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Eigenvalue varieties of Brunnian links

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In this article, it is proved that the eigenvalue variety of the exterior of a nontrivial, non-Hopf, Brunnian link in $\mathbb{S}^3$ contains a nontrivial component of maximal dimension. Eigenvalue varieties were first introduced to generalize the A–polynomial of knots in $\mathbb{S}^3$ to manifolds with nonconnected toric boundary. The result presented here generalizes, for Brunnian links, the nontriviality of the A–polynomial of nontrivial knots in $\mathbb{S}^3$.

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The A–polynomial of a knot in $\mathbb{S}^3$ is a two-variable polynomial constructed from the $\mathrm{SL}_2\mathbb{C}$–character variety of the knot exterior. Let K be a knot in $\mathbb{S}^3$ and let $\pi_1 K$ denote the fundamental group of the exterior of K; the peripheral subgroup $\mathbb{Z}^2$ is generated by a meridian μ and a longitude λ, and the zero-set of the A–polynomial A_K is the locus of eigenvalues for a common eigenvector of $\rho(\mu)$ and $\rho(\lambda)$ of representations ρ from $\pi_1 K$ to $\mathrm{SL}_2\mathbb{C}$. It was first introduced by Cooper, Culler, Gillet, Long and Shalen [2], where it is also proved that the A–polynomial of any knot contains the A–polynomial of the unknot as a factor. The A–polynomial of a knot is said to be *nontrivial* if it contains other factors, and it was also proved in the same article [2] that hyperbolic knots and nontrivial torus knots always have a nontrivial A–polynomial. This was later established in full generality for all nontrivial knots by Dunfield and Garoufalidis [4], and independently by Boyer and Zhang [1]; both proofs use a theorem by Kronheimer and Mrowka [5] on Dehn fillings on knots and representations in SU_2.

The notion of A–polynomial can be generalized to any 3–manifold M with connected toric boundary by specifying a *peripheral system* (generators of $\pi_1 \partial M \hookrightarrow \pi_1 M$). Stimulated by the work of Lash in [6], it was then extended to manifolds with nonconnected boundary by Tillmann. In his Ph D thesis [11] and the subsequent article [12], Tillmann presented the *eigenvalue variety* $\mathfrak{E}(M)$ associated to a 3–manifold M with toric boundary. If the boundary of M consists of n tori, the associated eigenvalue variety $\mathfrak{E}(M)$ is an algebraic subspace of $\mathbb{C}^{2n}$ corresponding to the closure of peripheral eigenvalues taken by representations (or equivalently, characters) of $\pi_1 M$ in $\mathrm{SL}_2\mathbb{C}$. Under these assumptions, Tillmann proved in [12] that the dimension of any component of $\mathfrak{E}(M)$ is at most n.

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In the same way as any A–polynomial is divisible by the A–polynomial of the unknot, any eigenvalue variety $\mathfrak{E}(M)$ contains components $\mathfrak{E}^{\text{red}}(M)$ corresponding to reducible characters. Components of $\mathfrak{E}^{\text{red}}(M)$ have maximal dimension, and any other component of $\mathfrak{E}(M)$ with maximal dimension is called a *nontrivially maximal* component of $\mathfrak{E}(M)$.

If M is hyperbolic, its character variety contains a family of distinguished components $Y_1, \dots, Y_k$ called the *geometric components*, each one containing the character of a discrete faithful representation. Using Thurston's results [10], Tillmann proved that each geometric component produces a nontrivially maximal component in $\mathfrak{E}(M)$, generalizing the result of [2] on hyperbolic knots. However, the question of classifying 3–manifolds M for which $\mathfrak{E}(M)$ contains a nontrivially maximal component, or even determining whether nontrivial exteriors of links in $\mathbb{S}^3$ have this property, remains open.

In this article, we answer this matter for a family of links in $\mathbb{S}^3$, the *Brunnian links*. A link in $\mathbb{S}^3$ is called *Brunnian* if any of its proper sublinks is trivial, and we prove:

Theorem 1 *The eigenvalue variety of any nontrivial non-Hopf Brunnian link contains a nontrivially maximal component.*

The defining property of Brunnian links makes them stable under $1/q$–Dehn fillings, which permits us to apply the Kronheimer–Mrowka theorem [5, Theorem 1] to produce irreducible characters in a similar fashion as in [4] and [1]. Then, an induction on the number of components of the links produces nontrivially maximal components of their eigenvalue varieties.

