

The mapping cone formula in Heegaard Floer homology and Dehn surgery on knots in S^3

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We write down an explicit formula for the $+$ version of the Heegaard Floer homology (as an absolutely graded vector space over an arbitrary field) of the results of Dehn surgery on a knot K in S^3 in terms of homological data derived from $\text{CFK}^\infty(K)$. This allows us to prove some results about Dehn surgery on knots in S^3 . In particular, we show that for a fixed manifold there are only finitely many alternating knots that can produce it by surgery. This is an improvement on a recent result by Lackenby and Purcell. We also derive a lower bound on the genus of knots depending on the manifold they give by surgery. Some new restrictions on Seifert fibred surgery are also presented.

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1 Introduction

Dehn surgery is a fundamental technique in 3–manifold topology. Indeed, we can construct any 3–manifold¹ beginning with any other 3–manifold and performing Dehn surgery enough times. However, it is a highly nontrivial and widely open problem to understand what manifolds can be obtained by doing Dehn surgery once (even starting from the “simplest” 3–manifold, namely S^3) and what knots yield a fixed manifold by surgery.

Heegaard Floer theory is a relatively recent collection of powerful tools in low-dimensional topology. It has many aspects and provides invariants in many different contexts. In this paper, we are only concerned with the 3–manifold and knot invariants (defined in Ozsváth and Szabó [18; 17] and Rasmussen [24]). The collections of 3–manifold invariants and knot invariants are connected via the surgery formula that expresses the Heegaard Floer homology of a 3–manifold obtained by surgery on a given knot in terms of the Heegaard Floer homology data of the knot (see Ozsváth and Szabó [23]). This makes Heegaard Floer homology an especially suitable tool for investigating questions about Dehn surgery.

¹In this paper, whenever we say “3–manifold” we mean “closed connected orientable 3–manifold”.

A natural question about Dehn surgery is whether there are manifolds that can be obtained by surgery on infinitely many distinct knots in S^3 . The answer is “yes”; see Osoinach [13] or Teragaito [27]. There is still hope, however, that perhaps this does not happen for some nice classes of knots.

One interesting and well-studied class of knots is that of alternating knots. At first sight, their diagrammatic definition seems to have little to do with the geometric-topological properties of these knots. However, this is not so; see, for example, Lackenby and Purcell [7]. In particular, they prove the following:

Theorem 1 [7, Theorem 1.3] *For any closed 3–manifold M with sufficiently large Gromov norm, there are at most finitely many prime alternating knots K and fractions p/q such that M is obtained by p/q surgery along K .*

In fact, the statement about fractions p/q can be deduced, for example, from Ni and Wu [11, Theorem 1.2]. We will also show in this paper that given any manifold Y there is a universal bound on q for such fractions, which also implies that they are finite in number. Using techniques that are very different from those used in [7] we are able to establish the following improvement of this theorem.

Theorem 2 *Let Y be a 3–manifold. There are at most finitely many alternating knots $K \subset S^3$ such that $Y = S_{p/q}^3(K)$.*

Heegaard Floer homology is also very useful in bounding genera of various surfaces. In particular, knot Floer homology determines the genus of a knot; see Ozsváth and Szabó [16]. Combining this with information about surgery often allows one to put restrictions on genera of knots admitting certain surgeries. For example, if surgery on a knot K produces an L –space Y (a generalisation of lens spaces; see below for the definition), then $2g(K) - 1 \leq |H_1(Y)|$, where by $g(K)$ we mean the genus of K (see eg [23, Corollary 1.4]).

We derive a bound which is in some respects “opposite” to the bound for L –spaces. It is a lower bound which can be nontrivial only for non- L –spaces. For the statement of the theorem below and the rest of the paper note that we work over an arbitrary field \mathbb{F} . Heegaard Floer homology is then an $\mathbb{F}[U]$ –module and we denote the action of U simply by multiplication. For a rational homology sphere Y , $\text{HF}_{\text{red}}(Y)$ denotes its reduced Floer homology.

Theorem 3 *For any knot $K \subset S^3$ and any $p/q \in \mathbb{Q}$ we have*

$$U^{g(K) + \lceil g_4(K)/2 \rceil} \cdot \text{HF}_{\text{red}}(S_{p/q}^3(K)) = 0.$$

We remark that if K is an L -space knot, then $U^{\lceil g_4(K)/2 \rceil} \cdot \text{HF}_{\text{red}}(S_{p/q}^3(K)) = 0$. Moreover, for any $N > 0$ and $p > 0$ there is a 3-manifold Y which can be obtained by a surgery on a knot in S^3 such that $U^N \cdot \text{HF}_{\text{red}}(Y) \neq 0$ and $|H_1(Y)| = p$.

Here $g_4(K)$ is the slice genus of K . We obviously have $\lceil \frac{1}{2}g_4(K) \rceil \leq \lceil \frac{1}{2}g(K) \rceil$, so the theorem does give a lower bound for $g(K)$.

A different lower bound for the knot genus producing non- L -spaces has been found by Jabuka [3, Theorem 1.3], but unlike our bound, it also depends on the denominator of the slope. Note also that there exists a manifold for which the genus of knots producing it is not bounded above [27].

More recently, Jabuka [4, Corollary 1.5] has produced a new lower bound on the genus that does not involve the denominator of the slope. He also obtained the ranks of $\widehat{\text{HF}}$ for the result of surgery on a knot in S^3 . His genus bound appears to be quite different from ours.

Using the genus bound of Theorem 3 and some other considerations we are able to prove results about Seifert fibred surgery on knots in S^3 . Wu (improving on the results of Ozsváth and Szabó [20]) has proven the following (the definitions of Seifert orientation and torsion coefficients will be provided later):

Theorem 4 [29, Theorems 1.2 and 1.3] *Let $K \subset S^3$ be a knot. Suppose there is a rational number $p/q > 0$ such that $Y = S_{p/q}^3(K)$ is Seifert fibred.*

If Y is a positively oriented Seifert fibred space, then all the torsion coefficients $t_i(K)$ are nonnegative and $\widehat{\text{HFK}}(K, g(K))$ is supported in even degrees. In particular, $\text{deg } \Delta_K = g(K)$.

If Y is a negatively oriented Seifert fibred space and $0 < p/q < 3$, then for all $i > 0$ the torsion coefficients $t_i(K)$ are nonpositive. If Y is a negatively oriented Seifert fibred space, $g(K) > 1$ and $2g(K) - 1 > p/q$, then $\widehat{\text{HFK}}(K, g(K))$ is supported in odd degrees. In particular, $\text{deg } \Delta_K = g(K)$.

We are able to prove the following.

Theorem 5 *Let $K \subset S^3$ be a knot. Suppose there is a rational number $p/q > 0$ such that $Y = S_{p/q}^3(K)$ is a negatively oriented Seifert fibred space. Then*

- $U^{g(K)} \cdot \text{HF}_{\text{red}}(Y) = 0$;
- if $0 < p/q \leq 3$, then all the torsion coefficients $t_i(K)$ are nonpositive (including $t_0(K)$) and $\text{deg } \Delta_K = g(K)$;

- more generally, if $i \geq \lfloor \frac{1}{2}(\lceil p/q \rceil - \sqrt{\lceil p/q \rceil}) \rfloor$, then t_i is nonpositive;
- if $g(K) > \lfloor \frac{1}{2}(\lceil p/q \rceil - \sqrt{\lceil p/q \rceil}) \rfloor$, then $\deg \Delta_K = g(K)$;
- if $U^{\lfloor |H_1(Y)|/2 \rfloor} \cdot \text{HF}_{\text{red}}(Y) \neq 0$ then $\deg \Delta_K = g(K)$.

In all statements where $\deg \Delta_K = g(K)$ we have that $\widehat{\text{HF}}\mathbb{K}(K, g(K))$ is supported in odd degrees.

After the proof of Theorem 3 in Section 5, we describe negatively oriented Seifert fibred spaces Y for which the power of U needed to annihilate $\text{HF}_{\text{red}}(Y)$ gets arbitrarily large compared to the order of the first homology group.

Theorem 5 combined with the result of Wu has the following straightforward corollary.

Corollary 6 *Suppose $Y = S^3_{p/q}(K)$ is a Seifert fibred rational homology sphere. If $|H_1(Y)| \leq 3$, then all the torsion coefficients of K have the same sign and $\deg \Delta_K = g(K)$.*

To prove Theorems 2 and 3 we need to study the mapping cone formula, which connects the Heegaard Floer data of the knot with the Heegaard Floer homology of the manifolds obtained by surgery on it. Given a knot K in S^3 there is a doubly filtered complex $C = \text{CFK}^\infty(K)$ associated to it. The doubly filtered homotopy type of this complex is a knot invariant, from which all the flavours of knot Floer homology are derived.

In fact, the mapping cone formula states that given C and a certain chain homotopy equivalence which identifies $C\{i \geq 0\}$ with $C\{j \geq 0\}$ we can determine $\text{HF}^+(S^3_{p/q}(K))$ completely for any rational p/q .

In Section 3 we explicitly describe $\text{HF}^+(S^3_{p/q}(K))$ as an absolutely graded vector space in terms of homological data from $\text{CFK}^\infty(K)$, with no reference to the chain homotopy equivalence mentioned above. For a large part this has already been done (see Ni and Zhang [12], Ni and Wu [11] and Ozsváth and Szabó [23]), but the results are scattered across multiple papers, sometimes not in explicit form, and we consider it useful to have them collected in one place. While all the results of this section concerning positive surgeries have been shown before, as far as we are aware, the results for negative and zero surgeries (contained in Sections 3.2 and 3.3, respectively) are new.

This allows us to derive some other applications as well, a few of which we mention here.

Theorem 7 *Suppose K is a nontrivial knot and $Y = S^3_{p/q}(K)$. Then*

$$|q| \leq |H_1(Y)| + \dim \text{HF}_{\text{red}}(Y).$$

The existence of such a bound for the denominator seems to be known to some experts in Heegaard Floer homology (it could be deduced from [23, Proposition 9.6]) but we have not seen it explicitly stated. We will use this fact in the proof of Theorem 2.

A bound on the number of slopes that produce a given manifold has also been obtained by Lackenby [6, Theorem 2.9] in quite a general but somewhat different setting (in particular, due to homological conditions, it does not deal with surgeries in S^3).

Theorem 8 *Let K be an L -space knot and $p/q \leq 1$ a rational number. Then $S_{p/q}^3(K)$ and p/q determine the Alexander polynomial of K .*

We remark that the Alexander polynomial determines the knot Floer homology of an L -space knot; see Ozsváth and Szabó [21]. Conversely, the Alexander polynomial of an L -space knot determines the HF^+ of all surgeries on it. In particular, if for two L -space knots K and K' we have that $\text{HF}^+(S_{p/q}^3(K)) \cong \text{HF}^+(S_{p/q}^3(K'))$ for some $p/q \leq 1$, then we also have $\text{HF}^+(S_{p'/q'}^3(K)) \cong \text{HF}^+(S_{p'/q'}^3(K'))$ for all other p'/q' .

This theorem also implies that if two torus knots $T_{r,s}, T_{r',s'}$ with $r, s, r', s' > 0$ have the same surgery with the same slope ≤ 1 , then they are the same. Presumably, this can also be obtained by more elementary methods. Note, however, that there do exist positive integral slopes for which there are two distinct torus knots with the same surgeries at these slopes; see Ni and Zhang [12, Example 1.1].

Teragaito [27] constructs a small Seifert fibred space Y and a sequence of knots $K_n \subset S^3$ such that $K_n(-4) = Y$.² Moreover, the genus of the knots K_n is unbounded. Incidentally, this shows that we cannot hope for an upper bound on knot genus for knots giving some arbitrary manifold by surgery. In Section 7 we show that Y can only be obtained by (-4) -surgery and we find the Alexander polynomial of the knots K_n . It is, in fact, possible to find the Heegaard Floer homology of all manifolds obtained by surgery on each K_n .

