A note on cobordisms of algebraic knots

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We use Heegaard Floer homology to study smooth cobordisms of algebraic knots and complex deformations of cusp singularities of curves. The main tool will be the concordance invariant ν^+: we study its behaviour with respect to connected sums, providing an explicit formula in the case of L–space knots and proving subadditivity in general.

1 Introduction

A *cobordism* between two knots K and K' in S^3 is a smoothly and properly embedded surface $F \subset S^3 \times [0, 1]$, with $\partial F = K \times \{0\} \cup K' \times \{1\}$. Carving along an arc connecting the two boundary components of F, one produces a slice surface for the connected sum $\overline{K} \# K'$, where $\overline{K}$ is the mirror of K. Two knots are *concordant* if there is a genus-0 cobordism between them; this is an equivalence relation, and the connected sum endows the quotient $\mathcal{C}$ of the set of knots with a group operation; $\mathcal{C}$ is therefore called the (smooth) *concordance group*. A knot is *smoothly slice* if it is concordant to the unknot.

Litherland [14] used Tristram–Levine signatures to show that torus knots are linearly independent in $\mathcal{C}$. In fact, Tristram–Levine signatures provide a lower bound for the slice genus of knots. Sharp lower bounds for the slice genus of torus knots are provided by the invariants τ in Heegaard Floer homology — see Ozsváth and Szabó [21] — and s in Khovanov homology; see Rasmussen [25].

More recently, Ozsváth, Stipsicz and Szabó [19] defined the concordance invariant Υ; Livingston and Van Cott [15] used Υ to improve on the bounds given by signatures along some families of connected sums of torus knots.

In this note we consider *algebraic knots*, ie links of irreducible curve singularities (*cusps*), and more generally L–*space knots*. Given two algebraic knots K and L, we give lower bounds on the genus of a cobordism between them by using the concordance

invariant ν^+ defined by Hom and Wu [12]. This is computed in terms of the semigroups of the two corresponding singularities, Γ_K and Γ_L, and the corresponding enumerating functions $\Gamma_K(\cdot)$ and $\Gamma_L(\cdot)$.

Theorem 1.1 *If K and L are algebraic knots with enumerating functions $\Gamma_K(\cdot)$ and $\Gamma_L(\cdot)$, respectively, then*

$$\nu^+(K \# \overline{L}) = \max\{g(K) - g(L) + \max_{n \geq 0}\{\Gamma_L(n) - \Gamma_K(n)\}, 0\}.$$

In Section 2.3 we define an appropriate enumerating function for L–space knots; Theorem 3.1 below mimics the statement above in this more general setting, and directly implies Theorem 1.1; the key of the definition and of the proofs is the reduced Floer complex defined by Krcatovich [13].

As an application of Theorem 1.1, we give a different proof of a result of Gorsky and Némethi [10] on the semicontinuity of the semigroup of an algebraic knot under deformations of singularities in the cuspidal case. A similar result was obtained by Borodzik and Livingston [7] under stronger assumptions (see Section 4 for details).

Theorem 1.2 *Assume there exists a deformation of an irreducible plane curve singularity with semigroup Γ_K to an irreducible plane curve singularity with semigroup Γ_L. Then for each nonnegative integer n*

$$\#(\Gamma_K \cap [0, n)) \leq \#(\Gamma_L \cap [0, n)).$$

In fact, there is an analogue of Theorem 1.2 when the deformation has multiple (not necessarily irreducible) singularities; see Theorem 5.2 below for a precise statement. As an immediate corollary, we obtain that the multiplicity decreases under deformations. More precisely, we have the following:

Corollary 1.3 *Let K and L be two links of irreducible singularities as above, and $m(K)$ and $m(L)$ denote their multiplicities. Then $m(L) \leq m(K)$.*

It is worth noting that the multiplicity of an irreducible singularity can be interpreted topologically as the *braid index* of the knot, ie the minimal number of strands among all braids whose closure is the given knot.

Finally, we turn to proving some properties of the function ν^+. The first one reflects analogous properties for other invariants (signatures, τ, s, etc.) and gives lower bounds for the unknotting number and related concordance invariants (see Section 5 below).

Theorem 1.4 *If K_+ is obtained from K_- by changing a negative crossing into a positive one, then*

$$\nu^+(K_-) \leq \nu^+(K_+) \leq \nu^+(K_-) + 1.$$

Theorem 1.5 *The function ν^+ is subadditive. Namely, for any two knots $K, L \subset S^3$,*

$$\nu^+(K \# L) \le \nu^+(K) + \nu^+(L).$$

As an application, we consider some concordance invariants, also studied by Owens and Strle [18]. Recall that the *concordance unknotting number $u_c(K)$* of a knot K is the minimum of unknotting numbers among all knots that are concordant to K; the *slicing number $u_s(K)$* of K is the minimal number of crossing changes needed to turn K into a slice knot; finally, the 4–*ball crossing number $c^*(K)$* is the minimal number of double points of an immersed disc in the 4–ball whose boundary is K. It is quite remarkable that there are knots for which these quantities disagree [18].

Proposition 1.6 *The unknotting number, concordance unknotting number, slicing number and 4–ball crossing number of K are all bounded below by $\nu^+(K) + \nu^+(\overline{K})$.*

1.1 Organisation of the paper

In Section 2 we recall some facts about Heegaard Floer correction terms and reduced knot Floer complex. In Section 3 we prove Theorem 1.1 as a corollary of Theorem 3.1, and in Section 4 we prove Theorem 1.2. In Section 5 we study cobordisms between arbitrary knots and prove Theorem 1.4 and Proposition 1.6; in Section 6 we prove Theorem 1.5. Finally, in Section 7 we study some concrete examples.

2 Singularities and Heegaard Floer homology

2.1 Links of curve singularities

In what follows, K and L will be two *algebraic knots*. We will recall briefly what this means and also what invariants can be associated with such knots. For further details, we refer to [8; 9; 26].

Assume $F \in \mathbb{C}[x, y]$ is an *irreducible* polynomial which defines an *isolated irreducible plane curve singularity*. This means that $F(0, 0) = 0$ and, in a sufficiently small neighbourhood $B_\varepsilon = \{|x|^2 + |y|^2 \leq \varepsilon^2\}$ for some $\varepsilon > 0$ of the origin, $\partial_1 F(x, y) = \partial_2 F(x, y) = 0$ holds if and only if $(x, y) = (0, 0)$. The *link* of the singularity is the zero set of F intersected with a sphere of sufficiently small radius: $K = \{F(x, y) = 0\} \cap \partial B_\varepsilon$. Since F is irreducible, K is a knot, rather than a link, in the 3–sphere ∂B_ε. A knot is called *algebraic* if its isotopy type arises in the above described way. All algebraic knots are *iterated torus knots*, ie they arise by iteratively cabling a torus knot.

The zero set of every isolated irreducible plane curve singularity admits a local parametrisation, ie there exist $x(t), y(t) \in \mathbb{C}[[t]]$ such that $F(x(t), y(t)) \equiv 0$ and $t \mapsto (x(t), y(t))$ is a bijection for $|t| < \eta \ll 1$ to a neighbourhood of the origin in the zero set of F. Consider the set of integers

$$\Gamma_K = \{ \mathrm{ord}_t\, G(x(t), y(t)) \mid G \in \mathbb{C}[[x, y]],\ F \text{ does not divide } G \}.$$

It can be seen easily that Γ_K is an additive semigroup. It depends only on the local topological type of the singularity; therefore, it can be seen as an invariant of the isotopy type of the knot K. We will say that Γ_K is the *semigroup of the algebraic knot* K.