This article is divided into two sections: First we recall the construction of the eigenvalue variety $\mathfrak{E}(L)$ for a link L in $\mathbb{S}^3$, its defining ideal $\mathcal{A}(L)$ and some of its properties, as presented in [12], to introduce notation for the following section. Then we study the family of Brunnian links in $\mathbb{S}^3$ and prove the main result of this article.

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1 Eigenvalue varieties of links in $\mathbb{S}^3$

First we briefly review the notion of *eigenvalue variety* associated to a link in $\mathbb{S}^3$, first introduced by Tillmann in [11, Section 3.2.4], and we reproduce the construction here (with a slightly different vocabulary) in order to set the notation for the next section.

1.1 Character varieties

Let π be a finitely generated group; the $SL_2\mathbb{C}$–*representation variety* of π is the algebraic affine set $\hom(\pi, SL_2\mathbb{C})$ and is denoted by $R(\pi)$. The algebraic Lie group $SL_2\mathbb{C}$ acts on $R(\pi)$ by conjugation, and the algebraic quotient under this action is the $SL_2\mathbb{C}$–*character variety* of π, denoted by $X(\pi)$. The ring $\mathbb{C}[X(\pi)]$ of regular functions on the character variety is equal to the subring $\mathbb{C}[R(\pi)]^{SL_2\mathbb{C}}$ of invariant functions. Dually, the inclusion $\mathbb{C}[X(\pi)] \hookrightarrow \mathbb{C}[R(\pi)]$ induces a natural algebraic epimorphism $t\colon R(\pi) \to X(\pi)$, and any regular function on $R(\pi)$ factors through t if and only if it is invariant under the conjugation action of $SL_2\mathbb{C}$. In particular, for any γ in π, the function $\tau_\gamma\colon R(\pi) \to \mathbb{C}$ mapping $\rho \mapsto \operatorname{tr}\rho(\gamma)$ defines a regular function I_γ on $X(\pi)$ called the *trace function at* γ; the trace functions finitely generate the ring $\mathbb{C}[X(\pi)]$; see [3] for example. Representation and character varieties are contravariant functors: any group morphism $\pi \to \pi'$ induces regular maps according to the following commutative diagram:

$$\begin{array}{ccc} R(\pi') & \longrightarrow & R(\pi) \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t} \\ X(\pi') & \longrightarrow & X(\pi) \end{array}$$

In the case where the group π is the fundamental group of a manifold M (resp. the exterior of a link L in $\mathbb{S}^3$), the representation and character varieties will be denoted by $R(M)$ and $X(M)$ (resp. $R(L)$ and $X(L)$).

1.2 Abelian characters

Any group π has an abelianization π^{ab} and a canonical projection $\pi \to \pi^{ab}$ which induces regular maps:

$$\begin{array}{ccc} R(\pi^{ab}) & \longrightarrow & R(\pi) \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t} \\ X(\pi^{ab}) & \longrightarrow & X(\pi) \end{array}$$

The image of $R(\pi^{ab})$ in $R(\pi)$ is precisely the closed set $R^{ab}(\pi)$ of abelian representations of π, and the image of $X(\pi^{ab})$ is a closed subset of $X(\pi)$ called the set of *abelian characters* of π and denoted by $X^{ab}(\pi)$.

Remark In $SL_2\mathbb{C}$, characters of reducible representations are characters of abelian representations. If $R^{red}(\pi)$ is the closed set of reducible representations and $X^{red}(\pi)$ is its image in $X(\pi)$, then $X^{red}(\pi) = X^{ab}(\pi)$.

