Algebraic & Geometric Topology 17 (2017) 2635–2685

Link homology and equivariant gauge theory

PRAYAT POUDEL
NIKOLAI SAVELIEV

Singular instanton Floer homology was defined by Kronheimer and Mrowka in connection with their proof that Khovanov homology is an unknot detector. We study this theory for knots and two-component links using equivariant gauge theory on their double branched covers. We show that the special generator in the singular instanton Floer homology of a knot is graded by the knot signature mod 4, thereby providing a purely topological way of fixing the absolute grading in the theory. Our approach also results in explicit computations of the generators and gradings of the singular instanton Floer chain complex for several classes of knots with simple double branched covers, such as two-bridge knots, some torus knots, and Montesinos knots, as well as for several families of two-component links.

57M27; 57R58

1 Introduction

This paper studies the Floer homology $I_*(\Sigma, \mathcal{L})$ of two-component links $\mathcal{L} \subset \Sigma$ in homology 3–spheres defined by Kronheimer and Mrowka [24] using singular SO(3) instantons. An important special case of this theory is the singular instanton knot Floer homology $I^\natural(k)$ for knots $k \subset S^3$ obtained by applying $I_*(S^3, \mathcal{L})$ to the link $\mathcal{L}$ which is a connected sum of k with the Hopf link. The Floer homology $I_*(\Sigma, \mathcal{L})$ has a relative $\mathbb{Z}/4$ grading, which can be upgraded to an absolute $\mathbb{Z}/4$ grading in the special case of $I^\natural(k)$. Kronheimer and Mrowka [24] used $I^\natural(k)$ and its close cousin $I^\sharp(k)$ to prove that the reduced Khovanov homology is an unknot-detector.

The definition of the groups $I_*(\Sigma, \mathcal{L})$ uses singular gauge theory, which makes them difficult to compute. We propose a new approach to these computations which uses equivariant gauge theory in place of the singular one. Given a two-component link $\mathcal{L}$ in an integral homology sphere Σ, we pass to the double branched cover $M \to \Sigma$ with branch set $\mathcal{L}$ and observe that the singular connections on Σ used in the definition of $I_*(\Sigma, \mathcal{L})$ pull back to equivariant smooth connections on M. The generators of the Floer chain complex $IC_*(\Sigma, \mathcal{L})$, whose homology is $I_*(\Sigma, \mathcal{L})$, are then derived from

Published: 19 September 2017 DOI: 10.2140/agt.2017.17.2635

the equivariant representations $\pi_1 M \to \mathrm{SO}(3)$, and their Floer gradings are computed using equivariant rather than singular index theory.[1]

As our first application of this approach, we determine the grading of the special generator in the Floer chain complex $IC^\natural(k)$ of a knot $k \subset S^3$; see Section 5. This fixes the absolute $\mathbb{Z}/4$ grading on $I^\natural(k)$ and confirms the conjecture of Hedden, Herald and Kirk [20, Section 12.6].

Theorem *For any knot $k \subset S^3$, the grading of the special generator in the Floer chain complex $IC^\natural(k)$ equals* $\operatorname{sign} k \pmod 4$.

We also achieve significant simplifications in computing the Floer chain complexes $IC^\natural(k)$ and $IC_*(\Sigma, \mathcal{L})$ for knots and links with simple double branched covers, such as torus and Montesinos knots and links, whose double branched covers are Seifert fibered manifolds. Explicit calculations for these knots and links are possible because the gauge theory on Seifert fibered manifolds is sufficiently well developed; see Fintushel and Stern [15] and, in the equivariant setting, Collin and Saveliev [11] and Saveliev [36]. Here are sample results of our calculations:

(1) The Floer chain complex $IC^\natural(k)$ of a two-bridge knot k is calculated in Section 7.1. For example, the Floer chain complex of the figure-eight knot consists of free abelian groups of ranks $(1,1,2,1)$. In fact, the Kronheimer–Mrowka spectral sequence [24] is known to collapse for all two-bridge knots k, which implies that $IC^\natural(k) = I^\natural(k)$ for all such knots.

(2) The Floer chain complex $IC^\natural(k)$ of a Montesinos knot $k = k(p,q,r)$ whose double branched cover is a Brieskorn homology sphere $\Sigma(p,q,r)$ consists of free abelian groups of ranks $(1+b, b, b, b)$, where b equals -2 times the Casson invariant of $\Sigma(p,q,r)$; see Section 7.2. General Montesinos knots are discussed in Section 7.3.

(3) The Floer chain complex $IC_*(S^3, \mathcal{L})$ of two-component Montesinos links $\mathcal{L} = K((a_1,b_1),\dots,(a_n,b_n))$ whose double branched cover is a homology $S^1 \times S^2$ is calculated in Section 8.3. For example, the chain complex of the pretzel link $\mathcal{L} = P(2,-3,-6)$ consists of free abelian groups of ranks $(2,0,2,0)$ up to cyclic permutation; see Section 8.2. It has zero differential, hence $IC_*(S^3,\mathcal{L}) = I_*(S^3,\mathcal{L})$.

(4) Our calculations for torus knots are less satisfactory because the equivariant index theory in this setting is less well developed. For instance, we prove that the Floer chain complex $IC^\natural(k)$ of a torus knot $k = T_{p,q}$ with odd coprime integers p and q has rank $1+4a$, where $a = -\operatorname{sign}(T_{p,q})/4$, and we conjecture that the Floer chain groups

[1] The theory $I_*(\Sigma,\mathcal{L})$ is different from $I^\natural(\Sigma,\mathcal{L})$ studied in [24]: the latter is a Floer homology of a three-component link obtained by summing $\mathcal{L}$ with the Hopf link.

have ranks $(1+a, a, a, a)$; see Section 7.4.[2] A complete calculation of the Floer chain complex of the torus knot $T_{3,4}$ can be found in Example 7.9.

Some of the above results concerning two-bridge and torus knots were obtained earlier by Hedden, Herald, and Kirk [20] using pillowcase techniques, which are completely different from our equivariant methods. We do not discuss the more difficult problem of computing the boundary operators in the Floer chain complexes $IC^{\natural}(k)$ and $IC_*(\Sigma, \mathcal{L})$. Such calculations are still out of reach except in a few special cases. However, it may be worth investigating whether our equivariant techniques can shed some light on this problem.

Here is an outline of the paper. It begins with a sketch of the definition of $I_*(\Sigma, \mathcal{L})$ mainly following Kronheimer and Mrowka [24] but using the language of projective representations developed in Ruberman and Saveliev [33]; see also Dostoglou and Salamon [13]. We obtain a purely algebraic description of the generators in $IC_*(\Sigma, \mathcal{L})$ as well as of a certain natural $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ action on them, which is crucial to the rest of the paper.

Equivariant gauge theory is developed in Section 3. The section begins with a computation of $\mathbb{Z}/2$ cohomology rings of double branched covers $M \to \Sigma$ of two-component links, followed by a computation of the characteristic classes of SO(3)–bundles on M pulled back from orbifold bundles on Σ. The results are used to establish a bijective correspondence between equivariant SO(3) representations of $\pi_1 M$ and orbifold SO(3) representations of $\pi_1 \Sigma$. In the rest of the section, we discuss equivariant index theory which is used later in the paper to compute Floer gradings of the generators in $IC_*(\Sigma, \mathcal{L})$. Our equivariant index theory approach is also used to recover the Kronheimer–Mrowka singular index formulas [24, Lemma 2.11] along the lines of Wang's paper [42].

The next five sections are dedicated to the singular knot Floer homology $I^{\natural}(k)$ for knots $k \subset S^3$. Section 4 describes generators in the chain complex $IC^{\natural}(k)$ in terms of equivariant representations $\pi_1 Y \to \mathrm{SO}(3)$ on the double branched cover $Y \to S^3$ with branch set the knot k. These representations fall into three categories: trivial, reducible nontrivial, and irreducible.

The trivial representation $\theta\colon \pi_1 Y \to \mathrm{SO}(3)$ gives rise to a special generator $\alpha \in IC^{\natural}(k)$ which is used in [24] to fix an absolute grading on $I^{\natural}(k)$. This generator is dealt with in Section 5. We pass to the double branched cover and use Taubes index theory [40] on manifolds with periodic ends to show that the Floer grading of α equals $\operatorname{sign} k \pmod 4$.

[2] Extensive calculations for torus knots have recently been done by Anvari [2] using similar equivariant techniques.

Having computed the absolute grading of α, we only need to compute the relative gradings of the remaining generators. We derive formulas for these gradings in Section 6 using equivariant index calculations on double branched covers, and apply these formulas to Montesinos and torus knots in Section 7.

Section 8 contains calculations of $IC_*(\Sigma, \mathcal{L})$ for several two-component links $\mathcal{L}$ not of the form $k^\natural$. For the pretzel link $\mathcal{L} = P(2,-3,-6)$ in the 3–sphere we obtain a complete calculation of the Floer homology groups of $P(2,-3,-6)$ and not just of the Floer chain complex. The same answer is independently confirmed by computing the Floer homology of Harper and Saveliev [19] for this two-component link: the latter theory is isomorphic to $I_*(\Sigma, \mathcal{L})$ but does not use singular connections in its definition.

Finally, Section 8.3 contains proofs of some topological results, which were postponed earlier in the paper for the sake of exposition.

Acknowledgements We are thankful to Ken Baker, Paul Kirk, and Daniel Ruberman for useful discussions. Both authors were partially supported by NSF Grant 1065905.

2 Link homology

In this section, we will sketch the definition of the singular instanton Floer homology $I_*(\Sigma, \mathcal{L})$ of a two-component link $\mathcal{L} \subset \Sigma$ in an integral homology 3–sphere. We will follow Kronheimer and Mrowka [24] closely, deviating in just two respects: we will use the language of projective representations to describe the generators in the Floer chain complex, and will introduce a canonical $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ action on these generators.

2.1 The Chern–Simons functional

Given a two-component link $\mathcal{L}$ in an integral homology sphere Σ, the second homology of its exterior $X = \Sigma - \operatorname{int} N(\mathcal{L})$ is isomorphic to a copy of $\mathbb{Z}$ spanned by either one of the boundary tori of X. Let $P \to X$ be the unique SO(3)–bundle with a nontrivial second Stiefel–Whitney class $w_2(P) \in H^2(X; \mathbb{Z}/2) = \mathbb{Z}/2$. The flat connections in this bundle serve as the starting point for building $I_*(\Sigma, \mathcal{L})$. Since $w_2(P)$ evaluates nontrivially on the boundary tori, these connections are necessarily irreducible and have order-2 holonomy along the meridians of the link components. Therefore, they give rise to flat connections in an orbifold SO(3)–bundle on Σ, which we again call P. The homology sphere Σ itself is viewed as an orbifold with the orbifold singularity $\mathcal{L}$, equipped with a Riemannian metric with cone angle π along the singular set.