The organisation of this paper is as follows. In Section 2 we review some definitions, notation and the mapping cone formula. In Section 3 we derive the expression for the Heegaard Floer homology of surgeries on a knot. In Section 4 we prove Theorem 2, in Section 5 we prove Theorem 3 and in Section 6 we prove Theorem 5. Finally, in Section 7 we present some other applications of the mapping cone formula to questions in Dehn surgery.

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²In fact, Teragaito constructs $-Y$, his knots are the mirror images of K_n and the slope he uses is 4. It is more convenient for us to work with this orientation.

Theorem 2 could be proven using techniques of this paper). I am also very grateful to Marco Marengon, Tom Hockenhull and V S Pyasetkii for many important comments on the structure and presentation of this paper. This paper greatly benefited from a visit to the University of Texas at Austin, and many interesting and enlightening conversations that I had there. For this opportunity, I am very thankful to the Doris Chen Award, and the help of my supervisor Dorothy Buck. I am particularly grateful to Dorothy for her continued encouragement and support over the course of my PhD studies. Finally, I would like to thank the referee for helpful comments on the drafts of this paper.

2 The mapping cone formula

In this section, we set up notation and review the rational surgery formula of Ozsváth and Szabó [23]. We largely follow the exposition and notation of Ni and Wu in [11].

Given a knot K in S^3 we can associate to it a doubly filtered complex $C = \text{CFK}^\infty(K)$. We denote generators of this complex by $[x, i, j]$, where this generator has filtration $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. By [24, Lemma 4.5] the complex C is homotopy equivalent (as a filtered complex) to a complex for which all filtration-preserving differentials are trivial. In other words, at each filtration level we replace the group, viewed as a chain complex with the filtration preserving differential, by its homology. From now on we work with this *reduced* complex.

The complex C is invariant under the shift by the vector $(-1, -1)$. There is an action of a formal variable U on C which is simply the translation by the vector $(-1, -1)$. In other words, the group at the filtration level (i, j) is the same as the one at the filtration level $(i - 1, j - 1)$ and U is the identity map from the first one to the second. Of course, U is a chain map. In C the map U is invertible (but note that it will not be in various subcomplexes and quotients), so C is an $\mathbb{F}[U, U^{-1}]$ -module.

This means that as an $\mathbb{F}[U, U^{-1}]$ -module C is generated by the elements with the first filtration level $i = 0$. In the reduced complex the group at filtration level $(0, j)$ is denoted $\widehat{\text{HFK}}(K, j)$ and is known as the knot Floer homology of K (at Alexander grading j).

The complex C is absolutely \mathbb{Z} -graded. In fact, the complex C is the complex used to compute the (∞ -flavour of the) Heegaard Floer homology of S^3 , and the knot provides an additional filtration for it. By grading the Heegaard Floer homology of S^3 we obtain the grading on C . The map U decreases this grading by 2.

Using the filtration on C we can define the following quotients of it:

$$\begin{aligned} A_k^+(K) &= C\{i \geq 0 \text{ or } j \geq k\}, \quad k \in \mathbb{Z}, \\ B^+ &= C\{i \geq 0\} \cong \text{CF}^+(S^3). \end{aligned}$$

We also define two chain maps $v_k, h_k: A_k^+(K) \rightarrow B^+$. The first one is just the projection (ie it sends to zero all generators with $i < 0$ and acts as the identity map for everything else). The second one is the composition of three maps: firstly we project to $C\{j \geq k\}$, then we multiply by U^k (this shifts everything by the vector $(-k, -k)$) and finally we apply a chain homotopy equivalence that identifies $C\{j \geq 0\}$ with $C\{i \geq 0\}$. Such a chain homotopy equivalence exists because the two complexes both represent $CF^+(S^3)$ and by general theory [17] there is a chain homotopy equivalence between them, induced by the moves between the Heegaard diagrams. We usually do not know the explicit form of this chain homotopy equivalence.

Genus detection, alluded to before, has the following form:

Theorem 9 (Ozsváth and Szabó [16, Theorem 1.2]) *Let $K \subset S^3$ be a knot.*

Then $g(K) = \max\{j \in \mathbb{Z} \mid \widehat{HF}(K, j) \neq 0\}$.

From this (together with symmetries of C) we can see that the maps v_k (resp. h_k) are isomorphisms if $k \geq g$ (resp. $k \leq -g$).

We define chain complexes

$$A_{i,p/q}^+(K) = \bigoplus_{n \in \mathbb{Z}} (n, A_{\lfloor (i+pn)/q \rfloor}^+(K)), \quad B^+ = \bigoplus_{n \in \mathbb{Z}} (n, B^+).$$

The first entry in the brackets here is simply a label used to distinguish different copies of the same group. There is a chain map $D_{i,p/q}^+$ from $A_{i,p/q}^+(K)$ to B^+ defined by taking sums of all maps v_k, h_k with appropriate domains and requiring that the map v_k goes to the group with the same label n and h_k increases the label by 1. Explicitly, $D_{i,p/q}^+(\{(k, a_k)\}_{k \in \mathbb{Z}}) = \{(k, b_k)\}_{k \in \mathbb{Z}}$, where

$$b_k = v_{\lfloor (i+pk)/q \rfloor}^+(a_k) + h_{\lfloor (i+p(k-1))/q \rfloor}^+(a_{k-1}).$$

Each of $A_k^+(K)$ and B^+ inherits a relative \mathbb{Z} -grading from the one on C . Let $\mathbb{X}_{i,p/q}^+$ denote the mapping cone of $D_{i,p/q}^+$. We fix a relative \mathbb{Z} -grading on the whole of it by requiring that the maps v_k, h_k (and so $D_{i,p/q}^+$) decrease it by 1. The following is proven in [23].

Theorem 10 (Ozsváth and Szabó [23, Theorem 1.1]) *There is a relatively graded isomorphism of $\mathbb{F}[U]$ -modules*

$$H_*(\mathbb{X}_{i,p/q}^+) \cong HF^+(S_{p/q}^3(K), i).$$

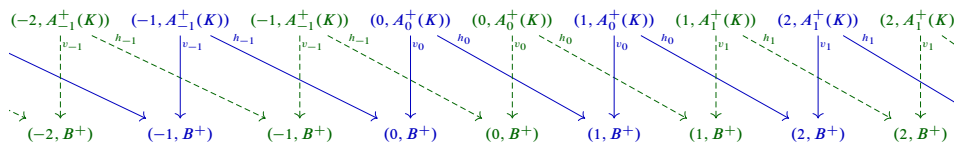


Figure 1: Schematic representation of the portion of the complex whose mapping cone gives the Heegaard Floer homology of the surgery on a knot. This case illustrates $\frac{2}{3}$ -surgery; the blue (solid) and green (dashed) subcomplexes represent two different Spin^c -structures on the resulting space.

The index i in $\text{HF}^+(S^3_{p/q}(K), i)$ stands for the Spin^c -structure. The numbering of Spin^c -structures we refer to is defined in [23], but we do not need precise details of how to obtain this numbering for our purposes.

We can also determine the absolute grading on the mapping cone. The group B^+ is independent of the knot. Now if we insist that the absolute grading on the mapping cone for the unknot should coincide with the grading of HF^+ of the surgery on it (ie $d(L(p, q), i)$), this fixes the grading on B^+ . We then use this grading to fix the grading on $\mathbb{X}^+_{i,p/q}$ for arbitrary knots; this grading then is the correct grading, ie it coincides with the one HF^+ should have.

The mental picture we have of the mapping cone theorem is illustrated in Figure 1. We have two rows of groups. The bottom row is just the row of identical groups B^+ . The upper row consists of the various ‘‘hook’’ groups $A_k^+(K)$. Specifically, if the surgery slope is p/q , in the upper row we repeat each group q times. We then have vertical arrows pointing down for the maps v_k , and the arrows for the maps h_k are slanted. More precisely, they go p groups to the right (if p is negative, this means $-p$ to the left). This creates $|p|$ subcomplexes, connected by a zig-zag set of arrows. Each such zig-zag subcomplex corresponds to a Spin^c -structure on the manifold that is the result of the surgery. To obtain the Heegaard Floer homology of this manifold we need to take the mapping cone of this chain map.

For our purposes, it suffices to pass to the homology of the mapping cone under consideration. Let

$$A_k^+(K) = H_*(A_k^+(K)), \quad B^+ = H_*(B^+),$$

$$\mathbb{A}^+_{i,p/q}(K) = H_*(\mathbb{A}^+_{i,p/q}(K)), \quad \mathbb{B}^+ = H_*(B^+)$$

and let $v_k, h_k, D^+_{i,p/q}$ denote the maps induced by $v_k, h_k, D^+_{i,p/q}$, respectively, in homology.

When we talk about $\mathbb{A}^+_{i,p/q}(K)$ as an absolutely graded group, we mean the grading that it inherits from the absolute grading of the mapping cone that we described above.

Since $B^+ \cong \text{CF}^+(S^3)$, we have $B^+ \cong \mathcal{T}_d^+$, where $\mathcal{T}_d^+ \cong \mathbb{F}[U^{-1}] = \mathbb{F}[U, U^{-1}]/U\mathbb{F}[U]$, d signifies the grading of 1 and multiplication by U decreases the grading by 2. We sometimes call this module the *tower*. When we are not interested in the absolute grading we omit the subscript.

Recall that the short exact sequence

$$0 \longrightarrow B^+ \xrightarrow{i} \mathbb{X}_{i,p/q}^+ \xrightarrow{j} \mathcal{A}_{i,p/q}^+(K) \longrightarrow 0$$

induces the exact triangle

$$(1) \quad \begin{array}{ccc} \mathcal{A}_{i,p/q}^+(K) & \xrightarrow{D_{i,p/q}^+} & B^+ \\ & \swarrow j_* & \downarrow i_* \\ & & H_*(\mathbb{X}_{i,p/q}^+) \cong \text{HF}^+(S_{p/q}^3(K), i) \end{array}$$

All maps in these sequences are U -equivariant. This triangle is the main tool in the calculations of the next section. In particular, if the surgery slope is positive, then the map $D_{i,p/q}^+$ will be surjective, so the triangle above implies that $\text{HF}^+(S_{p/q}^3(K), i) \cong \ker D_{i,p/q}^+$.

3 Calculations

In this section we want to use the mapping cone formula to calculate the Heegaard Floer homology for the results of surgery on a knot in S^3 . Given a rational homology sphere Y and a Spin^c -structure \mathfrak{s} , we have $\text{HF}^+(Y, \mathfrak{s}) = \mathcal{T}_d^+ \oplus \text{HF}_{\text{red}}(Y, \mathfrak{s})$, where $d = d(Y, \mathfrak{s})$ is called the *correction term* and $\text{HF}_{\text{red}}(Y, \mathfrak{s})$ is a finite-dimensional $\mathbb{F}[U]$ -module annihilated by a big enough power of U , called the reduced Floer homology of Y in Spin^c -structure \mathfrak{s} . The sum of these groups over all Spin^c -structures is called the reduced Floer homology of Y and is denoted by $\text{HF}_{\text{red}}(Y)$.

We state a weaker version of [22, Theorem 2.3].

Theorem 11 (Ozsváth and Szabó) *There is an integer N such that for all $m \geq N$ and all $i \in \mathbb{Z}/m\mathbb{Z}$ there is an isomorphism of relatively graded $\mathbb{F}[U]$ -modules*

$$\mathcal{A}_k^+(K) \cong \text{HF}^+(K_m, i),$$

where $k \equiv i \pmod{m}$ and $|k| \leq \frac{1}{2}m$.