Let Δ denote the map from $\mathbb{C}^*$ to $\mathrm{SL}_2\mathbb{C}$ mapping $z \mapsto \left[\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right]$; by composition, Δ defines maps:

$$\begin{array}{ccc} \hom(\pi, \mathbb{C}^*) & \xrightarrow{\Delta_*} & R^{\mathrm{ab}}(\pi) \\ & \searrow^{d} & \downarrow t \\ & & X^{\mathrm{ab}}(\pi) \end{array}$$

The map d is two-to-one onto $X^{\mathrm{ab}}(\pi)$ and invariant under inversion in $\hom(\pi, \mathbb{C}^*)$; for any φ in $\hom(\pi, \mathbb{C}^*)$ and γ in π,

$$I_\gamma \circ d(\varphi) = \varphi(\gamma) + \varphi(\gamma)^{-1}.$$

1.3 Eigenvalue varieties

Let L be a link in $\mathbb{S}^3$, let $|L|$ denote its number of components and let $\pi_1 L$ be the fundamental group of its exterior; the boundary of the exterior of L is a disjoint union of $|L|$ tori T_K, one for each component K of the link L. Each inclusion $\pi_1 T_K \hookrightarrow \pi_1 L$ induces a regular map $r_K \colon X(L) \to X(T_K)$. Since $\pi_1 T_K$ is abelian, $X(T_K) = X^{\mathrm{ab}}(T_K)$, and denoting $\hom(\pi_1 T_K, \mathbb{C}^*)$ by $E(T_K)$, we obtain the following diagram:

$$\begin{array}{ccc} & & \prod_{K\subset L} E(T_K) \\ & & \downarrow d \\ X(L) & \xrightarrow[r]{} & \prod_{K\subset L} X(T_K) \end{array}$$

Following Tillmann [11; 12], the *eigenvalue variety* of L is defined as the Zariski closure of the preimage by d of the image of r:

$$\mathfrak{E}(L) = \overline{d^{-1}(r(X(L)))}.$$

Dually, there are ring-maps

$$\begin{array}{ccc} & & \bigotimes_{K\subset L} \mathbb{C}[E(T_K)] \\ & & \uparrow d^* \\ \mathbb{C}[X(L)] & \xleftarrow[r^*]{} & \bigotimes_{K\subset L} \mathbb{C}[X(T_K)] \end{array}$$

and the defining ideal $\mathcal{A}(L)$ of $\mathfrak{E}(L)$ is called the $\mathcal{A}$*–ideal* of L and is the radical of the image by d^* of the kernel of r^*:

$$\mathcal{A}(L) = \sqrt{d^*(\ker\ r^*)}.$$

Each torus T_K is equipped with a *standard peripheral system* (μ_K, λ_K) of meridian and longitude of each component. This produces canonical coordinates (m_K, ℓ_K) in $\mathbb{C}^* \times \mathbb{C}^*$ for $E(T_K)$, and $\mathfrak{E}(L)$ is naturally a subset of $(\mathbb{C}^*)^{2|L|}$; dually, $\mathbb{C}[E(T_K)]$ is isomorphic to $\mathbb{C}[\mathfrak{m}_K^{\pm 1}, \mathfrak{l}_K^{\pm 1}]$, and $\mathcal{A}(L)$ is an ideal of $\mathbb{C}[\mathfrak{m}^{\pm}, \mathfrak{l}^{\pm}] = \bigotimes_{K\subset L} \mathbb{C}[\mathfrak{m}_K^{\pm 1}, \mathfrak{l}_K^{\pm 1}]$.

Proposition 2 *Let $\mathfrak{E}^{\mathrm{ab}}(L)$ denote the part of $\mathfrak{E}(L)$ corresponding to abelian characters and $\mathcal{A}^{\mathrm{ab}}(L)$ the corresponding defining ideal; $\mathfrak{E}^{\mathrm{ab}}(L)$ is a union of copies of $(\mathbb{C}^*)^{|L|}$, and $\mathcal{A}^{\mathrm{ab}}(L)$ is given in $\mathbb{C}[\mathfrak{m}^{\pm}, \mathfrak{l}^{\pm}]$ by*

$$\mathcal{A}^{\mathrm{ab}}(L) = \left\langle \mathfrak{l}_K - \prod_{K' \neq K} \mathfrak{m}_{K'}^{\pm \operatorname{lk}(K,K')} \right\rangle,$$

where $\operatorname{lk}(K, K')$ denotes the linking number of the components K and K'.

