row echelon form (ie those which

have exactly n elements), while the intersection of any subset of those complexes

$$C_{S_1\cap\cdots\cap S_k} = C_{S_1}\cap\cdots\cap C_{S_k}$$

has zero homology because

$$(1,0,\dots,0)\in S_1\cap\cdots\cap S_k$$

and hence $S_1\cap\cdots\cap S_k\neq\varnothing$. This implies (12).

Now the statement follows by induction on n using comparison theorems for spectral sequences. More concretely, if we denote by $C'\subset C$ the subcomplex generated by linearly independent subsets S with $|S|<n$, and $Q'\subset Q_n$ the subcomplex generated by sets S which span a subspace of dimension $<n$, then the induction hypothesis (given that an intersection of vector subspaces is a vector subspace), shows that the embedding $C\subset Q_n$ restricts to a quasi-isomorphism

$$C'\subset Q'. \tag{14}$$

Since the homologies of both C and Q_n are 0, we see that the homomorphism on degree n subcomplexes must induce an isomorphism on homology, thus implying that the degree n part of the E^1–term of our spectral sequence for Q_n is just the degree n part of C (which is, of course, isomorphic to $\mathbb{Z}/2$). □

Now by Lemma 1, the E^2–term of the spectral sequence of the cube (7) is

$$E^2=\bigoplus_{S\in F_n}\mathrm{Sym}_{\mathbb{Z}/2}\Big(\Big(G/\bigcap\{\mathrm{Ker}(\alpha)\mid\alpha\in S\}\Big)^*\Big)\cdot y_S \tag{15}$$

(where we make the identification $G^*\cong(\mathbb{Z}/2)^n$).

Now consider, for $0\neq\alpha\colon G\to\mathbb{Z}/2$, the map

$$f_\alpha\colon \Phi^{G/\mathrm{Ker}(\alpha)}H\mathbb{Z}/2_*=\Phi^{G/\mathrm{Ker}(\alpha)}(H\mathbb{Z}/2)_*^{\mathrm{Ker}(\alpha)}\to\Phi^G H\mathbb{Z}/2_*. \tag{16}$$

It is fairly obvious that for $n=1$ the spectral sequence associated with the (1–dimensional) cube (7) collapses to E^1 and that in fact

$$\Phi^{G/\mathrm{Ker}\,\alpha}H\mathbb{Z}/2_*=\mathbb{Z}/2[x_\alpha], \tag{17}$$

where in the spectral sequence, the element x_α is filtration degree 1 and is represented by the set $\{(1)\}$ if we make the identification $G/\mathrm{Ker}(\alpha)\cong\mathbb{Z}/2$. We will also denote the image under (16)

$$f_\alpha(x_\alpha)\in\Phi^G H\mathbb{Z}/2$$

by x_α.

Theorem 2 *We have*

$$\Phi^G_* H\mathbb{Z}/2 = \mathbb{Z}/2[x_\alpha \mid \alpha \in G^* \smallsetminus \{0\}]\,/(x_\alpha x_\beta + x_\alpha x_\gamma + x_\beta x_\gamma \mid \alpha+\beta+\gamma = 0), \tag{18}$$

where the classes x_α *are in dimension* 1.

Before proving the theorem, it is useful to record the following algebraic fact:

Proposition 3 *Let* $\{\alpha_1, \dots, \alpha_k\}$ *be a minimal* $\mathbb{Z}/2$*–linearly dependent subset of* $G^* \smallsetminus \{0\}$*, where* $k \geq 3$*. Then the ring* R_n *on the right-hand side of* (18) *satisfies*

$$\sigma_{k-1}(x_{\alpha_1}, \dots, x_{\alpha_k}) = 0. \tag{19}$$

(Here σ_i *denotes the* i^{th} *elementary symmetric polynomial.)*

Proof We will proceed by induction on k. For $k = 3$, this is by definition. Suppose $k > 3$ and suppose the statement is true with k replaced by $k-1$. Compute in R_n, where we denote $\beta = \alpha_{k-1} + \alpha_k$:

$$\begin{aligned}\sigma_{k-1}(x_{\alpha_1},\dots,x_{\alpha_k}) &= (x_{\alpha_k}+x_{\alpha_{k-1}})(x_{\alpha_1}\cdots x_{\alpha_{k-2}}) + x_{\alpha_k}x_{\alpha_{k-1}}\sigma_{k-3}(x_{\alpha_1},\dots,x_{\alpha_{k-2}}) \\ &= (x_{\alpha_k}+x_{\alpha_{k-1}})(x_{\alpha_1}\cdots x_{\alpha_{k-2}}) + (x_{\alpha_k}+x_{\alpha_{k-1}})x_\beta\sigma_{k-3}(x_{\alpha_1},\dots,x_{\alpha_{k-2}}) \\ &= (x_{\alpha_k}+x_{\alpha_{k-1}})\sigma_{k-2}(x_\beta, x_{\alpha_1},\dots,x_{\alpha_{k-2}}).\end{aligned} \tag{20}$$