Kronheimer and Mrowka [24] interpreted the gauge equivalence classes of the orbifold flat connections in P as the critical points of an orbifold Chern–Simons functional

$$\mathbf{cs}\colon \mathcal{B}(\Sigma, \mathcal{L}) \to \mathbb{R}/\mathbb{Z}, \tag{1}$$

and defined $I_*(\Sigma, \mathcal{L})$ as its Morse homology. An important feature of this construction is the use of the restricted orbifold gauge group $\mathcal{G}_S$ in the definition of the configuration space,

$$\mathcal{B}(\Sigma, \mathcal{L}) = \mathcal{A}(\Sigma, \mathcal{L})/\mathcal{G}_S,$$

where $\mathcal{A}(\Sigma, \mathcal{L})$ is an affine space of orbifold connections and $\mathcal{G}_S$ is the quotient of the determinant-1 orbifold gauge group $\mathcal{G}(\check{P})$ of Kronheimer and Mrowka [24, Section 2.6] by its center $\{\pm 1\}$. The group $\mathcal{G}_S$ is a normal subgroup of the full orbifold gauge group $\mathcal{G}$ with the quotient $\mathcal{G}/\mathcal{G}_S = H^1(X; \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. The full gauge group $\mathcal{G}$ acts on $\mathcal{A}(\Sigma, \mathcal{L})$ preserving the gradient of **cs**, thereby giving rise to the residual action of $H^1(X; \mathbb{Z}/2)$ on the configuration space $\mathcal{B}(\Sigma, \mathcal{L})$ and on the critical point set of the Chern–Simons functional.

We will next describe the critical points of the functional (1) algebraically using the holonomy correspondence between flat connections and representations of the fundamental group. A variant of this classical correspondence which applies to the situation at hand was described in [33, Section 3.2] using projective SU(2) representations. We will review these first; see [33, Section 3.1] for details.

2.2 Projective representations

Let G be a finitely presented group and view the center of SU(2) as $\mathbb{Z}/2 = \{\pm 1\}$. A map $\rho\colon G \to \mathrm{SU}(2)$ is called a projective representation if

$$c(g,h) = \rho(gh)\rho(h)^{-1}\rho(g)^{-1} \in \mathbb{Z}/2 \quad \text{for all } g,h \in G.$$

The function $c\colon G \times G \to \mathbb{Z}/2$ is a 2–cocycle on G defining a cohomology class $[c] \in H^2(G; \mathbb{Z}/2)$. This class has the following interpretation. The composition of $\rho\colon G \to \mathrm{SU}(2)$ with $\mathrm{Ad}\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a representation $\mathrm{Ad}\,\rho\colon G \to \mathrm{SO}(3)$. As such, it induces a continuous map $BG \to B\,\mathrm{SO}(3)$ which is unique up to homotopy. The pullback of the universal Stiefel–Whitney class $w_2 \in H^2(B\,\mathrm{SO}(3); \mathbb{Z}/2)$ via this map is our class $[c] = w_2(\mathrm{Ad}\,\rho) \in H^2(G; \mathbb{Z}/2)$. It serves as an obstruction to lifting $\mathrm{Ad}\,\rho\colon G \to \mathrm{SO}(3)$ to an SU(2) representation.

Let $\mathcal{PR}_c(G; \mathrm{SU}(2))$ be the space of conjugacy classes of projective representations $\rho\colon G \to \mathrm{SU}(2)$ whose associated cocycle is c. The topology on $\mathcal{PR}_c(G; \mathrm{SU}(2))$ is supplied by the algebraic set structure. One can easily see that $\mathcal{PR}_c(G; \mathrm{SU}(2))$ is determined uniquely up to homeomorphism by the cohomology class of c. The group $H^1(G; \mathbb{Z}/2) = \mathrm{Hom}(G, \mathbb{Z}/2)$ acts on $\mathcal{PR}_c(G; \mathrm{SU}(2))$ by sending ρ to $\chi \cdot \rho$ for any $\chi \in \mathrm{Hom}(G, \mathbb{Z}/2)$. The orbits of this action are in a bijective correspondence with the conjugacy classes of representations $G \to \mathrm{SO}(3)$ whose second Stiefel–Whitney class equals $[c]$. The bijection is given by taking the adjoint representation.

Projective representations $\rho\colon G \to SU(2)$ can also be described in terms of a presentation $G = F/R$. Consider a homomorphism $\gamma\colon R \to \mathbb{Z}/2$ defined by its values $\gamma(r) = \pm 1$ on the relators $r \in R$ and by the condition that it is constant on the orbits of the adjoint action of F on R. Also, choose a set-theoretic section $s\colon G \to F$ in the exact sequence

$$1 \to R \xrightarrow{i} F \xrightarrow{\pi} G \to 1$$

and let $r\colon G \times G \to R$ be the function defined by the formula $s(gh) = r(g,h)s(g)s(h)$.

Proposition 2.1 *A choice of a section $s\colon G \to F$ establishes a bijective correspondence between the conjugacy classes of projective representations $\rho\colon G \to SU(2)$ with the cocycle $c(g,h) = \gamma(r(g,h))$ and the conjugacy classes of homomorphisms $\sigma\colon F \to SU(2)$ such that $i^*\sigma = \gamma$. A different choice of s results in a cohomologous cocycle.*

Proof We begin by checking that $c(g,h) = \gamma(r(g,h))$ is a cocycle. For any $g, h, k \in G$, we have

$$s(ghk) = r(gh,k)s(gh)s(k) = r(gh,k)r(g,h)s(g)s(h)s(k),$$
$$s(ghk) = r(g,hk)s(g)s(hk) = r(g,hk)s(g)r(h,k)s(h)s(k),$$

which results in $r(gh,k)r(g,h) = r(g,hk)s(g)r(h,k)s(g)^{-1}$. Since the homomorphism γ is constant on the orbits of the adjoint action of F on R, its application to the above equality gives the cocycle condition $c(gh,k)c(g,h) = c(g,hk)c(h,k)$ as desired.

Now, given a homomorphism $\sigma\colon F \to SU(2)$ such that $i^*\sigma = \gamma$, define $\rho\colon G \to SU(2)$ by the formula $\rho(g) = \sigma(s(g))$. Then

$$\begin{aligned}\rho(gh) &= \sigma(s(gh)) = \sigma(r(g,h)s(g)s(h)) \\ &= \gamma(r(g,h))\sigma(s(g))\sigma(s(h)) = c(g,h)\rho(g)\rho(h),\end{aligned}$$

hence ρ is a projective representation with cocycle c. It is clear that conjugate representations σ define conjugate projective representations ρ, and that a different choice of s leads to a cohomologous cocycle c.

The inverse correspondence is defined as follows. Given a projective representation $\rho\colon G \to SU(2)$, write elements of F in the form $r \cdot s(g)$ with $r \in R$ and $g \in G$, and define $\sigma\colon F \to SU(2)$ by the formula $\sigma(r \cdot s(g)) = \gamma(r)\rho(g)$. That σ is a homomorphism can be checked by a straightforward calculation using the fact that $c(g,h) = \gamma(r(g,h))$. □

Example 2.2 Let $G = \pi_1 M$ be the fundamental group of a manifold M obtained by 0–surgery on a knot k in an integral homology sphere Σ. The group $\pi_1 M$ is obtained from $\pi_1 K$ by imposing the relation $\lambda = 1$, where λ is a canonical longitude of k. Therefore, $\pi_1 M$ admits a presentation $\pi_1 M = F/R$ with λ being one of the relators. Let $\gamma(\lambda) = -1$ and $\gamma(r) = 1$ for the rest of the relators $r \in R$. It has been known since Floer [16] that the action of $H^1(M;\mathbb{Z}/2) = \mathbb{Z}/2$ on the set of conjugacy classes of projective representations $\sigma\colon F \to \mathrm{SU}(2)$ with $i^*\sigma = \gamma$ is free, providing a two-to-one correspondence between this set and the set of the conjugacy classes of representations $\pi_1 M \to \mathrm{SO}(3)$ with nontrivial $w_2 \in H^2(M;\mathbb{Z}/2) = \mathbb{Z}/2$.

2.3 Holonomy correspondence

We will now apply the general theory of Section 2.2 to the group $G = \pi_1 X$, where X is the exterior of a two-component link $\mathcal{L}$ in an integral homology sphere Σ. We begin with the following simple observation.

Lemma 2.3 *Unless the link $\mathcal{L}$ is split, $H^2(X;\mathbb{Z}/2) = H^2(\pi_1 X;\mathbb{Z}/2) = \mathbb{Z}/2$. For split links, $I_*(\Sigma,\mathcal{L}) = 0$.*

Proof For a split link $\mathcal{L}$, the splitting sphere generates the group $H_2(X;\mathbb{Z}) = \mathbb{Z}$. Since there are no flat connections on this sphere with nontrivial $w_2(P)$, the group $I_*(\Sigma,\mathcal{L})$ must vanish. For a nonsplit link, the claimed equality follows from the Hopf exact sequence

$$\pi_2(X) \to H_2(X) \to H_2(\pi_1 X) \to 0$$

and the vanishing of the Hurewicz homomorphism $\pi_2(X) \to H_2(X)$. □

From now on, we will assume that the link $\mathcal{L} \subset \Sigma$ is not split. The holonomy correspondence of [33, Section 3.1] identifies the critical point set of the functional (1) with the set $\mathcal{PR}_c(X,\mathrm{SU}(2))$ of conjugacy classes of projective representations $\rho\colon \pi_1 X \to \mathrm{SU}(2)$, for any choice of cocycle c such that $0 \neq [c] = w_2(P) \in H^2(X;\mathbb{Z}/2) = \mathbb{Z}/2$. Note that this identification commutes with the $H^1(X;\mathbb{Z}/2)$ action, and that the orbits of this action on $\mathcal{PR}_c(X,\mathrm{SU}(2))$ are in bijective correspondence with the conjugacy classes of representations $\mathrm{Ad}\,\rho\colon \pi_1 X \to \mathrm{SO}(3)$ having $w_2(\mathrm{Ad}\,\rho) \neq 0$.