Proof The meridians form a basis of the homology group of the link exterior, and each longitude is given by the linking numbers

$$\lambda_K = \sum_{K' \neq K} \operatorname{lk}(K, K') \mu_{K'}.$$

Therefore, any morphism from $\pi_1 L$ to $\mathbb{C}^*$ is determined by the images of the meridians, and for any φ in $\hom(\pi_1 L, \mathbb{C}^*)$ and each longitude λ_K,

$$\varphi(\lambda_K) = \prod_{K \neq K'} \varphi(\mu_{K'})^{\operatorname{lk}(K,K')}.$$

By the invariance under inversion, any point $(m_K, \ell_K)_{K \subset L}$ of $\mathfrak{E}^{\mathrm{ab}}(L)$ then satisfies

$$\ell_K = \prod_{K \neq K'} m_K^{\pm \operatorname{lk}(K,K')}.$$

Conversely, for any $\xi = (m_K, \ell_K)_{K \subset L}$ satisfying these equations, there exists φ in $\hom(\pi_1 L, \mathbb{C}^*)$ such that $d(\xi) = r(\Delta_* \varphi)$, so $\mathcal{A}^{\mathrm{ab}}(L)$ is given by

$$\mathcal{A}^{\mathrm{ab}}(L) = \left\langle \mathfrak{l}_K - \prod_{K' \neq K} \mathfrak{m}_{K'}^{\pm \operatorname{lk}(K,K')} \right\rangle. \qquad \square$$

Remark For links with one component (knots), the $\mathcal{A}$–ideal is generated by the A–polynomial of the knot, and A^{ab} is the $\mathfrak{l} - 1$ factor corresponding to abelian characters.

By the defining equations of $\mathcal{A}^{\mathrm{ab}}(L)$, we have that $\mathfrak{E}^{\mathrm{ab}}(L)$ always has dimension $|L|$. As a matter of fact, by Tillmann [11, Proposition 3.10; 12, Proposition 13], any component of $\mathfrak{E}(L)$ has dimension at most $|L|$, which leads to the following definition:

Definition 3 A component of $\mathfrak{E}(L)$ is called *nontrivially maximal* if it has dimension $|L|$ and is not contained in $\mathfrak{E}^{\mathrm{ab}}(L)$.

Using Thurston's results on hyperbolic manifolds, Tillmann showed the following:

Theorem 4 [12, Proposition 13] *If L is a hyperbolic link in $\mathbb{S}^3$, then any geometric component of the character variety produces a nontrivially maximal component in the eigenvalue variety of L.*

Besides these cases, it is not known whether the eigenvalue variety of all (nontrivial) links admits a maximal nontrivial component. For knots, this is equivalent to the nontriviality of the A–polynomial (besides the $\mathfrak{l}-1$ factor) and was proven independently by Dunfield and Garoufalidis in [4], and Boyer and Zhang in [1]. In the next section, we answer this matter for Brunnian links in $\mathbb{S}^3$.

2 Characters of Brunnian links

In this section, we prove Theorem 1. First we recall some basic facts on $1/q$–Dehn fillings on links in $\mathbb{S}^3$; then we present Brunnian links and, after having studied their stability under these Dehn fillings, we use the Kronheimer–Mrowka theorem to create families of characters of exteriors of Brunnian links. Finally, we prove that these characters span a nontrivially maximal component in the eigenvalue varieties of nontrivial, non-Hopf, Brunnian links.

2.1 Dehn fillings

Any $1/q$–Dehn filling on the unknot in $\mathbb{S}^3$ produces $\mathbb{S}^3$ again; therefore, the $1/q$–Dehn filling over an unknotted component of a link in $\mathbb{S}^3$ produces the exterior of another link in $\mathbb{S}^3$.

Let $L = K \sqcup L'$ be a link with K an unknotted component of L, and let L_q denote the link obtained by $1/q$–surgery on K (so, in particular, $L' = L_0$). Any sublink L'' of L_q is obtained by $1/0$–Dehn filling along the other components. Because the meridians are unchanged by $1/q$–Dehn fillings, any proper sublink L'' of L_q is obtained by $1/q$–Dehn filling along K in the sublink $L'' \sqcup K$ of L.

Remark With this notation, if $L'' \sqcup K$ is trivial in $\mathbb{S}^3$, then so is L''.