We denote by $\mathbb{N} = \{0, 1, \dots\}$ the set of nonnegative integers. The semigroup Γ_K is a cofinite set in $\mathbb{N}$; in fact, $|\mathbb{N} \setminus \Gamma_K| = \delta_K < \infty$ and the greatest element not in Γ_K is $2\delta_K - 1$. The number δ_K is called the δ–*invariant* of the singularity. It is well-known that δ_K is the Seifert genus of K: $\delta_K = g(K)$.

We also write $\Gamma_K(n)$ for the nth element of Γ_K with respect to the standard ordering of $\mathbb{N}$, with the convention that $\Gamma_K(0) = 0$. The function $\Gamma_K(\cdot)$ will be called the *enumerating function* of Γ_K.

2.2 Heegaard Floer and concordance invariants

Heegaard Floer homology is a family of invariants of 3–manifolds introduced by Ozsváth and Szabó [22]; in this paper we are concerned with the "minus" version over the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ with two elements. It associates to a rational homology sphere Y equipped with a spin^c structure $\mathfrak{t}$ a $\mathbb{Q}$–graded $\mathbb{F}[U]$–module $\mathrm{HF}^-(Y, \mathfrak{t})$; the action of U decreases the degree by 2.

The group $\mathrm{HF}^-(Y, \mathfrak{t})$ further splits as a direct sum of $\mathbb{F}[U]$–modules $\mathbb{F}[U] \oplus \mathrm{HF}^-_{\mathrm{red}}(Y, \mathfrak{t})$. We call $\mathbb{F}[U]$ the *tower* of $\mathrm{HF}^-(Y, \mathfrak{t})$. The degree of the element $1 \in \mathbb{F}[U]$ is called the *correction term* of $(Y, \mathfrak{t})$, and it is denoted by[1] $d(Y, \mathfrak{t})$.

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[1]Note that our definition of $d(Y, \mathfrak{t})$ would differ by 2 from the original definition of [20]; however, it is more convenient for our purposes to use a shifted grading in HF^-.

When Y is obtained as an integral surgery along a knot K in S^3, one can recover the correction terms of Y in terms of a family of invariants introduced by Rasmussen [24] and then further studied by Ni and Wu [16] and Hom and Wu [12]. We call these invariants $\{V_i(K)\}_{i \geq 0}$, adopting Ni and Wu's notation instead of Rasmussen's — who used $h_i(K)$ — as this seems to have become more standard.

Recall that there is an indexing of spin^c structures on $S_n^3(K)$, as defined in [23, Section 2.4]: $S_n^3(K)$ is the boundary of the surgery handlebody $W_n(K)$ obtained by attaching a single 2–handle with framing n along $K \subset \partial B^4$. Notice that, if we orient K, there is a well-defined generator $[F]$ of $H_2(W_n(K); \mathbb{Z})$ obtained by capping off a Seifert surface of K with the core of the 2–handle. The spin^c structure $\mathfrak{t}_k$ on $S_n^3(K)$ is defined as the restriction of a spin^c structure $\mathfrak{s}$ on $W_n(K)$ such that

$$(1) \qquad \langle c_1(\mathfrak{s}), [F] \rangle \equiv n + 2k \pmod{2n}.$$

Theorem 2.1 [24; 16] *The sequence $\{V_i(K)\}_{i \geq 0}$ takes values in $\mathbb{N}$ and is eventually 0. Moreover, $V_i(K) - 1 \leq V_{i+1}(K) \leq V_i(K)$ for every i.*

With the spin^c labelling defined in (1) above, for every integer n we have

$$(2) \qquad d(S_n^3(K), \mathfrak{t}_i) = -2 \max\{V_i(K), V_{n-i}(K)\} + \frac{(n - 2i)^2 - n}{4n}.$$

Definition 2.2 [12] *The minimal index i such that $V_i(K) = 0$ is called $\nu^+(K)$.*

2.3 Reduced knot Floer homology

Krcatovich [13] introduced the *reduced knot Floer complex* $\underline{\mathrm{CFK}}^-(K)$ associated to a knot K in S^3. This complex is graded by the *Maslov grading* and filtered by the *Alexander grading*; the differential decreases the Maslov grading by 1 and respects the Alexander filtration.

Without going into technical details, for which we refer to [13], any knot Floer complex $\mathrm{CFK}^-(K)$ can be recursively simplified until the differential on the graded object associated to the Alexander filtration becomes trivial (while the differential on the filtered complex is, in general, nontrivial). Moreover, $\underline{\mathrm{CFK}}^-(K)$ still retains an $\mathbb{F}[U]$–module structure.

The power of Krcatovich's approach relies in the application to connected sums; if we need to compute $\mathrm{CFK}^-(K_1 \# K_2) \cong \mathrm{CFK}^-(K_1) \otimes_{\mathbb{F}[U]} \mathrm{CFK}^-(K_2)$ we can first reduce $\mathrm{CFK}^-(K_1)$, and then take the tensor product $\underline{\mathrm{CFK}}^-(K_1) \otimes_{\mathbb{F}[U]} \mathrm{CFK}^-(K_2)$.

This is particularly effective when dealing with L–*space knots*, ie knots that have a positive integral surgery Y such that $\mathrm{HF}^-(Y, \mathfrak{t}) = \mathbb{F}[U]$ for every spin^c structure $\mathfrak{t}$ on Y. Notice that all algebraic knots are L–space knots [11, Theorem 1.8].

In this case, $\underline{\mathrm{CFK}}^-(K)$ is isomorphic to $\mathbb{F}[U]$ as an $\mathbb{F}[U]$–module, and it has at most one generator in each Alexander degree. If we call x the homogeneous generator of $\underline{\mathrm{CFK}}^-(K)$ as an $\mathbb{F}[U]$–module, then $\underline{\mathrm{CFK}}^-(K) = \mathbb{F}[U]x$, and $\{U^n x\}_{n \geq 0}$ is a homogeneous basis of $\underline{\mathrm{CFK}}^-(K)$.

We denote by $\Gamma_K(n)$ the quantity $g(K) - A(U^n \cdot x)$, where A is the Alexander degree, and we call $\Gamma_K(\cdot)$ the *enumerating function* of K. As observed by Borodzik and Livingston [6, Section 4], when K is an algebraic knot, the function $\Gamma_K(\cdot)$ coincides with the enumerating function of the semigroup associated to K as defined above. Accordingly, for an arbitrary L–space knot K, we define the *semigroup* of K as the image of Γ_K.

Example 2.3 In general, $\Gamma_K(\cdot)$ is not the enumerating function of a semigroup; to this end, consider the pretzel knot $K = P(-2, 3, 7) = 12n_{242}$. K is an L–space knot with Alexander polynomial $t^{-5} - t^{-4} + t^{-2} - t^{-1} + 1 - t + t^2 - t^4 + t^5$, hence the function $\Gamma_K(\cdot)$ takes values $0, 3, 5, 7, 8, 10, 11, 12, \ldots$. Since 3 is in the image of $\Gamma_K(\cdot)$ but 6 is not, $\Gamma_K(\cdot)$ is not the enumerating function of a semigroup.