Now $\{\beta, \alpha_1, \dots, \alpha_{k-2}\}$ is also a minimal linearly dependent set (note that minimality is equivalent to the statement that $\alpha_1, \dots, \alpha_{k-1}$ are linearly independent and $\alpha_1 + \cdots + \alpha_k = 0$). Therefore, the right-hand side of (20) is 0 in R_n by the induction hypothesis. □

Proof of Theorem 2 We know that $\Phi^G_* H\mathbb{Z}/2$ is a ring, since $\Phi^G H\mathbb{Z}/2$ is an E_∞–ring spectrum. By (16), we know that the elements x_α represent elements of $\Phi^G_* H\mathbb{Z}/2$, and hence polynomials in the elements x_α do as well. Now it is important to note that (15) is not a spectral sequence of rings. However, there are maps arising from smashing n cubes (7) (over $H\mathbb{Z}/2$) for $n = 1$, and from this, it is not difficult to deduce that for S linearly independent, a monomial of the form

$$\prod_{s\in S} x_s^{r_s} \quad \text{where } r_s \geq 1 \tag{21}$$

is represented in (15) by

$$S \cdot \prod_{s\in S} x_s^{r_s - 1}. \tag{22}$$

(Note that by Lemma 1, for S not linearly independent, (22) does not survive to E^2.)

By Lemma 1, we know that such elements generate the E^2–term as a $\mathbb{Z}/2$–module, so we have already proved that the spectral sequence associated with the cube (7) collapses to E^2.

Now counting basis elements in filtration degree 2 shows that $\Phi^G_* H\mathbb{Z}/2$ must have a quadratic relation among the elements x_α, x_β, x_γ when

$$\alpha + \beta + \gamma = 0.$$

(It suffices to consider $n = 2$.) The relation must be symmetric and homogeneous for reasons of dimensions, so the possible candidates for the relation are

$$x_\alpha x_\beta + x_\alpha x_\gamma + x_\beta x_\gamma = 0 \tag{23}$$

or

$$x_\alpha x_\beta + x_\alpha x_\gamma + x_\beta x_\gamma + x_\alpha^2 + x_\beta^2 + x_\gamma^2 = 0. \tag{24}$$

We will prove the theorem by finding a basis of the monomials (21) of the ring on the right-hand side of (18) and matching them, in the form (22), with the E^2–term (15).

Before determining which of the relations (23), (24) is correct, we observe (by induction) that the ring R_n given by relation (23) satisfies (with the identification $G^* \cong (\mathbb{Z}/2)^n$)

$$R_n = R_{n-1} \otimes \mathbb{Z}/2[x_{(0,\dots,0,1)}] + \sum_{\alpha \in ((\mathbb{Z}/2)^{n-1} \smallsetminus \{0\}) \times \{1\}} R_{n-1} \otimes x_\alpha \cdot \mathbb{Z}/2[x_\alpha] \tag{25}$$

and that the ring R'_n obtained from the relations (24) satisfies a completely analogous statement with R_i replaced by R'_i. By the identification between (21) and (22), we see that we obtain a $\mathbb{Z}/2$–module of the same rank as the E^2–term of the spectral sequence of (7) in each dimension if and only if the sum (25) for each n is a direct sum (and similarly for the case of R'_n). Since we already know that the spectral sequence collapses to E_2, we know that this direct sum must occur for whichever relation (23) or (24) is correct, and also that the "winning" relation (23) (resp. (24)), ranging over all applicable choices of α, β and γ generates all the relations in $\Phi^G_* H\mathbb{Z}/2$.

We will complete the proof by showing that (24) generates a spurious relation, and hence is eliminated. This cannot be done for $n = 2$, as we actually have $R_2 \cong R'_2$ via the (nonfunctorial isomorphism) replacing the generators x_α, x_β and x_γ with $x_\alpha + x_\beta$, $x_\alpha + x_\gamma$ and $x_\beta + x_\gamma$.

We therefore must resort to $n = 3$. Let $\alpha_1 = (1,0,0)$, $\alpha_2 = (0,1,0)$, $\alpha_3 = (0,0,1)$, $\alpha_4 = (1,1,1)$. Applying the computation (20) in the proof of Proposition 3 to compute $\sigma_3(x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}, x_{\alpha_4})$ in the ring R'_3, we obtain

$$\sigma_3(x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}, x_{\alpha_4}) = (x_{\alpha_1} + x_{\alpha_2})(x_{\alpha_3}^2 + x_{\alpha_4}^2 + x_\beta^2) + (x_{\alpha_3} + x_{\alpha_4})(x_{\alpha_1}^2 + x_{\alpha_2}^2 + x_\beta^2).$$

As this is clearly not symmetrical in x_{α_1}, x_{α_2}, x_{α_3}, x_{α_4}, by permuting (say, using a 4–cycle) and adding both relations, we obtain a spurious relation in dimension 3 and filtration degree 2, which shows that the analogue of (25) with R_i replaced by R_i' fails to be a direct sum for $n = 3$, thereby excluding the relation (24), and completing the proof. □

From the fact that (25) is a direct sum, we obtain the following:

Corollary 4 *The Poincaré series of the ring* R_n *is*

$$\frac{1}{(1-x)^n}\prod_{i=1}^{n}(1+(2^{i-1}-1)x). \qquad \square$$

4 The coefficients of $H\mathbb{Z}/2$ suspended by a G–representation

In this section, we will compute explicitly the coefficients of $H\mathbb{Z}/2$ suspended by

$$V = \sum_{\alpha\in G^*\smallsetminus\{0\}} m_\alpha\alpha \tag{26}$$

with $m_\alpha \geq 0$.

