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Odd primary homotopy types of SU(n)–gauge groups

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Let $\mathcal{G}_k(\mathrm{SU}(n))$ be the gauge group of the principal $\mathrm{SU}(n)$–bundle with second Chern class k. If p is an odd prime and $n \le (p-1)^2+1$, we classify the p–local homotopy types of $\mathcal{G}_k(\mathrm{SU}(n))$.

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1 Introduction

Let G be a topological group, B be a space and $P \to B$ be a principal G–bundle over B. The *gauge group* $\mathcal{G}(P)$ is the group of G–equivariant automorphisms of P that fix B. Crabb and Sutherland [2] showed that, even though there may be infinitely many inequivalent principal G–bundles over B, their gauge groups have only finitely many distinct homotopy types. However, their argument did not give an explicit enumeration of the homotopy types. Using different methods, Kono [14] gave an explicit enumeration in the case of gauge groups of principal $\mathrm{SU}(2)$–bundles over S^4. He then asked whether this can be done more generally.

Since then there has been considerable effort to classify the homotopy types of gauge groups in specific cases. Let G be a simply connected, simple compact Lie group and let BG be its classifying space. The number of equivalence classes of principal G–bundles over S^4 is in one-to-one correspondence with homotopy classes of maps $[S^4, BG] \cong \mathbb{Z}$, and the correspondence in the case of $G = \mathrm{SU}(n)$ is given by the value of the second Chern class. Let $P_k \to S^4$ be the principal G–bundle corresponding to $k \in \mathbb{Z}$ and let $\mathcal{G}_k(G)$ be its gauge group. For integers a and b, let (a,b) be their greatest common divisor. Then:

- $\mathcal{G}_k(\mathrm{SU}(2)) \simeq \mathcal{G}_{k'}(\mathrm{SU}(2))$ if and only if $(12,k) = (12,k')$ (Kono [14]);
- $\mathcal{G}_k(\mathrm{SU}(3)) \simeq \mathcal{G}_{k'}(\mathrm{SU}(3))$ if and only if $(24,k) = (24,k')$ (Hamanaka and Kono [6]);
- $\mathcal{G}_k(\mathrm{Sp}(2)) \simeq_{(p)} \mathcal{G}_{k'}(\mathrm{Sp}(2))$ if and only if $(40,k) = (40,k')$ (Theriault [21]);
- $\mathcal{G}_k(\mathrm{SU}(5)) \simeq_{(p)} \mathcal{G}_{k'}(\mathrm{SU}(5))$ if and only if $(120,k) = (120,k')$ (Theriault [23]);

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where the homotopy equivalence in the third and fourth cases is p–local for any prime p or rational (using $p=0$ to indicate rational localization). Bounds, but not a classification, were obtained in the case of $\mathcal{G}_k(G_2)$ in Kishimoto, Theriault and Tsutaya [13], and classifications involving spheres of different dimensions or non-simply connected Lie groups can be found in Hamanaka and Kono [7], Kamiyama, Kishimoto, Kono and Tsukuda [10] and Kishimoto, Kono and Tsutaya [12]. In all cases the fixed number in the greatest common divisor is the order of the Samelson product $S^3 \wedge G \xrightarrow{\langle i,1\rangle} G$, where i is the inclusion of the bottom cell and 1 is the identity map on G.

Here we consider $\mathcal{G}_k(\mathrm{SU}(n))$ for all n. There is a canonical map $j_n\colon \Sigma\mathbb{C}\mathrm{P}^{n-1} \to \mathrm{SU}(n)$ that induces a projection onto the generating set in cohomology. In what follows, while spaces will be localized at a prime p, it is more illuminating to write the order of a map as an integer m rather than the p–component of m. We prove the following:

Theorem 1.1 *Localize at an odd prime p. Then:*

(a) *If $n \geq 2$, the composite $S^3 \wedge \Sigma\mathbb{C}\mathrm{P}^{n-1} \xrightarrow{1\wedge j_n} S^3 \wedge \mathrm{SU}(n) \xrightarrow{\langle i,1\rangle} \mathrm{SU}(n)$ has order at most $n(n^2-1)$.*

(b) *If $n \leq (p-1)^2+1$, the composite $S^3 \wedge \Sigma\mathbb{C}\mathrm{P}^{n-1} \xrightarrow{1\wedge j_n} S^3 \wedge \mathrm{SU}(n) \xrightarrow{\langle i,1\rangle} \mathrm{SU}(n)$ has order exactly $n(n^2-1)$.*

(c) *If $n \leq (p-1)^2+1$, the map $S^3 \wedge \mathrm{SU}(n) \xrightarrow{\langle i,1\rangle} \mathrm{SU}(n)$ has order $n(n^2-1)$.*

(d) *If $n \leq (p-1)^2+1$, there is a homotopy equivalence $\mathcal{G}_k(\mathrm{SU}(n)) \simeq \mathcal{G}_{k'}(\mathrm{SU}(n))$ if and only if $(n(n^2-1),k) = (n(n^2-1),k')$.*

Theorem 1.1(d) significantly improves on the known classifications of the homotopy types of gauge groups. It is the first general result; all the previous results held for specific Lie groups G and involved proofs that used particular properties of that Lie group. It recovers the known odd primary information for $\mathrm{SU}(2)$, $\mathrm{SU}(3)$ and $\mathrm{SU}(5)$ and gives exact information in a large range of previously unknown cases. For example, $\mathcal{G}_k(\mathrm{SU}(4)) \simeq_{(p)} \mathcal{G}_{k'}(\mathrm{SU}(4))$ at odd primes if and only if $(60,k) = (60,k')$.

Hamanaka and Kono [6], refining a result of Sutherland [19], showed that for any $n \geq 2$ the (integral) order of $\langle i,1\rangle$ is at least $n(n^2-1)$ by considering certain homotopy sets $[X,\mathrm{SU}(n)]$ where X is a sphere or a suspension of $\mathbb{C}\mathrm{P}^2$. Theorem 1.1(a)–(b) are much stronger reformulations of their result at odd primes. We obtain part (a) by closely examining a map constructed by Toda [24] to give a topological proof of Bott periodicity. The restriction to $n \leq (p-1)^2+1$ in parts (b) and (c) arises from the fact that for these n the space $\Sigma\mathbb{C}\mathrm{P}^{n-1}$ "generates" $\mathrm{SU}(n)$ in a sense that will be made precise in Section 5. Part (d) follows as a consequence of part (c) and other general results, to be discussed in Section 6.

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2 Gauge groups and function spaces

In this section we discuss some general results that translate the study of gauge groups into that of function spaces, which is more suited to topological methods. This holds for any simply connected, simple compact Lie group G, so it is stated that way. Since $[S^4, BG] \cong \mathbb{Z}$, principal G–bundles over S^4 are classified by their second Chern class, which can take any integer value. Let $P_k \to S^4$ be the principal G–bundle corresponding to $k \in \mathbb{Z}$, and let $\mathcal{G}_k(G)$ be its gauge group. As this is a group it has a classifying space $B\mathcal{G}_k(G)$.

Let $\mathrm{Map}(S^4, BG)$ and $\mathrm{Map}^*(S^4, BG)$ respectively be the spaces of freely continuous and pointed continuous maps between S^4 and BG. The components of each space are in one-to-one correspondence with the integers, where the integer is determined by the degree of a map $S^4 \to BG$. By [3] or [1], there is a homotopy equivalence $B\mathcal{G}_k(G) \simeq \mathrm{Map}_k(S^4, BG)$ between $B\mathcal{G}_k(G)$ and the component of $\mathrm{Map}(S^4, BG)$ consisting of maps of degree k. Evaluating a map at the basepoint of S^4, we obtain a map ev: $B\mathcal{G}_k(G) \xrightarrow{\mathrm{ev}} BG$ whose fibre is homotopy equivalent to $\mathrm{Map}^*_k(S^4, BG)$. It is well known that each component of $\mathrm{Map}^*(S^4, BG)$ is homotopy equivalent to $\Omega^3_0 G$, the component of $\Omega^3 G$ containing the basepoint. Putting all this together, for each $k \in \mathbb{Z}$ there is a homotopy fibration sequence

$$G \xrightarrow{\partial_k} \Omega^3_0 G \to B\mathcal{G}_k(G) \xrightarrow{\mathrm{ev}} BG, \tag{1}$$

where ∂_k is the fibration connecting map. In particular, the gauge group $\mathcal{G}_k(G)$ is the homotopy fibre of ∂_k, and it is by understanding the map ∂_k that information will be deduced about $\mathcal{G}_k(G)$.