Lemma 2.4 *Any representation $\mathrm{Ad}\,\rho\colon \pi_1 X \to \mathrm{SO}(3)$ with $w_2(\mathrm{Ad}\,\rho) \neq 0$ is irreducible, that is, its image is not contained in a copy of $\mathrm{SO}(2) \subset \mathrm{SO}(3)$.*

Proof The restriction of ρ to either boundary torus of X has nontrivial second Stiefel–Whitney class, which implies that it does not lift to an $\mathrm{SU}(2)$ representation. However, any reducible representation $\pi_1 T^2 \to \mathrm{SO}(3)$ admits an SU(2) lift, therefore, the image of ρ cannot be contained in a copy of $\mathrm{SO}(2) \subset \mathrm{SO}(3)$. It is essential here that $H_1(T^2)$ has no 2–torsion: a nontrivial SO(3) representation of $\mathbb{Z}/2$ is reducible but does not admit an SU(2) lift. □

2.4 Floer gradings

Given flat orbifold connections ρ and σ in the orbifold bundle $P \to \Sigma$, consider an arbitrary orbifold connection A in the pullback bundle on the product $\mathbb{R} \times \Sigma$ matching ρ and σ near the negative and positive ends, respectively. Equip $\mathbb{R} \times \Sigma$ with the orbifold product metric and consider the ASD operator

$$\mathcal{D}_A(\rho, \sigma) = -d_A^* \oplus d_A^+ \colon \Omega^1(\mathbb{R} \times \Sigma, \operatorname{ad} P) \to (\Omega^0 \oplus \Omega^2_+)(\mathbb{R} \times \Sigma, \operatorname{ad} P) \tag{2}$$

completed in the orbifold Sobolev L^2 norms as in [24, Section 3.1]. Since ρ and σ are irreducible, this operator will be Fredholm if we further assume that ρ and σ are nondegenerate as the critical points of the Chern–Simons functional (1). Define the relative Floer grading as

$$\operatorname{gr}(\rho, \sigma) = \operatorname{ind} \mathcal{D}_A(\rho, \sigma) \pmod 4. \tag{3}$$

This grading is well defined because replacing either ρ or σ by its gauge equivalent within the restricted gauge group $\mathcal{G}_S$ results in adding a multiple of four to the index of $\mathcal{D}_A$, see [24, Section 2.5]. This is no longer true if we use the full gauge group. The following lemma makes it precise; it will be proved in Section 3.7.

Lemma 2.5 *Let χ_1 and χ_2 be the generators of $H^1(X; \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ dual to the meridians of the link $\mathcal{L} = \ell_1 \cup \ell_2$. Then*

$$\operatorname{gr}(\chi_1 \cdot \rho, \sigma) = \operatorname{gr}(\chi_2 \cdot \rho, \sigma) = \operatorname{gr}(\rho, \sigma) + 2 \cdot \delta \pmod 4,$$

and similarly for the action on σ, where

$$\delta = \begin{cases} 0 & \text{if } \ell k(\ell_1, \ell_2) \text{ is odd}, \\ 1 & \text{if } \ell k(\ell_1, \ell_2) \text{ is even}. \end{cases}$$

2.5 Perturbations

The critical points of the Chern–Simons functional need not be nondegenerate, therefore we may have to perturb it to define $I_*(\Sigma, \mathcal{L})$. The perturbations used in [24, Section 3.4] are the standard Wilson loop perturbations along loops in Σ disjoint from the link $\mathcal{L}$. There are sufficiently many such perturbations to guarantee the nondegeneracy of the critical points of the perturbed Chern–Simons functional as well as the transversality properties for the moduli spaces of trajectories of its gradient flow. This allows us to define the boundary operator and to complete the definition of $I_*(\Sigma, \mathcal{L})$.

3 Equivariant gauge theory

In this section, we survey some equivariant gauge theory on the double branched cover $M \to \Sigma$ of a homology sphere Σ with branch set a two-component link $\mathcal{L}$. It will be used in the forthcoming sections to make headway in computing the link homology $I_*(\Sigma, \mathcal{L})$.

3.1 Topological preliminaries

Let Σ be an integral homology 3–sphere and $\mathcal{L} = \ell_1 \cup \ell_2$ a link of two components in Σ. The link exterior $X = \Sigma - \operatorname{int} N(\mathcal{L})$ is a manifold whose boundary consists of two tori, with $H_1(X; \mathbb{Z}) = \mathbb{Z}^2$ spanned by the meridians μ_1 and μ_2 of the link components. The homomorphism $\pi_1 X \to \mathbb{Z}/2$ sending μ_1 and μ_2 to the generator of $\mathbb{Z}/2$ gives rise to a regular double cover $\widetilde{X} \to X$, and also to a double branched cover $\pi\colon M \to \Sigma$ with branching set $\mathcal{L}$ and the covering translation $\tau\colon M \to M$. Denote by $\Delta(t)$ the one-variable Alexander polynomial of $\mathcal{L}$.

Proposition 3.1 *The first Betti number of M is 1 if $\Delta(-1) = 0$ and 0 otherwise. In the latter case, $H_1(M; \mathbb{Z})$ is a finite group of order $|\Delta(-1)|$. The induced involution $\tau_*\colon H_1(M) \to H_1(M)$ is multiplication by -1.*

Proof This is essentially proved in Kawauchi [21, Section 5.5]. The statement about τ_* follows from an isomorphism of $\mathbb{Z}[t, t^{-1}]$ modules $H_1(M) = H_1(E)/(1+t)H_1(E)$, where E is the infinite cyclic cover of X, established in [21, Theorem 5.5.1]. A completely different proof for the special case of double branched covers of S^3 with branch set a knot can be found in Ruberman [31, Lemma 5.5]. □

Proposition 3.2 *Let M be the double branched cover of an integral homology sphere with branch set a two-component link. Then $H_i(M; \mathbb{Z}/2) = H^i(M; \mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}/2$ if $i = 0, 1, 2, 3$, and is zero otherwise. The cup product*

$$H^1(M; \mathbb{Z}/2) \times H^1(M; \mathbb{Z}/2) \to H^2(M; \mathbb{Z}/2)$$

is given by the linking number $\ell k(\ell_1, \ell_2) \pmod 2$.

The proof of Proposition 3.2 will be postponed until Section 8.3 for the sake of exposition.

An important example of $\mathcal{L}$ to consider is the two-component link $k^\natural$ obtained as the connected sum of a knot $k \subset S^3$ with the Hopf link. The double branched cover $M \to S^3$ in this case is the connected sum $M = Y \# \mathbb{R}\mathrm{P}^3$, where Y is the double branched cover of k. Proposition 3.2 easily follows because $H_*(Y;\mathbb{Z}/2) = H_*(S^3;\mathbb{Z}/2)$.

3.2 The orbifold exact sequence

We will view $\Sigma = M/\tau$ as an orbifold with the singular set $\mathcal{L}$. To be precise, the regular double cover $\widetilde{X} \to X$ is a 3–manifold whose boundary consists of two tori, and

$$M = \widetilde{X} \cup_h N(\mathcal{L}),$$

where the gluing homeomorphism $h\colon \partial\widetilde{X} \to \partial N(\mathcal{L})$ identifies $\pi^{-1}(\mu_i)$ with the meridian μ_i for $i = 1, 2$. The involution $\tau\colon M \to M$ acts by meridional rotation on $N(\mathcal{L})$, thereby fixing the link $\mathcal{L}$, and by covering translation on $\widetilde{X}$. Define the orbifold fundamental group

$$\pi_1^V(\Sigma,\mathcal{L}) = \pi_1 X/\langle {\mu_1}^2 = {\mu_2}^2 = 1\rangle.$$

Then the homotopy exact sequence of the covering $\widetilde{X} \to X$ gives rise to a split short exact sequence, called the orbifold exact sequence,

$$1 \to \pi_1 M \xrightarrow{\pi_*} \pi_1^V(\Sigma,\mathcal{L}) \xrightarrow{j} \mathbb{Z}/2 \to 1. \tag{4}$$

The homomorphism j maps the meridians μ_1, μ_2 to the generator of $\mathbb{Z}/2$ and one obtains a splitting by sending this generator to either μ_1 or μ_2.

It follows from the definition of the orbifold fundamental group $\pi_1^V(\Sigma,\mathcal{L})$ that its abelianization is given by

$$H_1(X)/\langle {\mu_1}^2 = {\mu_2}^2 = 1\rangle = H_1(X;\mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

with the canonical generators μ_1 and μ_2. The homomorphism π_* of the orbifold exact sequence (4) then induces a map $\pi_*\colon H_1(M;\mathbb{Z}/2) \to H_1(X;\mathbb{Z}/2)$, which can be described as follows.

Lemma 3.3 *The homomorphism $\pi_*\colon H_1(M;\mathbb{Z}/2) \to H_1(X;\mathbb{Z}/2)$ sends the generator of $H_1(M;\mathbb{Z}/2) = \mathbb{Z}/2$ to the sum of the meridians $\mu_1 + \mu_2 \in H_1(X;\mathbb{Z}/2)$.*

Proof That $H_1(M;\mathbb{Z}/2) = \mathbb{Z}/2$ follows from Proposition 3.2. An explicit generator of this group is described in the proof of Proposition A.2 as the circle $\pi^{-1}(w)$, where w is an embedded arc in Σ with endpoints on the two different components of $\mathcal{L}$. The commutative diagram

$$\begin{array}{ccc} \pi_1 \widetilde{X} & \xrightarrow{\pi_*} & \pi_1 X \\ \big\downarrow & & \big\downarrow \\ \pi_1 M & \xrightarrow{\pi_*} & \pi_1^V(\Sigma, \mathcal{L}) \end{array}$$

gives rise to the commutative diagram in homology

$$\begin{array}{ccc} H_1(\widetilde{X}; \mathbb{Z}/2) & \xrightarrow{\pi_*} & H_1(X; \mathbb{Z}/2) \\ \big\downarrow & & \big\downarrow \\ H_1(M; \mathbb{Z}/2) & \xrightarrow{\pi_*} & H_1(X; \mathbb{Z}/2) \end{array}$$

The cycle $\pi^{-1}(w)$ in M is homologous to a cycle in $\widetilde{X}$ which consists of the two arcs $\pi^{-1}(w) \cap \widetilde{X}$ whose endpoints on each of the tori in $\partial\widetilde{X}$ are connected by an arc. The map $\pi_*\colon H_1(\widetilde{X}; \mathbb{Z}/2) \to H_1(X; \mathbb{Z}/2)$ takes the homology class of this cycle to $\mu_1 + \mu_2$, and the result follows. □

3.3 Pulled-back bundles

Let $P \to \Sigma$ be the orbifold SO(3)–bundle used in the definition of $I_*(\Sigma, \mathcal{L})$ in Section 2. It pulls back to an orbifold SO(3)–bundle $Q \to M$ because the projection map $\pi\colon M \to \Sigma$ is regular in the sense of Chen and Ruan [10]. The bundle Q is in fact smooth because orbifold connections on P with order-2 holonomy along the meridians of $\mathcal{L}$ lift to connections in Q with trivial holonomy along the meridians of the two-component link $\widetilde{\mathcal{L}} = \pi^{-1}(\mathcal{L})$.

Proposition 3.4 *The bundle $Q \to M$ is nontrivial.*

The rest of this section is dedicated to the proof of this proposition. We will accomplish it by showing that $w_2(Q) \in H^2(M; \mathbb{Z}/2) = \mathbb{Z}/2$ is nonvanishing. Our argument will split into two cases, corresponding to the parity of the linking number between the components of $\mathcal{L}$.