Theorem 5 (i) *For* $m_\alpha \geq 0$ *and* $G^* \cong (\mathbb{Z}/2)^n$, *recalling* (26), *the Poincaré series of*

$$\Sigma^V H\mathbb{Z}/2_*$$

is

$$\frac{1}{(1-x)^n}\Bigg(\sum_{(\mathbb{Z}/2)^k\cong H\subseteq G^*}(-1)^k\Bigg(\prod_{i=1}^{n-k}(1+(2^{i-1}-1)x)\Bigg)x^l\Bigg) \tag{27}$$

where

$$l = k + \sum_{\alpha\in H\smallsetminus\{0\}} m_\alpha.$$

(ii) *For* $m_\alpha \geq 0$, *the canonical map*

$$\Sigma^V H\mathbb{Z}/2 \to \widetilde{E\mathcal{F}[G]} \wedge H\mathbb{Z}/2$$

(given by the smash product of the inclusions $S^{m_\alpha\alpha} \to S^{\infty\alpha}$*) induces an injective map on* $\mathbb{Z}$*–graded homotopy groups.*

We need the following purely combinatorial result. Let

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(2^n-1)\cdot(2^{n-1}-1)\cdots(2^{n-k+1}-1)}{(2^k-1)\cdot(2^{k-1}-1)\cdots(2^1-1)}.$$

Note that this is the number of k–dimensional $\mathbb{Z}/2$–vector subspaces of $(\mathbb{Z}/2)^n$. The following statement amounts to part (i) of Theorem 5 for $m_\alpha = 0$.

Lemma 6 *We have*

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}x^k\prod_{i=1}^{n-k}(1+(2^{i-1}-1)x) = (1-x)^n.$$

Proof Induction on n. We have

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + 2^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix},$$

so by the induction hypothesis,

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}x^k\prod_{i=1}^{n-k}(1+(2^{i-1}-1)x)$$
$$= \sum_{k=0}^{n}(-1)^k\left(\begin{bmatrix} n-1 \\ k \end{bmatrix} + 2^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\right)x^k\prod_{i=1}^{n-k}(1+(2^{i-1}-1)x).$$

Splitting the right-hand side into two sums, we get

$$\sum_{k=0}^{n-1}(-1)^k\begin{bmatrix} n-1 \\ k \end{bmatrix}x^k\prod_{i=1}^{n-k}(1+(2^{i-1}-1)x)$$
$$+\sum_{k=1}^{n}(-1)^k 2^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}x^k\prod_{i=1}^{n-k}(1+(2^{i-1}-1)x)$$
$$= (1-x)^n + \sum_{k=0}^{n-1}(-1)^k\begin{bmatrix} n-1 \\ k \end{bmatrix}x^k\prod_{i=1}^{n-k-1}(1+(2^{i-1}-1)x)2^{n-k-1}$$
$$+\sum_{k=1}^{n}(-1)^k 2^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}x^k\prod_{i=1}^{n-k}(1+(2^{i-1}-1)x)$$
$$= (1-x)^n. \qquad \square$$

Proof of Theorem 5 We will proceed by induction on n. Assume (i) and (ii) are true for lower values of n. Then, for the given n, we proceed by induction on

$$\ell = |\{\alpha \in G^* \mid m_\alpha > 0\}|.$$

For $\ell = 0$, (i) follows from Lemma 6 and (ii) is obvious (by ring structure of $\Phi^G H\mathbb{Z}/2$). Suppose $\ell \geq 1$ and (i), (ii) are true for lower values of ℓ. Setting

$$V_\ell = \sum_{i=1}^{\ell} m_{\alpha_i} \alpha_i, \tag{28}$$

we will study the effect on coefficients $(?)_*$ of the cofibration sequence

$$S(m_\ell \alpha_\ell)_+ \wedge \Sigma^{V_{\ell-1}} H\mathbb{Z}/2 \longrightarrow \Sigma^{V_{\ell-1}} H\mathbb{Z}/2 \longrightarrow \Sigma^{V_\ell} H\mathbb{Z}/2. \tag{29}$$

First, we observed that the first map factors through the top row of the diagram:

$$\begin{array}{ccc} (EG/\operatorname{Ker}\alpha_\ell)_+ \wedge \Sigma^{V_{\ell-1}} H\mathbb{Z}/2 & \longrightarrow & \Sigma^{V_{\ell-1}} H\mathbb{Z}/2 \\ \downarrow & & \downarrow \\ (EG/\operatorname{Ker}\alpha_\ell)_+ \wedge \widetilde{E\mathcal{F}[G]} \wedge H\mathbb{Z}/2 & \longrightarrow & \widetilde{E\mathcal{F}[G]} \wedge H\mathbb{Z}/2 \end{array} \tag{30}$$

Next, the right column of (30) is injective on $(?)_*$ by (ii) for $\ell - 1$, and hence the top row, and hence also the first map (29), is 0 on $(?)_*$.