Note that, while the components of $\mathrm{Map}^*(S^4, BG)$ are all homotopy equivalent, the same need not be true for the components of $\mathrm{Map}(S^4, BG)$. For example, in [25; 26] it was shown that there is a homotopy equivalence

$$\mathrm{Map}_k(S^4, B\mathrm{SU}(2)) \simeq \mathrm{Map}_{k'}(S^4, B\mathrm{SU}(2))$$

if and only if $k = \pm k'$. However, many components become homotopy equivalent after looping. In the SU(2) example, Kono [14] showed that $\Omega\mathrm{Map}_k(S^4, B\mathrm{SU}(2)) \simeq \Omega\mathrm{Map}_{k'}(S^4, B\mathrm{SU}(2))$ (ie $\mathcal{G}_k(\mathrm{SU}(2)) \simeq \mathcal{G}_{k'}(\mathrm{SU}(2))$) if and only if $(12, k) = (12, k')$. This example also shows that Theorem 1.1(d) likely cannot be upgraded to a statement about the classifying spaces of the relevant gauge groups.

The triple adjoint of ∂_k was described in [15, Theorem 2.7]. More precisely, the homotopy class of a homotopy fibration connecting map is determined only up to self-homotopy equivalences of its domain and range; in [15, Theorem 2.7] it was shown that choices of self-homotopy equivalences could be made which allow for the triple adjoint of ∂_k to be described in terms of Samelson products. In fact, in [15] a four-fold adjoint is taken using the fact that $G \simeq \Omega BG$, and this four-fold adjoint is described in terms of Whitehead products. We choose for ease of presentation to use only a three-fold adjoint, which is described in terms of a Samelson product.

Let $i\colon S^3 \to G$ be the inclusion of the bottom cell and let $1\colon G \to G$ be the identity map. In general, for an H–space Y, let $k\colon Y \to Y$ be the k^{th}–power map.

Lemma 2.1 *The triple adjoint of the map $G \xrightarrow{\partial_k} \Omega_0^3 G$ is homotopic to the Samelson product $S^3 \wedge G \xrightarrow{\langle k\circ i, 1\rangle} G$.* □

The linearity of the Samelson product implies that $\langle k\circ i, 1\rangle \simeq k\circ\langle i, 1\rangle$. Taking adjoints therefore implies the following:

Corollary 2.2 *There is a homotopy $\partial_k \simeq k\circ\partial_1$.* □

In what follows we will prove results about the order of ∂_1 by proving them about the order of $\langle i, 1\rangle$.

3 Properties of Toda's map

Take cohomology with $\mathbb{Z}$–coefficients. Recall that $H^*(\mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}[x]$, where x has degree 2. Write σx^i for the image of x^i under the suspension isomorphism $H^{2i}(\mathbb{C}\mathrm{P}^\infty) \xrightarrow{\cong} H^{2i+1}(\Sigma\mathbb{C}\mathrm{P}^\infty)$. In his topological proof of Bott periodicity, Toda [24] constructed a map

$$f\colon \Sigma^3\mathbb{C}\mathrm{P}^\infty \to \Sigma\mathbb{C}\mathrm{P}^\infty$$

with the property that $f^*(\sigma x^m) = m\sigma^3 x^{m-1}$ for $m \geq 2$. Composing, we obtain a composite

$$g\colon \Sigma^5\mathbb{C}\mathrm{P}^\infty \xrightarrow{\Sigma^2 f} \Sigma^3\mathbb{C}\mathrm{P}^\infty \xrightarrow{f} \Sigma\mathbb{C}\mathrm{P}^\infty$$

with the property that $g^*(\sigma x^m) = m(m-1)\sigma^5 x^{m-2}$ for $m \geq 3$.

Let

$$g_{2n+1}\colon \Sigma^5\mathbb{C}\mathrm{P}^{n-2} \to \Sigma\mathbb{C}\mathrm{P}^n$$

be the restriction of g to $(2n+1)$–skeletons. Then skeletal inclusions give a commutative square:

$$\begin{array}{ccc} \Sigma^5\mathbb{C}\mathrm{P}^{n-3} & \xrightarrow{g_{2n-1}} & \Sigma\mathbb{C}\mathrm{P}^{n-1} \\ \downarrow & & \downarrow \\ \Sigma^5\mathbb{C}\mathrm{P}^{n-1} & \xrightarrow{g_{2n+3}} & \Sigma\mathbb{C}\mathrm{P}^{n+1} \end{array}$$

Let $X^{n+1} = \mathbb{C}\mathrm{P}^{n+1}/\mathbb{C}\mathrm{P}^{n-1}$ be the stunted complex projective space. The commutativity of the preceding square implies that there is a homotopy cofibration diagram

$$\begin{array}{ccc} \Sigma^5\mathbb{C}\mathrm{P}^{n-3} & \xrightarrow{g_{2n-1}} & \Sigma\mathbb{C}\mathrm{P}^{n-1} \\ \downarrow & & \downarrow \\ \Sigma^5\mathbb{C}\mathrm{P}^{n-1} & \xrightarrow{g_{2n+3}} & \Sigma\mathbb{C}\mathrm{P}^{n+1} \\ \downarrow & & \downarrow \\ \Sigma^5 X^{n-1} & \xrightarrow{\overline{g}_{2n+3}} & \Sigma X^{n+1} \\ \downarrow & & \downarrow \\ \Sigma^6\mathbb{C}\mathrm{P}^{n-3} & \xrightarrow{\Sigma g_{2n-1}} & \Sigma^2\mathbb{C}\mathrm{P}^{n-1} \end{array} \tag{2}$$

for some map $\overline{g}_{2n+3}$.

We describe some properties of X^{n+1} and $\overline{g}_{2n+3}$. The quotient map $\mathbb{C}\mathrm{P}^{n+1} \to X^{n+1} = \mathbb{C}\mathrm{P}^{n+1}/\mathbb{C}\mathrm{P}^{n-1}$ induces a map $H^*(X^{n+1}) \to H^*(\mathbb{C}\mathrm{P}^{n+1})$. A straightforward long exact sequence argument immediately shows that $H^*(X^{n+1}) \cong \mathbb{Z}\{y_n, y_{n+1}\}$, where y_n and y_{n+1} are the images of x^n and x^{n+1}, respectively. The homotopy commutativity of the middle square in (2) and the description of g^* immediately imply the following:

Lemma 3.1 *We have:*

(a) $(\overline{g}_{2n+3})^*(\sigma y_n) = n(n-1)\sigma^5 y_{n-2}$.