Suppose that $\ell k(\ell_1, \ell_2)$ is even and consider the regular double cover $\pi\colon M - \widetilde{\mathcal{L}} \to \Sigma - \mathcal{L}$. It gives rise to the Gysin exact sequence

$$\begin{aligned} \cdots \to H^1(\Sigma - \mathcal{L}; \mathbb{Z}/2) &\xrightarrow{\cup w_1} H^2(\Sigma - \mathcal{L}; \mathbb{Z}/2) \xrightarrow{\pi^*} H^2(M - \widetilde{\mathcal{L}}; \mathbb{Z}/2) \\ \longrightarrow H^2(\Sigma - \mathcal{L}; \mathbb{Z}/2) &\xrightarrow{\cup w_1} H^3(\Sigma - \mathcal{L}; \mathbb{Z}/2) \longrightarrow \cdots \end{aligned}$$

where $\cup w_1$ means taking the cup product with the first Stiefel–Whitney class of the cover. The cup product on $H^*(\Sigma - \mathcal{L}; \mathbb{Z}/2)$ can be determined from the following commutative diagram:

$$\begin{array}{ccc} H_2(\Sigma,\mathcal{L};\mathbb{Z}/2)\times H_2(\Sigma,\mathcal{L};\mathbb{Z}/2) & \xrightarrow{\ \cdot\ } & H_1(\Sigma,\mathcal{L};\mathbb{Z}/2) \\ \Big\uparrow \mathrm{PD} & & \Big\uparrow \mathrm{PD} \\ H^1(\Sigma-\mathcal{L};\mathbb{Z}/2)\times H^1(\Sigma-\mathcal{L};\mathbb{Z}/2) & \xrightarrow{\ \cup\ } & H^2(\Sigma-\mathcal{L};\mathbb{Z}/2) \end{array}$$

where PD stands for the Poincaré duality isomorphism and the dot in the upper row for the intersection product. Note that Seifert surfaces of knots ℓ_1 and ℓ_2 generate $H_2(\Sigma,\mathcal{L};\mathbb{Z}/2)=\mathbb{Z}/2\oplus\mathbb{Z}/2$, and any arc in Σ with one endpoint on ℓ_1 and the other on ℓ_2 generates $H_1(\Sigma,\mathcal{L};\mathbb{Z}/2)=\mathbb{Z}/2$. An easy calculation shows that, with respect to these generators, the intersection product is given by the matrix

$$\begin{pmatrix} 0 & \ell k(\ell_1,\ell_2) \\ \ell k(\ell_1,\ell_2) & 0 \end{pmatrix}.$$

Since $\ell k(\ell_1,\ell_2)$ is even, this gives a trivial cup product structure on the link complement $\Sigma-\mathcal{L}$. Therefore, the map $\cup w_1$ in the Gysin sequence is zero and the map $\pi^*\colon H^2(\Sigma-\mathcal{L};\mathbb{Z}/2)\to H^2(M-\widetilde{\mathcal{L}};\mathbb{Z}/2)$ is injective. Since $w_2(P)\in H^2(\Sigma-\mathcal{L};\mathbb{Z}/2)$ is nonzero we conclude that $\pi^*(w_2(P))\neq 0$. This implies that $w_2(Q)\neq 0$ because $Q=\pi^*P$ over $M-\widetilde{\mathcal{L}}$.

Now suppose that $\ell k(\ell_1,\ell_2)$ is odd. The above calculation implies that the second Stiefel–Whitney class of π^*P vanishes in $H^2(M-\widetilde{\mathcal{L}};\mathbb{Z}/2)$. We will prove, however, that $w_2(Q)\in H^2(M;\mathbb{Z}/2)$ is nonzero, by showing that Q carries a flat connection with nonzero w_2.

Note that the orbifold bundle P carries a flat SO(3) connection whose holonomy is a representation $\alpha\colon \pi_1^V(\Sigma,\mathcal{L})\to \mathrm{SO}(3)$ of the orbifold fundamental group $\pi_1^V(\Sigma,\mathcal{L})=\pi_1X/\langle {\mu_1}^2={\mu_2}^2=1\rangle$ sending the two meridians to $\mathrm{Ad}\,i$ and $\mathrm{Ad}\,j$. This flat connection pulls back to a flat connection on Q with holonomy $\pi^*\alpha\colon \pi_1M\to\mathrm{SO}(3)$. We wish to compute the second Stiefel–Whitney class of $\pi^*\alpha$.

Lemma 3.5 *The representation $\pi^*\alpha\colon \pi_1M\to\mathbb{Z}/2\oplus\mathbb{Z}/2$ is nontrivial.*

Proof Our proof will rely on the orbifold exact sequence (4). Assume that $\pi^*\alpha$ is trivial. Then $\pi_1M\subset\ker(\pi^*\alpha)$, hence α factors through $\pi_1^V(\Sigma,\mathcal{L})/\pi_*(\pi_1M)\to\mathbb{Z}/2\oplus\mathbb{Z}/2$. Since $\pi_1^V(\Sigma,\mathcal{L})/\pi_*(\pi_1M)=\mathbb{Z}/2$, we obtain a contradiction with the surjectivity of α. □

Since the group $\mathbb{Z}/2\oplus\mathbb{Z}/2$ is abelian, the representation $\pi^*\alpha\colon\pi_1M\to\mathbb{Z}/2\oplus\mathbb{Z}/2$ factors through a homomorphism $H_1(M)\to\mathbb{Z}/2\oplus\mathbb{Z}/2$ which is uniquely determined by its two components $\xi,\ \eta\in\mathrm{Hom}(H_1(M),\mathbb{Z}/2)=H^1(M;\mathbb{Z}/2)=\mathbb{Z}/2$;

see Proposition 3.2. A calculation identical to that in [33, Proposition 4.3] shows that $w_2(\pi^*\alpha) = \xi \cup \xi + \xi \cup \eta + \eta \cup \eta$ (note that, unlike in [33], the classes $\xi \cup \xi$ and $\eta \cup \eta$ need not vanish). Since ξ and η cannot both be trivial by Lemma 3.5, we may assume without loss of generality that $\xi \neq 0$. If $\eta = 0$ then $w_2(\pi^*\alpha) = \xi \cup \xi$. If $\eta \neq 0$ then $\xi = \eta$ due to the fact that $H^1(M;\mathbb{Z}/2) = \mathbb{Z}/2$, and therefore again $w_2(\pi^*\alpha) = \xi \cup \xi$. Since $\ell k(\ell_1, \ell_2)$ is odd, it follows from Proposition 3.2 that $w_2(\pi^*\alpha) \neq 0$.

3.4 Pulled-back representations

Assuming that $\mathcal{L} \subset \Sigma$ is nonsplit, we identified in Section 2.3 the critical point set of the Chern–Simons functional (1) with the space $\mathcal{PR}_c(X, \mathrm{SU}(2))$ of the conjugacy classes of projective representations $\pi_1 X \to \mathrm{SU}(2)$ on the link exterior, for any choice of cocycle c not cohomologous to zero. We further identified the quotient of $\mathcal{PR}_c(X, \mathrm{SU}(2))$ by the natural $H^1(X;\mathbb{Z}/2)$ action with the subspace $\mathcal{R}_w(X;\mathrm{SO}(3))$ of the $\mathrm{SO}(3)$ character variety of $\pi_1 X$ cut out by the condition $w_2 \neq 0$. The latter condition implies that both meridians μ_1 and μ_2 are represented by $\mathrm{SO}(3)$ matrices of order 2, which leads to a natural identification of this subspace with

$$\mathcal{R}_\omega(\Sigma, \mathcal{L}; \mathrm{SO}(3)) = \{\rho\colon \pi_1^V(\Sigma, \mathcal{L}) \to \mathrm{SO}(3) \mid w_2(\rho) \neq 0\}/\operatorname{Ad}\mathrm{SO}(3),$$

where the condition $w_2(\rho) \neq 0$ applies to the representation ρ restricted to X. To summarize, the group $H^1(X;\mathbb{Z}/2)$ acts on $\mathcal{PR}_c(X, \mathrm{SU}(2))$ with the quotient map

$$\mathcal{PR}_c(X, \mathrm{SU}(2)) \to \mathcal{R}_\omega(\Sigma, \mathcal{L}; \mathrm{SO}(3)).$$

We now wish to study $\mathcal{R}_\omega(\Sigma, \mathcal{L}; \mathrm{SO}(3))$ using representations on the double branched cover $M \to \Sigma$ equivariant with respect to the covering translation $\tau\colon M \to M$.

Lemma 3.6 *Let $\rho\colon \pi_1^V(\Sigma, \mathcal{L}) \to \mathrm{SO}(3)$ be a representation with $w_2(\rho) \neq 0$ and $\pi^*\rho\colon \pi_1 M \to \mathrm{SO}(3)$ its pullback via the homomorphism π_* of the orbifold exact sequence* (4)*. Then there exists an element $u \in \mathrm{SO}(3)$ of order 2 such that $\tau^*(\pi^*\rho) = u \cdot (\pi^*\rho) \cdot u^{-1}$.*

Proof Let $\widetilde{X} \to X$ be the regular double cover as in Section 3.2. Choose a basepoint b in one of the boundary tori of $\widetilde{X}$ and consider the commutative diagram

$$\begin{array}{ccccc}
\pi_1(\widetilde{X}, b) & \xrightarrow{\ \tau_*\ } & \pi_1(\widetilde{X}, \tau(b)) & \xrightarrow{\ \psi_f\ } & \pi_1(\widetilde{X}, b) \\
{\scriptstyle \pi_*}\searrow & & \swarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\
& \pi_1(X, \pi(b)) & \xrightarrow{\ \ \varphi\ \ } & & \pi_1(X, \pi(b))
\end{array}$$

whose maps ψ_f and φ are defined as follows. Given a path $f\colon [0,1] \to X$ from b to $\tau(b)$, take its inverse $\bar{f}(s) = f(1-s)$ and define the map ψ_f by the formula $\psi_f(\beta) = f \cdot \beta \cdot \bar{f}$. Since $\pi(b) = \pi(\tau(b))$, the path f projects to a loop in X based at $\pi(b)$, and the map φ is the conjugation by that loop. In fact, one can choose the path f to project onto the meridian μ_i of the boundary torus on which $\pi(b)$ lies so that $\varphi(x) = \mu_i \cdot x \cdot \mu_i^{-1}$. After filling in the solid tori, we obtain the commutative diagram

$$\begin{array}{ccc} \pi_1 M & \xrightarrow{\tau_*} & \pi_1 M \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ \pi_1^V(\Sigma, \mathcal{L}) & \xrightarrow{\varphi} & \pi_1^V(\Sigma, \mathcal{L}) \end{array}$$

which tells us that, for any $\rho\colon \pi_1^V(\Sigma, \mathcal{L}) \to \mathrm{SO}(3)$, the pullback representation $\pi^*\rho$ has the property that $\tau^*(\pi^*\rho) = u \cdot (\pi^*\rho) \cdot u^{-1}$ with $u = \rho(\mu_i)$ of order 2. □

Example 3.7 Let $\mathcal{L} \subset S^3$ be the Hopf link. Then $M = \mathbb{RP}^3$ and the orbifold exact sequence (4) takes the form

$$1 \to \mathbb{Z}/2 \xrightarrow{\pi_*} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{j} \mathbb{Z}/2 \to 1$$

with the two copies of $\mathbb{Z}/2$ in the middle group generated by the meridians μ_1 and μ_2. Define $\rho\colon \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathrm{SO}(3)$ on the generators by $\rho(\mu_1) = \operatorname{Ad} i$ and $\rho(\mu_2) = \operatorname{Ad} j$; up to conjugation, this is the only representation $\mathbb{Z}/2 \to \mathrm{SO}(3)$ with $w_2(\rho) \neq 0$. The pullback representation $\pi^*\rho\colon \mathbb{Z}/2 \to \mathrm{SO}(3)$ sends the generator to $\operatorname{Ad} i \cdot \operatorname{Ad} j = \operatorname{Ad} k$. Since $\tau^*(\pi^*\rho) = \pi^*\rho$, the identity $\tau^*(\pi^*\rho) = u \cdot (\pi^*\rho) \cdot u^{-1}$ holds for multiple choices of u, including the second-order u of the form $u = \operatorname{Ad} q$, where q is any unit quaternion such that $-qk = kq$.