In particular, each $A_k^+(K)$ is an HF^+ of a rational homology sphere in a certain Spin^c -structure, hence by the previous paragraph we can decompose it as $A_k^+(K) \cong A_k^T(K) \oplus A_k^{\text{red}}(K)$, where $A_k^{\text{red}}(K)$ is a finite-dimensional vector space in the kernel of some power of U and $A_k^T(K) \cong \mathcal{T}^+$.

We will need to talk about the Euler characteristic of the groups $A_k^{\text{red}}(K)$, so we need to fix an absolute $\mathbb{Z}/2\mathbb{Z}$ -grading for them. We do so by requiring that for the purposes of this grading each group $A_k^T(K)$ lies entirely in grading 0 and then using the relative $\mathbb{Z}/2\mathbb{Z}$ -grading (induced by the parity of the relative \mathbb{Z} -grading) on $A_k^+(K)$.

A rational homology sphere Y is called an L -space if $\text{HF}_{\text{red}}(Y, \mathfrak{s}) = 0$ for all Spin^c -structures \mathfrak{s} . A knot $K \subset S^3$ is called an L -space knot if some positive surgery on it is an L -space. In fact, it is known that a p/q surgery on an L -space knot is an L -space if and only if $p/q \geq 2g(K) - 1$ (here $g(K)$ is, as usual, the genus of K). In particular, all large surgeries on L -space knots are L -spaces, hence for any L -space knot K we have $A_k^{\text{red}}(K) = 0$ for all k .

In the same way we can decompose the complexes of the exact triangle (1):

$$\mathbb{A}_{i,p/q}^T(K) = \bigoplus_{n \in \mathbb{Z}} (n, A_{\lfloor (i+pn)/q \rfloor}^T(K)), \quad \mathbb{A}_{i,p/q}^{\text{red}}(K) = \bigoplus_{n \in \mathbb{Z}} (n, A_{\lfloor (i+pn)/q \rfloor}^{\text{red}}(K)).$$

We can also decompose the map $D_{i,p/q}^+ = D_{i,p/q}^T \oplus D_{i,p/q}^{\text{red}}$, where the first map is the restriction of $D_{i,p/q}^+$ to $\mathbb{A}_{i,p/q}^T(K)$ and the second one is the restriction to $\mathbb{A}_{i,p/q}^{\text{red}}(K)$. Note that $D_{i,p/q}^+ = D_{i,p/q}^T$ for L -space knots.

Now the restrictions of v_k and h_k to $A_k^T(K)$ are multiplications by some powers of U , which we denote by V_k and H_k , respectively. (This is because at large gradings these maps are isomorphisms.) The following are some useful properties of V_k and H_k (see [11]):

- $V_k \geq V_{k+1}$ for any $k \in \mathbb{Z}$;
- $H_k \leq H_{k+1}$ for any $k \in \mathbb{Z}$;
- $V_k = H_{-k}$ for any $k \in \mathbb{Z}$;
- $V_k \rightarrow +\infty$ as $k \rightarrow -\infty$;
- $H_k \rightarrow +\infty$ as $k \rightarrow +\infty$;
- $V_k = 0$ for $k \geq g(K)$;
- $H_k = 0$ for $k \leq -g(K)$.

In other words, the V_k form a nonincreasing unbounded sequence of nonnegative numbers which become zero at $g(K)$, and $H_k = V_{-k}$.

We now separate into three different cases. Firstly, we cover the case of positive surgery slope. Secondly, we treat negative surgeries. The third case is the zero surgery.

3.1 Positive surgeries

The next lemma is used to establish that $D_{i,p/q}^T$ is surjective when $p/q > 0$. We state it in a slightly more general form because in this form it also applies to other manifolds.

Lemma 12 *Let $X = Y = \bigoplus_{i \in \mathbb{Z}} (i, \mathcal{T}^+)$, $X' = \bigoplus_{i \neq 0} (i, \mathcal{T}^+)$ and let the maps*

$$\alpha_i: (i, \mathcal{T}^+) \rightarrow (i, \mathcal{T}^+), \quad \beta_i: (i, \mathcal{T}^+) \rightarrow (i + 1, \mathcal{T}^+)$$

be multiplications by U^{a_i} and U^{b_i} , respectively. Suppose further that

- *there is a number N such that $a_i = 0$ for $i \geq N$ and $b_i = 0$ for $i \leq -N$, and*
- *$a_i \rightarrow +\infty$ as $i \rightarrow -\infty$ and $b_i \rightarrow +\infty$ as $i \rightarrow +\infty$.*

Define D to be the sum of the maps α_i and β_i . Then the restriction of D to X' is surjective.

The setting here is very similar to the one described by Figure 1, only we choose one of the zig-zag complexes and all the groups in both the top and the bottom row are the towers; see Figure 2.

Proof This is essentially what Ni and Wu prove in [11, Lemma 2.8]. We will show that, for any $n \geq 0$ and $i \leq 0$, (i, U^{-n}) is in the image of the restriction of D to X' . The conclusion will then follow by symmetry and linearity.

We clearly have $(i, U^{-n}) = \beta_{i-1}(i - 1, U^{-n-b_{i-1}})$. Define $\xi = \{(i, \xi_i)\}_{i \in \mathbb{Z}} \in X'$ recursively by

$$\xi_s = \begin{cases} 0 & \text{if } s \geq i, \\ U^{-n-b_{i-1}} & \text{if } s = i - 1, \\ (-1)^{s-i+1} U^{a_{s+1}-b_s} \cdot \xi_{s+1} & \text{otherwise.} \end{cases}$$

In a way, after we set that $\xi_s = 0$ for $s \geq i$, this is the only possible definition (up to the kernel of D). This is because the arrow “slanted to the right” has to be used to cancel the rightmost element in the lower row, hence we know what element in its codomain we have to choose so that it indeed cancels. This tells us what the image of the “vertical” arrow is and hence what the next “slanted” arrow has to cancel, etc.

Since we have $a_{s+1} - b_s \rightarrow +\infty$ as $s \rightarrow -\infty$, ξ only has a finite number of nonzero coordinates and hence is a well-defined element of X' . It is also easy to see that $D(\xi) = (i, U^{-n})$. □

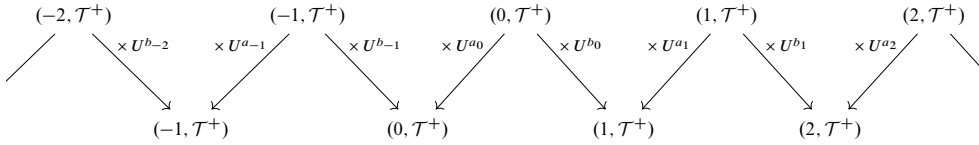


Figure 2: Maps and groups of Lemmas 12 and 13

Let $\tau_d(N)$ be a submodule of \mathcal{T}_d^+ generated by $\{U^{-n}\}_{0 \leq n \leq N-1}$. As before, we omit the subscript in the absence of the absolute grading.

The setting of the next lemma is less general; indeed, we use more information about the numbers V_k and H_k .

Lemma 13 *To the assumptions of Lemma 12 add the following:*

- (a_i) is a nonincreasing sequence;
- (b_i) is a nondecreasing sequence;
- $a_i \leq b_i$ for $i \geq 0$;
- $a_i \geq b_i$ for $i < 0$.

Put absolute gradings on X and Y by the rule that the maps α_i and β_i decrease it by 1, the multiplication by U decreases it by 2 and $1 \in (0, \mathcal{T}^+) \subset Y$ has grading $d - 1$, where d is some rational number.

Then if $a_0 \geq b_{-1}$ we have

$$\ker D \cong \mathcal{T}_{d-2a_0}^+ \oplus \bigoplus_{n \geq 1} \tau_{d_n}^-(b_{-n}) \oplus \bigoplus_{n \geq 1} \tau_{d_n}^+(a_n).$$

Otherwise

$$\ker D \cong \mathcal{T}_{d-2b_{-1}}^+ \oplus \bigoplus_{n \geq 2} \tau_{d_n}^-(b_{-n}) \oplus \bigoplus_{n \geq 0} \tau_{d_n}^+(a_n).$$

The isomorphisms are as absolutely graded $\mathbb{F}[U]$ -modules. The numbers d_n^\pm are defined by

$$\begin{aligned} d_0^\pm &= d - 2 \max\{a_0, b_{-1}\}, \\ d_{n+1}^- &= d_n^- + 2(a_{-n} - b_{-(n+1)}), \\ d_{n+1}^+ &= d_n^+ + 2(b_n - a_{n+1}). \end{aligned}$$

Proof The two cases are completely analogous, so we will assume $a_0 \geq b_{-1}$. First, following [11, proof of Proposition 1.6] we define $\rho^T: \mathcal{T}_{d-2a_0}^+ \rightarrow \ker D$ as follows. If we write $\rho^T(\eta) = \{(s, \xi_s)\}_{s \in \mathbb{Z}}$, we set $\xi_0 = \eta$ and determine the other components by

$$\xi_s = \begin{cases} -U^{b_{s-1}-a_s} \xi_{s-1} & \text{if } s > 0, \\ -U^{a_{s+1}-b_s} \xi_{s+1} & \text{if } s < 0. \end{cases}$$

In effect, we want to simply send the tower to the tower in the 0-component of the upper group. But it is not in the kernel of D , so we need to correct for that. In fact, we also want the map to be an $\mathbb{F}[U]$ -module homomorphism, which is the reason for considering the cases $a_0 \geq b_{-1}$ and $a_0 < b_{-1}$ separately.

Notice that we always multiply by a nonnegative power of U : if $s > 0$, then $b_{s-1} \geq a_{s-1} \geq a_s$; if $s = -1$, this is the assumption $a_0 \geq b_{-1}$; if $s < -1$, then $a_{s+1} \geq b_{s+1} \geq b_s$. Thus the map is indeed an $\mathbb{F}[U]$ -module homomorphism.

As before, $\xi_s = 0$ if $|s|$ is very big, so the map is well-defined. The map ρ^T is one-to-one because its 0-component is (ie $\xi_0 = \eta$). It is also graded correctly (ie the map ρ^T sends homogeneous elements of absolute grading d to homogeneous elements of grading d) because $(0, U^{-a_0}) \in X$ is sent to $(0, 1) \in Y$ by α_0 , which has grading $d - 1$. Thus $(0, 1) \in X$ has grading $d - 2a_0$, since to descend from $(0, U^{-a_0}) \in X$ to $(0, 1) \in X$ we need to multiply by U^{a_0} and multiplication by U has grading -2 .

We have identified the tower in the kernel. Now we need to deal with the rest of it. Below we prove that the rest of the kernel consist of the kernels of the maps $\alpha_i + \beta_i$ for each i , except the one at which the tower is situated (ie $i = 0$). It is easy to see that the kernel of $\alpha_i + \beta_i$ is isomorphic to $\tau(\min(a_i, b_i))$.

If $v = \{(s, v_s)\}_{s \in \mathbb{Z}} \in \ker D$, by subtracting elements in the image of ρ^T we may assume that $v \in X'$, ie $v_0 = 0$. Without loss of generality, there exists $s < 0$ such that $v_s \neq 0$. To finish the proof we need to show that $U^{b_s} \cdot v_s = 0$ (recall that in this range $b_s \leq a_s$). Suppose this is not so and $0 \neq U^{b_s} \cdot v_s$. Since v is in the kernel, it has to be cancelled by something. It follows that we must have $\beta_s(v_s) + \alpha_{s+1}(v_{s+1}) = 0$. Thus $0 \neq U^{b_s} \cdot v_s = -U^{a_{s+1}} v_{s+1}$ implies that $0 \neq U^{b_{s+1}} v_{s+1}$, as $a_{s+1} \geq b_{s+1}$ if $s < -1$. By proceeding in this way it follows that $v_0 \neq 0$, ie $v \notin X'$, a contradiction. \square

The two lemmas above can be readily translated into results about surgery. The d -invariant formula (2) from the corollary below is [11, Proposition 1.6].