The meridians are unchanged by $1/q$–Dehn fillings, but the longitudes are changed according to the linking numbers. With the same notation as above, if (μ, λ) is a standard peripheral system for a component J of L, then the new longitude λ_q of J in L_q is

$$\lambda_q = \lambda + q\,\mathrm{lk}(K, J)^2 \mu,$$

and the linking number $\mathrm{lk}_q(J, J')$ of any two components J and J' of L_q is given by

$$\mathrm{lk}_q(J, J') = \mathrm{lk}(J, J') - q\,\mathrm{lk}(K, J)\,\mathrm{lk}(K, J').$$

A link is called *homologically trivial* if all the linking numbers between components vanish. By the previous discussion, the link obtained by $1/q$–Dehn fillings on an unknotted component of a homologically trivial link is still homologically trivial and has the same longitudes.

The proof of Theorem 1 uses Dehn fillings to produce closed 3–manifolds which admit irreducible representations; this will be done by iterating $1/q$–Dehn fillings along the components of the link. However, even if all the components of a link L in $\mathbb{S}^3$ are unknotted, a $1/q$–Dehn filling along a component generally knots the other components, thus making impossible to continue the process while remaining in $\mathbb{S}^3$. In other words, to achieve this goal, we need a family of links $\mathcal{L}$ satisfying

- if $L \in \mathcal{L}$ has two or more components, each is individually unknotted;
- for any $K \sqcup L_0$ in $\mathcal{L}$, we have that L_q is also in $\mathcal{L}$.

In the next section, we show that the family of *Brunnian links* in $\mathbb{S}^3$ satisfies these conditions. Moreover, nontriviality can be preserved in the process, making it possible to reason by induction on the number of components of the link.

2.2 Brunnian links

Definition 5 A link is called *Brunnian* if any of its proper sublinks is trivial.

Remark Any knot is considered Brunnian; for links with more components, we have:

- If a Brunnian link has two or more components, they are individually unknotted.
- Any Brunnian link with three or more components is homologically trivial.
- By Section 2.1, if $L = K \sqcup L_0$ is Brunnian, L_q is also Brunnian for any integer q.

Given $L = K \sqcup L_0$ Brunnian, we can perform a $1/p$–surgery on a component of L_q to obtain another Brunnian link, and so on, until obtaining a knot in $\mathbb{S}^3$. However, any $1/q$–Dehn filling on a component of the Hopf link or the unlink produces the unlink. Therefore, given a Brunnian link $L = K \sqcup L_0$, we need to prevent L_q from being the Hopf link or the unlink in order to obtain, in fine, a nontrivial knot in $\mathbb{S}^3$.

If $L = K \sqcup K'$ is a Brunnian link with two components, this is a special case of Mathieu's theorem from [9]. This more general result on knots in a solid torus (links with one unknotted component) asserts that, besides the Hopf link, for any $|q| \geqslant 2$, any $1/q$–Dehn filling on an unknotted component of a 2–component link in $\mathbb{S}^3$ produces a nontrivial knot. For our concern, this implies that, for any $|q| \geqslant 2$, the $1/q$–Dehn filling on any component of a Brunnian, non-Hopf, nontrivial 2–link may never produce the trivial knot.

On the other hand, if L has three components or more, it is homologically trivial, and the work of Mangum and Stanford [8, Theorem 2 and its proof] ensures that, for any integer q and any homologically trivial Brunnian link $L = K \sqcup L_0$, if L is nontrivial, then L_q is trivial if and only if $q = 0$. Otherwise, it is a nontrivial, homologically trivial Brunnian link (in particular, it is never the Hopf link).

Therefore, we obtain the following result for the stability of nontrivial non-Hopf Brunnian links under $1/q$–Dehn fillings:

Proposition 6 *Let $L = K \sqcup L_0$ be a nontrivial, non-Hopf, Brunnian link in $\mathbb{S}^3$. Then for any $|q| \geqslant 2$, the link L_q is a Brunnian link in $\mathbb{S}^3$, nontrivial and non-Hopf.*

We will use the stability of nontrivial non-Hopf Brunnian links to apply the Kronheimer–Mrowka theorem on some Dehn fillings of the link exteriors to produce nontrivially maximal components in the eigenvalue varieties. On the other hand, for the Hopf link and the trivial link, no such component exists:

Proposition 7 *The eigenvalue varieties of the Hopf link and the trivial link do not admit any nontrivially maximal component.*

Proof The fundamental group of the exterior of the Hopf link is abelian, so all the characters are abelian, and $\mathfrak{E} = \mathfrak{E}^{\mathrm{ab}}$.