(b) $(\overline{g}_{2n+3})^*(\sigma y_{n+1}) = (n+1)n\,\sigma^5 y_{n-1}$. □

It is useful to identify the homotopy type of X^{n+1}. Observe that X^{n+1} has a CW–structure consisting of two cells, one each in dimensions $2n$ and $2n+2$. The structure of the Steenrod algebra on $\mathbb{C}\mathrm{P}^{n+1}$ (see [27, Chapter VIII, Theorem 9.2], for example) implies that there is a Sq^2 connecting the two generators in $H^*(X^{n+1};\mathbb{Z}/2\mathbb{Z})$ if and only if n is odd. Also, Sq^2 detects the stable generator η_m of $\pi_{m+1}(S^m) \cong \mathbb{Z}/2\mathbb{Z}$ (see [27, Chapter VIII, Corollary 8.8], for example). As the cofibre of η_m is $\Sigma^{m-2}\mathbb{C}\mathrm{P}^2$, we obtain the following:

Lemma 3.2 *We have:*

(a) *If n is odd then $X^{n+1} \simeq \Sigma^{2n-2}\mathbb{C}P^2$.*

(b) *If n is even then $X^{n+1} \simeq S^{2n} \vee S^{2n+2}$.* □

At this point we localize at an odd prime p; the explanation as to why this is done will be deferred to Remark 4.5. At odd primes, the map η_m generating the stable group $\pi_{m+1}(S^m) \cong \mathbb{Z}/2\mathbb{Z}$ is null homotopic. Thus $\Sigma^{m-2}\mathbb{C}P^2 \simeq S^m \vee S^{m+2}$ for $m \geq 3$. Consequently, Lemma 3.2 implies the following:

Corollary 3.3 *Localize at an odd prime p. Then $X^{n+1} \simeq S^{2n} \vee S^{2n+2}$.* □

By Corollary 3.3, the map $\Sigma^5 X^{n-1} \xrightarrow{\bar{g}_{2n+3}} \Sigma X^{n+1}$ is a self-map of $S^{2n+1} \vee S^{2n+3}$. This lets us determine the homotopy class of $\bar{g}_{2n+3}$. In general, suppose that there is a map $h\colon S^m \vee S^{m+2} \to S^m \vee S^{m+2}$, where $m \geq 3$. Let h_1 and h_2 be the restrictions of h to S^m and S^{m+2}, respectively. The map $h_1\colon S^m \to S^m \vee S^{m+2}$ is determined by pinching to S^m and S^{m+2}. The pinch map to S^m is a map of some degree, say d_1, and the pinch to S^{m+2} is null homotopic for dimensional reasons. Thus $h_1 \simeq d_1 + *$. Since $m \geq 2$, the Hilton–Milnor theorem implies that $\pi_{m+2}(S^m \vee S^{m+2}) \cong \pi_{m+2}(S^m) \oplus \pi_{m+2}(S^{m+2})$. Therefore, the map $h_2\colon S^{m+2} \to S^m \vee S^{m+2}$ is also determined by pinching it to S^m and S^{m+2}. The pinch map to S^m is an element of $\pi_{m+2}(S^m) \cong 0$ (at odd primes) and the pinch map to S^{m+2} is a map of some degree, say d_2. Therefore $h_2 \simeq * + d_2$. Thus $h \simeq d_1 \vee d_2$. In particular, the homotopy class of h is determined by the map it induces in cohomology. Hence, from Lemma 3.1 we immediately obtain the following:

Lemma 3.4 *Localize at an odd prime p. The map $\Sigma^5 X^{n-1} \xrightarrow{\bar{g}_{2n+3}} \Sigma X^{n+1}$ is homotopic to the wedge of degree maps $d_1 \vee d_2$, where $d_1 = n(n-1)$ and $d_2 = (n+1)n$.* □

4 An upper bound for the order of ∂_1 in the unitary case

We wish to estimate the order of the map $SU(n) \xrightarrow{\partial_1} \Omega_0^3 SU(n)$. By Lemma 2.1, it is equivalent to calculate the order of the Samelson product $S^3 \wedge SU(n) \xrightarrow{\langle i,1\rangle} SU(n)$. Let $SU(\infty)$ be the infinite special unitary group and let t be the standard inclusion $t\colon SU(n) \to SU(\infty)$. There is a homotopy fibration sequence

$$\Omega(SU(\infty)/SU(n)) \to SU(n) \xrightarrow{t} SU(\infty) \to SU(\infty)/SU(n).$$

Since t is a group homomorphism, it is an H–map, and so it commutes with Samelson products. That is, $t \circ \langle i, 1\rangle \simeq \langle t \circ i, t\rangle$. Since $SU(\infty)$ is an infinite loop space, it is homotopy commutative. Therefore the Samelson product $\langle t \circ i, t\rangle$ is null homotopic, implying that there is a lift

$$\begin{array}{ccc} & & \Omega(\mathrm{SU}(\infty)/\mathrm{SU}(n)) \\ & \overset{\lambda}{\nearrow} & \downarrow \\ S^3 \wedge \mathrm{SU}(n) & \xrightarrow{\langle i,1\rangle} & \mathrm{SU}(n) \end{array}$$

for some map λ.

There is a canonical map

$$j_n\colon \Sigma\mathbb{C}\mathrm{P}^{n-1} \to \mathrm{SU}(n),$$

which induces an epimorphism in cohomology (see [27, Chapter VII, Section 4.6], for example). Let

$$q_n\colon \mathbb{C}\mathrm{P}^{n-1} \to X^{n-1} = \mathbb{C}\mathrm{P}^{n-1}/\mathbb{C}\mathrm{P}^{n-3}$$

be the quotient map. Observe that $\Omega(\mathrm{SU}(\infty)/\mathrm{SU}(n))$ is $(2n-1)$–connected. Therefore, the restriction of the composite $S^3\wedge\Sigma\mathbb{C}\mathrm{P}^{n-1}\xrightarrow{1\wedge j_n}S^3\wedge\mathrm{SU}(n)\xrightarrow{\lambda}\Omega(\mathrm{SU}(\infty)/\mathrm{SU}(n))$ to $S^3\wedge\Sigma\mathbb{C}\mathrm{P}^{n-3}$ is null homotopic. This implies that there is a homotopy commutative diagram

(3)
$$\begin{array}{ccccc} & & S^3\wedge\Sigma X^{n-1} & \xrightarrow{\nu} & \Omega(\mathrm{SU}(\infty)/\mathrm{SU}(n)) \\ & \overset{\Sigma^4 q_n}{\nearrow} & & \overset{\lambda}{\nearrow} & \downarrow \\ S^3\wedge\Sigma\mathbb{C}\mathrm{P}^{n-1} & \xrightarrow{1\wedge j_n} & S^3\wedge\mathrm{SU}(n) & \xrightarrow{\langle i,1\rangle} & \mathrm{SU}(n) \end{array}$$

for some map ν. Hamanaka and Kono [5, Proposition 5.2] showed that λ can be chosen so that $\lambda^*\circ(1\wedge j_n)*$ is a degree 1 isomorphism in dimensions $2n$ and $2n+2$. The homotopy commutativity of the left square in (3) therefore implies that ν^* is a degree 1 isomorphism in dimensions $2n$ and $2n+2$.

Let Y be the $(2n{+}2)$–skeleton of $\Omega(\mathrm{SU}(\infty)/\mathrm{SU}(n))$ and let $\overline{\nu}\colon \Sigma^4X^{n-1}\to Y$ be the factorization of ν through the $(2n{+}2)$–skeleton. An integral homology Serre spectral sequence calculation for the homotopy fibration $\Omega\mathrm{SU}(\infty)\to\Omega(\mathrm{SU}(\infty)/\mathrm{SU}(n))\to\mathrm{SU}(n)$ immediately shows that Y can be given a CW–structure with two cells, one each in dimensions $2n$ and $2n+2$. Localized at an odd prime this implies that $Y\simeq S^{2n}\vee S^{2n+2}$. Thus $\overline{\nu}$ is a self-map of $S^{2n}\vee S^{2n+2}$, so arguing as for Lemma 3.4 shows that $\overline{\nu}$ is homotopic to a wedge of degree maps. Since ν^* is a degree 1 isomorphism in dimensions $2n$ and $2n+2$. Hence, including Y into $\Omega(\mathrm{SU}(\infty)/\mathrm{SU}(n))$ we obtain the following:

Lemma 4.1 *The map $\Sigma^4X^{n-1}\xrightarrow{\nu}\Omega(\mathrm{SU}(\infty)/\mathrm{SU}(n))$ induces a degree 1 isomorphism in cohomology in dimensions $2n$ and $2n+2$.* □

We now assemble several pieces of information, aimed at establishing the homotopy commutativity of (10). The naturality of the Samelson product implies that the composite $S^3 \wedge \Sigma\mathbb{C}P^{n-1} \xrightarrow{1\wedge j_n} S^3 \wedge SU(n) \xrightarrow{\langle i,1\rangle} SU(n)$ is the Samelson product $\langle i, j_n\rangle$. Notice that the map $\Omega(SU(\infty)/SU(n)) \to SU(n)$ is a loop map. So we can take the adjoint of diagram (3) to obtain a homotopy commutative diagram

$$\begin{array}{ccc} \Sigma^5 X^{n-1} & \xrightarrow{\widetilde{\nu}} & SU(\infty)/SU(n) \\ \uparrow{\scriptstyle \Sigma^5 q_n} & & \downarrow \\ \Sigma^5\mathbb{C}P^{n-1} & \xrightarrow{\widetilde{\langle i,j_n\rangle}} & BSU(n) \end{array} \tag{4}$$

where $\widetilde{\langle i, j_n\rangle}$ and $\widetilde{\nu}$ are the adjoints of $\langle i, j_n\rangle$ and ν, respectively.

On the other hand, by Corollary 3.3, $\Sigma^5 X^{n-1} \simeq \Sigma X^{n+1} \simeq S^{2n+1} \vee S^{2n+3}$ and, by Lemma 4.1, $\widetilde{\nu}$ induces a degree 1 isomorphism in cohomology in dimensions $2n+1$ and $2n+3$. So we may choose a homotopy equivalence $\Sigma^5 X^{n-1} \simeq \Sigma X^{n+1}$ so that there is a homotopy commutative diagram

$$\begin{array}{cccc} \Sigma^5\mathbb{C}P^{n-1} & \longrightarrow & \Sigma^5 X^{n-1} & \\ & & \downarrow{\scriptstyle \wr} & \searrow{\scriptstyle \widetilde{\nu}} \\ & & \Sigma X^{n+1} & \xrightarrow{\iota} SU(\infty)/SU(n) \end{array} \tag{5}$$

where ι is the inclusion.

Next, the map $\Sigma\mathbb{C}P^{n-1} \xrightarrow{j_n} SU(n)$ is natural with respect to the usual inclusion of $SU(n)$ into $SU(n+1)$. Let j be the composite $\Sigma\mathbb{C}P^{n+1} \xrightarrow{j_{n+2}} SU(n+2) \to SU(\infty)$. Consider the diagram:

$$\begin{array}{ccccccc} \Sigma\mathbb{C}P^{n-1} & \longrightarrow & \Sigma\mathbb{C}P^{n+1} & \xrightarrow{\Sigma q_{n+2}} & \Sigma X^{n+1} & \longrightarrow & \Sigma^2\mathbb{C}P^{n-1} \\ \downarrow{\scriptstyle j_n} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ SU(n) & \longrightarrow & SU(\infty) & \longrightarrow & SU(\infty)/SU(n) & \longrightarrow & BSU(n) \end{array}$$

The left square commutes by the naturality of j_n. As the top row is a homotopy cofibration and the bottom row is a homotopy fibration, the homotopy commutativity of the left square induces the middle square and right square for some maps a and b. It is standard that the CW–structure of $SU(\infty)/SU(n)$ is taken so that a is the inclusion of the $(2n+3)$–skeleton. The Peterson–Stein formulas (see [8, Formula 3.4.2]) imply that the map b may be taken to be the adjoint $\tilde{j}_n$ of j_n. Therefore, reorienting the right square in the previous diagram, we obtain a homotopy commutative square:

$$
\begin{array}{ccc}
\Sigma X^{n+1} & \xrightarrow{a} & \mathrm{SU}(\infty)/\mathrm{SU}(n) \\
\downarrow & & \downarrow \\
\Sigma^2\mathbb{C}\mathrm{P}^{n-1} & \xrightarrow{\tilde{j}_n} & B\mathrm{SU}(n)
\end{array}
\tag{6}
$$

The last preparatory step is to manipulate degree maps on $\Sigma^5\mathbb{C}\mathrm{P}^{n-1}$. By [17], there is a p–local homotopy equivalence

$$\Sigma\mathbb{C}\mathrm{P}^{n-1} \simeq \bigvee_{i=1}^{p-1} A_i,$$

where $H^*(A_i;\mathbb{Z}/p\mathbb{Z})$ consists of those elements in $H^*(\Sigma\mathbb{C}\mathrm{P}^{n-1};\mathbb{Z}/p\mathbb{Z})$ that are in degrees of the form $2i+2(p-1)k+1$ for some $k\geq 0$. Select indices i_0 and i_1 such that $2i_0+2(p-1)k_0+1=2n-3$ and $2i_1+2(p-1)k_1+1=2n-1$ for some integers k_0 and k_1. Then A_{i_0} and A_{i_1} inherit the mod-p cohomology generators of $\Sigma\mathbb{C}\mathrm{P}^{n-1}$ in degrees $2n-3$ and $2n-1$, respectively. For dimension and connectivity reasons, there is a homotopy commutative diagram

$$
\begin{array}{ccccc}
\Sigma\mathbb{C}\mathrm{P}^{n-1} & & \xrightarrow{\Sigma q_n} & & \Sigma X^{n-1} \\
\downarrow\wr & & & & \downarrow\wr \\
\bigvee_{i=1}^{p-1} A_i & \xrightarrow{Q} & A_{i_0}\vee A_{i_1} & \xrightarrow{q_{i_0}\vee q_{i_1}} & S^{2n-3}\vee S^{2n-1}
\end{array}
\tag{7}
$$

where Q is the pinch map onto the two designated summands and q_{i_0} and q_{i_1} are the pinch maps onto the top cells of A_{i_0} and A_{i_1}, respectively. Let $\overline{\gamma}$ be the composite

$$\overline{\gamma}\colon \Sigma^5\mathbb{C}\mathrm{P}^{n-1} \xrightarrow{\simeq} \bigvee_{i=1}^{p-1}\Sigma^4 A_i \xrightarrow{\Sigma^4 Q} \Sigma^4 A_{i_0}\vee\Sigma^4 A_{i_1} \xrightarrow{d_1\vee d_2} \Sigma^4 A_{i_0}\vee\Sigma^4 A_{i_1} \hookrightarrow \bigvee_{i=1}^{p-1}\Sigma^4 A_i \xrightarrow{\simeq} \Sigma^5\mathbb{C}\mathrm{P}^{n-1},$$

where $d_1=n(n-1)$ and $d_2=(n+1)n$ are the degree maps. Notice that all the maps in the composite defining $\overline{\gamma}$ are suspensions and so they commute with degree maps. Therefore, the definition of $\overline{\gamma}$ and (7) imply that there is a homotopy commutative diagram:

$$(8)\qquad \begin{array}{ccc}
\Sigma^5\mathbb{C}\mathrm{P}^{n-1} & \xrightarrow{\quad\overline{\gamma}\quad} & \Sigma^5\mathbb{C}\mathrm{P}^{n-1} \\
\big\downarrow{\scriptstyle \Sigma^5 q_n} & & \big\downarrow{\scriptstyle \Sigma^5 q_n} \\
\Sigma^5 X_{n-1} & & \Sigma^5 X_{n-1} \\
\big\downarrow{\scriptstyle \wr} & & \big\downarrow{\scriptstyle \wr} \\
S^{2n+1}\vee S^{2n+3} & \xrightarrow{d_1\vee d_2} & S^{2n+1}\vee S^{2n+3}
\end{array}$$