Now the Poincaré series of

$$(S(m_\ell \alpha_\ell)_+ \wedge \Sigma^{V_{\ell-1}} H\mathbb{Z}/2)^G_* \tag{31}$$

is

$$\frac{1-x^{m_\ell}}{1-x}$$

times the Poincaré series of

$$(\Sigma^{V_{\ell-1}} H\mathbb{Z}/2)^{\operatorname{Ker}\alpha_\ell}_*, \tag{32}$$

which, when multiplied by x and added to the Poincaré series of

$$(\Sigma^{V_{\ell-1}} H\mathbb{Z}/2)^G_*,$$

is (27) by the induction hypothesis. This proves (i).

To prove (ii), we observe that the elements of (31) are generated by powers of x_{α_ℓ} multiplied by elements of (32), so again, we are done by the induction hypothesis. □

5 The complex calculating RO(G)–graded coefficients

To calculate the RO(G)–graded coefficients of $H\mathbb{Z}/2_G$ in dimensions given by virtual representations, we introduce another spectral sequence. In fact, we will again use the cofibration sequence (6), but we will rewrite it as

$$S^0 \to S^{\infty\alpha} \to \Sigma EG/\operatorname{Ker}(\alpha)_+. \tag{33}$$

We will smash the second maps of (33) over all $\alpha \in G^* \smallsetminus \{0\}$, to obtain a cube

$$\bigwedge_{\alpha \in G^* \smallsetminus \{0\}} (S^{\infty\alpha} \to \Sigma EG/\mathrm{Ker}(\alpha)_+) \tag{34}$$

whose iterated fiber is S^0. Our method is to smash with $H\mathbb{Z}/2_G$ and take $\mathrm{RO}(G)$–graded coefficients to obtain

$$\Big(\bigwedge_{\alpha \in G^* \smallsetminus \{0\}} (S^{\infty\alpha} \to \Sigma EG/\mathrm{Ker}(\alpha)_+) \wedge H\mathbb{Z}/2\Big)_\star, \tag{35}$$

thus yielding a spectral sequence calculating $H\mathbb{Z}/2_\star$.

However, there is a key point to notice which drastically simplifies this calculation. Namely, smashing (6) with $EG/\mathrm{Ker}(\alpha)_+$, the first morphism becomes an equivalence, thus showing that

$$EG/\mathrm{Ker}(\alpha)_+ \wedge S^{\infty\alpha} \simeq *. \tag{36}$$

Together with (8), this shows that the only vertices of the cube (34) which are nonzero are actually those of the form where all the elements α for which we take the term $S^{\infty\alpha}$ in (34) are those *not vanishing* on some subgroup $A \subseteq G$, while those elements α for which we take the term $\Sigma EG/\mathrm{Ker}(\alpha)_+$ are those nonzero elements of G^* which *do vanish* on A, ie nonzero elements of $(G/A)^*$. The corresponding vertex of (34) is then a suspension of

$$\mathrm{gr}_A(S^0) = EG/A_+ \wedge \widetilde{E\mathcal{F}[A]}. \tag{37}$$

We also put

$$\mathrm{gr}_A(H\mathbb{Z}/2) = \mathrm{gr}_A(S^0) \wedge H\mathbb{Z}/2.$$

Because of the high number of zero terms, the spectral sequence may be regraded by $\mathrm{rank}_{\mathbb{Z}/2}(A)$, thus having only n, instead of $2^n - 1$, filtration degrees. (Note that the cube (34) may be reinterpreted as a "filtration" of the spectrum S^0; from this point of view, we have simply observed that many of the filtered parts coincide.)

It is now important, however, to discuss the grading seriously. Since we index coefficients homologically, we will write the spectral sequence in homological indexing. Additionally, we want the term $\mathrm{gr}_G(S^0)$ be in filtration degree 0 (since that is where the unit is). Thus, the (homologically indexed) filtration degree of (37) is

$$p = \mathrm{rank}(A) - n$$

(a nonpositive number). Thus,

$$\pi_k\Big(\sum^{\sum_{\alpha \in G^* \smallsetminus \{0\}} m_\alpha \alpha} \mathrm{gr}_A(H\mathbb{Z}/2)\Big) \subseteq E^1_{\mathrm{rank}(A)-n,\,k+n-\mathrm{rank}(A)},$$

or, put differently, for a given choice of the elements m_α,

$$E^1_{p,q} = \bigoplus_{\operatorname{rank}(A)=n+p} \pi_{q+p-\sum m_\alpha \alpha}\, \mathrm{gr}_A(H\mathbb{Z}/2) \quad \text{for } p=-n,\dots,0. \tag{38}$$

We will next describe explicitly the differential

$$d^1\colon E^1_{p,q} \to E^1_{p-1,q}. \tag{39}$$

Let us first introduce some notation. To this end, we need to start out by describing the E^1–term more explicitly.