On the other hand, for the trivial link, all the longitudes are nullhomotopic and are therefore trivialized by any representation, so $\mathcal{A} = \langle \mathfrak{l}_K - 1,\ K \subset L \rangle = \mathcal{A}^{\mathrm{ab}}$. □

2.3 Kronheimer–Mrowka characters

By the Kronheimer–Mrowka theorem from [5], any nontrivial $1/q$–Dehn filling along a nontrivial knot in $\mathbb{S}^3$ produces a closed 3–manifold which admits an irreducible representation in SU_2. By Proposition 6, if $L = K \sqcup L_0$ is a nontrivial Brunnian link in $\mathbb{S}^3$, L_q is nontrivial for any $|q| \geqslant 2$. Performing another $1/p$–Dehn filling on a component of L_q (in the new standard peripheral system if the link is not homologically trivial) will produce again a nontrivial Brunnian link; this process may be continued until a nontrivial knot is produced, on which a final $1/k$–Dehn filling may be performed to obtain a closed 3–manifold which admits an irreducible representation in SU_2.

For any Brunnian link $L = K_1 \sqcup \cdots \sqcup K_n$ in $\mathbb{S}^3$, and any $\underline{q} = (q_1, \ldots, q_k)$ in $\mathbb{Z}^k$ for $k \leqslant n$, we denote by $L(\underline{q})$ the 3–manifold obtained by performing $1/q_i$–Dehn fillings on the components of L, where each $1/q_i$–Dehn filling is performed in the standard peripheral system given after the Dehn fillings $1/q_j$ for $j < i$.

Remark As already pointed out, the meridians never change, and since L is assumed Brunnian, longitudes change only if L is a Brunnian link with two components $L = K_1 \sqcup K_2$ with nonzero linking number α; in that case, denoting by $(\mu_i, \lambda_i)_{i=1,2}$ the respective standard peripheral systems, any $1/q_1$–Dehn filling on K_1 changes the longitude λ_2 into $\lambda_2 + q_1\alpha^2\mu_2$. Therefore, a $1/q_2$–Dehn filling on K_2 is performed along the slope

$$(1 + q_1 q_2 \alpha^2)\mu_2 + q_2\lambda_2 \in H_1(T_{K_2}).$$

Proposition 8 *Let $L = K_1 \sqcup \cdots \sqcup K_n$ be a nontrivial Brunnian link in $\mathbb{S}^3$ different from the Hopf link, and let $\underline{q} = (q_1, \dots, q_n)$ be a family of integers.*

- *If $q_i = 0$ for some $1 \leqslant i \leqslant n$, then $L_{\underline{q}} = \mathbb{S}^3$.*
- *If $|q_i| \geqslant 2$ for all $1 \leqslant i \leqslant n$, then there exists an irreducible representation*

$$\rho_{\underline{q}}\colon \pi_1 L_{\underline{q}} \to \mathrm{SU}_2.$$

Proof First, if one of the q_i is zero, the link $L_{(q_1,\dots,q_i)}$ is trivial, so performing $1/q_k$–Dehn fillings for $i < k \leqslant n$ produces the standard 3–sphere.

On the other hand, if all the $|q_i|$ are greater than 1, each $L_{(q_1,\dots,q_k)}$ for $k \leqslant n$ is nontrivial by Proposition 6, so $L_{(q_1,\dots,q_{n-1})}$ is a nontrivial knot in $\mathbb{S}^3$. The Kronheimer–Mrowka theorem concludes the proof. □

By inclusion of SU_2 in $\mathrm{SL}_2\mathbb{C}$, we can consider $\rho_{\underline{q}}$ as an irreducible representation of $R(L_{\underline{q}})$ (with no nontrivial parabolic image). Moreover, composing with the group homomorphism $\pi_1 L \to \pi_1 L_{\underline{q}}$, we may also consider $\rho_{\underline{q}}$ as an irreducible representation of $R(L)$. The irreducible characters $\chi_{\underline{q}} = t(\rho_{\underline{q}})$ obtained this way are called *Kronheimer–Mrowka characters*, and we denote by $X_{\mathrm{KM}}(L)$ the Zariski closure in $X(L)$ of all Kronheimer–Mrowka characters:

$$X_{\mathrm{KM}}(L) = \overline{\{\chi_{\underline{q}},\ \underline{q} \in (\mathbb{Z} \setminus \{-1, 0, 1\})^{|L|}\}}.$$

Remark The subset $X_{\mathrm{KM}}(L)$ of $X(L)$ may contain several algebraic components.