It is also useful at this point to define the map γ by the composite

$$\gamma\colon \Sigma^5\mathbb{C}\mathrm{P}^{n-1} \xrightarrow{\simeq} \bigvee_{i=1}^{p-1}\Sigma^4 A_i \xrightarrow{\Sigma^4 Q} \Sigma^4 A_{i_0}\vee\Sigma^4 A_{i_1} \xrightarrow{d\vee d} \Sigma^4 A_{i_0}\vee\Sigma^4 A_{i_1} \hookrightarrow \bigvee_{i=1}^{p-1}\Sigma^4 A_i \xrightarrow{\simeq} \Sigma^5\mathbb{C}\mathrm{P}^{n-1},$$

where $d = n(n^2-1)$. As before, the definition of γ and (7) imply that there is a homotopy commutative diagram:

$$(9)\qquad \begin{array}{ccc}
\Sigma^5\mathbb{C}\mathrm{P}^{n-1} & \xrightarrow{\ \gamma\ } & \Sigma^5\mathbb{C}\mathrm{P}^{n-1} \\
\big\downarrow{\scriptstyle \Sigma^5 q_n} & & \big\downarrow{\scriptstyle \Sigma^5 q_n} \\
\Sigma^5 X_{n-1} & \xrightarrow{\ d\ } & \Sigma^5 X_{n-1}
\end{array}$$

Now we put things together. Consider the diagram:

$$(10)\qquad \begin{array}{ccccc}
\Sigma^5\mathbb{C}\mathrm{P}^{n-1} & \xrightarrow{\ \overline{\gamma}\ } & \Sigma^5\mathbb{C}\mathrm{P}^{n-1} & & \\
\Big\downarrow & & \big\downarrow{\scriptstyle \Sigma^5 q_n} & & \\
{\scriptstyle \Sigma^5 q_n} & & \Sigma^5 X^{n-1} & \searrow{\scriptstyle \widetilde{\nu}} & \\
\Big\downarrow & & \big\downarrow{\scriptstyle \wr} & & \\
\Sigma^5 X^{n-1} & \xrightarrow{\ \overline{g}_{2n+3}\ } & \Sigma X^{n+1} & \xrightarrow{\ a\ } & \mathrm{SU}(\infty)/\mathrm{SU}(n) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\Sigma^6\mathbb{C}\mathrm{P}^{n-3} & \xrightarrow{\Sigma g_{2n-1}} & \Sigma^2\mathbb{C}\mathrm{P}^{n-1} & \xrightarrow{\ \tilde{j}_n\ } & B\mathrm{SU}(n)
\end{array}$$

The upper left rectangle homotopy commutes by (8) and the description of $\overline{g}_{2n+3}$ in Lemma 3.4; the upper right triangle homotopy commutes by (5); the lower left square homotopy commutes by (2); and the lower right square homotopy commutes by (6). Thus the entire diagram homotopy commutes. In the upper direction around the diagram, by (4) the composite

$$\Sigma^5\mathbb{C}P^{n-1} \xrightarrow{\Sigma^5 q_n} \Sigma^5 X^{n-1} \xrightarrow{\widetilde{\nu}} SU(\infty)/SU(n) \to BSU(n)$$

is homotopic to $\widetilde{\langle i, j_n\rangle}$, so the upper direction around the diagram is homotopic to $\widetilde{\langle i, j_n\rangle} \circ \overline{\gamma}$. On the other hand, in the lower direction around the diagram the left column is null homotopic since it is two consecutive maps in a homotopy cofibration. Therefore $\widetilde{\langle i, j_n\rangle} \circ \overline{\gamma}$ is null homotopic.

Lemma 4.2 *Localize at an odd prime* p*. The map* $\Sigma^5\mathbb{C}P^{n-1} \xrightarrow{\widetilde{\langle i,j_n\rangle}} BSU(n)$ *has order dividing* $n(n^2-1)$*.*

Proof We wish to show that the composite $\Sigma^5\mathbb{C}P^{n-1} \xrightarrow{d} \Sigma^5\mathbb{C}P^{n-1} \xrightarrow{\widetilde{\langle i,j_n\rangle}} BSU(n)$ is null homotopic, where $d = n(n^2-1)$. For convenience, label the map from $SU(\infty)/SU(n)$ to $BSU(n)$ by δ. Consider the string of homotopies

$$\widetilde{\langle i, j_n\rangle} \circ d \simeq \delta \circ \widetilde{\nu} \circ \Sigma^5 q_n \circ d \simeq \delta \circ \widetilde{\nu} \circ d \circ \Sigma^5 q_n \simeq \delta \circ \widetilde{\nu} \circ \Sigma^5 q_n \circ \gamma \simeq \widetilde{\langle i, j_n\rangle} \circ \gamma.$$

From left to right, the first homotopy holds by (4), the second holds since maps which are suspensions commute with degree maps, the third holds by (9), and the fourth holds by (4). Observe that, by its definition, γ factors through $\overline{\gamma}$. Therefore, as $\widetilde{\langle i, j_n\rangle} \circ \overline{\gamma}$ is null homotopic, so is $\widetilde{\langle i, j_n\rangle} \circ \gamma$. Hence $\widetilde{\langle i, j_n\rangle} \circ d$ is null homotopic, as asserted. □

Recall that, by definition, $\widetilde{\langle i, j_n\rangle}$ is the adjoint of the Samelson product $\langle i, j_n\rangle$, which is homotopic to the composite $S^3 \wedge \Sigma\mathbb{C}P^{n-1} \xrightarrow{1\wedge j_n} S^3 \wedge SU(n) \xrightarrow{\langle i,1\rangle} SU(n)$. Therefore Lemma 4.2 immediately implies the following:

Corollary 4.3 *Localize at an odd prime* p*. The composite* $S^3 \wedge \Sigma\mathbb{C}P^{n-1} \xrightarrow{1\wedge j_n} S^3 \wedge SU(n) \xrightarrow{\langle i,1\rangle} SU(n)$ *has order at most* $n(n^2-1)$*.* □

Also, the map $\langle i, 1\rangle$ is the triple adjoint of the map $SU(n) \xrightarrow{\partial_1} \Omega^3_0 SU(n)$. Therefore, Corollary 4.3 implies the following:

Proposition 4.4 *Localize at an odd prime* p*. The composite* $\Sigma\mathbb{C}P^{n-1} \xrightarrow{j_n} SU(n) \xrightarrow{\partial_1} \Omega^3_0 SU(n)$ *has order at most* $n(n^2-1)$*.* □

Remark 4.5 It is unclear whether an analogue of Proposition 4.4 holds at the prime 2. At odd p the splitting $\Sigma\mathbb{C}P^{n-1} \simeq \bigvee_{i=1}^{p-1} A_i$ lets us work with the two different degree maps $d_1 = n(n-1)$ and $d_2 = (n+1)n$ on S^{2n+1} and S^{2n+3} separately. However, at 2 there is no such splitting of $\Sigma\mathbb{C}P^{n-1}$, so it is unclear how a map $\overline{\gamma}$ can be defined so as to obtain a diagram as in (8).

5 Retractile Lie groups and an upper bound for the order of ∂_1

This section is aimed at proving parts (b) and (c) of Theorem 1.1. To do so we consider those n for which $\mathrm{SU}(n)$ has the special property of being retractile (a term defined in a moment). The results in this section hold in more generality than just the $\mathrm{SU}(n)$ case, so they are stated this way. Throughout this section spaces and maps have been localized at an odd prime p and homology is taken with mod-p coefficients.

Definition 5.1 An H–space B is *retractile* if there is a space A and a map $i\colon A \to B$ with the following properties:

(i) There is an algebra isomorphism $H_*(B) \cong \Lambda(\widetilde{H}_*(A))$.

(ii) i_* is the inclusion of the generating set in homology.

(iii) The map $\Sigma A \xrightarrow{\Sigma i} \Sigma B$ has a left homotopy inverse.