In effect, we can calculate $\mathrm{gr}_A(H\mathbb{Z}/2)_\star$ by taking first the A–fixed points using Theorem 2 with G replaced by A, and then applying the Borel homology spectral sequence for G/A. This spectral sequence collapses because there exists a splitting:

$$\begin{array}{ccc} A & \xrightarrow{\subseteq} & G \\ & \searrow^{=} & \downarrow r \\ & & A \end{array} \tag{40}$$

However, the splitting is not canonical, and this is reflected by the choice of generators we observe. More explicitly, the splitting determines for each representation

$$0 \neq \beta\colon A \to \mathbb{Z}/2$$

an extension

$$\widetilde{\beta}\colon G \to \mathbb{Z}/2.$$

One difficulty with describing Borel homology is that it does not naturally form a ring. Because of that, it is more convenient to describe first the coefficients of

$$\gamma_A(H\mathbb{Z}/2) := F(EG/A_+, \widetilde{E\mathcal{F}[A]}) \wedge H\mathbb{Z}/2. \tag{41}$$

This is an (E_∞–) ring spectrum, and its ring of coefficients is given by

$$\gamma_A(H\mathbb{Z}/2)_\star = M[(y_\alpha u_\alpha^{-1})^{\pm 1}][\![y_\alpha \mid \alpha \in (G/A)^* \smallsetminus \{0\}]\!]/(y_{\alpha+\alpha'} - y_\alpha - y_{\alpha'}) \tag{42}$$

where

$$M = \mathbb{Z}/2[x_{\widetilde{\beta}}, u_{\widetilde{\beta}}^{\pm 1}, u_{\widetilde{\beta}+\alpha}^{\pm 1} \mid \beta \in A^* \smallsetminus \{0\}, \alpha \in (G/A)^* \smallsetminus \{0\}] \\ /(x_{\widetilde{\alpha}} x_{\widetilde{\beta}} + x_{\widetilde{\alpha}} x_{\widetilde{\gamma}} + x_{\widetilde{\beta}} x_{\widetilde{\gamma}} \mid \alpha+\beta+\gamma = 0)$$

and the RO(G)–graded dimensions of the generators are

$$|u_\gamma| = -\gamma, \quad |x_\gamma| = 1 \quad \text{and} \quad |y_\gamma| = -1.$$

We may then describe $\mathrm{gr}_A(H\mathbb{Z}/2)_\star$ as the $(\dim_{\mathbb{Z}/2}(G/A))^{\text{th}}$ (= only nontrivial) local cohomology module of the ring $\gamma_A(H\mathbb{Z}/2)$ with respect to the ideal generated by the elements y_α. Note that after taking A–fixed points first, this is the usual computation of G/A–Borel homology from the corresponding Borel cohomology. Recall that $H^*_I(R)$ for a finitely generated ideal I of a commutative ring R is obtained by choosing finitely many generators $y_1, \dots, y_\ell$ of I, tensoring, over R, the cochain complexes

$$R \to y_i^{-1} R$$

(with R in degree 0) and taking cohomology. It is, canonically, independent of the choice of generators. In the present case, we are simply dealing with the power series ring R in $\dim_{\mathbb{Z}/2}(G/A)$ generators over a $\mathbb{Z}/2$–algebra, and the augmentation ideal. Taking the defining generators of the power series ring, we see immediately that only the top local cohomology group survives.

We note that the basic philosophy of our notation is

$$\text{“}y_\alpha = x_\alpha^{-1}\text{”}. \tag{43}$$

As a first demonstration of this philosophy, let us investigate the effect of a change of the splitting (40). Writing metaphorically

$$x_{\widetilde{\beta+\alpha}} x_{\widetilde{\beta}} + x_{\widetilde{\beta+\alpha}} x_\alpha + x_{\widetilde{\beta}} x_\alpha = 0, \tag{44}$$

we get

$$x_{\widetilde{\beta+\alpha}} x_{\widetilde{\beta}} y_\alpha + x_{\widetilde{\beta+\alpha}} + x_{\widetilde{\beta}} = 0, \tag{45}$$

from which we calculate

$$x_{\widetilde{\beta+\alpha}} = x_{\widetilde{\beta}}(1 + x_{\widetilde{\beta}} y_\alpha)^{-1} = \sum_{k=0}^{\infty} x_{\widetilde{\beta}}^{k+1} y_\alpha^k. \tag{46}$$

This formula is correct in $\gamma_A(H\mathbb{Z}/2)_\star$ and hence can also be used in the module $\mathrm{gr}_A(H\mathbb{Z}/2)_\star$.