Given a double branched cover $\pi\colon M \to \Sigma$ with branch set $\mathcal{L}$ and the covering translation $\tau\colon M \to M$, define

$$\mathcal{R}_\omega(M; \mathrm{SO}(3)) = \{\beta\colon \pi_1 M \to \mathrm{SO}(3) \mid w_2(\beta) \neq 0\} / \operatorname{Ad} \mathrm{SO}(3).$$

Since $w_2(\tau^*\beta) = w_2(\beta) \in H^2(M; \mathbb{Z}/2) = \mathbb{Z}/2$, the pullback of representations via τ gives rise to a well defined involution

$$\tau^*\colon \mathcal{R}_\omega(M; \mathrm{SO}(3)) \to \mathcal{R}_\omega(M; \mathrm{SO}(3)). \tag{5}$$

Its fixed point set $\operatorname{Fix}(\tau^*)$ consists of those conjugacy classes of representations $\beta\colon \pi_1 M \to \mathrm{SO}(3)$ such that $w_2(\beta) \neq 0$ and there exists an element $u \in \mathrm{SO}(3)$ having the property that $\tau^*\beta = u \cdot \beta \cdot u^{-1}$. Consider the subvariety

$$\mathcal{R}_w^\tau(M; \mathrm{SO}(3)) \subset \operatorname{Fix}(\tau^*) \tag{6}$$

defined by the condition that the conjugating element u can be chosen to be of order 2. This subvariety is well defined because all elements of order 2 in $\mathrm{SO}(3)$ are conjugate to each other. The following proposition is the main result of this section.

Proposition 3.8 *The homomorphism $\pi_*\colon \pi_1 M \to \pi_1^V(\Sigma, \mathcal{L})$ of the orbifold exact sequence* (4) *induces via the pullback a homeomorphism*

$$\pi^*\colon \mathcal{R}_\omega(\Sigma, \mathcal{L}; \mathrm{SO}(3)) \to \mathcal{R}_\omega^\tau(M; \mathrm{SO}(3)).$$

Proof Orbifold representations $\pi_1^V(\Sigma, \mathcal{L}) \to \mathrm{SO}(3)$ with nontrivial w_2 pull back to representations $\pi_1 M \to \mathrm{SO}(3)$ with nontrivial w_2; see Section 3.3. In addition, these pullback representations are equivariant in the sense of Lemma 3.6. Therefore, the map

$$\pi^*\colon \mathcal{R}_\omega(\Sigma, \mathcal{L}; \mathrm{SO}(3)) \to \mathcal{R}_\omega^\tau(M; \mathrm{SO}(3))$$

is well defined. To finish the proof, we will construct an inverse of π^*. Given $\beta\colon \pi_1 M \to \mathrm{SO}(3)$ whose conjugacy class belongs to $\mathcal{R}_\omega^\tau(M; \mathrm{SO}(3))$, there exists an element $u \in \mathrm{SO}(3)$ of order 2 such that $\tau^*\beta = u\cdot\beta\cdot u^{-1}$. The pair (β, u) then defines an $\mathrm{SO}(3)$ representation of $\pi_1^V(\Sigma, \mathcal{L}) = \pi_1 M \rtimes \mathbb{Z}/2$ by the formula $\rho(x, t^\ell) = \beta(x)\cdot u^\ell$, where $x \in \pi_1 M$ and t is the generator of $\mathbb{Z}/2$. □

3.5 Equivariant index

All orbifolds we encounter in this paper are obtained by taking the quotient of a smooth manifold by an orientation-preserving involution. The orbifold elliptic theory on such global quotient orbifolds is equivalent to the equivariant elliptic theory on their branched covers; see for instance [42]. In particular, the orbifold index of the ASD operator (2) can be computed as an equivariant index as explained below.

Let X be a smooth oriented Riemannian 4–manifold without boundary, which may or may not be compact. If X is not compact, we assume that its only noncompactness comes from a product end $(0, \infty) \times Y$ equipped with a product metric. Let $\tau\colon X \to X$ be a smooth orientation-preserving isometry of order 2 with nonempty fixed point set F making X into a double branched cover over X' with branch set F'. Let $P \to X$ be an $\mathrm{SO}(3)$–bundle to which τ lifts so that its action on the fibers over the fixed point set of τ has order 2. This lift will be denoted by $\tilde{\tau}\colon P \to P$. The quotient of P by the involution $\tilde{\tau}$ is naturally an orbifold $\mathrm{SO}(3)$–bundle $P' \to X'$, and any equivariant connection A in P gives rise to an orbifold connection A' in P'. The ASD operator

$$\mathcal{D}_A(X) = -d_A^* \oplus d_A^+\colon \Omega^1(X, \mathrm{ad}\, P) \to (\Omega^0 \oplus \Omega^2_+)(X, \mathrm{ad}\, P)$$

associated with A is equivariant in that the diagram

$$\begin{array}{ccc} \Omega^1(X,\operatorname{ad}P) & \xrightarrow{\mathcal{D}_A(X)} & (\Omega^0\oplus\Omega^2_+)(X,\operatorname{ad}P) \\ \tilde{\tau}^*\big\downarrow & & \big\downarrow\tilde{\tau}^* \\ \Omega^1(X,\operatorname{ad}P) & \xrightarrow{\mathcal{D}_A(X)} & (\Omega^0\oplus\Omega^2_+)(X,\operatorname{ad}P) \end{array}$$

commutes, giving rise to the orbifold operator

$$\mathcal{D}_{A'}(X')\colon \Omega^1(X',\operatorname{ad}P')\to(\Omega^0\oplus\Omega^2_+)(X',\operatorname{ad}P').$$

From this we immediately conclude that

$$\operatorname{ind}\mathcal{D}_{A'}(X')=\operatorname{ind}\mathcal{D}^\tau_A(X), \tag{7}$$

where $\mathcal{D}^\tau_A(X)$ is the operator $\mathcal{D}_A(X)$ restricted to the $(+1)$–eigenspaces of the involution $\tilde{\tau}^*$. If X is closed, the operators in (7) are automatically Fredholm. If X has a product end, we ensure Fredholmness by completing with respect to the weighted Sobolev norms

$$\|\varphi\|_{L^2_{k,\delta}(X)}=\|h\cdot\varphi\|_{L^2_k(X)},$$

where $h\colon X\to\mathbb{R}$ is a smooth function which is τ–invariant and which, over the end, takes the form $h(t,y)=e^{\delta t}$ for a sufficiently small positive δ. We choose to work with these particular norms to match the global boundary conditions of Atiyah, Patodi and Singer [4].

In particular, if ρ and σ are nondegenerate critical points of the orbifold Chern–Simons functional on Σ, they pull back to the flat connections $\pi^*\rho$ and $\pi^*\sigma$ on the double branched cover $M\to\Sigma$. The formula (3) for the relative Floer grading can then be written as

$$\operatorname{gr}(\rho,\sigma)=\operatorname{ind}\mathcal{D}^\tau_A(\pi^*\rho,\pi^*\sigma)\pmod 4,$$

where A is an equivariant connection on $\mathbb{R}\times Y$ whose limits at the negative and positive ends are $\pi^*\rho$ and $\pi^*\sigma$, respectively. The index in the above formula can be understood as the L^2_δ index for any sufficiently small $\delta\ge 0$ because the operator $\mathcal{D}^\tau_A(\pi^*\rho,\pi^*\sigma)$ is Fredholm in the usual L^2 Sobolev completion.

3.6 Index formulas

Let us continue with the setup of the previous subsection. One can easily see that

$$\operatorname{ind}\mathcal{D}^\tau_A(X)=\tfrac12\operatorname{ind}\mathcal{D}_A(X)+\tfrac12\operatorname{ind}(\tau,\mathcal{D}_A)(X),$$

where

$$\operatorname{ind}(\tau,\mathcal{D}_A)(X)=\operatorname{tr}(\tilde{\tau}^*\,|\ker\mathcal{D}_A(X))-\operatorname{tr}(\tilde{\tau}^*\,|\operatorname{coker}\mathcal{D}_A(X)).$$

We will use this observation together with the standard index theorems to obtain explicit formulas for the index of the operators in question.

Proposition 3.9 *Let X be a closed manifold. Then*

$$\operatorname{ind}\mathcal{D}^{\tau}_A(X) = -p_1(P) - \tfrac{3}{4}(\sigma(X)+\chi(X)) + \tfrac{1}{4}(\chi(F)+F\cdot F).$$

Proof The index of $\mathcal{D}_A(X)$ can be expressed topologically using the Atiyah–Singer index theorem [6]. Since the operator $\mathcal{D}_A$ has the same symbol as the positive chiral Dirac operator twisted by $S^+ \otimes (\operatorname{ad} P)_{\mathbb{C}}$ (see [3]), we obtain

$$\begin{aligned}\operatorname{ind}\mathcal{D}_A(X) &= \int_X \widehat{A}(X)\operatorname{ch}(S^+)\operatorname{ch}(\operatorname{ad} P)_{\mathbb{C}}\\ &= \int_X -2p_1(A) - \tfrac{1}{2}p_1(TX) - \tfrac{3}{2}e(TX)\\ &= -2p_1(P) - \tfrac{3}{2}(\sigma(X)+\chi(X)),\end{aligned}$$

using the Hirzebruch signature theorem in the last line. A similar expression for $\operatorname{ind}(\tau,\mathcal{D}_A)(X)$ is obtained using the G–index theorem of Atiyah and Singer [6]. For the twisted Dirac operator in question, an explicit calculation in Shanahan [39, Section 19] leads us to the formula

$$\operatorname{ind}(\tau,\mathcal{D}_A)(X) = -\frac{1}{2}\int_F (e(TF)+e(NF))\operatorname{ch}_g(\operatorname{ad} P)_{\mathbb{C}} = \tfrac{1}{2}(\chi(F)+F\cdot F).$$

Here TF and NF are the tangent and the normal bundle of the fixed point set $F\subset X$, and the zero-order term in $\operatorname{ch}_g(\operatorname{ad} P)_{\mathbb{C}}$ equals -1 because this is the trace of the second-order SO(3) operator acting on the fiber. Adding these formulas together, we obtain the desired formula. □

Remark 3.10 Our formula matches the formulas for $\operatorname{ind}\mathcal{D}_{A'}(X')$ of Kronheimer and Mrowka [24, Lemma 2.11] and Wang [42, Theorem 18],

$$\operatorname{ind}\mathcal{D}_{A'}(X') = -p_1(P) - \tfrac{3}{2}(\sigma(X')+\chi(X')) + \chi(F') + \tfrac{1}{2}F'\cdot F',$$

after taking into account that $F'\cdot F' = 2(F\cdot F)$, $\chi(F)=\chi(F')$, $2\chi(X')=\chi(X)+\chi(F)$, and $2\sigma(X') = \sigma(X)+F\cdot F$; see for instance Viro [41].