Many simply connected, simple compact Lie groups are retractile. In [20] it was shown that the retractile cases are:

$$\begin{array}{llll} \mathrm{SU}(n), & n \le (p-1)^2+1; & G_2, F_4, E_6, & p \ge 3; \\ \mathrm{Sp}(n), & 2n \le (p-1)^2+1; & E_7, & p \ge 5; \\ \mathrm{Spin}(2n+1), & 2n \le (p-1)^2+1; & E_8, & p \ge 7; \\ \mathrm{Spin}(2n), & 2(n-1) \le (p-1)^2+1; & & \end{array}$$

further, in each of these cases the space A is a co-H–space.

Let G be a simply connected, simple compact Lie group which is retractile. Recall the map $G \xrightarrow{\partial_1} \Omega_0^3 G$ from Section 2. Define the map $\overline{\partial}_1$ by the composite

$$\overline{\partial}_1\colon A \xrightarrow{i} G \xrightarrow{\partial_1} \Omega_0^3 G.$$

We will prove the following, which relates the order of ∂_1 to that of $\overline{\partial}_1$:

Proposition 5.2 *Let G be a retractile, simply connected, simple compact Lie group. If $p^r \circ \overline{\partial}_1$ is null homotopic then so is $p^r \circ \partial_1$.*

Proposition 5.2 is very useful, in practise it tends to be much easier to prove facts about $\overline{\partial}_1$ rather than ∂_1. The proposition says that this is good enough. Granting Proposition 5.2, we obtain the following corollary, which lets us go on to prove Theorem 1.1(a)–(c).

Corollary 5.3 *Let G be a retractile simply connected, simple compact Lie group. Then ∂_1 has order p^r if and only if $\overline{\partial}_1$ has order p^r.*

Proof If ∂_1 has order p^r then, as $\overline{\partial}_1$ factors through ∂_1, the order of $\overline{\partial}_1$ is at most p^r. Conversely, if $\overline{\partial}_1$ has order p^r then by Proposition 5.2 ∂_1 has order at most p^r. Hence, if a and b are the orders of ∂_1 and $\overline{\partial}_1$, respectively, then $a \le b \le a$, implying that $a = b$. □

Proof of Theorem 1.1(a)–(c) Part (a) is the statement of Corollary 4.3. For part (c), SU(n) is retractile if $n \le (p-1)^2+1$, and in this case the space A is $\Sigma\mathbb{C}\mathrm{P}^{n-1}$ and the map i is the map $\Sigma\mathbb{C}\mathrm{P}^{n-1}\xrightarrow{j_n}\mathrm{SU}(n)$. Therefore the composite $\Sigma\mathbb{C}\mathrm{P}^{n-1}\xrightarrow{j_n}\mathrm{SU}(n)\xrightarrow{\partial_1}\Omega^3_0\mathrm{SU}(3)$ is $\overline{\partial}_1$. Let p^r be the p–component of $n(n^2-1)$. By Proposition 4.4, $p^r\circ\overline{\partial}_1$ is null homotopic, so Proposition 5.2 implies that $p^r\circ\partial_1$ is null homotopic. By [6, Theorem 2.4 and Lemma 2.5], ∂_1 has order at least p^r. Therefore the order of ∂_1 equals p^r. This proves part (c), and part (b) now follows by Corollary 5.3. □

It remains to prove Proposition 5.2. To do so we describe a homotopy decomposition of ΣG which is designed to behave well with respect to $p^r\circ\partial_1$. In [4] it was shown that if G is retractile then $\Sigma G \simeq \Sigma A \vee C$, where C can be chosen so that it factors through the Hopf construction on G. We need a refinement of this, so the construction will be described in more detail.

In general, let X and Y be path-connected, pointed spaces and let I be the unit interval. The *reduced join* of X and Y is the quotient space $X * Y = (X\times Y\times I)/{\sim}$, where $(x,y,0)\sim(x,y',0)$, $(x,y,1)\sim(x',y,1)$ and $(*,*,t)\sim(*,*,0)$ for all x, $x'\in X$, y, $y'\in Y$ and $t\in I$. The reduced suspension $\Sigma(X\times Y)$ is also a quotient of $X\times Y\times I$, given by $\Sigma(X\times Y)=(X\times Y\times I)/{\sim'}$, where $(x,y,0)\sim'(x',y',0)$, $(x,y,1)\sim'(x',y',1)$ and $(*,*,t)\sim'(*,*,0)$. The relations imply that the quotient map from $X\times Y\times I$ to $\Sigma(X\times Y)$ factors through $X*Y$. Thus there is a map $X*Y\to\Sigma(X\times Y)$. Note that this map is natural in both variables. Note also that the composites $X*Y\to\Sigma(X\times Y)\xrightarrow{\Sigma\pi_1}\Sigma X$ and $X*Y\to\Sigma(X\times Y)\xrightarrow{\Sigma\pi_2}\Sigma Y$ are null homotopic, where π_1 and π_2 are the projections onto the first and second factors, respectively. There is also a natural homotopy equivalence $\Sigma X\wedge Y\simeq X*Y$ (see [18, Proposition 7.7.1], for example), so we obtain a natural map

$$\Sigma X\wedge Y\to\Sigma(X\times Y), \tag{11}$$

which composes trivially with $\Sigma\pi_1$ and $\Sigma\pi_2$. Iterating, if $X_1,\dots,X_k$ are path-connected, pointed spaces then there is a map

$$\Sigma X_1\wedge\cdots\wedge X_k\to\Sigma(X_1\times\cdots\times X_k) \tag{12}$$

which is natural in all k variables. In particular, if $X^{\times k}$ is the Cartesian product of k copies of X with itself and $X^{\wedge k}$ is the k–fold smash product of X with itself, then there is a natural map $\Sigma X^{\wedge k} \to \Sigma X^{\times k}$.

Suppose now that X is an H–space with multiplication $\mu\colon X \times X \to X$. Let μ^* be the composite

$$\mu^*\colon \Sigma X \wedge X \to \Sigma(X \times X) \xrightarrow{\Sigma\mu} \Sigma X.$$

This composite is known as the *Hopf construction* on X. In particular, if G is a simply connected, simple compact Lie group then there is a Hopf construction $\Sigma G \wedge G \xrightarrow{\mu^*} \Sigma G$.

Now we bring in the space A. As G is homotopy associative, the James construction [9] implies that the map $A \xrightarrow{i} G$ extends to an H–map $j\colon \Omega\Sigma A \to G$. By the Bott–Samelson theorem, there is an algebra isomorphism $H_*(\Omega\Sigma A) \cong T(\widetilde{H}_*(A))$, where $T(\,\cdot\,)$ is the free tensor algebra functor. Since G is retractile, there is an algebra isomorphism $H_*(G) \cong \Lambda(\widetilde{H}_*(A))$; observe also that $\Lambda(\widetilde{H}_*(A)) \cong S(\widetilde{H}_*(A))$, where $S(\,\cdot\,)$ is the free symmetric algebra functor. Since j is an H–map, j_* is an algebra map and so is determined by its restriction to the generating set $\widetilde{H}_*(A)$ of $T(\widetilde{H}_*(A))$. Since j is an extension of i and i_* is the inclusion of the generating set into $\Lambda(\widetilde{H}_*(A)) \cong S(\widetilde{H}_*(A))$, the map j_* is therefore the abelianization of the tensor algebra. Let m be the number of generators in $\widetilde{H}_*(A)$. For $1 \le k \le m$, let M_k be the submodule of $H_*(G) \cong \Lambda(\widetilde{H}_*(A))$ consisting of monomials of length k. Observe that there is a module isomorphism $H_*(G) \cong \bigoplus_{k=1}^m M_k$.