Next, we will describe the differential d^1 of (38). These connecting maps will be the sums of maps of degree -1 of the form

$$d^{AB}\colon \mathrm{gr}_A(H\mathbb{Z}/2)_\star \to \mathrm{gr}_B(H\mathbb{Z}/2)_\star, \tag{47}$$

where $B \subset A$ is a subgroup with quotient isomorphic to $\mathbb{Z}/2$. Let $\beta\colon A \to \mathbb{Z}/2$ be the unique nontrivial representation which vanishes when restricted to A. The key point is to observe that the canonical map

$$EG/A_+ \wedge \widetilde{E\mathcal{F}[B]} \wedge S^{\infty\widetilde{\beta}} \xrightarrow{\simeq} EG/A_+ \wedge \widetilde{E\mathcal{F}[A]} \tag{48}$$

is an equivalence, and hence (47) can be calculated by smashing with $H\mathbb{Z}/2$ the connecting map

$$EG/A_+ \wedge \widetilde{E\mathcal{F}[B]} \wedge S^{\infty\widetilde{\beta}} \to \Sigma EG/B_+ \wedge \widetilde{E\mathcal{F}[B]}. \tag{49}$$

Consequently, (47) is a homomorphism of $\gamma_A H\mathbb{Z}/2_\star$–modules, and is computed, just like in dimension 1, by making the replacement

$$x_{\widetilde{\beta}} \mapsto y_{\widetilde{\beta}}^{-1}$$

and multiplying by $y_{\widetilde{\beta}}$. (Note that independence of the splitting $\widetilde{\beta}$ at this point follows from topology; it is a nontrivial fact to verify purely algebraically.)

We have thereby finished describing the differential d^1 of the spectral sequence (38). The main result of the present section is the following:

Theorem 7 *The spectral sequence* (38) *collapses to* E^2.

We will first prove some auxiliary results.

Lemma 8 *The Borel homology spectral sequence of any cell* $H\mathbb{Z}/2_G$*–module with cells*

$$\Sigma^? G_+ \wedge H\mathbb{Z}/2 \tag{50}$$

collapses to E^2.

Proof Taking G–fixed points, we obtain a cell $H\mathbb{Z}/2$–module with one cell for each cell (50). Now the homotopy category of $H\mathbb{Z}/2$–modules is equivalent to the derived category of $\mathbb{F}_2$–vector spaces, and a chain complex of $\mathbb{F}_2$–modules is isomorphic to a sum of an acyclic module and suspensions of $\mathbb{F}_2$. □

Lemma 9 *Let* G *and* H *be finite groups, let* X *be an* G*–cell spectrum, and let* Y *be an* H*–cell spectrum (all indexed over the complete universe). Then*

$$(H\mathbb{Z}/2_{G\times H} \wedge i_\sharp X \wedge j_\sharp Y)^{G\times H} \simeq (H\mathbb{Z}/2_G \wedge X)^G \wedge_{H\mathbb{Z}/2} (H\mathbb{Z}/2_H \wedge Y)^H.$$

Here on the left-hand side, $i_\sharp$ *is the functor introducing trivial* H*–action on a* G*–spectrum and pushing forward to the complete universe, while* $j_\sharp$ *is the functor introducing trivial* G*–action on an* H*–spectrum and pushing forward to the complete universe.*

Proof First consider $Y = S^0$. Then we have the forgetful map

$$(i_\sharp X \wedge H\mathbb{Z}/2)^{G\times H} \to (X \wedge H\mathbb{Z}/2)^G$$

which is an equivalence because it is true on cells.

In general, we have a map

$$Z^\Gamma \wedge T^\Gamma \to (Z \wedge T)^\Gamma,$$

so take the composition

$$\begin{aligned}(X \wedge H\mathbb{Z}/2)^G \wedge (Y \wedge H\mathbb{Z}/2)^H &= (i_\sharp X \wedge H\mathbb{Z}/2)^{G\times H} \wedge (j_\sharp Y \wedge H\mathbb{Z}/2)^{G\times H}\\ &\to (i_\sharp H\mathbb{Z}/2 \wedge j_\sharp Y \wedge H\mathbb{Z}/2)^{G\times H}\\ &\to (i_\sharp X \wedge j_\sharp Y)^{G\times H}\end{aligned}$$

(the last map coming from the ring structure on $H\mathbb{Z}/2$). Then again this map is an equivalence on cells, and hence an equivalence. □

Lemma 10 *Recalling again the notation* (26)*, we have:*

(a) *The spectral sequence* (35) *for*

$$\pi_* \Sigma^V H\mathbb{Z}/2 \tag{51}$$

with all $m_\alpha \geq 0$ collapses to the E^2–term in filtration degree 0.