2.4 An example

We are going to show an application of the reduced knot Floer complex in a concrete case. Consider the knot $K = T_{3,7} \# \overline{T_{4,5}}$. The genera, signatures and υ–function [19] of $T_{3,7}$ and $T_{4,5}$ all agree: $g(T_{3,7}) = g(T_{4,5}) = 6$, $\sigma(T_{3,7}) = \sigma(T_{4,5}) = 8$, and $\upsilon(T_{3,7}) = \upsilon(T_{4,5}) = -4$. It follows that $\tau(K) = s(K) = \sigma(K) = \upsilon(K) = 0$. However, we can show the following:

Proposition 2.4 *The knot K satisfies $v^+(K) = v^+(\overline{K}) = 1$.*

Proof We need to compute a Floer complex of $T_{3,7}$, $T_{4,5}$ and their mirrors, as well as the reduced Floer complex of $T_{3,7}$ and $T_{4,5}$. Let $K_1 = T_{3,7}$ and $K_2 = T_{4,5}$.

For an L–space knot L, and in particular for every positive torus knot, each of the knot Floer complexes $\mathrm{CFK}^-(L)$ and $\mathrm{CFK}^-(\overline{L})$ is determined by a *staircase complex*; the staircase for $\overline{L}$ is obtained by reflecting the one for L across the diagonal of the second and fourth quarters, and switching the direction of all arrows. For example, when $L = T_{2,3}$ the two staircases are:

$$\mathrm{CFK}^-(L) \qquad \mathrm{CFK}^-(\overline{L})$$
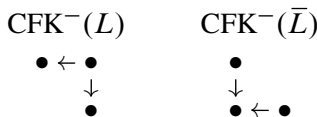
In the case of K_1 and K_2, we have:
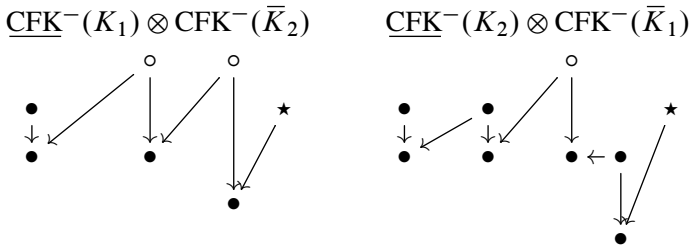

$$\text{CFK}^-(K_1) \qquad\qquad \text{CFK}^-(K_2)$$
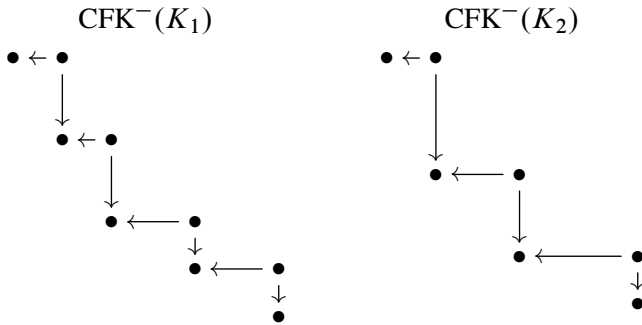
The reduced complex $\underline{\text{CFK}}^-(K_1)$, on the other hand, has a single generator in each of the following bidegrees $(-i, j)$ (where $-i$ records the U–power and j records the Alexander grading):

$$(0, 6), \;\; (-1, 3), \;\; (-2, 0), \;\; (-3, -1), \;\; (-4, -3), \;\; (-5, -4), \;\; (-6-n, -6-n), \;\; n \geq 0.$$

The reduced complex $\underline{\text{CFK}}^-(K_2)$ has a generator in each of the bidegrees

$$(0, 6), \;\; (-1, 2), \;\; (-2, 1), \;\; (-3, -2), \;\; (-4, -3), \;\; (-5, -4), \;\; (-6-n, -6-n), \;\; n \geq 0.$$

In both cases, the U–action carries a generator with i–coordinate k to one with i–coordinate $k-1$. Taking the tensor product over $\mathbb{F}[U]$, one gets twisted staircases as follows, with a generator in bidegree $(0, 0)$ (marked with a $\star$):

$$\underline{\text{CFK}}^-(K_1) \otimes \text{CFK}^-(\overline{K}_2) \qquad\qquad \underline{\text{CFK}}^-(K_2) \otimes \text{CFK}^-(\overline{K}_1)$$

The generators marked with a $\circ$ exhibit the fact that $V_0(K_1 \# \overline{K}_2)$ and $V_0(K_2 \# \overline{K}_1)$ are both strictly positive (see [13, Section 4] for details). $\qquad\square$

3 Computing the invariant

In this section we are going to prove a version of Theorem 1.1 for L–space knots. Given an integer x we denote by $(x)_+$ the quantity $(x)_+ = \max\{0, x\}$.

Theorem 3.1 *Let K and L be two L–space knots with enumerating functions $\Gamma_K(\cdot), \Gamma_L(\cdot) \colon \mathbb{N} \to \mathbb{N}$. Then*

$$\nu^+(K \# \overline{L}) = \left(g(K) - g(L) + \max_{n \geq 0}\{\Gamma_L(n) - \Gamma_K(n)\}\right)_+.$$

Notice that, since algebraic knots are L–space knots, Theorem 1.1 is an immediate corollary. Theorem 3.1 will in turn be a consequence of the following proposition:

Proposition 3.2 *In the notation of Theorem 3.1, let $\{0 = a_1 < \cdots < a_d = g(L)\}$ be the image of the function $n \mapsto \Gamma_L(n) - n$, and define $a'_k = g(L) - a_{d+1-k}$ for $k = 1, \ldots, d$. Then*

$$\nu^+(K \# \overline{L}) = \left(g(K) - g(L) + \max_{1 \leq k \leq d}\{a_k + a'_k - \Gamma_K(a'_k)\}\right)_+.$$

Proof Let $\delta_K = g(K)$, $\delta_L = g(L)$. Consider the complex $\underline{\mathrm{CFK}}^-(K) \otimes_{\mathbb{F}[U]} \mathrm{CFK}^-(\overline{L})$ that computes the knot Floer homology of $K \# \overline{L}$. Recall that the function $\Gamma_K(\cdot)$ describes the reduced Floer complex: $\underline{\mathrm{CFK}}^-(K)$ has a generator x_k in each bidegree $(-k, \delta_K - \Gamma_K(k))$. Moreover, $U \cdot x_k = x_{k+1}$.

As observed by Krcatovich [13, Section 4], the sequences $\{a_k\}$ and $\{a'_k\}$ determine a "twisted staircase" knot Floer complex $\mathrm{CFK}^-(\overline{L})$ for $\overline{L}$: the generator of the tower $\mathbb{F}[U]$ in $\mathrm{HFK}^-(\overline{L})$ is represented by the sum of d generators $U^{a'_1} y_1, \ldots, U^{a'_d} y_d$, where y_k sits in bidegree $(0, a'_k + a_k - \delta_L)$. In more graphical terms, a_k will be the Alexander grading of $U^{a'_k} y_k$, ie its j–coordinate, and $-a'_k$ will be its i–coordinate.