Next, let X be a manifold with a product end $(0,\infty)\times Y$, where Y need not be connected, and work with the L^2_δ norms for sufficiently small $\delta>0$. In a temporal gauge over the end, the operator $\mathcal{D}_A(X)$ takes the form $\mathcal{D}_A(X) = \partial/\partial t + K_{A(t)}$.

Proposition 3.11 *Let X be a manifold with product end as above, and A an equivariant connection whose limit over the end is a flat connection β. Then*

$$\operatorname{ind}\mathcal{D}^{\tau}_A(X) = \frac{1}{2}\int_X \widehat{A}(X)\operatorname{ch}(S^+)\operatorname{ch}(\operatorname{ad}P)_{\mathbb{C}} + \tfrac{1}{4}(\chi(F)+F\cdot F) - \tfrac{1}{4}(h_\beta - \eta_\beta(0)) - \tfrac{1}{4}(h^{\tau}_\beta - \eta^{\tau}_\beta(0)).$$

The notation here is as follows:

- h_β is the dimension of $H^0(Y;\operatorname{ad}\beta)\oplus H^1(Y;\operatorname{ad}\beta)$;
- h^{τ}_β is the trace of the map induced by $\widetilde{\tau}^*$ on $H^0(Y;\operatorname{ad}\beta)\oplus H^1(Y;\operatorname{ad}\beta)$;
- $\eta_\beta(0)$ is the Atiyah–Patodi–Singer spectral asymmetry of K_β; and
- $\eta^{\tau}_\beta(0)$ its equivariant version, defined as follows. For any eigenvalue λ of the operator K_β, the λ–eigenspace W^{β}_λ is acted upon by $\widetilde{\tau}^*$ with trace $\operatorname{tr}(\widetilde{\tau}^*|W^{\beta}_\lambda)$. The infinite series
$$\eta^{\tau}_\beta(s) = \sum_{\lambda\neq 0}\operatorname{sign}\lambda\cdot\operatorname{tr}(\widetilde{\tau}^*|W^{\beta}_\lambda)|\lambda|^{-s}$$
converges for $\operatorname{Re}(s)$ large enough and has a meromorphic continuation to the entire complex s–plane with no pole at $s=0$; see Donnelly [12]. This makes $\eta^{\tau}_\beta(0)$ a well-defined real number.

Proof of Proposition 3.11 The index $\operatorname{ind}\mathcal{D}_A(X)$ can be computed using the index theorem of Atiyah, Patodi and Singer [4] as

$$\operatorname{ind}\mathcal{D}_A(X) = \int_X \widehat{A}(X)\operatorname{ch}(S^+)\operatorname{ch}(\operatorname{ad}P)_{\mathbb{C}} - \tfrac{1}{2}(h_\beta - \eta_\beta(0))(Y),$$

and $\operatorname{ind}(\tau,\mathcal{D}_A)(X)$ using its equivariant counterpart, the G–index theorem of Donnelly [12], as

$$\operatorname{ind}(\tau,\mathcal{D}_A)(X) = \frac{1}{2}\int_F (e(TF)+e(NF)) - \tfrac{1}{2}(h^{\tau}_\beta - \eta^{\tau}_\beta(0))(Y).$$

The desired formula now follows because, according to the Gauss–Bonnet theorem,

$$\int_F e(TF) = \chi(F) \quad\text{and}\quad \int_F e(NF) = F\cdot F. \qquad\square$$

Example 3.12 Let $P\to Y$ be a trivial $\mathrm{SO}(3)$–bundle with an involution $\widetilde{\tau}$ acting as a second-order operator on the fibers. Application of Proposition 3.11 to the product connection A on the manifold $X=\mathbb{R}\times Y$ results in the formula $\operatorname{ind}\mathcal{D}^{\tau}_\theta(X)=-1$, which corresponds to the fact that the $(+1)$–eigenspace of the involution $\widetilde{\tau}^*\colon H^0(X;\operatorname{ad}\theta)\to H^0(X;\operatorname{ad}\theta)$ is 1–dimensional.

3.7 Proof of Lemma 2.5

Since both ρ and σ are irreducible and nondegenerate, we have $\operatorname{gr}(\chi_1 \cdot \rho, \sigma) = \operatorname{gr}(\chi_1 \cdot \rho, \rho) + \operatorname{gr}(\rho, \sigma)$. Therefore, we only need to compute $\operatorname{gr}(\chi_1 \cdot \rho, \rho) \pmod 4$.

Let $g \in \mathcal{G}$ be a gauge transformation matching ρ and $\chi_1 \cdot \rho$. The mapping torus of g is an orbifold bundle P_0 over $S^1 \times \Sigma$, and

$$\operatorname{gr}(\chi_1 \cdot \rho, \rho) = \operatorname{ind} \mathcal{D}_A \,(S^1 \times \Sigma) \pmod 4$$

for any choice of orbifold connection A in P_0. Let M be the double branched cover of Σ with branch set $\mathcal{L}$. Then the index in the above formula, treated as an equivariant index on $S^1 \times M$, equals $-p_1(Q_0)$ by the formula of Proposition 3.9 applied to the pullback bundle $Q_0 = \pi^* P_0$. This reduces the above formula to

$$\operatorname{gr}(\chi_1 \cdot \rho, \rho) = -p_1(Q_0) \pmod 4.$$

To compute the Pontryagin number $p_1(Q_0)$ we observe that the bundle Q_0 on $S^1 \times M$ can be obtained from the bundle $Q = \pi^* P$ on M as the mapping torus of a gauge transformation matching $\pi^*\rho$ with $\pi^*(\chi_1 \cdot \rho) = \eta \cdot \pi^*\rho$, where $\eta = \pi^*\chi_1 \in H^1(M; \mathbb{Z}/2)$. According to Braam and Donaldson [8, Part II, Propositions 1.9 and 1.13],

$$p_1(Q_0) = 2 \cdot (\eta \cup w_2(Q) + \eta \cup \eta \cup \eta)[M] \pmod 4.$$

We already know that $w_2(Q)$ is a generator of $H^2(M; \mathbb{Z}/2) = \mathbb{Z}/2$; see Proposition 3.4. It follows from Lemma 3.3 that the class η is a generator of $H^1(M; \mathbb{Z}/2) = \mathbb{Z}/2$. The desired formula now follows from the calculation of the cohomology ring $H^*(M; \mathbb{Z}/2)$ in Proposition 3.2.

4 Knot homology: the generators

We will now use the equivariant theory of Section 3 to better understand the chain complex $IC^\natural(k)$ which computes the singular instanton knot homology $I^\natural(k) = I_*(S^3, k^\natural)$ of Kronheimer and Mrowka [24]. In this section, we will describe the conjugacy classes of projective SU(2) representations on the exterior of $k^\natural$ with nontrivial $[c]$ and separate them into the orbits of the canonical $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ action. The next two sections will be dedicated to computing Floer gradings.

4.1 Projective representations

Given a knot $k \subset S^3$, denote by $K = S^3 - \operatorname{int} N(k)$ its exterior and by $K^\natural = S^3 - \operatorname{int} N(k^\natural)$ the exterior of the two-component link $k^\natural = k \cup \ell$ obtained as the connected sum of k with the Hopf link. The Wirtinger presentation

$$\pi_1 K = \langle a_1, a_2, \dots, a_n \mid r_1, \dots, r_m \rangle$$

with meridians a_i and relators r_j gives rise to the Wirtinger presentation

$$\pi_1 K^\natural = \langle a_1, a_2, \dots, a_n, b \mid r_1, \dots, r_m,\ [a_1, b] = 1 \rangle,$$

where b stands for the meridian of the component ℓ. Since the link $k^\natural$ is not split, it follows from Lemma 2.3 that $H^2(\pi_1 K^\natural; \mathbb{Z}/2) = H^2(K^\natural; \mathbb{Z}/2) = \mathbb{Z}/2$. The generator of the latter group evaluates nontrivially on both boundary components of $K^\natural$, which makes it Poincaré dual to any arc connecting these two boundary components. It follows from Proposition 2.1 that the projective representations with nontrivial $[c]$ which we are interested in are precisely the homomorphisms $\rho\colon F \to \mathrm{SU}(2)$ of the free group F generated by the meridians $a_1, \dots, a_n, b$ such that

$$\rho(r_1) = \dots = \rho(r_n) = 1 \quad \text{and} \quad \rho([a_1, b]) = -1.$$

Representations ρ are uniquely determined by the SU(2) matrices $A_i = \rho(a_i)$ and $B = \rho(b)$ subject to the above relations, and the space $\mathcal{PR}_c(K^\natural, \mathrm{SU}(2))$ consists of all such tuples $(A_1, \dots, A_n; B)$ up to conjugation.

The relation $A_1 B = -B A_1$ implies that, up to conjugation, $A_1 = i$ and $B = j$. Since the Wirtinger relations $r_1 = 1, \dots, r_m = 1$ are of the form $a_i a_j a_i^{-1} = a_k$, all the matrices A_i must have zero trace. In particular, the matrices $A_1 = \dots = A_n = i$ and $B = j$ satisfy all of the relations, thereby giving rise to the special projective representation $\alpha = (i, i, \dots, i; j)$. On the other hand, if we assume that not all A_i commute with each other, we have an entire circle of projective representations,

$$(i, e^{i\varphi} A_2 e^{-i\varphi}, \dots, e^{i\varphi} A_n e^{-i\varphi}; j). \tag{8}$$

It is parametrized by $e^{2i\varphi} \in S^1$ because the center of SU(2) is the stabilizer of the adjoint action of SU(2) on itself. Note that two tuples like (8) are conjugate if and only if they are equal to each other. One can easily see that the formula $\psi(A_1, \dots, A_n; B) = (A_1, \dots, A_n)$ defines a surjective map

$$\psi\colon \mathcal{PR}_c(K^\natural, \mathrm{SU}(2)) \to \mathcal{R}_0(K, \mathrm{SU}(2)), \tag{9}$$

where $\mathcal{R}_0(K, \mathrm{SU}(2))$ is the space of the conjugacy classes of traceless representations $\rho_0\colon \pi_1 K \to \mathrm{SU}(2)$. If ρ_0 is irreducible, the fiber $C(\rho_0) = \psi^{-1}([\rho_0])$ is a circle of the form (8). The special projective representation α is a fiber of (9) in its own right over the unique (up to conjugation) reducible traceless representation $\pi_1 K \to H_1(K) \to \mathrm{SU}(2)$ sending all the meridians to the same traceless matrix i. Therefore, assuming that $\mathcal{R}_0(K, \mathrm{SU}(2))$ is nondegenerate, the space $\mathcal{PR}_c(K^\natural, \mathrm{SU}(2))$ consists of an isolated point and finitely many circles, one for each conjugacy class of irreducible representations in $\mathcal{R}_0(K, \mathrm{SU}(2))$. The same result holds in general after perturbation.