(b) *Let $m_\alpha \leq 0$ for all α and let*

$$S = \{\alpha \in G^* \smallsetminus \{0\} \mid m_\alpha \neq 0\}.$$

Suppose the subgroup of G^* spanned by S has rank m. Then the spectral sequence (35) *for* (51) *collapses to E^2 in filtration degree $-m$.*

Proof Recall the notation (28). Let $G^* \smallsetminus \{0\} = \{\alpha_1, \dots, \alpha_{2^n-1}\}$. When α_k is linearly independent of $\alpha_1, \dots, \alpha_{k-1}$, we have

$$\pi_* \Sigma^{V_k} H\mathbb{Z}/2 \cong \pi_*(\Sigma^{V_{k-1}} H\mathbb{Z}/2)^G \otimes \pi_*(\Sigma^{m_{\alpha_k}\gamma} H\mathbb{Z}/2)^{\mathbb{Z}/2}, \tag{52}$$

where γ is the sign representation of $\mathbb{Z}/2$ by Lemma 9. Note that in the case (b), we may, without loss of generality, assume $m = n$ (ie that S spans G^*) and that what we just said occurs for $k = 1, \dots, n$ and additionally that $m_{\alpha_i} < 0$ for $i = 1, \dots, n$.

When α_k is a linear combination of $\alpha_1, \dots, \alpha_{k-1}$, and $m_{\alpha_k} \neq 0$, we use the cofibration sequence

$$S(m_{\alpha_k}\alpha_k)_+ \wedge \Sigma^{V_{k-1}} H\mathbb{Z}/2 \to \Sigma^{V_{k-1}} H\mathbb{Z}/2 \to \Sigma^{V_k} H\mathbb{Z}/2 \tag{53}$$

in the case (a) and

$$\Sigma^{V_k} H\mathbb{Z}/2 \to \Sigma^{V_{k-1}} H\mathbb{Z}/2 \to DS(-m_{\alpha_k}\alpha_k)_+ \wedge \Sigma^{V_{k-1}} H\mathbb{Z}/2 \tag{54}$$

in the case (b). If we denote each of these cofibration sequences symbolically as

$$A \to B \to C,$$

then in the case (a), (53) gives a short exact sequence of the form

$$0 \to E^1 A \to E^1 B \to E^1 C \to 0 \tag{55}$$

of the spectral sequence of (35) where in the A–term, we replace G by $\mathrm{Ker}(\alpha_k)$ and $H\mathbb{Z}/2$ by $S(m_k\alpha_k)_+ \wedge H\mathbb{Z}/2$. By the induction hypothesis, however, the homology of $E^1 A$ is concentrated in the top filtration degree, which is -1 from the point of view of G, and the homology of $E^1 B$ is concentrated in filtration degree 0, so the long exact sequence in homology gives

$$0 \to E^2 \to E^2 C \to \Sigma E^2 A \to 0, \tag{56}$$

which is all in filtration degree 0, so our statement follows.

In the case (b), by our assumptions, we have $k > n$. Additionally, (54) gives a short exact sequence

$$0 \to \Sigma^{-1} E^1 C \to E^1 A \to E^1 B \to 0, \tag{57}$$

but by the induction hypothesis (using the fact that a set of generators of G^* projects to a set of generators of the factor group $\mathrm{Ker}(\alpha_k)^*$), the homology of the first and last term is concentrated in filtration degree $-n$, so (57) translates to the same short exact sequence with E^1 replaced by E^2, which is entirely in filtration degree $-n$, and the statement follows. □

To continue the proof of Theorem 7, let again

$$G^* \smallsetminus \{0\} = \{\alpha_1, \dots, \alpha_{2^n-1}\}.$$

Consider

$$\Sigma^{V_{2^n-1}} H\mathbb{Z}/2, \tag{58}$$

and let, this time, without loss of generality,

$$m_{\alpha_1}, \dots, m_{\alpha_q} < 0 \quad \text{and} \quad m_{\alpha_{q+1}}, \dots, m_{\alpha_{2^n-1}} \geq 0.$$

Let $A = \mathrm{Ker}(\alpha_1) \cap \cdots \cap \mathrm{Ker}(\alpha_q)$. We will consider the sequence of cofibrations (53) with $q \leq k < 2^n - 1$. Resolving this recursively, we may consider this as a cell

object construction in the category of $H\mathbb{Z}/2_G$–modules, with "cells" of the form of suspensions (by an integer) of

(59) $G/(\mathrm{Ker}(\alpha_{j_1})\cap\cdots\cap\mathrm{Ker}(\alpha_{j_p}))_+ \wedge \Sigma^{V_q} H\mathbb{Z}/2 \quad \text{where } q<j_1<\cdots<j_p\leq 2^n-1.$

By the *degree* of a cell c, we shall mean the number

$$\deg(c) = n - \mathrm{rank}(\mathrm{Ker}(\alpha_{j_1}) \cap \cdots \cap \mathrm{Ker}(\alpha_{j_p})),$$

and by the *A–relative degree* of c, we shall mean

$$\deg_A(c) = \mathrm{rank}(G/A) - \mathrm{rank}\big(\mathrm{Ker}(\alpha_{j_1})\cap\cdots\cap\mathrm{Ker}(\alpha_{j_p}) \,/\, \mathrm{Ker}(\alpha_{j_1})\cap\cdots\cap\mathrm{Ker}(\alpha_{j_p})\cap A\big).$$

We see easily from the construction that cells of a given degree are attached to cells of strictly lower degree, and that cells of a given A–relative degree are attached to cells of lesser or equal A–relative degree. (Roughly speaking, "more free" cells are attached to "less free" ones.)