The tensor product $\underline{\mathrm{CFK}}^-(K) \otimes \mathrm{CFK}^-(\overline{L})$ has a staircase in Maslov grading 0 generated by the chain $z = \sum_{k=1}^d x_0 \otimes U^{a'_k} y_k$. Notice that $x_0 \otimes U^{a'_k} y_k = x_{a'_k} \otimes y_k$ sits in Alexander degree $A(x_{a'_k}) + A(y_k) = \delta_K - \Gamma(a'_k) + a_k + a'_k - \delta_L$. Therefore, the maximal Alexander degree in the chain z is precisely $M = \delta_K - \delta_L + \max\{a_k + a'_k - \Gamma_K(a_k)\}$, and we claim that if $M \geq 0$, then $\nu^+(K \# \overline{L}) = M$.

We let A_k^- be the filtration sublevel of $C_\# = \underline{\mathrm{CFK}}^-(K) \otimes \mathrm{CFK}^-(\overline{L})$ defined by $j \leq k$, ie generated by all elements with Alexander filtration level at most k.

If $M \leq k$, the entire staircase is contained in the subcomplex A_k^-. That is, the inclusion $A_k^- \to C_\#$ induces a surjection onto the tower, hence $\nu^+(K \# \overline{L}) \leq k$. In particular, if $M \leq 0$, then $\nu^+(K \# \overline{L}) = 0 = (M)_+$.

If $M > 0$, for each $k < M$ the complex A_k^- misses at least one of the generators of the chain; this implies that the inclusion $A_k^- \to \mathrm{CFK}^-(K)$ does not induce a surjection onto the tower. It follows that $V_k(K \# \overline{L}) > 0$. Hence, by definition of ν^+, we have $\nu^+(K \# \overline{L}) = M = (M)_+$, as desired. $\qquad\square$

Proof of Theorem 3.1 As remarked above, the values of a_k and a'_k determine the positions of the generators in the staircase. By the symmetry of the Alexander polynomial (and hence of the staircases), $\Gamma_L(a'_k) - a'_k = a_k$ for each k (compare with [13, Section 4]).

Moreover, for any $a'_k \le n < a'_{k+1}$, we have $\Gamma_L(n) - n = a_k$, and for every $a'_d \le n$ we have $\Gamma_L(n) - n = a_d$. Furthermore, as $\Gamma_K(\cdot)$ is strictly increasing, $n \mapsto \Gamma_K(n) - n$ is nondecreasing; therefore, for any $a'_k \le n < a'_{k+1}$ we have $\Gamma_K(a'_j) - a'_j \le \Gamma_K(n) - n$, so

$$
\begin{aligned}
a_k + a'_k - \Gamma_K(a'_k) &= (\Gamma_L(a'_k) - a'_k) - (\Gamma_K(a'_k) - a'_k) \\
&= \max_{a'_k \le n < a'_{k+1}} \{(\Gamma_L(n) - n) - (\Gamma_K(n) - n)\} \\
&= \max_{a'_k \le n < a'_{k+1}} \{\Gamma_L(n) - \Gamma_K(n)\}. \qquad \square
\end{aligned}
$$

Remark The same argument shows that, for every $m \le V_0(K \# \overline{L})$,

$$
\min\{ i \mid V_i(K \# \overline{L}) = m \} = \left(g(K) - g(L) + \max_{n \ge 0}\{\Gamma_L(n) - \Gamma_K(n+m)\} \right)_+,
$$

thus allowing one to compute all correction terms of $K \# \overline{L}$ from the enumerating functions of K and L.

4 Semicontinuity of the semigroups

In this section we prove Theorem 1.2 about the deformations of plane curve singularities. We note here that our Theorem 1.2 differs slightly from both of the results mentioned in the introduction: it reproves [10, Proposition 4.5.1] in the special case when both the central and the perturbed singularity are irreducible, but (in the spirit of [7]) using only smooth topological (not analytic) methods; however, we do not restrict ourselves to δ–constant deformations, as opposed to [7, Theorem 2.16].

In the context of deformations, inequalities which hold for certain invariants are usually referred to as *semicontinuity* of that particular invariant. Our result can be viewed as the semicontinuity of the semigroups (resembling the spectrum semicontinuity; see also [7, Section 3.1.B]).

For a brief introduction to the topic of deformations, we follow mainly [7, Section 1.5] and adapt the notions and definitions from there. By a *deformation* of a singularity with link K we mean an analytic family $\{F_s\}$ of polynomials parametrised by $|s| < 1$ such that there exists a ball $B \subset \mathbb{C}^2$ with the following properties:

- the only singular point of F_0 inside B is at the origin;

- $\{F_s = 0\}$ intersects ∂B transversely and $\{F_s = 0\} \cap \partial B$ is isotopic to K for every $|s| < 1$;

- the zero set of F_s has only isolated singular points in B for every $|s| < 1$;

- all the singular points of F_s inside B are irreducible for every $|s| < 1$;

- all fibres F_s with $s \neq 0$ have the same collection of local topological type of singularities.

For simplicity, we also assume that there is only one singular point of F_s inside B for each s. If such an analytic family of polynomials $\{F_s\}$ exists, we say that the singularity of F_0 at the origin *has a deformation* to the singularity of $F_{1/2}$.

Consider now a sufficiently small ball B_2 around the singular point of $F_{1/2}$ such that $\{F_{1/2} = 0\} \cap \partial B_2$ is isotopic to L, the link of the perturbed singular point. Then $V = \{F_{1/2} = 0\} \cap \bar{B} \setminus B_2$ is a genus-g cobordism between K and L, where $g = g(K) - g(L)$. By a slight abuse of notation, we also say that L is a deformation of K.

Let K and L be two L–space knots, with corresponding semigroups Γ_K and Γ_L, respectively. We define the *semigroup counting functions* $R_K, R_L \colon \mathbb{N} \to \mathbb{N}$ as $R_K(n) = \#[0, n) \cap \Gamma_K$ and $R_L(n) = \#[0, n) \cap \Gamma_L$. For simplicity, we allow n to run on negative numbers as well: if $n < 0$, then we define $R_K(n) = R_L(n) = 0$. In this section, we will assume that $g(K) = \delta_K \geq \delta_L = g(L)$.

Proposition 4.1 *Assume there is a genus-g cobordism between two L–space knots K and L. Then for any $a \in \mathbb{Z}$ we have*

$$R_K(a + \delta_K) \leq R_L(a + \delta_L + g).$$

Proof Since ν^+ is a lower bound for the cobordism genus, by Theorem 1.1 for any $m \in \mathbb{N}$ we have

$$\delta_K - \delta_L + \Gamma_L(m) - \Gamma_K(m) \leq g,$$

equivalently,

$$\Gamma_L(m) - \delta_L - g \leq \Gamma_K(m) - \delta_K.$$

Notice that since $\Gamma_K(m) = a$ implies $R_K(a) = m$, and the largest a for which $R_K(a) = m$ is $a = \Gamma_K(m)$ (and analogously for Γ_L), the above inequality can be interpreted as

$$R_K(a + \delta_K) \leq R_L(a + \delta_L + g). \qquad \square$$

The proposition above should be compared with [7, Theorem 2.14]. There, Borodzik and Livingston introduced the concept of *positively self-intersecting concordance*, and their result is the counterpart of Proposition 4.1 above: their assumption is on the

double point count of the positively self-intersecting concordance, while ours is on the cobordism genus. The former is related to the 4–ball crossing number considered in Proposition 1.6.