Let $E\colon A \to \Omega\Sigma A$ be the suspension map. Let E_k be the composite

$$E_k\colon A^{\times k} \xrightarrow{E^{\times k}} (\Omega\Sigma A)^{\times k} \to \Omega\Sigma A,$$

where the left map is the iterated loop space multiplication. In [4] it was shown that if $1 \le k \le m$ then there is a retract $S_k(A)$ of $\Sigma A^{\wedge k}$ such that $S_1(A) = \Sigma A$, $H_*(S_k(A)) \cong \Sigma M_k$ and the composite

$$\phi_k\colon S_k(A) \to \Sigma A^{\wedge k} \to \Sigma A^{\times k} \xrightarrow{\Sigma E_k} \Sigma\Omega\Sigma A \xrightarrow{\Sigma j} \Sigma G$$

induces an isomorphism onto the submodule ΣM_k of $H_*(\Sigma G)$. Taking the wedge sum over $1 \le k \le m$ gives a map

$$\phi\colon \bigvee_{k=1}^{m} S_k(A) \to \Sigma G$$

which induces an isomorphism in homology and so is a homotopy equivalence. Notice that, when $k = 1$, we have $S_1(A) = \Sigma A$ and $\phi_1 \simeq \Sigma i$. Thus if we let $C = \bigvee_{k=2}^m S_k(A)$

then there is a homotopy equivalence

$$\Sigma A \vee C \xrightarrow{\phi} \Sigma B,$$

where ϕ restricted to ΣA is $\phi_1 \simeq \Sigma i$. The refinement on ϕ we need is to show that its restriction to C factors not just through the Hopf construction $\Sigma G \wedge G \xrightarrow{\mu^*} \Sigma G$, as stated in [4], but through the composite

$$\overline{\mu}^*\colon \Sigma G \wedge A \xrightarrow{\Sigma 1 \wedge i} \Sigma G \wedge G \xrightarrow{\mu^*} \Sigma G.$$

Lemma 5.4 *The restriction of the homotopy equivalence $\Sigma A \vee C \xrightarrow{\phi} G$ to C factors through $\overline{\mu}^*$.*

Proof Consider the diagram

(13)
$$\begin{array}{ccccccc}
S_k(A) & \longrightarrow & \Sigma A^{\wedge k} & \xrightarrow{\Sigma E^{\wedge k}} & \Sigma(\Omega\Sigma A)^{\wedge k} & \xrightarrow{\Sigma j^{\wedge k}} & \Sigma G^{\wedge k} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \Sigma A^{\times k} & \xrightarrow{\Sigma E^{\times k}} & \Sigma(\Omega\Sigma A)^{\times k} & \xrightarrow{\Sigma j^{\times k}} & \Sigma G^{\times k} \\
 & & & \searrow^{\Sigma E_k} & \downarrow{\scriptstyle \Sigma\mu_k} & & \downarrow{\scriptstyle \Sigma\mu_k} \\
 & & & & \Sigma\Omega\Sigma A & \xrightarrow{\Sigma j} & \Sigma G
\end{array}$$

where μ_k is the iterated multiplication on $\Omega\Sigma A$ and G (both of which are homotopy associative, so the order in which the multiplication is taken is irrelevant). The two upper squares homotopy commute by the naturality of (12), the left lower triangle homotopy commutes by the definition of E_k, and the lower right square homotopy commutes since j is an H–map. By the definition of ϕ_k, the lower direction around the diagram is $\Sigma\phi_k$. On the other hand, since $j \circ E \simeq i$, if $k \geq 2$ then the upper direction around the diagram can be rewritten as the composite

$$S_k(A) \to \Sigma A^{\wedge k} \xrightarrow{i^{\wedge(k-1)}\wedge 1} \Sigma G^{\wedge(k-1)} \wedge A \to \Sigma G^{\times(k-1)} \wedge A \xrightarrow{\Sigma\mu_{k-1}\wedge 1} \Sigma G \wedge A$$
$$\xrightarrow{\Sigma 1\wedge i} \Sigma G \wedge G \xrightarrow{\mu^*} \Sigma G.$$

Notice that the last two maps in this composite define $\overline{\mu}^*$, so the homotopy commutativity of (13) shows that ϕ_k factors through $\overline{\mu}^*$. □

Now we relate the homotopy equivalence for ΣG to the order of the boundary map $G \xrightarrow{\partial_1} \Omega^3_0 G$. In general, if $\Omega B \xrightarrow{\partial} F \to E \to B$ is a homotopy fibration sequence then there is a homotopy action

$$\theta\colon \Omega B \times F \to F$$

such that the restriction of θ to ΩB is ∂, the restriction of θ to F is the identity map, and there is a homotopy commutative diagram:

$$\begin{array}{ccc} \Omega B \times \Omega B & \xrightarrow{\mu} & \Omega B \\ \downarrow{\scriptstyle 1\times\partial} & & \downarrow{\scriptstyle \partial} \\ \Omega B \times F & \xrightarrow{\theta} & F \end{array} \tag{14}$$

In our case, since $G \xrightarrow{\partial_1} \Omega_0^3 G$ is a homotopy fibration connecting map there is a homotopy action $\theta\colon G \times \Omega_0^3 G \to \Omega_0^3 G$.

Let ev: $\Sigma\Omega_0^3 G \to \Omega^2 G$ be the canonical evaluation map. For an H–space X, recall that $p^r\colon X \to X$ is the p^r–power map. For a co-H–space Y, let $\underline{p}^r\colon Y \to Y$ be the map of degree p^r. Recall that $\overline{\partial}_1$ is the composite

$$\overline{\partial}_1\colon A \xrightarrow{i} G \xrightarrow{\partial_1} \Omega_0^3 G.$$

Let

$$\phi_C\colon C \to \Sigma G$$

be the restriction of the homotopy equivalence $\Sigma A \vee C \xrightarrow{\phi} \Sigma G$ to C.

Lemma 5.5 *Let G be a retractile, simply connected, simple compact Lie group. Suppose that $p^r \circ \overline{\partial}_1$ is null homotopic. Then for $2 \le k \le m$ the composite*

$$C \xrightarrow{\phi_C} \Sigma G \xrightarrow{\Sigma\partial_1} \Sigma\Omega_0^3 G \xrightarrow{\text{ev}} \Omega^2 G \xrightarrow{p^r} \Omega^2 G$$

is null homotopic.

Proof Consider the diagram

$$\begin{array}{ccc} G \times A & \xrightarrow{\pi_1} & G \\ \downarrow{\scriptstyle 1\times\underline{p}^r} & & \\ G \times A & & \downarrow{\scriptstyle i_1} \\ \downarrow{\scriptstyle 1\times i} & & \\ G \times G & \xrightarrow{1\times\partial_1} & G \times \Omega_0^3 G \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \theta} \\ G & \xrightarrow{\partial_1} & \Omega_0^3 G \end{array}$$

where π_1 is the projection onto the first factor and i_1 is the inclusion of the first factor. Since $\overline{\partial}_1 = \partial_1 \circ i$, the hypothesis that $p^r \circ \overline{\partial}_1$ is null homotopic implies that the upper rectangle homotopy commutes. The lower square homotopy commutes by (14). Suspending and using the naturality of (11) and the definition of $\overline{\mu}^*$, we obtain a homotopy commutative diagram:

$$\begin{array}{ccccccc}
\Sigma G\wedge A & \longrightarrow & \Sigma(G\times A) & \xrightarrow{\Sigma\pi_1} & \Sigma G & & \\
\downarrow{\scriptstyle \Sigma(1\wedge \underline{p}^r)} & & \downarrow{\scriptstyle \Sigma(1\times \underline{p}^r)} & & \Big\downarrow{\scriptstyle \Sigma i_1} & & \\
\Sigma G\wedge A & \longrightarrow & \Sigma(G\times A) & & & & \\
& \searrow{\scriptstyle \overline{\mu}^*} & \downarrow{\scriptstyle \Sigma(1\times i)} & & & & \\
& & \Sigma(G\times G) & \xrightarrow{\Sigma(1\times\partial_1)} & \Sigma(G\times\Omega^3_0 G) & & \\
& & \downarrow{\scriptstyle \Sigma\mu} & & \downarrow{\scriptstyle \Sigma\theta} & & \\
& & \Sigma G & \xrightarrow{\Sigma\partial_1} & \Sigma\Omega^3_0 G & \xrightarrow{\text{ev}} & \Omega^2 G
\end{array}$$