Lemma 11 *The spectral sequence arising from the cube* (35) *with $H\mathbb{Z}/2$ replaced by the complex formed by our "cells" of A–relative degree d collapses to E^2 concentrated in filtration degree $d-\mathrm{rank}(G/A)$.*

Proof Within a given A–relative degree d, attaching cells of each consecutive degree results in a short exact sequence of the form (55) where the first two terms collapse to E^2 in filtration degree $d-\mathrm{rank}(G/A)-1$ and $d-\mathrm{rank}(G/A)$, respectively. Thus, there results a short exact sequence of the form (56) in filtration degree $d-\mathrm{rank}(G/A)$, as claimed. □

(The rest of) the proof of Theorem 7 Filtering cells of (58) by A–relative degree, we obtain a spectral sequence $\mathcal{E}$ converging to E^2 of the spectral sequence of the cube (35) for (58). By Lemma 11, all the terms will be of the same (35)-filtration degree $-\mathrm{rank}(G/A)$, which is the complementary degree of $\mathcal{E}$. (Note that in this discussion, we completely ignore the original topological degree.) Thus, being concentrated in one complementary degree, $\mathcal{E}$ collapses to E^2 in that complementary degree.

However, by precisely the same arguments, we can write a variant $\widetilde{\mathcal{E}}$ of the spectral sequence $\mathcal{E}$ in homotopy groups (rather than (35) E^1–terms) of the filtered pieces of (58) by A–relative degree. By Lemma 11, $\widetilde{\mathcal{E}}^1 \cong \mathcal{E}^1$, and $d^1_{\widetilde{\mathcal{E}}}$, $d^1_{\mathcal{E}}$ have the same rank (since they are computed by the same formula). It follows that $\widetilde{\mathcal{E}}^2 \cong \mathcal{E}^2$, both collapsing to a single complementary degree. Therefore, it follows that E^2 (of the spectral sequence associated with (35) for (58)) is isomorphic to the homotopy of (58), and hence the spectral sequence collapses to E^2 by a counting argument. □

6 Example: $n = 2$

In the case $n = 2$, there are only three sign representations α, β, γ which play a symmetrical role and satisfy

$$\alpha + \beta + \gamma = 0 \in G^*, \tag{60}$$

which means that the Poincaré series of the homotopy

$$\pi_*(\Sigma^{k\alpha+\ell\beta+m\gamma} H\mathbb{Z}/2) \tag{61}$$

can be written down explicitly.

First recall that by Theorem 5, for $k, \ell, m \geq 0$, the Poincaré series is

$$\frac{1}{(1-x)^2}(1 + x - x^{1+k} - x^{1+\ell} - x^{1+m} + x^{2+k+\ell+m}). \tag{62}$$

If $k, \ell < 0$ and $m \leq 0$, by the proof of Lemma 10, the formula (62) is still valid when multiplied by x^{-2} (since all the homotopy classes are in filtration degree -2).

If $k, \ell < 0$ and $m > 0$, in the proof of Theorem 7, $A = 0$, so the A–relative degree and the degree coincide. Further, by (60) and our formula for the differential d^1 of the spectral sequence of (35), the differential $d^1_{\mathcal{E}}$ has maximal possible rank (ie "everything that can cancel dimensionwise will"). We conclude that the E^2 is concentrated in filtration degrees -1 and -2. By the cancellation principle we just mentioned, the Poincaré series can still be recovered from the formula (62). If we write the expression (62) as

$$P_+(x) - P_-(x), \tag{63}$$

where $P_+(x)$ (resp. $-P_-(x)$) is the sum of monomial summands with a positive coefficient (resp. with a negative coefficient) then the correct Poincaré series in this case is

$$x^{-2}P_+(x) + x^{-1}P_-(x),$$

the two summands of which represent classes in filtration degree -2 and -1, respectively.

Similarly, one shows that if $k, \ell \geq 0$ and $m < 0$, the E^2 collapses to filtration degrees 0 and -1, and the Poincaré series in this case is

$$P_+(x) + x^{-1}P_-(x).$$

All other cases are related to these by a symmetry of $(\mathbb{Z}/2)^2$.

Remark It might seem natural to conjecture that the classes of different filtration degrees in E^2 may be of different dimensions, with a gap between them (evoking the

"gap condition" which was proved for $\mathbb{Z}/2$ in [13], and made famous for the group $\mathbb{Z}/8$ by the Hill, Hopkins and Ravenel [12] work on the Kervaire invariant 1 problem). However, one easily sees that for $n \geq 3$, classes of different filtration degrees may occur in the same dimension. For example, by Lemma 9 and by what we just proved, such a situation always occurs for $\pi_* \Sigma^{4\alpha+4\beta-2\gamma+4\delta} H\mathbb{Z}/2$ where α, β, γ are the three sign representations of $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ factoring through the projections to the first two copies of $\mathbb{Z}/2$, and δ is the sign representation which factors through the projection onto the last $\mathbb{Z}/2$.

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