The assumption in [7] allowed Borodzik and Livingston to treat δ–constant deformations (because irreducible singularities can be perturbed to transverse intersections). However, equipped with Proposition 4.1, we can prove the semigroup semicontinuity even if the deformation is not δ–constant (but assuming that there is only one singularity in the perturbed curve $\{F_{1/2} = 0\}$).

Recall that, with the definition of the function R in place, Theorem 1.2 asserts that if L is a deformation of K then $R_K(n) \leq R_L(n)$ for each nonnegative integer n.

Proof of Theorem 1.2 Apply Proposition 4.1 with $a = n - \delta_K$ and recall that $g = \delta_K - \delta_L$ in this case. $\square$

Remark In [7, Section 3], the example of torus knots $T_{6,7}$ and $T_{4,9}$ was extensively studied. The semigroup semicontinuity proved in Theorem 1.2 obstructs the existence of a deformation between the corresponding singularities. Since the difference of the δ–invariants is 3, a deformation from $T_{6,7}$ to $T_{4,9}$ would produce a genus-3 cobordism between the two knots. However, the bound coming from ν^+ is 4 (compare with [7, Remark 3.1]).

We now turn to proving Corollary 1.3, ie that the braid index/multiplicity is nonincreasing under deformations.

Proof of Corollary 1.3 The multiplicity $m(L)$ of the singularity whose link is L is the minimal positive element in Γ_L. In particular, $R_L(m(L)) = 2$, and $R_L(m) = 1$ for $0 \leq m < m(L)$; symmetrically, $R_K(m) \geq 2$ for every $m \geq m(K)$. Let us apply Theorem 1.2 with $n = m(L)$; we obtain $R_K(m(L)) \leq R_L(m(L)) = 2$, hence $m(L) \leq m(K)$, as desired. $\square$

5 Bounds on the slice genus and concordance unknotting number

Recall that $\nu^+(K) \leq g_*(K)$ for every knot K; as outlined in the introduction, this shows that $\nu^+(K \# \bar{L})$ gives a lower bound on the genus of cobordisms between K and L. Notice that $\nu^+(L \# \bar{K})$ gives a bound, too, and the two bounds are often different.

We now state a preliminary lemma that we will use to prove Theorem 1.4, ie that trading a negative crossing for a positive one does not decrease ν^+, nor does it increase it by more than 1.

Lemma 5.1 *If there is a genus-g cobordism between two knots K and L then, for each $m \geq 0$,*

$$V_{m+g}(K) \leq V_m(L).$$

As a consequence, $\nu^+(K) \leq \nu^+(L) + g$.

Before proving the lemma, we observe some of its consequences. Most notably, it allows us to generalise Theorem 1.2 to the case of more than one irreducible singularity (both in the central and deformed fibre).

Remark In the case of algebraic knots, the lemma is equivalent to Proposition 4.1. Indeed, using the symmetry property of the semigroup, one has that $R_K(a + \delta_K) = R_K(\delta_K - a) + a$ and $R_L(a + \delta_L + g) = R_L(\delta_L - a - g) + a + g$ for every integer a. Using these substitutions in both sides of the statement of Proposition 4.1, we obtain

$$R_K(\delta_K - a) \leq R_L(\delta_L - a - g) + g.$$

If we now set $a = -g - m$, we get

$$R_K(\delta_K + m + g) \leq R_L(\delta_L + m) + g,$$

and by [3, Equation (5.1)] we have $R_K(\delta_K + m + g) = V_{m+g}(K) + m + g$ and $R_L(\delta_L + m) = V_m(L) + m$, thus proving the equivalence of the two statements.

Since the proof of Theorem 1.2 relies on Proposition 4.1, which is in turn equivalent to Lemma 5.1, we can use the latter to generalise its statement. In order to do so, we introduce the concept of the infimum convolution of two functions [6; 4]: given $R_1, R_2 \colon \mathbb{N} \to \mathbb{N}$ bounded below, we define the *infimum convolution* of R_1 and R_2 as

$$(R_1 \diamond R_2)(n) := \min_{i+j=n} R_1(i) + R_2(j).$$

Theorem 5.2 *Let $\{F_s\}_{|s|<1}$ be a deformation of F_0, and suppose that F_0 has only irreducible singularities $K_1, \ldots, K_a$, while $F_{1/2}$ has irreducible singularities $L_1, \ldots, L_b$ (and possibly other reducible singularities). Then, for each nonnegative integer n,*

$$(R_{K_1} \diamond \cdots \diamond R_{K_a})(n) \leq (R_{L_1} \diamond \cdots \diamond R_{L_b})(n).$$

Proof (sketch) Similarly as how we argued in Section 4, it is easy to show that a deformation gives rise to a cobordism Σ_0 from the link $K_1 \cup \cdots \cup K_a$ in the disjoint union of a copies of S^3 to the link $L_1 \cup \cdots \cup L_b$ in the disjoint union of b copies of S^3, living in S^4 with $a + b$ open balls removed. This cobordism will be singular if there are reducible singularities in $F_{1/2}$.

Figure 1: The Borromean knot $K_{B,1}$. The Borromean knot $K_{B,g}$ is the connected sum of g copies of $K_{B,1}$.

We resolve all singularities of Σ_0, replacing each of them with the Milnor fibre of the corresponding reducible singularity and obtain a smooth cobordism, Σ_1; note that the difference $g(\Sigma_1) - g(\Sigma_0)$ is the sum of all δ–invariants of the reducible singularities of $F_{1/2}$.

We can now carve paths along the cobordism Σ_1 connecting all the boundary components containing a K_i and all boundary components containing an L_j, thus obtaining a smooth cobordism Σ from $K = K_1 \# \cdots \# K_a$ to $L = L_1 \# \cdots \# L_b$. Note that this does not change the genus, ie $g(\Sigma) = g(\Sigma_1)$.

Similarly as in the irreducible case, we have $g(\Sigma) = g(K) - g(L)$; using [6, Theorems 5.4 and 5.6] we know that $V_i(K) + i = (R_{K_1} \diamond \cdots \diamond R_{K_a})(g(K) + i)$ and $V_i(L) + i = (R_{L_1} \diamond \cdots \diamond R_{L_b})(g(L) + i)$.

The statement now follows from Lemma 5.1 and the remark following the lemma, as in the proof of Theorem 1.2.   □

Proof of Lemma 5.1 Consider the 4–manifold W obtained by attaching a 4–dimensional 2–handle to $S^3 \times [0, 1]$ along $L \times \{1\} \subset S^3 \times \{1\}$, with framing $n \geq 2\nu^+(L)$.

The cobordism is a genus-g embedded surface F in $S^3 \times [0, 1]$, whose boundary components are $K \times \{0\}$ and $L \times \{1\}$. Capping off the latter boundary component in W, and taking the cone over $(S^3 \times \{0\}, K)$, we obtain a singular genus-g surface $\widehat{F} \subset W' = W \cup B^4$, whose only singularity is a cone over K.