Corollary 14 *If $p/q > 0$ the map $D_{i,p/q}^T$ is surjective. It follows that so is $D_{i,p/q}^+$, and we conclude that $\text{HF}^+(S_{p/q}^3(K), i) \cong \ker D_{i,p/q}^+$.*

If $\lfloor i/q \rfloor \leq -\lfloor (i-p)/q \rfloor$, then

$$\ker D_{i,p/q}^T \cong \mathcal{T}_d^+ \oplus \bigoplus_{n \geq 1} \tau_{d_n}^-(H_{\lfloor (i-np)/q \rfloor}) \oplus \bigoplus_{n \geq 1} \tau_{d_n}^+(V_{\lfloor (i+np)/q \rfloor}).$$

Otherwise

$$\ker D_{i,p/q}^T \cong \mathcal{T}_d^+ \oplus \bigoplus_{n \geq 2} \tau_{d_n}^-(H_{\lfloor (i-np)/q \rfloor}) \oplus \bigoplus_{n \geq 0} \tau_{d_n}^+(V_{\lfloor (i+np)/q \rfloor}).$$

Here

$$(2) \quad d = d(S_{p/q}^3(K), i) = d(L(p, q), i) - 2 \max\{V_{\lfloor i/q \rfloor}, H_{\lfloor (i-p)/q \rfloor}\},$$

and

$$d_n^- = d + 2 \sum_{k=0}^{n-1} (V_{\lfloor (i-kp)/q \rfloor} - H_{\lfloor (i-(k+1)p)/q \rfloor}),$$

$$d_n^+ = d + 2 \sum_{k=0}^{n-1} (H_{\lfloor (i+kp)/q \rfloor} - V_{\lfloor (i+(k+1)p)/q \rfloor}).$$

Proof This is a straightforward application of Theorem 10 and Lemmas 12 and 13 after renumbering of the groups and maps; objects numbered with $\lfloor (i + np)/q \rfloor$ correspond to the ones numbered with n in Lemmas 12 and 13.

To fix the grading, note that the grading of \mathbb{B}^+ does not depend on the knot, but only on the surgery slope. Thus to grade it we can take the unknot U . For the unknot we have $V_i = 0$ for $i \geq 0$ and $V_i = i$ for $i < 0$. Hence $0 = V_{\lfloor i/q \rfloor} \geq H_{\lfloor (i-p)/q \rfloor} = 0$, and by the same argument as we used for an arbitrary knot, the grading of 1 in $(0, \mathbf{A}_{\lfloor i/q \rfloor}^+(U))$ is the d -invariant of the surgery, which we know to be $d(L(p, q), i)$ in this case. Since $V_{\lfloor i/q \rfloor} = 0$, we find that the grading of 1 in $(0, \mathbf{B}^+)$ is $d(L(p, q), i) - 1$. This allows us to fix the d -invariants for all other knots.

We can fix d_n^\pm by the fact that the maps v_k and h_k reduce it by 1 and the multiplication by U reduces it by 2. □

As we noted before, for L -space knots, $\mathbf{D}_{i,p/q}^+ = \mathbf{D}_{i,p/q}^T$. Let K be a knot and $\Delta_K(T) = a_0 + \sum_i a_i(T^i + T^{-i})$ be its symmetrised Alexander polynomial, with normalisation convention $\Delta_K(1) = 1$. Define its *torsion coefficients* $t_i(K)$ for $i \geq 0$ by

$$t_i(K) = \sum_{j \geq 1} j a_{i+j}.$$

Clearly, if we know all the torsion coefficients, we know the Alexander polynomial. For L -space knots, $V_k = t_k$ for $k \geq 0$ (this follows, for example, from [23, Theorem 1.2]), so Corollary 14 determines the Heegaard Floer homology of positive surgeries on an L -space knot in terms of its Alexander polynomial.

The next proposition expresses the Heegaard Floer homology of positive surgeries for arbitrary knots in terms of data from CFK^∞ . This proposition is essentially [12, Proposition 3.5].

Proposition 15 As absolutely graded vector spaces,

$$\ker \mathbf{D}_{i,p/q}^+ \cong \ker \mathbf{D}_{i,p/q}^T \oplus \mathbb{A}_{i,p/q}^{\text{red}}(K).$$

Moreover, $\ker \mathbf{D}_{i,p/q}^T$ is actually a submodule of $\ker \mathbf{D}_{i,p/q}^+$.

Proof This is a straightforward exercise in linear algebra.

Given vector spaces U, V, W and linear maps $\rho_U: U \rightarrow W$ and $\rho_V: V \rightarrow W$ such that ρ_U is surjective, $\ker(\rho_U \oplus \rho_V) \cong \ker \rho_U \oplus V$.

There exists a map $\rho_U^*: W \rightarrow V$ such that $\rho_U \circ \rho_U^* = \text{id}_W$. In the graded situation we can make ρ_U^* send homogeneous elements to homogeneous elements. Then we can define $T: \ker \rho_U \oplus V \rightarrow \ker(\rho_U \oplus \rho_V)$ by $T(x \oplus y) = (x - \rho_U^* \circ \rho_V(y)) \oplus y$. Since in our case $\rho_U \oplus \rho_V$ is graded, T is an isomorphism of graded vector spaces. \square

Let

$$\mathbb{A}^s(K) = \bigoplus_{k \in \mathbb{Z}} A_k^{\text{red}}(K).$$

This is a finite-dimensional vector space, as each $A_k^{\text{red}}(K)$ is and $A_k^{\text{red}}(K) = 0$ for $|k| \geq g(K)$. We define $\delta(K) = \dim \mathbb{A}^s(K)$. Note that $\delta(K) = 0$ if and only if K is an L -space knot. The following proposition generalises [11, Proposition 5.3]:

Proposition 16 [12, Corollary 3.6] Let $K \subset S^3$ be a knot and $p/q > 0$. Then

$$(3) \dim \text{HF}_{\text{red}}(S_{p/q}^3(K)) = q\delta(K) + qV_0 + 2q \sum_{i=1}^{g-1} V_i - \sum_{i=0}^{p-1} \max(V_{\lfloor i/q \rfloor}, H_{\lfloor (i-p)/q \rfloor}).$$

Proof Since

$$\dim \text{HF}_{\text{red}}(S_{p/q}^3(K)) = \sum_{i=0}^{p-1} \dim \text{HF}_{\text{red}}(S_{p/q}^3(K), i),$$

combining Proposition 15 and Corollary 14 we see that

$$\begin{aligned} \dim \text{HF}_{\text{red}}(S_{p/q}^3(K)) &= \sum_{i \in \mathbb{Z}} \dim A_{\lfloor i/q \rfloor}^{\text{red}}(K) + \sum_{i \geq 0} V_{\lfloor i/q \rfloor} + \sum_{i \geq 1} H_{\lfloor -i/q \rfloor} - \sum_{i=0}^{p-1} \max(V_{\lfloor i/q \rfloor}, H_{\lfloor (i-p)/q \rfloor}) \\ &= q \sum_{k \in \mathbb{Z}} \dim A_k^{\text{red}}(K) + q \sum_{i=0}^{g-1} V_i + q \sum_{i=-(g-1)}^{-1} H_i - \sum_{i=0}^{p-1} \max(V_{\lfloor i/q \rfloor}, H_{\lfloor (i-p)/q \rfloor}) \\ &= q\delta(K) + qV_0 + 2q \sum_{i=1}^{g-1} V_i - \sum_{i=0}^{p-1} \max(V_{\lfloor i/q \rfloor}, H_{\lfloor (i-p)/q \rfloor}). \end{aligned} \quad \square$$

Now we are ready to prove Theorem 7.

Theorem 7 *Suppose K is a nontrivial knot and $Y = S^3_{p/q}(K)$. Then*

$$|q| \leq |H_1(Y)| + \dim \text{HF}_{\text{red}}(Y).$$

Proof This is an easy consequence of Ni and Zhang’s formula of Proposition 16 (by taking the mirror image we may assume $p/q > 0$). We have

$$\begin{aligned} \dim \text{HF}_{\text{red}}(S^3_{p/q}(K)) + \sum_{i=0}^{p-1} \max(V_{\lfloor i/q \rfloor}, H_{\lfloor (i-p)/q \rfloor}) \\ = q\delta(K) + qV_0 + 2q \sum_{i=1}^{g-1} V_i \geq q(\delta(K) + V_0). \end{aligned}$$

Recall that $\delta(K) = \dim \mathbb{A}^s(K)$, so it is nonnegative and $\delta(K) = 0$ if and only if K is an L -space knot, in which case $V_k = 0$ if and only if $k \geq g(K)$, so for nontrivial L -space knots $V_0 \neq 0$. If $V_0 = 0$ then all V ’s (and H ’s) are zero and as $\delta(K) \neq 0$ by the previous sentence, we clearly get $q \leq \dim \text{HF}_{\text{red}}(S^3_{p/q}(K))$.

So suppose $V_0 \neq 0$. Then

$$\begin{aligned} \dim \text{HF}_{\text{red}}(S^3_{p/q}(K)) + pV_0 &\geq \dim \text{HF}_{\text{red}}(S^3_{p/q}(K)) + \sum_{i=0}^{p-1} \max(V_{\lfloor i/q \rfloor}, H_{\lfloor (i-p)/q \rfloor}) \\ &\geq q(\delta(K) + V_0). \end{aligned}$$

Finally, we have

$$\begin{aligned} q &\leq \frac{\dim \text{HF}_{\text{red}}(S^3_{p/q}(K)) + pV_0}{\delta(K) + V_0} \\ &= \frac{\dim \text{HF}_{\text{red}}(S^3_{p/q}(K))}{\delta(K) + V_0} + \frac{pV_0}{\delta(K) + V_0} \\ &\leq \dim \text{HF}_{\text{red}}(S^3_{p/q}(K)) + p. \end{aligned} \quad \square$$

3.2 Negative surgeries

In the case when $p/q < 0$ the map $D^+_{i,p/q}$ is no longer surjective. However, we can show that the cokernel consists of exactly the tower part and the kernel is the reduced Floer homology $\text{HF}_{\text{red}}(S^3_{p/q}(K), i)$. We start with a general lemma, which is similar to Lemmas 12 and 13. The main difference is in that the β_i maps go to the groups labelled with a smaller index.

Lemma 17 Let $X = Y = \bigoplus_{i \in \mathbb{Z}} (i, \mathcal{T}^+)$ and let the maps

$$\alpha_i: (i, \mathcal{T}^+) \rightarrow (i, \mathcal{T}^+), \quad \beta_i: (i, \mathcal{T}^+) \rightarrow (i - 1, \mathcal{T}^+)$$

be multiplications by U^{a_i} and U^{b_i} , respectively. Suppose further that a_i and b_i have the following properties:

- There is a number N such that $a_i = 0$ for $i \geq N$ and $b_i = 0$ for $i \leq -N$;
- $a_i \rightarrow +\infty$ as $i \rightarrow -\infty$ and $b_i \rightarrow +\infty$ as $i \rightarrow +\infty$;
- $a_i \geq b_i$ for $i < 0$ and $a_i \leq b_i$ for $i \geq 0$.