4.2 The action of $H^1(K^\natural;\mathbb{Z}/2)$

The group $H^1(K^\natural;\mathbb{Z}/2)=\mathbb{Z}/2\oplus\mathbb{Z}/2$ generated by the duals χ_k and χ_ℓ of the meridians of the link $k^\natural=k\cup\ell$ acts on the space of projective representations $\mathcal{PR}_c(K^\natural,\mathrm{SU}(2))$ as explained in Section 2.2. In terms of the tuples (8), the generators χ_k and χ_ℓ send $(i,e^{i\varphi}A_2e^{-i\varphi},\dots,e^{i\varphi}A_ne^{-i\varphi};j)$ to

$$(-i,-e^{i\varphi}A_2e^{-i\varphi},\dots,-e^{i\varphi}A_ne^{-i\varphi};j)\quad\text{and}\quad(i,e^{i\varphi}A_2e^{-i\varphi},\dots,e^{i\varphi}A_ne^{-i\varphi};-j),$$

respectively. The isolated point $\alpha=(i,i,\dots,i;j)$ is a fixed point of this action since

$$(-i,-i,\dots,-i;j)=j\cdot(i,i,\dots,i;j)\cdot j^{-1},\quad(i,i,\dots,i;-j)=i\cdot(i,i,\dots,i;j)\cdot i^{-1}.$$

To describe the action of χ_ℓ on the circle $C(\rho_0)$ for an irreducible ρ_0, conjugate $(i,e^{i\varphi}A_2e^{-i\varphi},\dots,e^{i\varphi}A_ne^{-i\varphi};-j)$ by i to obtain

$$(i,e^{i(\varphi+\pi/2)}A_2e^{-i(\varphi+\pi/2)},\dots,e^{i(\varphi+\pi/2)}A_ne^{-i(\varphi+\pi/2)};j).$$

Since the circle $C(\rho_0)$ is parametrized by $e^{2i\varphi}$, we conclude that the involution χ_ℓ acts on $C(\rho_0)$ via the antipodal map.

The action of χ_k on the circle $C(\rho_0)$ for an irreducible ρ_0 will depend on whether ρ_0 is a binary dihedral representation or not. Recall that a representation $\rho_0\colon\pi_1K\to\mathrm{SU}(2)$ is called *binary dihedral* if it factors through a copy of the binary dihedral subgroup $S^1\cup j\cdot S^1\subset\mathrm{SU}(2)$, where S^1 stands for the circle of unit complex numbers. Equivalently, ρ_0 is binary dihedral if its adjoint representation $\mathrm{Ad}(\rho_0)\colon\pi_1K\to\mathrm{SO}(3)$ is *dihedral* in that it factors through a copy of $O(2)$ embedded into $\mathrm{SO}(3)$ via the map $A\to(A,\det A)$.

One can show that a representation ρ_0 is binary dihedral if and only if $\chi\cdot\rho_0$ is conjugate to ρ_0, where $\chi\colon\pi_1K\to\mathbb{Z}/2$ is the generator of $H^1(K;\mathbb{Z}/2)=\mathbb{Z}/2$. Note that χ defines an involution on $\mathcal{R}_0(K,\mathrm{SU}(2))$ which makes the following diagram commute:

$$\begin{array}{ccc}\mathcal{PR}_c(K^\natural,\mathrm{SU}(2)) & \xrightarrow{\pi} & \mathcal{R}_0(K,\mathrm{SU}(2))\\ \Big\downarrow{\scriptstyle\chi_k} & & \Big\downarrow{\scriptstyle\chi}\\ \mathcal{PR}_c(K^\natural,\mathrm{SU}(2)) & \xrightarrow{\pi} & \mathcal{R}_0(K,\mathrm{SU}(2))\end{array}$$

The action of χ_k can now be described as follows. If an irreducible $\rho_0\colon\pi_1K\to\mathrm{SU}(2)$ is not binary dihedral, the involution χ_k takes the circle $C(\rho_0)$ to the circle $C(\chi\cdot\rho_0)$. Since $\chi\cdot\rho_0$ is not conjugate to ρ_0, these two circles are disjoint from each other, and χ_k permutes them. If an irreducible $\rho_0\colon\pi_1K\to\mathrm{SU}(2)$ is binary dihedral, there exists $u\in\mathrm{SU}(2)$ such that $uiu^{-1}=-i$ and $uA_iu^{-1}=-A_i$ for $i=2,\dots,n$. The irreducibility of ρ_0 also implies that $u^2=-1$, so after conjugation we may assume that $u=k$. Now conjugate

$$\chi_k\cdot(i,e^{i\varphi}A_2e^{-i\varphi},\dots,e^{i\varphi}A_ne^{-i\varphi};j)=(-i,-e^{i\varphi}A_2e^{-i\varphi},\dots,-e^{i\varphi}A_ne^{-i\varphi};j)$$

by j to obtain

$$\begin{aligned}(i,j(-e^{i\varphi}A_2e^{-i\varphi})j^{-1},\dots,j(-e^{i\varphi}A_ne^{-i\varphi})j^{-1};j)\\ &=(i,-e^{-i\varphi}jA_2j^{-1}e^{i\varphi},\dots,-e^{-i\varphi}jA_nj^{-1}e^{i\varphi};j)\\ &=(i,-(ie^{-i\varphi})kA_2k^{-1}(i^{-1}e^{i\varphi}),\dots,-(ie^{-i\varphi})kA_nk^{-1}(i^{-1}e^{i\varphi});j)\\ &=(i,e^{i(\pi/2-\varphi)}A_2e^{-i(\pi/2-\varphi)},\dots,e^{i(\pi/2-\varphi)}A_ne^{-i(\pi/2-\varphi)};j).\end{aligned}$$

Therefore, χ_k acts on $C(\rho_0)$ by sending $e^{2i\varphi}$ to $-e^{-2i\varphi}$, which is an involution on the complex unit circle with two fixed points, i and $-i$.

Finally, observe that the quotient of $\mathcal{R}_0(K,\mathrm{SU}(2))$ by the involution χ is precisely the space $\mathcal{R}_0(K,\mathrm{SO}(3))$ of the conjugacy classes of representations $\mathrm{Ad}\,\rho_0\colon \pi_1K\to\mathrm{SO}(3)$. Since $H^2(K;\mathbb{Z}/2)=0$, every $\mathrm{SO}(3)$ representation lifts to an $\mathrm{SU}(2)$ representation, hence $\mathcal{R}_0(K,\mathrm{SO}(3))$ can also be described as the space of the conjugacy classes of representations $\pi_1K\to\mathrm{SO}(3)$ sending the meridians to $\mathrm{SO}(3)$ matrices of trace -1. Compose (9) with the projection $\mathcal{R}_0(K,\mathrm{SU}(2))\to\mathcal{R}_0(K,\mathrm{SO}(3))$ to obtain a surjective map $\psi\colon \mathcal{PR}_c(K^\natural,\mathrm{SU}(2))\to\mathcal{R}_0(K,\mathrm{SO}(3))$. The above discussion can now be summarized as follows.

Proposition 4.1 *The group $H^1(K^\natural;\mathbb{Z}/2)=\mathbb{Z}/2\oplus\mathbb{Z}/2$ acts on $\mathcal{PR}_c(K^\natural,\mathrm{SU}(2))$ preserving the fibers of the map $\psi\colon \mathcal{PR}_c(K^\natural,\mathrm{SU}(2))\to\mathcal{R}_0(K,\mathrm{SO}(3))$. Furthermore:*

(a) *For the unique reducible in $\mathcal{R}_0(K,\mathrm{SO}(3))$, the fiber of ψ consists of just one point, which is the conjugacy class of the special projective representation α. This point is fixed by both χ_k and χ_ℓ.*

(b) *For any dihedral representation in $\mathcal{R}_0(K,\mathrm{SO}(3))$, the fiber of ψ is a circle. The involution χ_k is a reflection of this circle with two fixed points, while χ_ℓ is the antipodal map.*

(c) *Otherwise, the fiber of ψ consists of two circles. The involution χ_k permutes these circles, while χ_ℓ acts as the antipodal map on both.*

It should be noted that perturbing the Chern–Simons functional (1) may easily break the $\mathbb{Z}/2\oplus\mathbb{Z}/2$ symmetry. Finding a perturbation which preserves this symmetry runs as usual into the equivariant transversality problem, which we do not try to address here. It should be noted, however, that such a problem was successfully solved in [33] in a similar setting.

4.3 Double branched covers

Next, we would like to describe the space $\mathcal{PR}_c(K^\natural, \mathrm{SU}(2))$ using the equivariant theory of Section 3. We could proceed as in that section, by passing to the double branched cover $M \to S^3$ with branch set the link $k^\natural$ and working with the equivariant representations $\pi_1 M \to \mathrm{SO}(3)$. However, in the special case at hand, one can observe that M is simply the connected sum $Y \# \mathbb{R}\mathrm{P}^3$, where Y is the double branched cover of S^3 with branch set the knot k, hence the same information about $\mathcal{PR}_c(K^\natural, \mathrm{SU}(2))$ can be extracted more easily by working directly with Y and using Proposition 4.1. The only missing step in this program is a description of $\mathcal{R}_0(K, \mathrm{SO}(3))$ in terms of equivariant representations $\pi_1 Y \to \mathrm{SO}(3)$, which we will take up next.

Every representation $\rho\colon \pi_1 K \to \mathrm{SO}(3)$ gives rise to a representation of the orbifold fundamental group $\pi_1^V(S^3, k) = \pi_1 K / \langle \mu^2 = 1\rangle$, where we choose $\mu = a_1$ to be our meridian. The latter group can be included into the split orbifold exact sequence

$$1 \to \pi_1 Y \xrightarrow{\pi_*} \pi_1^V(S^3, k) \xrightarrow{j} \mathbb{Z}/2 \to 1.$$

Proposition 4.2 *Let Y be the double branched cover of S^3 with branch set a knot k and let $\tau\colon Y \to Y$ be the covering translation. The pullback of representations via the map π_* in the orbifold exact sequence establishes a homeomorphism*

$$\pi^*\colon \mathcal{R}_0(K, \mathrm{SO}(3)) \to \mathcal{R}^\tau(Y, \mathrm{SO}(3)),$$

where $\mathcal{R}^\tau(Y)$ is the fixed point set of the involution $\tau^*\colon \mathcal{R}(Y, \mathrm{SO}(3)) \to \mathcal{R}(Y, \mathrm{SO}(3))$. The unique reducible representation in $\mathcal{R}_0(K, \mathrm{SO}(3))$ pulls back to the trivial representation of $\pi_1 Y$, and the dihedral representations in $\mathcal{R}_0(K, \mathrm{SO}(3))$ are precisely those that pull back to reducible representations of $\pi_1 Y$.