As argued in [3, Section 4; 5, Theorem 3.1], the boundary ∂N of a regular neighbourhood N of $\widehat{F}$ in W' is diffeomorphic to the 3–manifold Y_n obtained as n–surgery along the connected sum of K and the Borromean knot $K_{B,g}$ in $\#^{2g}(S^2 \times S^1)$. It follows that $Z = -(W' \setminus N)$ can be viewed as a cobordism from $S^3_n(L)$ to Y_n.

We can view N as the surgery cobordism from $\#^{2g}(S^2 \times S^1)$ to Y_n, filled with a 1–handlebody; since the class of $[\widehat{F}]$ generates both $H_2(N)$ and $H_2(-W')$, we obtain that the restriction of any spin^c structure on $-W'$ to Z induces an isomorphism between (torsion) spin^c structure on its two boundary components that respects the surgery-induced labelling. Moreover, we also obtain that $b_2^+(Z) = 0$.

The 3–manifold Y_n has standard HF^∞ [3; 5], and its bottom-most correction terms have been computed in [3, Proposition 4.4; 5, Theorem 6.10]:

$$d_b(Y_n, \mathfrak{t}_m) = \min_{0 \leq k \leq g} \{2k - g - 2V_{m+g-2k}(K)\} - \frac{n - (2m - n)^2}{4n}.$$

We observe that, choosing $k = 0$ in the minimum, we obtain the inequality

$$d_b(Y_n, \mathfrak{t}_m) \leq -g - 2V_{m+g}(K) - \frac{n - (2m - n)^2}{4n}.$$

Applying the last inequality and [2, Theorem 4.1] to Z, seen as a negative semidefinite cobordism from $S_n^3(L)$ to Y_n, we get

$$d(S_n^3(L), \mathfrak{t}_m) \leq d_b(Y_n, \mathfrak{t}_m) + g,$$

from which we get

$$-2V_m(L) \leq -g - 2V_{m+g}(K) + g \iff V_{m+g}(K) \leq V_m(L).$$

The last part of the statement now follows from the observation that $V_{\nu^+(L)+g}(K) \leq V_{\nu^+(L)}(L) = 0$, hence $\nu^+(K) \leq \nu^+(L) + g$, as desired. □

We are now in position to prove Theorem 1.4, which asserts that, if K_+ and K_- differ at a single crossing (which is positive for K_+ and negative for K_-), then $\nu^+(K_-) \leq \nu^+(K_+) \leq \nu^+(K_-) + 1$.

Proof of Theorem 1.4 The inequality $\nu^+(K_-) \leq \nu^+(K_+)$ readily follows from [4, Theorem 6.1]: the latter states that for each nonnegative integer n we have $V_n(K_-) \leq V_n(K_+)$. Applying the inequality with $n = \nu^+(K_+)$, we obtain $V_{\nu^+(K_+)}(K_-) \leq 0$, hence $\nu^+(K_-) \leq \nu^+(K_+)$, as desired.

The inequality $\nu^+(K_+) \leq \nu^+(K_-) + 1$ follows from Lemma 5.1 above; in fact, there is a genus-1 cobordism from K_- to K_+ obtained by smoothing the double point of the regular homotopy associated with the crossing change, and the previous lemma concludes the proof. □

Remark In fact, the second inequality follows from [4, Theorem 6.1] as well: Borodzik and Hedden prove that, in the notation of the proposition, $V_{m+1}(K_+) \leq V_m(K_-)$, and the claim about ν^+ follows as in the proof of Lemma 5.1. However, Lemma 5.1 is stronger than [4, Theorem 6.1], and we think it might be of independent interest.

We now turn to applications to other, more geometrically defined, concordance invariants, and we prove Proposition 1.6.

Proof of Proposition 1.6 We need at least $v^+(K)$ negative crossing changes and at least $v^+(\overline{K})$ positive crossing changes to turn K into a knot K_0 such that $v^+(K_0) = v^+(\overline{K}_0) = 0$. In particular, we need to change at least $v^+(K) + v^+(\overline{K})$ crossings to make K slice, hence $u_s(K) \geq v^+(K) + v^+(\overline{K})$.

As for the concordance unknotting number, one simply observes that $v^+(K)$ and $v^+(\overline{K})$ are concordance invariants, hence every knot in the same concordance class of K has unknotting number at least $v^+(K) + v^+(\overline{K})$.

Finally, [18, Proposition 2.1] asserts that every immersed concordance can be factored into two concordances and a sequence of crossing changes. That is, given an immersed concordance from K to the unknot with c double points, there exist knots K_0 and K_1 such that K_0 is slice, K_1 is concordant to K, and there is a sequence of c crossing changes from K_0 to K_1; from the proposition above, it follows that $c \geq v^+(K_0 \,\#\, K_1) + v^+(\overline{K_0 \,\#\, K_1}) = v^+(K) + v^+(\overline{K})$. □

6 Subadditivity of v^+

The goal of this section is to prove Theorem 1.5. We start with a preliminary proposition. In the course of the proof, we will make use of twisted correction terms, as defined in [2]. These are a generalisation of ordinary and bottom-most correction terms to arbitrary 3–manifolds; specifically, given a torsion spin^c structure $\mathfrak{t}$ on a 3–manifold Y, there is an associated rational number $\underline{d}(Y, \mathfrak{t})$, which is a rational homology cobordism invariant of the pair $(Y, \mathfrak{t})$.

When Y is a rational homology sphere, $\underline{d}(Y, \mathfrak{t}) = d(Y, \mathfrak{t})$. If, on the other hand, Y is obtained as 0–surgery along a knot in S^3, equipped with its unique torsion spin^c structure $\mathfrak{t}$, then $\underline{d}(Y, \mathfrak{t}) = d_b(Y, \mathfrak{t})$ (see [2, Section 3.3]).

Moreover, much like d_b, the twisted correction term $\underline{d}$ behaves well under negative semidefinite cobordisms (see [2, Section 4]).

Proposition 6.1 *For any two knots $K, L \subset S^3$ and any two nonnegative integers m and n, we have*

$$V_{m+n}(K \,\#\, L) \leq V_m(K) + V_n(L).$$

Proof Consider the surgery diagrams in Figure 2 and Figure 3, representing a closed 4–manifold X and a 4–dimensional cobordism W from $-S^3_{2(m+n)}(K \,\#\, L)$ to $-(S^3_{2m}(K) \,\#\, S^3_{2n}(L))$. One should be careful with orientation reversals here; in particular, notice that in Figure 3 we represent the cobordism $\overline{W}$ obtained by turning W upside down.
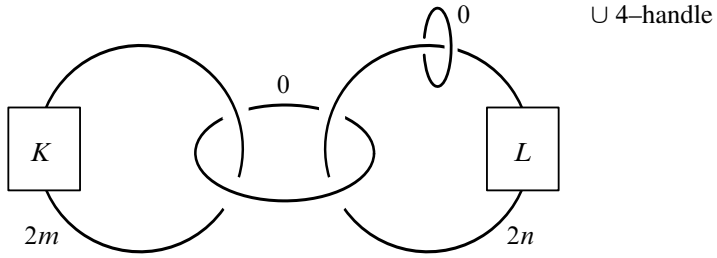
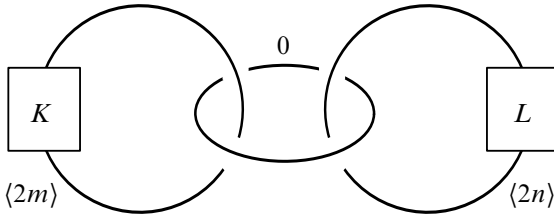Figure 2: The surgery diagram for the closed 4–manifold X

Figure 3: The surgery diagram for the upside-down cobordism $\overline{W}$ from $S^3_{2m}(K) \# S^3_{2n}(L)$ to $S^3_{2(m+n)}(K \# L)$. The coefficients in brackets represent the negative boundary $\partial_- W$.