Then no element of $(-1, \mathcal{T}^+) \subset Y$ is in the image of D and $(-1, \mathcal{T}^+) \subset Y$ generates the cokernel of D . The kernel of D has the form

$$\ker D \cong \bigoplus_{i \in \mathbb{Z}} \tau(\min(a_i, b_i)).$$

Proof As all of the maps α_i, β_i are surjective, it is easy to see that the cokernel of D is generated by the (equivalence classes of) elements in any one of $(i, \mathcal{T}^+) \subset Y$. Suppose $\eta = \{(s, \eta_s)\}_{s \in \mathbb{Z}} = D(\xi)$ with $\eta_s = 0$ for $s \neq -1$. Let $\xi = \{(s, \xi_s)\}_{s \in \mathbb{Z}}$.

Without loss of generality (by symmetry) we may assume that $\alpha_{-1}(\xi_{-1}) \neq 0$. Since $a_{-1} \geq b_{-1}$ it follows that $\beta_{-1}(\xi_{-1}) \neq 0$. Since $\eta_{-2} = 0 = \beta_{-1}(\xi_{-1}) + \alpha_{-2}(\xi_{-2})$, we have $\alpha_{-2}(\xi_{-2}) \neq 0$, and hence $\xi_{-2} \neq 0$. Continuing in the same way we conclude that ξ is not supported on a finite set and hence no such ξ can exist.

Similarly to the proof of Lemma 13, we want to show that the kernel of D separates into the kernels of maps $\alpha_i + \beta_i$. This will finish the proof.

Now let $\xi = \{(s, \xi_s)\}_{s \in \mathbb{Z}} \in \ker D$. As before, without loss of generality we assume there is $n < 0$ such that $\beta_n(\xi_n) \neq 0$. Then $\alpha_{n-1}(\xi_{n-1}) \neq 0$, so $\beta_{n-1}(\xi_{n-1}) \neq 0$. Proceeding inductively we again reach a contradiction to ξ being finitely supported. \square

The previous lemma describes the action of $D_{i,p/q}^T$ when $p/q < 0$. We make this explicit in the next lemma.

Lemma 18 Let $p < 0, q > 0$. Then

$$\text{coker } D_{i,p/q}^T \cong \mathcal{T}_d^+,$$

where $d = d(L(p, q), i)$, and

$$\ker D_{i,p/q}^T \cong \bigoplus_{n \geq 1} \tau_{d_n^-}(H_{\lfloor (i-np)/q \rfloor}) \bigoplus_{n \geq 0} \tau_{d_n^+}(V_{\lfloor (i+np)/q \rfloor}).$$

Here

$$d_0^+ = d + 1 - 2H_{\lfloor i/q \rfloor},$$

$$d_n^- = d_0^+ + 2 \sum_{k=0}^{n-1} (V_{\lfloor (i-kp)/q \rfloor} - H_{\lfloor (i-(k+1)p)/q \rfloor}),$$

$$d_n^+ = d_0^+ + 2 \sum_{k=0}^{n-1} (H_{\lfloor (i+kp)/q \rfloor} - V_{\lfloor (i+(k+1)p)/q \rfloor}).$$

Proof This is a straightforward application of Lemma 17. Objects that are labelled with $\lfloor (i + np)/q \rfloor$ in the mapping cone correspond to the ones labelled with $-n$ in Lemma 17. In particular, take $a_n = V_{\lfloor (i-np)/q \rfloor}$ and $b_n = H_{\lfloor (i-np)/q \rfloor}$. The grading comes from the fact that this works in the same way for the unknot (the towers in the cokernel coincide for all knots). Just as in Corollary 14, we get the values of d_n^\pm by the fact that the maps v_k, h_k have grading -1 and the multiplication by U has grading -2 . \square

Just as Corollary 14 is sufficient for positive surgeries on L -space knots, so is Lemma 18 for negative surgeries on L -space knots. We observe that in this case the Alexander polynomial also determines the Heegaard Floer homology of the surgeries. Lemma 18 also implies that negative p/q surgeries on L -space knots have the same d -invariants as the lens spaces $L(p, q)$, so do not depend on the particular L -space knot. The next proposition extends our analysis to arbitrary knots.

Proposition 19 *Let $p < 0, q > 0$. As absolutely graded $\mathbb{F}[U]$ -modules, we have*

$$\text{coker } \mathbf{D}_{i,p/q}^+ \cong \mathcal{T}_d^+.$$

As absolutely graded vector spaces, we have

$$\text{HF}_{\text{red}}(S_{p/q}^3(K), i) \cong \ker \mathbf{D}_{i,p/q}^+ \cong \ker \mathbf{D}_{i,p/q}^T \oplus \mathcal{A},$$

where $\mathbb{A}_{i,p/q}^{\text{red}}(K) \cong \mathcal{A} \oplus \tau_\delta(N_{i,p/q})$, $\delta = d(L(p, q), i) + 1$, and $N_{i,p/q}$ is characterised by

$$d = d(S_{p/q}^3(K), i) = d(L(p, q), i) + 2N_{i,p/q}.$$

In fact, $N_{i,p/q} = \max\{\bar{V}_{\lfloor i/q \rfloor}, \bar{H}_{\lfloor (i+p)/q \rfloor}\}$, where \bar{V}_k, \bar{H}_k are for the mirror image of K the same as V_k, H_k are for K .

Proof Recall that no element in $(-1, \mathbf{B}^+)$ is in the image of the map $\mathbf{D}_{i,p/q}^T$. Since $\mathbb{A}_{i,p/q}^{\text{red}}(K)$ lies in the kernel of the multiplication by a big enough power of U , so does its image under $\mathbf{D}_{i,p/q}^+$. Hence $\mathbf{D}_{i,p/q}^+$ only ‘‘chops off’’ a finite piece of the tower.

More precisely, let N be the largest integer such that $U^{-N+1} \in (-1, \mathbf{B}^+)$ appears as a term of some element η in the image of $\mathbf{D}_{i,p/q}^+$.

We claim that then U^{-N+k} is also in the image for all $k \geq 1$. This is easily seen by an inductive argument: 1 is in the image, as $1 = U^{N-1}\eta$; U^{-1} is, because 1 is and $U^{N-2}\eta$ is. Proceeding in the same way, we establish the claim.

Thus the cokernel of $\mathbf{D}_{i,p/q}^+$ is generated by U^{-N-k} for $k \geq 0$, none of which are in its image. Thus the map i_* from the exact triangle (1) injects $\langle \{U^{-N-k}\}_{k \geq 0} \rangle_{\mathbb{F}}$ into $\text{HF}^+(S_{p/q}^3(K), i)$. Since $U^{-N+1} \in (-1, \mathbf{B}^+)$ is in the image of $\mathbf{D}_{i,p/q}^+$, it is in the kernel of i_* and we have $U \cdot i_*(U^{-N}) = 0$. Hence the image of i_* is exactly the tower \mathcal{T}_d^+ with $d = d(S_{p/q}^3(K), i)$. By Lemma 18, $1 \in (-1, \mathbf{B}^+)$ has grading $d(L(p, q), i)$, so $d(S_{p/q}^3(K), i) = d(L(p, q), i) + 2N$.

By the first isomorphism theorem and exactness of (1), we have

$$\ker \mathbf{D}_{i,p/q}^+ = \text{im } j_* \cong \text{HF}^+(S_{p/q}^3(K), i) / \ker j_* = \text{HF}^+(S_{p/q}^3(K), i) / \text{im } i_*.$$

Since $\text{im } i_*$ is the tower, we have

$$\ker \mathbf{D}_{i,p/q}^+ \cong \text{HF}^+(S_{p/q}^3(K), i) / \text{im } i_* \cong \text{HF}_{\text{red}}(S_{p/q}^3(K), i).$$

The rest is just linear algebra again. We can split $\mathbb{A}_{i,p/q}^{\text{red}}(K)$ into the part that goes isomorphically to the base of the tower, which is not in the image of $\mathbf{D}_{i,p/q}^T$ (ie $(-1, \mathbf{B}^+) \cap \text{im } \mathbf{D}_{i,p/q}^+$) and the part that goes into the image of $\mathbf{D}_{i,p/q}^T$. We then proceed as in the proof of Proposition 15.

The fact that $N_{i,p/q} = \max\{\bar{V}_{\lfloor i/q \rfloor}, \bar{H}_{\lfloor (i-p)/q \rfloor}\}$ follows from taking the mirror image of K and comparing with the formula already obtained for the correction terms from Corollary 14. We have

$$\begin{aligned} 2N_{i,p/q} &= d(S_{p/q}^3(K), i) - d(L(p, q), i) \\ &= -d(S_{-p/q}^3(m(K)), i) + d(L(-p, q), i) \\ &= \max\{\bar{V}_{\lfloor i/q \rfloor}, \bar{H}_{\lfloor (i+p)/q \rfloor}\}, \end{aligned}$$

where $m(K)$ is the mirror image of K . □

We can also express the total rank of $\text{HF}_{\text{red}}(S_{p/q}^3(K), i)$ as follows.

Proposition 20 *We have*

$$\dim \text{HF}_{\text{red}}(S_{p/q}^3(K)) = q\delta(K) + qV_0 + 2q \sum_{i=1}^{g-1} V_i - \sum_{i=0}^{p-1} N_{i,p/q}.$$

Proof The proof is virtually the same as for Proposition 16. □

3.3 Zero surgeries

We now treat the case of zero surgeries. For the case of L -space knots the formula for the Heegaard Floer homology of the zero surgery was derived in [14, Theorem 7.2]. The main tool we use is due to Ozsváth and Szabó:

Theorem 21 [19, Theorem 9.19] *There is a U -equivariant exact triangle*

$$(4) \quad \begin{array}{ccc} \mathrm{HF}^+(S^3) & \xrightarrow{F_{\cdot;i}^+} & \bigoplus_{j \equiv i \pmod{m}} \mathrm{HF}^+(S_0^3(K), j) \\ & \swarrow F_{m;i}^+ & \downarrow F_{0;i}^+ \\ & & \mathrm{HF}^+(S_m^3(K), i) \end{array}$$

Moreover, the map $F_{m;i}^+$ is equal to the one induced by the surgery cobordism.

Given i , we can make m in (4) so big that

$$\bigoplus_{j \equiv i \pmod{m}} \mathrm{HF}^+(S_0^3(K), j) = \mathrm{HF}^+(S_0^3(K), i).$$

From now on we assume that m is at least that large.

The group $A_0^+(K) \cong A_0^T(K) \oplus A_0^{\mathrm{red}}(K)$ is relatively \mathbb{Z} -graded. If we fix an absolute \mathbb{Q} -grading for any element of $A_0^+(K)$, the relative grading will fix the absolute grading for all the elements. In particular, it will absolutely grade $A_0^{\mathrm{red}}(K)$.

In the statement of the next proposition (but not necessarily in the proof), we use the grading of $A_0^{\mathrm{red}}(K)$ induced by grading the tower $A_0^T(K)$ in such a way that the grading of 1 is $\frac{1}{2} - 2V_0$.

Proposition 22 *Let $k \neq 0$. Then, as $\mathbb{Z}/2\mathbb{Z}$ -graded vector spaces, we have*

$$(5) \quad \mathrm{HF}^+(S_0^3(K), k) \cong \tau(V_{|k|}) \oplus A_k^{\mathrm{red}}(K).$$

As absolutely \mathbb{Q} -graded vector spaces, we have

$$(6) \quad \mathrm{HF}^+(S_0^3(K), 0) \cong \mathcal{T}_{-1/2+2\bar{V}_0}^+ \oplus \mathcal{T}_{1/2-2V_0}^+ \oplus \mathcal{A}.$$

Here $\mathcal{A} \oplus \tau_{1/2}(\bar{V}_0) \cong A_0^{\mathrm{red}}(K)$ as absolutely graded vector spaces, where the absolute grading of $A_0^{\mathrm{red}}(K)$ is as described above.