Remark For any nontrivial, non-Hopf, Brunnian link $L = K \sqcup L_0$, the group homomorphism $i_q\colon \pi_1 L \to \pi_1 L_q$ induces an algebraic map

$$i_q^*\colon X(L_q) \to X(L),$$

and if $|q| \geqslant 2$, then $i_q^* X_{\mathrm{KM}}(L_q) \subset X_{\mathrm{KM}}(L)$.

Any representation $\rho_{\underline{q}}$ satisfies the $1/q_K$–Dehn filling relations for each component K of L. On the other hand, no $\rho_{\underline{q}}(\mu_K)$ is trivial since, otherwise, it would satisfy the $1/0$ relation on K; it would then factor as a representation of $\mathbb{S}^3$ and therefore be trivial. Since $\rho_{\underline{q}}$ factors in SU_2, this is equivalent to $\operatorname{tr}\rho_{\underline{q}}(\mu_K\lambda_K^{q_K}) = 2$ and $\operatorname{tr}\rho_{\underline{q}}(\mu_K) \neq 2$.

It follows that any Kronheimer–Mrowka character $\chi_{\underline{q}}$ satisfies, for any $K \subset L$,

$$I_{\mu_K\lambda_K^{q_K}}(\chi_{\underline{q}}) = 2, \tag{1}$$

$$I_{\mu_K}(\chi_{\underline{q}}) \neq 2. \tag{2}$$

Finally, following Section 1, we denote by $\mathfrak{E}_{\mathrm{KM}}(L)$ the part corresponding to $X_{\mathrm{KM}}(L)$ in $\mathfrak{E}(L)$. For any $\xi_{\underline{q}} \in \mathfrak{E}_{\mathrm{KM}}(L)$ corresponding to a Kronheimer–Mrowka character $\chi_{\underline{q}}$ in $X_{\mathrm{KM}}(L)$, and any component K of L, (1) and (2) imply that

$$\mathfrak{m}_K \mathfrak{l}_K^{q_K}(\xi_{\underline{q}}) = 1, \tag{3}$$

$$\mathfrak{m}_K(\xi_{\underline{q}}) \neq 1. \tag{4}$$

Remark Together with the equations for $\mathcal{A}^{\mathrm{red}}(L)$, this implies that no such point $\xi_{\underline{q}}$ is in $\mathfrak{E}^{\mathrm{red}}(L)$, so no component of $\mathfrak{E}_{\mathrm{KM}}(L)$ is contained in $\mathfrak{E}^{\mathrm{red}}(L)$.

2.4 Maximal components

In this last section, we prove the following result which implies Theorem 1:

Theorem 9 *For any nontrivial Brunnian link L different from the Hopf link, $\mathfrak{E}_{\mathrm{KM}}(L)$ contains a maximal component.*

Proof This is proved by induction on the number of components of L.

For the base case, L is a knot K, and the proof is the same as the one for the nontriviality of the A–polynomial of nontrivial knots from Dunfield and Garoufalidis in [4] or Boyer and Zhang in [1].

For any $|q| \geqslant 2$, performing $1/q$–surgery produces an irreducible character χ_q in $X(K)$ and a point $\xi_q = (m_q, \ell_q)$ in $\mathfrak{E}(K)$. They show that there are infinitely many distinct ℓ_q obtained this way, so $\mathfrak{E}_{\mathrm{KM}}(K)$ contains a curve different from the line $\mathfrak{l} - 1$. We do not reproduce this proof here, but very similar ideas are used in the induction step.

For the induction step, let $L = K \sqcup L_0$ be a nontrivial, non-Hopf, Brunnian link in $\mathbb{S}^3$. For any $|q| \geqslant 2$, L_q is nontrivial, non-Hopf and Brunnian, so we can assume, by induction, that $\mathfrak{E}_{\mathrm{KM}}(L_q)$ contains a maximal component.