As observed by Owens and Strle [17], if $m, n > 0$, then $W \subset X$ is a negative definite cobordism from $S^3_{2m}(K) \# S^3_{2n}(L)$ to $S^3_{2(m+n)}(K \# L)$ with $H_2(W; \mathbb{Z}) = \mathbb{Z}$ and $\chi(W) = 1$. When $m = 0$ or $n = 0$, W has signature $\sigma(W) = 0$; therefore, regardless of positivity of m and n, W is negative semidefinite. Moreover, W is obtained from $\partial_- W$ by attaching a single 2–handle, and this does not decrease the first Betti number of the boundary. It follows that we are in the right setup to apply [2, Theorem 4.1].

The 4–manifold X is even; since 0 is a characteristic vector, it is the first Chern class of a spin^c structure $\mathfrak{s}_0$ on X. The spin^c structure $\mathfrak{s}_0$ restricts to the spin^c structure on W with trivial first Chern class, hence $c_1(\mathfrak{s}_0)^2 = 0$.

Notice also that $X \setminus W$ is the disjoint union of two 4–manifolds: one is the boundary connected sum of the surgery handlebodies for $S^3_{2m}(K)$ and $S^3_{2n}(L)$, and the other is the surgery handlebody for $S^3_{2(m+n)}(K \# L)$ with the reversed orientation. In particular, labelling of the restriction of $\mathfrak{s}_0$ onto the two boundary components of W is determined by the evaluation of $c_1(\mathfrak{s}_0)$ on the generators of the second cohomology of the two pieces [23, Section 2.4].

With the chosen convention for labelling spin^c structures (1), since $c_1(\mathfrak{s}_0) = 0$, the spin^c structure $\mathfrak{s}_0$ restricts to $\mathfrak{t}_m$ on $S^3_{2m}(K)$, to $\mathfrak{t}_n$ on $S^3_{2n}(L)$, and to $\mathfrak{t}_{m+n}$ on $S^3_{2(m+n)}(K)$.

We observe here that

$$\underline{d}(-S_0^3(\overline{K}), t_0) + \tfrac{1}{2}b_1(-S_0^3(\overline{K})) = 2V_0(K),$$

and the same holds for L and $K \# L$ (compare with [20, Proposition 4.12] and [2, Example 3.9]). When $m > 0$, however,

$$\underline{d}(-S_{2m}^3(\overline{K}), t_m) + \tfrac{1}{2}b_1(-S_{2m}^3(\overline{K})) = d(-S_{2m}^3(\overline{K}), t_m) = \tfrac{1}{4} + 2V_m(K),$$

and analogous formulae hold for L and $K \# L$.

We now apply additivity of $\underline{d}$ [2, Proposition 3.7] and [2, Theorem 4.1] to W to obtain the inequality

$$(3) \quad b_2^-(W) + 4d\left(-(S_{2m}^3(\overline{K}) \# S_{2n}^3(\overline{L})), t_m \# t_n\right) + 2b_1(-(S_{2m}^3(\overline{K}) \# S_{2n}^3(\overline{L})))$$
$$\leq 4\underline{d}(-S_{2m}^3(\overline{K}), t_m) + 2b_1(-S_{2m}^3(\overline{K})) + 4\underline{d}(-S_{2n}^3(\overline{L}), t_n) + 2b_1(-S_{2n}^3(\overline{L})).$$

When m and n are both positive, (3) becomes

$$1 + 1 + 8V_{m+n}(K \# L) \leq 1 + 8V_m(K) + 1 + 8V_n(L).$$

When exactly one among m and n vanishes, say $m = 0$, (3) turns into

$$1 + 8V_n(K \# L) \leq 8V_0(K) + 1 + 8V_n(L).$$

Finally, when $m = n = 0$, (3) reads

$$8V_0(K \# L) \leq 8V_0(K) + 8V_0(L).$$

In all cases, we have proved that $V_{m+n}(K \# L) \leq V_m(K) + V_n(L)$, as desired. □

We are now ready to prove Theorem 1.5, ie that ν^+ is subadditive.

Proof of Theorem 1.5 This now follows from Proposition 6.1 by setting $m = \nu^+(K)$ and $n = \nu^+(L)$. In fact, since $V_m(K) = V_n(L) = 0$,

$$V_{m+n}(K \# L) \leq V_m(K) + V_n(L) = 0;$$

that is, $\nu^+(K \# L) \leq m + n = \nu^+(K) + \nu^+(L)$. □

7 Examples

In this section we study a 3–parameter family of pairs of torus knots on which the lower bound given by ν^+ is sharp. We first start with a 1–parameter subfamily that we study in some detail, and we then turn to the whole family. The techniques used here are inspired by Baader's "scissor equivalence" [1].

Example 7.1 We are going to present an example in which the bound provided by ν^+ on the genus of a cobordism between torus knots is stronger than the ones given by the Tristram–Levine signature function, τ, s and Υ, and moreover it is sharp.
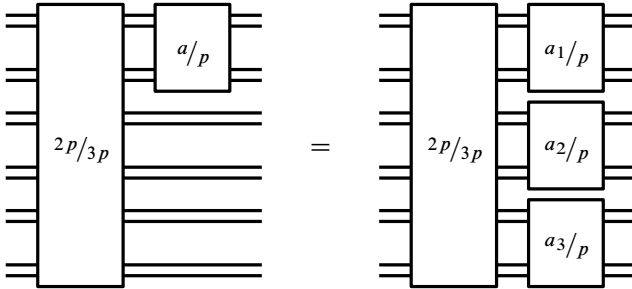

Figure 4: The knot $K_{a,p}$; the boxes indicate the number of full twists. Equality holds whenever $a = a_1 + a_2 + a_3$.
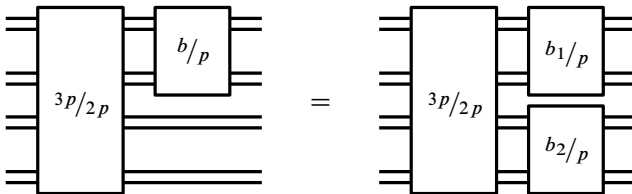

Figure 5: The knot $K'_{b,p}$; the boxes indicate the number of full twists. Equality holds whenever $b = b_1 + b_2$.