Proof The first part is immediate from [14, proof of Theorem 7.2]. Note that $\mathrm{HF}^+(S_m^3(K), k) \cong \mathcal{T} \oplus A_k^{\mathrm{red}}(K)$ (recall that we are assuming that m is large). In the

proof of [14, Theorem 7.2] Ozsváth and Szabó show that the restriction of $F_{m;i}^+$ to the tower part is surjective and its kernel is $\mathbb{F}[U^{-1}]/U^{-V_{|k|}}$. So we are done by the same elementary linear algebra as in the proof of Proposition 15.

For the second part, note that we can assign absolute gradings, as we are dealing with a torsion Spin^c -structure. As shown in [19, Theorem 10.4], $\text{HF}^\infty(S_0^3(K), 0)$ is a direct sum of two copies of $\mathbb{Z}[U, U^{-1}]$ that lie in different relative $\mathbb{Z}/2\mathbb{Z}$ -gradings. This is equivalent to saying that the difference of the absolute gradings between the elements from the different summands is always odd. As in the case of rational homology spheres, the exact sequence

$$\dots \rightarrow \text{HF}^-(Y, \mathfrak{s}) \rightarrow \text{HF}^\infty(Y, \mathfrak{s}) \rightarrow \text{HF}^+(Y, \mathfrak{s}) \rightarrow \dots$$

establishes that

$$\text{HF}^+(S_0^3(K), 0) \cong \mathcal{T}_{d_1} \oplus \mathcal{T}_{d_2} \oplus \mathcal{A},$$

where $\mathcal{A} = \text{HF}_{\text{red}}(S_0^3(K), 0)$ is a finitely generated $\mathbb{F}[U]$ -module in the kernel of some large enough power of U .

In fact, combining [14, Proposition 4.12] with the d -invariant formula of Ni and Wu stated in Corollary 14, we obtain $d_1 = -\frac{1}{2} + 2\bar{V}_0$ and $d_2 = \frac{1}{2} - 2V_0$.

The last step in the proof is determining \mathcal{A} . The maps $F_{;0}^+$ and $F_{0;0}^+$ from the exact triangle (4) have gradings $-\frac{1}{2}$ and $\frac{1}{4}(m-3)$, respectively, by [14, Lemma 7.11]. The map $F_{m;0}^+$ is not graded but is a sum of graded maps, and the set of grading shifts of these maps is $\{\frac{1}{4}(1-m(2k-1)^2)\}_{k \in \mathbb{Z}}$.

Since $\text{HF}^+(S^3) \cong \mathcal{T}_0^+$ and the grading of the map $F_{;0}^+$ is $-\frac{1}{2}$, $\mathcal{T}_{1/2-2V_0}^+$ is not in the image of $F_{;0}^+$, hence the map $F_{0;0}^+$ is an isomorphism between $\mathcal{T}_{1/2-2V_0}^+$ and the tower part of $\text{HF}^+(S_m^3(K), 0)$, which is equal to $\mathcal{T}_{(m-1)/4-2V_0}^+$ by Proposition 15. Hence the restriction of the map $F_{m;0}^+$ to the tower part of $\text{HF}^+(S_m^3(K), 0)$ is zero. As in the proof of Proposition 19, the restriction of $F_{m;0}^+$ to $\text{HF}_{\text{red}}(S_m^3(K), 0)$ maps a subgroup of the form $\tau(N)$ isomorphically to the base of the tower $\text{HF}^+(S^3) \cong \mathcal{T}_0^+$. By the grading considerations again we see that $N = \bar{V}_0$.

Recall from Proposition 15 that $\text{HF}^+(S_m^3(K), 0) \cong \mathcal{T}_{(m-1)/4-2V_0}^+ \oplus A_0^{\text{red}}(K)$ (the grading here is such that the relative grading is as it should be). Let the maximal grading of a nontrivial element in $A_0^{\text{red}}(K)$ be $\frac{1}{4}(m-1) - 2V_0 + C$.

Consider one homogeneous summand of $F_{m;0}^+$ with grading $\frac{1}{4}(1-m(2k-1)^2)$. It maps the element of $A_0^{\text{red}}(K)$ of maximal grading to an element with grading

$$\frac{1}{4}(m-1) - 2V_0 + C + \frac{1}{4}(1-m(2k-1)^2) = \frac{1}{4}(m(1-(2k-1)^2) - 8V_0 + 4C).$$

If $k \neq 0, 1$ we have $1 - (2k - 1)^2 < 0$ and so by making m sufficiently large we can make sure that $\frac{1}{4}(m(1 - (2k - 1)^2) - 8V_0 + 4C) < 0$, and, as all nontrivial elements in the image have grading ≥ 0 , this means that all components with $k \neq 0, 1$ are zero.

Thus we can assume that the map $F_{m;0}^+$ has grading $\frac{1}{4}(1 - m)$. As discussed above the map $F_{m;0}^+$ maps a subgroup of $A_0^{\text{red}}(K)$ of the form $\tau(\bar{V}_0)$ isomorphically to such a subgroup at the lower end of the tower $\text{HF}^+(S^3) \cong \mathcal{T}_0^+$. Therefore 1 in $\tau(\bar{V}_0)$ must have grading $\frac{1}{4}(m - 1)$.

The rest of $A_0^{\text{red}}(K)$ will be in the kernel of $F_{m;0}^+$ and thus in the image of \mathcal{A} by $F_{0;0}^+$. Now noting that the grading of the map $F_{0;0}^+$ is $\frac{1}{4}(m - 3)$ finishes the proof. \square

Torsion coefficients of the Alexander polynomial of a knot describe the Euler characteristics of the groups $A_k^{\text{red}}(K)$, which we can see for example by combining Theorems 10.14 and 10.17 of [19] (though a more direct proof is also possible). This has also been shown in [12, Lemma 3.2].

Lemma 23 *For $k \geq 0$ we have*

$$(7) \quad t_k(K) = V_k + \chi(A_k^{\text{red}}(K)).$$

Recall that the absolute $\mathbb{Z}/2\mathbb{Z}$ grading used to calculate the Euler characteristics here is fixed by the requirement that the tower $A_k^T(K)$ lies entirely in grading 0.

4 Proof of Theorem 2

In this section we prove:

Theorem 2 *Let Y be a 3-manifold. There are at most finitely many alternating knots $K \subset S^3$ such that $Y = S_{p/q}^3(K)$.*

The strategy of our proof is as follows. We first want to restrict the possible Alexander polynomials of knots that yield a given 3-manifold Y by surgery. We then want to show that, out of this restricted set, only finitely many can be Alexander polynomials of alternating knots. This will finish the proof, due to the next proposition, which was proved by Moore and Starkston. We provide the proof for the reader's convenience (and since it is nice and short).

Proposition 24 [9, Proposition 5.1] *There is only a finite number of alternating knots with a given Alexander polynomial.*

Proof By the Bankwitz theorem [1, Theorem 5.5] the determinant $\det(K)$ of an alternating knot K is greater than or equal to the minimal crossing number of K . Thus there are only finitely many alternating knots with a given determinant. The classical result [26, page 213] (or definition) $\det(K) = |\Delta_K(-1)|$ finishes the proof. \square

For a knot $K \subset S^3$, let $m(K)$ be its mirror image. Clearly, K is alternating if and only if $m(K)$ is. Since $S^3_{p/q}(K) = -S^3_{-p/q}(m(K))$ we can assume that the surgery slope is positive (if nonzero).

For Y a rational homology sphere and $q > 0$ a natural number, define

$$M(Y, q) = \frac{1}{2} \left(\sum_{0 \leq i \leq p-1} d(L(p, q), i) - \sum_{s \in \text{Spin}^c(Y)} d(Y, s) \right),$$

where $p = |H_1(Y)|$.

Theorem 7 shows that for any rational homology sphere Y there is some number $n(Y)$ such that $Y \neq S^3_{p/q}(K)$ for any K and $|q| > n(Y)$.

If Y is obtained by $p/q > 0$ surgery on K , then by (2) the numbers V_k for K satisfy

$$M(Y, q) = \sum_{i=0}^{p-1} \max\{V_{\lfloor i/q \rfloor}, V_{-\lfloor (i-p)/q \rfloor}\}.$$

Combining this with Proposition 16 we get

$$\dim \text{HF}_{\text{red}}(S^3_{p/q}(K)) + M(S^3_{p/q}(K), q) = q \left(\delta(K) + V_0 + 2 \sum_{i \geq 1} V_i \right).$$

This formula implies the inequality

$$\frac{\dim \text{HF}_{\text{red}}(S^3_{p/q}(K)) + M(S^3_{p/q}(K), q)}{q} \geq \sum_{k \geq 0} (V_k + \dim A_k^+(K)).$$

Now let

$$c(Y) = \max_{1 \leq q \leq n(Y)} \left\{ \frac{\dim \text{HF}_{\text{red}}(Y) + M(Y, q)}{q} \right\}.$$

The inequality above implies that if a rational homology sphere Y is obtained by surgery on a knot K with associated sequence $\{V_k\}_{k \geq 0}$, then

$$(8) \quad c(Y) \geq \sum_{k \geq 0} (V_k + \dim A_k^+(K)).$$

Lemma 25 Suppose Y is a rational homology sphere obtained by a $p/q > 0$ surgery on a knot $K \subset S^3$. Then

$$(9) \quad \sum_{i \geq 0} |t_i(K)| \leq c(Y).$$

Proof It follows from Lemma 23 that for each $k \geq 0$ we have

$$|t_k(K)| = |V_k + \chi(A_k^+(K))| \leq V_k + |\chi(A_k^+(K))| \leq V_k + \dim A_k^+(K).$$

Combining with (8) yields the result. \square

Let S_Y be some set of knots in S^3 that give a rational homology sphere Y by surgery (not necessarily all such knots and not necessarily alternating). Denote by $g(S_Y)$ and $\Delta(S_Y)$ the sets of genera and of Alexander polynomials, respectively, of knots in S_Y .

Lemma 26 If $g(S_Y)$ is finite, then so is $\Delta(S_Y)$.

Proof We clearly have $t_i(K) = 0$ for all $K \in S_Y$ and all $i \geq \max(g(S_Y))$. By Lemma 25, $\sum_{i \geq 0} |t_i(K)|$ is bounded above, so we clearly have finitely many sequences $\{t_i(K)\}$ for $K \in S_Y$. Now observe that the torsion coefficients determine the Alexander polynomial, so this results in at most finitely many possible Alexander polynomials. \square

A theorem of Murasugi is crucial for our proof:

Theorem 27 [10, Theorem 1.1] Let $K \subset S^3$ be an alternating knot and

$$\Delta_K(T) = a_0 + \sum_{i=1}^{g(K)} a_i(T^i + T^{-i})$$

be its Alexander polynomial. Then $a_i \neq 0$ for $0 \leq i \leq g(K)$.

The next lemma is the last step before we can prove Theorem 2.

Lemma 28 Let $K \subset S^3$ be an alternating knot that gives a rational homology sphere Y by surgery. Then

$$g(K) \leq 3c(Y).$$

Proof Suppose $g(K) \geq 3c(Y) + 1$. Note that $a_g = t_{g-1}(K) \neq 0$. We claim that there are three consecutive indices i , $i + 1$ and $i + 2 \leq g$ with $t_i(K) = t_{i+1}(K) = t_{i+2}(K) = 0$. It then follows that $a_{i+1} = 0$, which is a contradiction to Theorem 27.

To prove the claim, suppose there is no such consecutive triple of zero torsion coefficients. Then

$$\sum_{i \geq 0} |t_i(K)| = \sum_{k \geq 0} (|t_{3k}(K)| + |t_{3k+1}(K)| + |t_{3k+2}(K)|) \geq \lfloor \frac{1}{3}(g-1) \rfloor + 1 \geq c(Y) + 1,$$

which contradicts Lemma 25.