By (11), the composite along the top row is null homotopic. Thus the lower direction around the diagram is null homotopic. Observe that the map $\Sigma 1\wedge \underline{p}^r$ is homotopic to the degree p^r map on $\Sigma G\wedge A$, so we obtain a null homotopy for $\text{ev}\circ\Sigma\partial_1\circ\overline{\mu}^*\circ\underline{p}^r$. In $[\Sigma X,\Omega Y]$ the group structure induced by the comultiplication equals that induced by the multiplication, so we obtain a null homotopy for the composite $\Sigma G\wedge A\xrightarrow{\overline{\mu}^*}\Sigma G\xrightarrow{\Sigma\partial_1}\Sigma\Omega^3_0 G\xrightarrow{\text{ev}}\Omega^2 G\xrightarrow{p^r}\Omega^2 G$. By Lemma 5.4, the restriction of the homotopy equivalence $\Sigma A\vee C\xrightarrow{\phi}\Sigma G$ to C factors through $\overline{\mu}^*$. Thus the composite

$$C\xrightarrow{\phi_C}\Sigma G\xrightarrow{\Sigma\partial_1}\Sigma\Omega^3_0 G\xrightarrow{\text{ev}}\Omega^2 G\xrightarrow{p^r}\Omega^2 G$$

is null homotopic. □

Putting all this together, we prove Proposition 5.2:

Proof of Proposition 5.2 Consider the composite

$$\psi\colon \Sigma A\vee C\xrightarrow{\phi}\Sigma G\xrightarrow{\Sigma\partial_1}\Omega^3_0 G\xrightarrow{\text{ev}}\Omega^2 G\xrightarrow{p^r}\Omega^2 G.$$

The restriction of ϕ to ΣA is homotopic to Σi, and by definition $\overline{\partial}_1=\partial_1\circ i$. So the restriction of ψ to ΣA is homotopic to $p^r\circ\text{ev}\circ\Sigma\overline{\partial}_1$, which is null homotopic since $p^r\circ\overline{\partial}_1$ is null homotopic. By Lemma 5.5, the hypothesis that $p^r\circ\overline{\partial}_1$ is null homotopic implies that the restriction of ψ to C is null homotopic. Thus ψ is null homotopic. As ϕ is a homotopy equivalence, this implies that $p^r\circ\text{ev}\circ\Sigma\partial_1$ is null homotopic. Taking adjoints, we obtain that $p^r\circ\partial_1$ is null homotopic. □

6 Applications to gauge groups

The first application is to prove Theorem 1.1(d) by counting the number of distinct homotopy types of gauge groups. This requires two known results. Hamanaka and Kono [6] proved the following:

Theorem 6.1 *If $\mathcal{G}_k(\mathrm{SU}(n)) \simeq \mathcal{G}_{k'}(\mathrm{SU}(n))$ then $(n(n^2-1,k) = (n(n^2-1),k')$.* □

Theorem 6.1 improved an earlier result of Sutherland [19] by a factor of 2 in the case where n is odd. On the other hand, the author [21] proved the following. For positive integers a and b, let (a,b) be their greatest common divisor.

Theorem 6.2 *Let X be a space and Y be an H–space with a homotopy inverse. Suppose there is a map $X \xrightarrow{f} Y$ of order m, where m is finite. For a positive integer $k \geq 0$, let F_k be the homotopy fibre of $k \circ f$. If $(m,k) = (m,k')$ then F_k and $F_{k'}$ are homotopy equivalent when localized rationally or at any prime.* □

Proof of Theorem 1.1(d) Consider the map $\mathrm{SU}(n) \xrightarrow{\partial_1} \Omega_0^3\mathrm{SU}(n)$. By Lemma 2.1, $k \circ \partial_1 \simeq \partial_k$, and recall that the homotopy fibre of ∂_k is $\mathcal{G}_k(\mathrm{SU}(n))$. Localize at an odd prime p. If $n \leq (p-1)^2+1$ then Theorem 1.1(c) says that ∂_1 has order $n(n^2-1)$. So, by Theorem 6.2, if $(n(n^2-1),k) = (n(n^2-1),k')$ then $\mathcal{G}_k(\mathrm{SU}(n)) \simeq \mathcal{G}_{k'}(\mathrm{SU}(n))$ when localized at any odd prime. The converse is Theorem 6.1. □

The next application is to a p–local homotopy decomposition of $\mathcal{G}_k(\mathrm{SU}(n))$. Assume from now on that all spaces and maps are localized at p. In [16] it was shown that there is a homotopy equivalence $\mathrm{SU}(n) \simeq \prod_{i=1}^{p-1} B_i$, where the generators of $H^*(B_i;\mathbb{Z}/p\mathbb{Z})$ are those of $H^*(\mathrm{SU}(n);\mathbb{Z}/p\mathbb{Z})$ which occur in dimensions of the form $2i+2(p-1)t+1$ for some $t \geq 0$. Let i_0 and i_1 be such that $2n-3 = 2i_0+2(p-1)t_0+1$ and $2n-1 = 2i_1+2(p-1)t_1+1$ for some integers t_0 and t_1. In what follows, when we write B_{i+2} and $i = p-2$ or $p-1$ we mean B_1 or B_2, respectively. In [22] in the retractile case and in [11] in the general case it was shown that there is a homotopy equivalence

$$\mathcal{G}_k(\mathrm{SU}(n)) \simeq \left(\prod_{\substack{i=1 \\ i \neq i_0,i_1}}^{p-1} B_i \times \Omega_0^4 B_{i+2} \right) \times X_{i_0} \times X_{i_1}, \tag{15}$$

where for $j \in \{i_0,i_1\}$ there are homotopy fibrations $\Omega_0^4 B_{j+2} \to X_j \to B_j$.

The extra information we can now add is when the spaces X_{i_0} and X_{i_1} decompose as products. If $\mathrm{SU}(n)$ is retractile and k is a multiple of $n(n^2-1)$ then, by Theorem 1.1(c) and Lemma 2.1, $\partial_k \simeq k \circ \partial_1$ is null homotopic. Therefore, $\mathcal{G}_k(\mathrm{SU}(n)) \simeq \mathrm{SU}(n) \times \Omega_0^4 \mathrm{SU}(n)$, implying that there are homotopy equivalences $X_{i_0} \simeq B_{i_0} \times \Omega^4 B_{i_0+2}$ and $X_{i_1} \simeq B_{i_1} \times \Omega_0^4 B_{i_1+2}$. On the other hand, by Theorem 1.1(d), there is a homotopy equivalence $\mathcal{G}_k(\mathrm{SU}(n)) \simeq \mathrm{SU}(n) \times \Omega_0^4 \mathrm{SU}(n))$ if and only if k is a multiple of $n(n^2-1)$. Hence if k is not a multiple of $n(n^2-1)$ then there cannot be simultaneous homotopy equivalences $X_{i_0} \simeq B_{i_0} \times \Omega^4 B_{i_0+2}$ and $X_{i_1} \simeq B_{i_1} \times \Omega_0^4 B_{i_1+2}$. Thus we obtain the following:

Theorem 6.3 *Localize at an odd prime p and suppose $n \leq (p-1)^2+1$. Then in* (15) *there are homotopy equivalences $X_{i_0} \simeq B_{i_0} \times \Omega^4 B_{i_0+2}$ and $X_{i_1} \simeq B_{i_1} \times \Omega_0^4 B_{i_1+2}$ if and only if k is a multiple of $n(n^2-1)$.* □

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