Define the two families of links $K_{a,p}$ and $K'_{b,p}$ as the closure of the braids pictured in Figures 4 and 5. Notice that $K_{a,p}$ and $K'_{b,p}$ are (p, s)–cables of the trefoil for some s, and that they are knots if and only if $\gcd(a, p) = 1$ and $\gcd(b, p) = 1$, respectively. Moreover, $K_{a,p}$ is the product of $2p(3p - 1) + a(p - 1)$ positive generators of the braid group on $3p$ strands, hence its closure represents a transverse knot in the standard contact 3–sphere with self-linking number $6p^2 + (a - 5)p - a$. Since for closures of positive braids the self-linking number agrees with the Seifert genus, we can compute the cabling parameter $s = 6p + a$. In conclusion, we have shown that $K_{a,p}$ is the $(p, 6p + a)$–cable of $T_{2,3}$.

The same argument applies to $K'_{b,p}$, the self-linking number computation yields $6p^2 + (b - 5)p - b$, hence showing that $K'_{b,p}$ is the $(p, 6p + b)$–cable of $T_{2,3}$. In particular $K_{a,p}$ and $K'_{b,p}$ are isotopic if and only if $a = b$.

Now consider the knots $K_{12,p} = K'_{12,p}$. Denote by σ_i the ith elementary generator of the braid group, and, whenever $i < j$, denote by $\sigma_{i,j}$ the product $\sigma_i \sigma_{i+1} \cdots \sigma_j$.

Setting $a_1 = a_2 = a_3 = 4$ in the right-hand side of Figure 4 exhibits $K_{12,p}$ as the closure of the braid

$$\sigma_{1,3p-1}^{2p} \cdot \overbrace{\underbrace{\sigma_{1,p-1}^4}_{\Sigma_1^4} \cdot \underbrace{\sigma_{p+1,2p-1}^4}_{\Sigma_2^4} \cdot \underbrace{\sigma_{2p+1,3p-1}^4}_{\Sigma_3^4}}^{\Pi} \cdot$$

The three elements Σ_1, Σ_2 and Σ_3 commute, hence $\Pi = (\Sigma_1 \Sigma_2 \Sigma_3)^4$. Now, notice that $\Sigma_1 \sigma_p \Sigma_2 \sigma_{2p} \Sigma_3 = \sigma_{1,3p-1}$. Since adding a generator σ_i corresponds to attaching a band between two strands, we produce a cobordism built out of 8 bands from $K_{12,p}$ to $T_{2p+4,3p}$; if p is coprime with 6, both ends of the cobordism are connected, and its genus is 4.

An analogous argument, setting $b_1 = b_2 = 6$ in the right-hand side of Figure 5 produces a 6–band, genus-3 cobordism from $K'_{12,p}$ to $T_{2p,3p+6}$ whenever p is coprime with 6. Suppose now that $p \equiv 5 \pmod{6}$ and $p \geq 11$. Gluing the two cobordisms above yields a genus-7 cobordism between $K = T_{2p+4,3p}$ and $L = T_{2p,3p+6}$.

Applying Proposition 3.2 above we obtain a sharp bound on the slice genus; in fact, in the same notation as in Proposition 3.2, we have

- $2\delta_K = 2g(K) = 6p^2 + 7p - 3$ and $2\delta_L = 2g(L) = 6p^2 + 7p - 5$;
- $\Gamma_K(2) = 3p$ and $\Gamma_L(2) = 3p + 6$;
- $\Gamma_K(3) = 4p + 8$ while $\Gamma_L(3) = 4p$.

It follows that
$$\nu^+(K \# \bar{L}) \geq 1 + \Gamma_L(2) - \Gamma_K(2) = 7,$$
$$\nu^+(L \# \bar{K}) \geq -1 + \Gamma_K(3) - \Gamma_L(3) = 7.$$

A direct computation using [19, Theorem 1.15] shows that for $p = 11, 17, 23, 29$ the bound given by Υ is 3, the one given by the Tristram–Levine signatures is either 2 or 5, and the one given by τ and s is 1.

Moreover, we need at least 7 positive and 7 negative crossing changes to turn K into L, hence their Gordian distance is at least 14. Additionally, suppose that we have a factorisation of the cobordism above into genus-1 cobordisms, and suppose that one of these cobordisms goes from K_1 to K_2. Then both $\nu^+(K_2) = \nu^+(K_1) - 1$ and $\nu^+(\bar{K}_2) = \nu^+(\bar{K}_1) - 1$.

Example 7.2 We can promote the family above to a family parametrised by suitable triples of integers (p, q, r) as follows: Instead of considering the $(p, 6p + 12)$–cable of the trefoil $K_{12,p} = K'_{12,p}$ we can consider the $(p, qr(p + 2))$–cable $K_{q,r}^p$ of $T := T_{q,r}$. The first condition we impose on the triple (p, q, r) is that $q < r$ and $\gcd(q, r) = 1$.

By looking at $K_{q,r}^p$ as a cable of T seen as the closure of an r–braid, we can glue $2q \cdot (r-1)$ bands to $K_{q,r}^p$ and obtain $K = T_{q(p+2),rp}$. Call $x_1 = q(p+2)$ and $x_2 = rp$ the two generators of the semigroup Γ_K.

By viewing $K_{q,r}^p$ as a cable of T seen as the closure a q–braid instead, we see that we can glue $2r \cdot (q-1)$ bands to $K_{q,r}^p$ and obtain $L = T_{qp,r(p+2)}$. Call $y_1 = qp$ and $y_2 = r(p+2)$ the two generators of the semigroup Γ_L.

If $\gcd(p, 2qr) = \gcd(p+2, 2qr) = 1$, both K and L have one component, ie they are torus knots; eg both equalities hold if $p \equiv -1 \pmod{2qr}$. Moreover, $\delta_K - \delta_L = g(K) - g(L) = r - q$, and above we produced a cobordism of genus $2qr - q - r$ between K and L, made of $4qr - 2q - 2r$ bands. Hence, $\nu^+(K \# \bar{L}), \nu^+(L \# \bar{K}) \leq 2qr - q - r$.

Choose p sufficiently large; it is elementary to check that if $p \geq 2qr - 1$ then, for $n_1 = \delta_T + q - 1$ and $n_2 = \delta_T + r - 1$, we have

$$\Gamma_T(n_1) = (q-1)r, \qquad\qquad \Gamma_T(n_2) = (r-1)q,$$
$$\Gamma_K(n_1) = (q-1)x_2 = (q-1)rp, \qquad \Gamma_K(n_2) = (r-1)x_1 = (r-1)q(p+2),$$
$$\Gamma_L(n_1) = (q-1)y_2 = (q-1)r(p+2), \quad \Gamma_L(n_2) = (r-1)y_1 = (r-1)qp.$$

If we set $n = n_1$ in Theorem 1.1 we obtain

$$\nu^+(K \# \bar{L}) \geq \delta_K - \delta_L + \Gamma_L(n_1) - \Gamma_K(n_1) = 2qr - q - r.$$

Reversing the roles of K and L and setting $n = n_2$ yields

$$\nu^+(L \# \bar{K}) \geq \delta_L - \delta_K + \Gamma_K(n_2) - \Gamma_L(n_2) = 2qr - q - r.$$

The lower bound for the genus given by ν^+ is in this case is tight, as the upper and lower bounds match, and moreover the Gordian distance between K and L is at least $4qr - 2q - 2r$.

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