We have thus established that $g \leq 3c(Y)$. □

Proof of Theorem 2 Suppose Y is a rational homology sphere. Then by Lemma 28 there is a genus bound for alternating knots that give Y by surgery, so by Lemma 26 the set of Alexander polynomials of such alternating knots is finite.

If Y is obtained by 0–surgery on K , then Propositions 10.14 and 10.17 of [19] show that the Alexander polynomial of K can be deduced directly from the Heegaard Floer homology of Y .

Proposition 24 now finishes the proof. □

5 The genus bound

We now turn to the proof of Theorem 3, which we restate here.

Theorem 3 For any knot $K \subset S^3$ and any $p/q \in \mathbb{Q}$ we have

$$U^{g(K) + \lceil g_4(K)/2 \rceil} \cdot \text{HF}_{\text{red}}(S^3_{p/q}(K)) = 0.$$

Lemma 29 Let K be a knot in S^3 with genus g . Then for any $k \in \mathbb{Z}$ we have

$$U^g \cdot A_k^{\text{red}}(K) = 0.$$

Proof By the conjugation symmetry we may assume that $k \geq 0$. Let $C = \text{CFK}^\infty(K)$, $\Delta_k = C\{i < 0 \text{ and } j \geq k\}$. This is a subquotient of C (ie a subcomplex of a quotient). Note that $U^g \cdot \Delta_k = 0$, as this is the maximal possible “height” of this complex. We illustrate the complexes $\Delta_k, A_k^+(K), B^+$ in Figure 3.

We have an exact sequence

$$0 \rightarrow \Delta_k \rightarrow A_k^+(K) \rightarrow B^+ \rightarrow 0$$

which leads to an exact U –equivariant triangle

$$(10) \quad \begin{array}{ccc} H_*(\Delta_k) & \xrightarrow{i_*} & A_k^+(K) \\ & \swarrow & \downarrow v_k \\ & & B^+ \end{array}$$

Since v_k is surjective, we in fact have a short exact sequence

$$0 \rightarrow H_*(\Delta_k) \rightarrow A_k^+(K) \rightarrow B^+ \rightarrow 0,$$

so $H_*(\Delta_k) \cong \ker v_k$ and hence $U^g \cdot \ker v_k = 0$.

Recall that

$$A_k^+(K) = A_k^T(K) \oplus A_k^{\text{red}}(K),$$

and similarly we can decompose the map $v_k = v_k^T \oplus v_k^{\text{red}}$ into components. The map v_k^T is surjective. We claim that $A_k^{\text{red}}(K) \cong \ker v_k / \ker v_k^T$. From this the conclusion of the lemma follows immediately.

To prove the claim we construct an isomorphism from $\ker v_k / \ker v_k^T$ to $A_k^{\text{red}}(K)$. Let $x \in \ker v_k \setminus \ker v_k^T$. Then send an equivalence class of x to $A_k^{\text{red}}(K)$ by projection. This map is well defined, because two different elements with the same projection are in $\ker v_k^T$. Clearly this is also a surjective $\mathbb{F}[U]$ -module homomorphism. \square

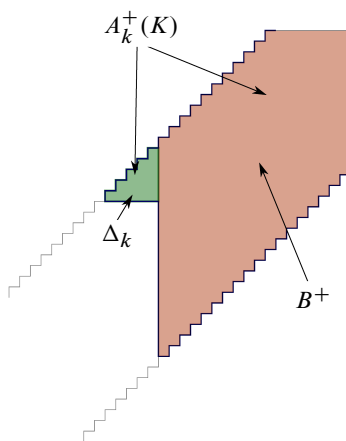


Figure 3: Complexes Δ_k , $A_k^+(K)$ and B^+ inside CFK^∞

The previous lemma clearly implies:

Corollary 30 *The following relation holds:*

$$U^g \cdot \mathbb{A}_{i,p/q}^{\text{red}}(K) = 0.$$

Proof of Theorem 3 Note that by [25, Theorem 2.3] we have $V_0 \leq \lceil \frac{1}{2}g_4(K) \rceil$.

If the slope is negative the reduced part is exactly equal to the kernel of $D_{i,p/q}^+$. So suppose $x \in \ker D_{i,p/q}^+$. By Corollary 30, $U^g \cdot x \in \ker D_{i,p/q}^T$. But by Lemma 18 the

kernel of $D_{i,p/q}^T$ consist of the summands of the type $\tau(N)$ with $N \leq V_0 \leq \lceil \frac{1}{2}g_4(K) \rceil$, so

$$U^{\lceil g_4(K)/2 \rceil} \cdot \ker D_{i,p/q}^T = 0.$$

Now suppose the slope is positive. If we assume that $x \in \ker D_{i,p/q}^+$ then we still have $U^g \cdot x \in \ker D_{i,p/q}^T$, and $U^{\lceil g_4(K)/2 \rceil} \cdot \ker D_{i,p/q}^T$ is contained in the tower part by Corollary 14.

Similarly, the case of zero surgery follows immediately from Proposition 22. This finishes the proof. □

Since by Corollary 14 and Lemma 18 the reduced Floer homology of surgeries on L -space knots consists only of a direct sum of $\mathbb{F}[U]$ -modules of the form $\tau(V_k)$, we see that if K is an L -space knot, then $U^{\lceil g_4(K)/2 \rceil} \cdot \text{HF}_{\text{red}}(S_{p/q}^3(K)) = 0$.

In order to construct examples for which this genus bound gets arbitrarily large, note that every negative surgery on a knot contains a summand of the form $\tau(V_0)$. So if V_0 is large, the genus bound will also be large, independent of the absolute value of the negative slope we use. In particular, we can choose any order of the first homology we like.

For L -space knots, $V_0 = t_0$ can be read from the Alexander polynomial; in particular, this is true for torus knots $T_{p,q}$ with $p, q > 0$.

Suppose we have an L -space knot K with Alexander polynomial

$$\Delta_K(T) = a_0 + \sum_{i=1}^g a_i(T^i + T^{-i}).$$

Then the coefficients alternate between 1 and -1 , with the first nontrivial coefficient being 1 [21, Theorem 1.2]. So we clearly have

$$t_0 \geq \#\{a_i = 1, i > 0\} \geq \frac{1}{4}(\#\{a_i \neq 0\} - 1) \geq \frac{1}{4}(\Delta_K(-1) - 1).$$

Consider the torus knots $T_{p,2}$ for p positive odd. They have Alexander polynomials of the form

$$\frac{(T^{2p} - 1)(T - 1)}{(T^p - 1)(T^2 - 1)} = T^{p-1} - T^{p-2} + \dots + 1,$$

which evaluates to p at -1 .

Moreover, these examples are actually negatively oriented (see next section) small Seifert fibred spaces, which is interesting in light of the next section.

We note that a result similar to Theorem 3 can be obtained for a knot in any L -space rational homology sphere, the bound being in terms of the width of the knot Floer homology rather than the genus.

6 Seifert fibred surgery

The aim of this section is to prove:

Theorem 5 *Let $K \subset S^3$ be a knot. Suppose there is a rational number $p/q > 0$ such that $Y = S^3_{p/q}(K)$ is a negatively oriented Seifert fibred space. Then*

- $U^{g(K)} \cdot \text{HF}_{\text{red}}(Y) = 0$;
- if $0 < p/q \leq 3$, then all the torsion coefficients $t_i(K)$ are nonpositive (including $t_0(K)$) and $\text{deg } \Delta_K = g(K)$;
- more generally, if $i \geq \lfloor \frac{1}{2}(\lceil p/q \rceil - \sqrt{\lceil p/q \rceil}) \rfloor$, then t_i is nonpositive;
- if $g(K) > \lfloor \frac{1}{2}(\lceil p/q \rceil - \sqrt{\lceil p/q \rceil}) \rfloor$, then $\text{deg } \Delta_K = g(K)$;
- if $U^{\lfloor |H_1(Y)|/2 \rfloor} \cdot \text{HF}_{\text{red}}(Y) \neq 0$, then $\text{deg } \Delta_K = g(K)$.

In all statements where $\text{deg } \Delta_K = g(K)$ we have that $\widehat{\text{HFK}}(K, g(K))$ is supported in odd degrees.

Proof First we need to define the Seifert orientation for Seifert fibred spaces. Following [20] we say that Y has *positive Seifert orientation* if $-Y$ bounds $W(\Gamma)$, where Γ is a weighted tree which has either negative definite or negative semidefinite intersection form. For the construction of the 4–manifold $W(\Gamma)$ from the weighted tree Γ , see [15]. We say that Y has *negative Seifert orientation* if $-Y$ has positive Seifert orientation.

Using [15, Corollary 1.4] (together with the inversion of the absolute $\mathbb{Z}/2\mathbb{Z}$ –grading on the reduced homology upon reversing the orientation) we can see that if Y has a negative Seifert orientation, then its reduced Floer homology is concentrated in the odd $\mathbb{Z}/2\mathbb{Z}$ –grading and that it bounds a negative definite 4–manifold with torsion-free first homology group.

Lemma 31 *Let $K \subset S^3$ be a knot. Suppose there is a rational number $p/q > 0$ such that $Y = S^3_{p/q}(K)$ is a negatively oriented Seifert fibred space. Then $A_k^{\text{red}}(K)$ is supported in odd $\mathbb{Z}/2\mathbb{Z}$ –grading for every k .*

Proof of Lemma 31 As an absolutely $\mathbb{Z}/2\mathbb{Z}$ –graded group, each $A_k^{\text{red}}(K)$ is a subgroup of $\text{HF}_{\text{red}}(S^3_{p/q}(K))$ by Proposition 15. Since $\text{HF}_{\text{red}}(S^3_{p/q}(K))$ is supported in odd grading, so must be each $A_k^{\text{red}}(K)$. \square

Denote by \tilde{g} the minimal index i for which $V_i = 0$. As above, denote by a_i the coefficient of the Alexander polynomial of K corresponding to T^i . If $\tilde{g} < g(K)$, then by Lemma 23 we have

$$(11) \quad a_{g(K)} = t_{g(K)-1} = \chi(A_{g(K)-1}^{\text{red}}),$$

so, in particular, $a_{g(K)} \neq 0$ if all $A_k^{\text{red}}(K)$ are supported in the same $\mathbb{Z}/2\mathbb{Z}$ -grading, since in this case

$$A_{g(K)-1}^{\text{red}} \cong \text{HF}^+(S_0^3(K), g-1) \cong \widehat{\text{HFK}}(K, g(K)) \neq 0.$$

It follows that in this case $\deg(\Delta_K) = g(K)$ and $\widehat{\text{HFK}}(K, g(K))$ is supported in odd degrees.

Moreover, if $\tilde{g} = 0$, then $V_k = 0$ for all $k \geq 0$, so that

$$t_k = \chi(A_k^{\text{red}}(K)) \leq 0.$$

We now need to establish conditions which ensure that $\tilde{g} = 0$ or $\tilde{g} < g(K)$.

McCoy [8, Lemma 2.3] slightly modified the proof of [2, Theorem 1.1] by Greene to show that if $S_{p/q}^3(K)$ bounds a negative-definite 4-manifold with torsion-free first homology, then

$$2\tilde{g} \leq n - \sqrt{n},$$

where $n = \lceil p/q \rceil$.

It follows that if $p/q \leq 3$ then $\tilde{g} = 0$.