We have the commutative diagram

$$\begin{array}{ccc}
X_{\mathrm{KM}}(L_q) & \longrightarrow & X_{\mathrm{KM}}(L) \\
{\scriptstyle r_q}\big\downarrow & & \big\downarrow{\scriptstyle r} \\
\prod_{J \neq K} X(T_J) & \longleftarrow & \prod_{J \subset L} X(T_J) \\
{\scriptstyle d}\big\uparrow & & \big\uparrow{\scriptstyle d} \\
\prod_{J \neq K} E(T_J) & \longleftarrow & \prod_{J \subset L} E(T_J)
\end{array}$$

so there exists X_q in $X_{\mathrm{KM}}(L)$ corresponding to $\mathfrak{E}_q$ in $\mathfrak{E}_{\mathrm{KM}}(L)$ such that $\dim \mathfrak{E}_q \geqslant |L|-1$. If $\dim \mathfrak{E}_q = |L|$ for some q, then there is nothing more to prove.

Let us assume now that all the components $\mathfrak{E}_q$ have dimension $|L|-1$. We will show that $\mathfrak{E}_{\mathrm{KM}}(L)$ contains infinitely many different such subspaces $\mathfrak{E}_q$; by algebraicity, this means that $\mathfrak{E}_{\mathrm{KM}}(L)$ must contain a component of dimension $|L|$, which will conclude the proof of Theorem 9.

The subspaces $\mathfrak{E}_q$ will be separated using the following lemma:

Lemma 10 *For any integers q, q',*

$$\mathfrak{E}_q \subset \mathfrak{E}_{q'} \implies \mathfrak{l}_K^{q-q'}|_{\mathfrak{E}_q} \equiv 1.$$

Moreover, for any $|q| \geqslant 2$, the set $\{p \in \mathbb{Z} \mid \mathfrak{l}_K^p|_{\mathfrak{E}_q} \equiv 1\}$ is an ideal $d_q\mathbb{Z}$ with $q \notin d_q\mathbb{Z}$.

Proof For any ξ in $\mathfrak{E}_q$, we have $\mathfrak{m}_K \mathfrak{l}_K^q(\xi) = 1$ by (3), so if ξ also belongs to $\mathfrak{E}_{q'}$, then $\mathfrak{m}_K \mathfrak{l}_K^{q'}(\xi) = 1$ and $\mathfrak{l}_K^{q-q'}(\xi) = 1$. Therefore, if $\mathfrak{E}_q \subset \mathfrak{E}_{q'}$, then $\mathfrak{l}_K^{q-q'} \equiv 1$ on $\mathfrak{E}_q$.

If q is in the ideal $d_q\mathbb{Z}$, the surgery relation implies that $\mathfrak{m}_K|_{\mathfrak{E}_q} \equiv 1$, in contradiction with (4). □

If $S = \{q \in \mathbb{Z} \setminus \{-1, 0, 1\} \mid d_q = 0\}$ is infinite, then by Lemma 10, $\mathfrak{E}_q \neq \mathfrak{E}_{q'}$ for $q \neq q'$ in S, so $(\mathfrak{E}_q)_{q \in S}$ is a family of infinitely many distinct subspaces.

Otherwise, there exists N in $\mathbb{N}$ such that, for any $q \geqslant N$, $d_q \geqslant 2$. Let $(q_i)_{i \in \mathbb{N}}$ be a family of integers such that

- $q_0 \geqslant N$;
- for any j in $\mathbb{N}$, we have $q_{j+1} \geqslant q_j$ and $q_{j+1} \in \bigcap_{i=1}^{j} d_{q_j}\mathbb{Z}$.

Then $(\mathfrak{E}_{q_i})_{i \in \mathbb{N}}$ contains infinitely many different subspaces, since

$$\mathfrak{E}_{q_i} \neq \mathfrak{E}_{q_j} \quad \text{for all } i < j.$$

Indeed, for any j in $\mathbb{N}$, let us assume that $\mathfrak{E}_{q_i} = \mathfrak{E}_{q_j}$ for some $i < j$. By Lemma 10, this would imply that $q_j - q_i \in d_{q_i}\mathbb{Z}$. But $q_j \in d_{q_i}\mathbb{Z}$ by construction, so this would imply $q_i \in d_{q_i}\mathbb{Z}$, a contradiction.

We have proved that $\mathfrak{E}_{\mathrm{KM}}(L)$ contains infinitely many different subsets of dimension $|L|-1$; by algebraicity, it must contain a component of dimension $|L|$, which concludes the proof of Theorem 9. □

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