More generally, if $i \geq \lfloor (n - \sqrt{n})/2 \rfloor$, where $n = \lceil p/q \rceil$, then $i \geq \tilde{g}$ and hence $V_i = 0$. It follows that $t_i = \chi(A_i^{\text{red}}(K)) \leq 0$. If $g(K) > \lfloor \frac{1}{2}(\lceil p/q \rceil - \sqrt{\lceil p/q \rceil}) \rfloor$, then $g(K) > \tilde{g}$ as well.

For the improvement of the genus bound, note that all the summands of $\text{HF}_{\text{red}}(S_{p/q}^3(K))$ coming from the V_i (ie of the form $\tau(V_i)$) are situated in the even grading and therefore must vanish. Now the proof of Theorem 3 shows that $U^{g(K)} \cdot \text{HF}_{\text{red}}(S_{p/q}^3(K)) = 0$.

Now if

$$U^{\lfloor |H_1(Y)|/2 \rfloor} \cdot \text{HF}_{\text{red}}(S_{p/q}^3(K)) \neq 0,$$

then $\lfloor \frac{1}{2}|H_1(Y)| \rfloor \leq g(K) - 1$, so $\frac{1}{2}(|H_1(Y)| + 1) \leq g(K)$. On the other hand,

$$\tilde{g} \leq \frac{1}{2}(\lceil p/q \rceil - \sqrt{\lceil p/q \rceil}) < \frac{1}{2}(p/q + 1) \leq \frac{1}{2}(p + 1) = \frac{1}{2}(|H_1(Y)| + 1) \leq g(K).$$

It follows from (11) that $\deg(\Delta_K) = g(K)$. □

We end this section with the following question:

Question 32 Does there exist a knot $K \subset S^3$ with $\deg(\Delta_K) \neq g(K)$ and with a Seifert fibred surgery?

7 Some other applications of the mapping cone formula

In this section, we demonstrate some other applications of the results obtained in Section 3.

Theorem 8 *Let K be an L -space knot and $p/q \leq 1$ a rational number. Then $S_{p/q}^3(K)$ and p/q determine the Alexander polynomial of K .*

Proof If the slope is zero this is immediate from Proposition 22. If the slope is negative this also easily follows from Lemma 18; by looking at $\text{HF}_{\text{red}}(S_{p/q}^3(K))$ we can work out a sequence of numbers that represents all the torsion coefficients with some repetitions (they are orders of cyclic $\mathbb{F}[U]$ -modules). But we know the number of repetitions because we know the slope. From this we deduce all the torsion coefficients (in the correct order, as they form a monotone sequence), and hence the Alexander polynomial.

If the slope is in the interval $(0, 1]$ the reasoning is the same; Corollary 14 allows us to work out the torsion coefficients, since we know how many times each occurs. The only torsion coefficient we might not be able to work out from the module structure of $\text{HF}_{\text{red}}(S_{p/q}^3(K))$ is t_0 if the slope is 1. But in this case, it can be worked out from the d -invariant formula of Ni and Wu from Corollary 14. \square

Sometimes we can work out a lot about the Heegaard Floer homology associated to a knot from a surgery on it even if it is not an L -space knot.

Proposition 33 *The small Seifert fibred space $Y = S^2((2, 1), (6, -1), (7, -2))$ can only be obtained by (-4) -surgery. All knots producing it are non- L -space knots.*

Proof We find the HF^+ of this space using the computer program `HFNem2` by Çağrı Karakurt [5]. There are four Spin^c -structures $\{\mathfrak{s}_i\}_{i=0}^3$, and HF^+ in them have the form

$$\text{HF}^+(Y, \mathfrak{s}_0) \cong \mathcal{T}_{-3/4},$$

$$\text{HF}^+(Y, \mathfrak{s}_1) \cong \mathcal{T}_0 \oplus \tau_0(1),$$

$$\text{HF}^+(Y, \mathfrak{s}_2) \cong \mathcal{T}_{1/4},$$

$$\text{HF}^+(Y, \mathfrak{s}_3) \cong \mathcal{T}_0 \oplus \tau_0(1).$$

Using Theorem 7 we can restrict the possible slopes to $\{\pm 4, \pm \frac{4}{3}, \pm \frac{4}{5}\}$. Calculating the correction terms of $L(4, 1) = L(4, -3) = L(4, 5)$ and $L(4, -1) = L(4, 3) = L(4, -5)$ we notice that only $L(4, -1)$ has correction terms such that the difference of each of them with some correction term of Y is an integer. This means that the slope has to be in $\{-4, \frac{4}{3}, -\frac{4}{5}\}$.

We also notice that the d -invariants of Y coincide exactly with the d -invariants of the lens space $L(4, -1)$. By the d -invariant formula (2) we conclude that $V_0 = 0$. A similar argument using the d -invariant formula for negative surgeries in Proposition 19 establishes that $\bar{V}_0 = 0$.

Now using the total dimension formulas of Propositions 16 and 20 we conclude

$$2 = \dim \text{HF}_{\text{red}}(S_{p/q}^3(K)) = q\delta(K),$$

which is impossible for $q = 3$ or $q = -5$.

Comparing the labelling of Spin^c -structures we see that the order in which we listed $\text{HF}^+(Y, \mathfrak{s}_i)$ above corresponds to $i = 0, 1, 2$ and 3 .

If Y could be obtained by (-4) -surgery on an L -space knot, then the fact that $V_0 = 0$ would imply that its genus is zero, ie it is the unknot. However, Y is not a lens space. \square

It seems worth noticing that in fact there are infinitely many knots K_n that produce Y from the proposition above; see [27]. In fact, $K_0 = \overline{9_{42}}$. The spaces resulting from p/q -surgeries on these knots have rather similar Floer homologies; in particular, all the correction terms are the same (and coincide with the correction terms of the lens space $L(p, q)$) and the total rank of reduced Floer homology is $2q$.

Moreover, we can work out the Heegaard Floer homology of all surgeries on these knots and their Alexander polynomials. Teragaito [27, Remark 6.1] mentions that K_n has genus $2n + 2$. In [17, Corollary 4.5] it is shown that

$$\widehat{\text{HFK}}(K, g(K)) \cong \text{HF}^+(S_0^3(K), g - 1),$$

so it is nontrivial by Theorem 9, and thus by Proposition 22 and the fact that $V_0 = 0$ for the present examples, we get that $A_{\pm(g(K)-1)}^{\text{red}}$ have to be nontrivial. By description of the Heegaard Floer homology of Y in the proof of Proposition 33 we conclude that $A_{2n+1}^{\text{red}}(K_n) = A_{-(2n+1)}^{\text{red}}(K_n) = \tau(1)$ and $A_k^{\text{red}}(K_n) = 0$ for any $k \neq \pm 2n + 1$. Using Proposition 19 we can also fix the gradings, and then using results from Section 3 deduce the Heegaard Floer homology of all surgeries on these knots.

Proposition 34 *The Alexander polynomial of K_0 is $-1 + 2(T + T^{-1}) - (T^2 + T^{-2})$. For $n \neq 0$ the Alexander polynomial is given by*

$$\Delta_{K_n}(T) = 1 - (T^{2n} + T^{-2n}) + 2(T^{2n+1} + T^{-(2n+1)}) - (T^{2n+2} + T^{-(2n+2)}).$$

Proof From the discussion above, $V_0 = 0$ and the only nontrivial $A_k^{\text{red}}(K)$ are $A_{2n+1}^{\text{red}}(K_n) = A_{-(2n+1)}^{\text{red}}(K_n) = \tau(1)$. Moreover, since the reduced parts of the Heegaard Floer homology of (-4) -surgery are in absolute $\mathbb{Z}/2\mathbb{Z}$ -grading 0, it means

that $A_{\pm(2n+1)}^{\text{red}}$ are in grading 1. (We can see from the description of the absolute grading on the mapping cone and Lemma 18 that for negative surgeries the $\mathbb{Z}/2\mathbb{Z}$ -grading of $\mathbb{A}_{i,p/q}^+(K)$ switches from what we have defined it to be in the mapping cone.) Now Lemma 23 implies that $t_{2n+1} = -1$ and $t_i = 0$ for all other $i \geq 0$. \square

By a straightforward argument involving $\mathbb{Z}/2\mathbb{Z}$ -grading considerations and dimension count it is not difficult to establish that in fact for $n > 0$ we have

$$\widehat{\text{HFK}}(K_n, 2n + 2) \cong \widehat{\text{HFK}}(K_n, 2n) \cong \mathbb{F} \quad \text{and} \quad \widehat{\text{HFK}}(K_n, 2n + 1) \cong \mathbb{F}^2.$$

7.1 Property S

Heegaard Floer homology has been very successful in restricting cosmetic surgeries on knots in S^3 (see [11; 23; 28]). In this subsection, we define a class of knots that do not admit purely cosmetic surgeries.

Definition 35 Let $r_1, r_2 \in \mathbb{Q}$ and let $K \subset S^3$ be a knot. The surgeries on K with slopes r_1 and r_2 are called cosmetic if $S_{r_1}^3(K)$ is homeomorphic to $S_{r_2}^3(K)$. They are called purely cosmetic if $S_{r_1}^3(K) \cong S_{r_2}^3(K)$, by which we mean that there exists an orientation-preserving homeomorphism between them.

We now begin defining the property that will imply the nonexistence of purely cosmetic surgeries.

Definition 36 We say that a rational homology sphere Y has property S if $\text{HF}_{\text{red}}(Y)$ is all concentrated in the same absolute $\mathbb{Z}/2\mathbb{Z}$ -grading.

Definition 37 We say that a knot $K \subset S^3$ has property S if $S_{p/q}^3(K)$ has property S for some $p/q \neq 0$.

Proposition 38 A knot K has property S if and only if $S_{p/q}^3(K)$ has property S for any $p/q \geq 2g(K) - 1$.

Proof Suppose $S_{p/q}^3(K)$ has property S. By taking the mirror of the knot, we may assume $p/q > 0$.

Then by looking at Corollary 14 and Proposition 15 we see that for all k all elements of $A_k^{\text{red}}(K)$ are in the same $\mathbb{Z}/2\mathbb{Z}$ -grading. This is enough for all elements of $\text{HF}^+(S_{p/q}^3(K))$ for $p/q \geq 2g(K) - 1$ to be concentrated in the same $\mathbb{Z}/2\mathbb{Z}$ -grading. \square

Corollary 39 A nontrivial knot with property S admits no purely cosmetic surgeries.

Proof The proof is completely analogous to the proof of [11, Corollary 3.12]. In fact, Ni and Wu show that if Y can be obtained by a purely cosmetic surgery, then the Euler

characteristic of $\text{HF}_{\text{red}}(S_{p/q}^3(K))$ has to be 0. They also show that V_0 and \bar{V}_0 have to be zero for a knot that admits cosmetic surgeries. This implies that $V_i = H_i = 0$ for $i \geq 0$, so we do not have any $\tau(V_i)$ groups in the reduced Floer homology. A knot with property S has all the $A_k^+(K)$ groups concentrated in the same $\mathbb{Z}/2\mathbb{Z}$ -grading, and in the case at hand these are the groups that constitute the reduced Floer homology. Therefore, in this case, the Euler characteristic of $\text{HF}_{\text{red}}(S_{p/q}^3(K))$ is equal to (plus or minus) its rank, so it is an L -space. However, if an L -space knot has $V_0 = 0$, then it is trivial. \square

Ni and Wu [11, Corollary 3.12] show that Seifert fibred spaces cannot be obtained by purely cosmetic surgeries. We can extend this result as follows.

Corollary 40 *There are no purely cosmetic surgeries on knots with nonzero Seifert fibred surgeries.*

Proof By [15] Seifert fibred rational homology spheres have property S. \square

We remark that there are knots which do not have this property, for example 9_{44} . Indeed, $+1$ and -1 -surgeries on this knot have the same HF^+ , but are not homeomorphic [23, Section 9].

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