Proof A slight modification of the argument of Proposition 3.8 (see also [11, Proposition 3.3]), establishes a homeomorphism between $\mathcal{R}_0(K, \mathrm{SO}(3))$ and the subspace of $\mathcal{R}^\tau(Y, \mathrm{SO}(3))$ consisting of the conjugacy classes of representations $\beta\colon \pi_1 Y \to \mathrm{SO}(3)$ such that $\tau^*\beta = u \cdot \beta \cdot u^{-1}$ for some $u \in \mathrm{SO}(3)$ of order 2. The proof of the first statement of the proposition will be complete after we show that this subspace in fact comprises the entire space $\mathcal{R}^\tau(Y, \mathrm{SO}(3))$.

If $\beta\colon \pi_1 Y \to \mathrm{SO}(3)$ is reducible, it factors through a representation $H_1(Y) \to \mathrm{SO}(2)$. According to Proposition 3.1, the involution τ_* acts on $H_1(Y)$ as multiplication by -1. Therefore, $\tau^*\beta = \beta^{-1}$, and the latter representation can obviously be conjugated to β by an element $u \in \mathrm{SO}(3)$ of order 2. If $\beta\colon \pi_1 Y \to \mathrm{SO}(3)$ is irreducible, the condition $\beta \in \mathrm{Fix}(\tau^*)$ implies that there exists a unique $u \in \mathrm{SO}(3)$ such that $\tau^*\beta = u \cdot \beta \cdot u^{-1}$ and $u^2 = 1$. If $u = 1$, then $\tau^*\beta = \beta$, which implies that β is the pullback of a representation of $\pi_1^V(S^3,k)$ which sends the meridian μ to the identity matrix and hence factors through $\pi_1 S^3 = 1$. This contradicts the irreducibility of β.

To prove the second statement of the proposition, observe that the homomorphism j in the above orbifold exact sequence sending μ to the generator of $\mathbb{Z}/2$ is in fact the abelianization homomorphism. This implies that the unique reducible representation in $\mathcal{R}_0(K,\mathrm{SO}(3))$ pulls back to the trivial representation of $\pi_1 Y$. Since $\pi_1 Y$ is the commutator subgroup of $\pi_1^V(S^3,k)$, any dihedral representation $\rho\colon \pi_1^V(S^3,k)\to O(2)$ must map $\pi_1 Y$ to the commutator subgroup of $O(2)$, which happens to be $\mathrm{SO}(2)$. This ensures that the pullback of ρ is reducible. Conversely, if the pullback of ρ is reducible, its image is contained in a copy of $\mathrm{SO}(2)$, and the image of ρ itself in its 2–prime extension. The latter group is of course just a copy of $O(2)\subset \mathrm{SO}(3)$. □

Remark 4.3 For future use note that, for any projective representation $\rho\colon \pi_1 K^\natural \to \mathrm{SU}(2)$ in $C(\rho_0)$ described by a tuple (8), the adjoint representation $\mathrm{Ad}\,\rho\colon \pi_1 K^\natural\to \mathrm{SO}(3)$ pulls back to an $\mathrm{SO}(3)$ representation of $\pi_1(Y\,\#\,\mathbb{RP}^3) = \pi_1 Y * \mathbb{Z}/2$ of the form $\beta * \gamma\colon \pi_1 Y * \mathbb{Z}/2 \to \mathrm{SO}(3)$, where $\beta = \pi^*\,\mathrm{Ad}\,\rho_0$ and $\gamma\colon \mathbb{Z}/2\to\mathrm{SO}(3)$ sends the generator of $\mathbb{Z}/2$ to $\mathrm{Ad}\,i\cdot\mathrm{Ad}\,j = \mathrm{Ad}\,k$. The representation $\beta*\gamma$ is equivariant, $\tau^*(\beta*\gamma) = u\cdot(\beta*\gamma)\cdot u^{-1}$, with the conjugating element u given by $\mathrm{Ad}\,\rho_0(a_1) = \mathrm{Ad}\,i$.

5 Knot homology: grading of the special generator

Given a knot $k\subset S^3$, we will continue using the notation K for its exterior and $K^\natural$ for the exterior of the two-component link $k^\natural = k\cup \ell$ obtained as the connected sum of k with the Hopf link h. The special projective representation $\alpha\colon \pi_1 K^\natural \to \mathrm{SU}(2)$, which sends all the meridians of k to i and the meridian of ℓ to j, is a generator in the chain complex $IC^\natural(k)$. In this section, we compute its absolute Floer grading.

Theorem 5.1 *For any knot k in S^3, we have* $\mathrm{gr}(\alpha) = \mathrm{sign}\,k \pmod 4$.

Before we go on to prove this theorem, recall the definition of $\mathrm{gr}(\alpha) \pmod 4$. Let (W',S) be a cobordism of pairs (S^3,u) and (S^3,k), where u is an unknot in S^3. The manifold W' is required to be oriented but the surface S is not. Construct a new cobordism (W',S') of the pairs (S^3,h) and $(S^3,k^\natural)$ by letting S' be the disjoint union of S with the normal circle bundle along a path in S connecting the two boundary components (the surface S' is called $S^\natural$ in [24, Section 4.3]). According to [24, Proposition 4.4], the generator α has grading

(10) $$\mathrm{gr}(\alpha) = -\mathrm{ind}\,\mathcal{D}_{A'}(\alpha_u,\alpha) - \tfrac{3}{2}(\chi(W') + \sigma(W')) - \chi(S') \pmod 4,$$

where α_u stands for the special generator in the Floer chain complex of u, and we use the fact that $\chi(S) = \chi(S')$. The operator $\mathcal{D}_{A'}(\alpha_u,\alpha)$ refers to the ASD operator on the noncompact manifold obtained from W' by attaching cylindrical ends to the two boundary components; this manifold is again called W'. The connection A' can be any connection on W' which is singular along the surface S' and whose limits on the two ends are flat connections with holonomies α_u and α. The index of $\mathcal{D}_{A'}(\alpha_u, \alpha)$ is understood as the L^2_δ index for a small positive δ.

5.1 Constructing the cobordism

Our calculation of the Floer index $\operatorname{gr}(\alpha)$ will use a specific cobordism (W', S') constructed as follows.

Let Σ be the double branched cover of S^3 with branch set the knot k. Choose a Seifert surface F' of k and push its interior slightly into the ball D^4 so that the resulting surface, which we still call F', is transverse to $\partial D^4 = S^3$. Let V be the double branched cover of D^4 with branch set the surface F'. Then V is a smooth simply connected spin 4–manifold with boundary Σ, which admits a handle decomposition with only 0– and 2–handles; see Akbulut and Kirby [1, page 113].

Next, choose a point in the interior of the surface $F' \subset D^4$. Excising a small open 4–ball containing that point from (D^4, F') results in a manifold W_1' diffeomorphic to $I \times S^3$ together with the surface $F_1' = F' - \operatorname{int} D^2$ properly embedded into it, thereby providing a cobordism (W_1', F_1') from an unknot to the knot k. The double branched cover $W_1 \to W_1'$ with branch set F_1' is a cobordism from S^3 to Σ. The manifold W_1 is simply connected because it can be obtained from the simply connected manifold V by excising an open 4–ball.

Similarly, consider the manifold $W_2' = I \times S^3$ and surface $F_2' = I \times h \subset W_2'$ providing a product cobordism from the Hopf link h to itself. The double branched cover $W_2 \to W_2'$ with branch set F_2' is then a cobordism $W_2 = I \times \mathbb{R}\mathrm{P}^3$ from $\mathbb{R}\mathrm{P}^3$ to itself.

As the final step of the construction, consider a path γ_1' in the surface F_1' connecting its two boundary components. Similarly, consider a path γ_2' of the form $I \times \{p\}$ in the surface $F_2' = I \times H$. Remove tubular neighborhoods of these two paths and glue the resulting manifolds and surfaces together using an orientation-reversing diffeomorphism $1 \times h\colon I \times S^2 \to I \times S^2$. The resulting pair (W', S') is the desired cobordism of the pairs (S^3, h) and $(S^3, k^\natural)$. One can easily see that

$$\chi(W') = \sigma(W') = 0 \quad \text{and} \quad \chi(S') = \chi(F') - 1. \tag{11}$$

Note that the double branched cover $W \to W'$ with branch set S' is a cobordism from $\mathbb{R}\mathrm{P}^3$ to $\Sigma \# \mathbb{R}\mathrm{P}^3$ which can be obtained from the cobordisms W_1 and W_2 by taking a connected sum along the paths $\gamma_1 \subset W_1$ and $\gamma_2 \subset W_2$ lifting, respectively, the paths γ_1' and γ_2'. To be precise,

$$W = W_1^\circ \cup W_2^\circ, \tag{12}$$

where W_1° and W_2° are obtained from W_1 and W_2 by removing tubular neighborhoods of γ_1 and γ_2. The identification in (12) is done along a copy of $I \times S^2$. In particular, we see that $\pi_1 W = \mathbb{Z}/2$.

5.2 L^2–index

We will rely on Ruberman [32] and Taubes [40] in our index calculations.

Let $\pi\colon W \to W'$ be the double branched cover with branch set S' constructed in the previous section, and $\tau\colon W \to W$ the covering translation. Let us consider a representation $\rho\colon \pi_1^V(W', S') \to \mathrm{SO}(3)$ sending the two sets of meridians of S' to $\mathrm{Ad}\, i$ and $\mathrm{Ad}\, j$. Then the representation $\pi^*\rho\colon \pi_1 W \to \mathrm{SO}(3)$ sends the generator of $\pi_1 W$ to $\mathrm{Ad}\, k$ and it is equivariant in that $\tau^*(\pi^*\rho) = u \cdot \pi^*\rho \cdot u^{-1}$ with $u = \mathrm{Ad}\, i$; compare with Example 3.7. The representation ρ restricts to α_u and α over the two ends of W', therefore $\pi^*\rho$ makes W into a flat cobordism between $\gamma\colon \pi_1(\mathbb{R}\mathrm{P}^3) \to \mathrm{SO}(3)$ and $\theta * \gamma\colon \pi_1 \Sigma * \pi_1(\mathbb{R}\mathrm{P}^3) \to \mathrm{SO}(3)$, where γ is the representation of Example 3.7.

Let A and A' be flat connections on W and W' whose holonomies are, respectively, $\pi^*\rho$ and ρ. We will use A' as the twisting connection of the operator $\mathcal{D}_{A'}(\alpha_u, \alpha)$. Instead of computing the index of this operator, we will compute the equivariant index $\mathrm{ind}\, \mathcal{D}_A^\tau(\gamma, \theta * \gamma)$ of its pullback to W. The latter index equals minus the equivariant index of the elliptic complex

$$\Omega^0(W, \mathrm{ad}\, P) \xrightarrow{-d_A} \Omega^1(W, \mathrm{ad}\, P) \xrightarrow{d_A^+} \Omega^2_+(W, \mathrm{ad}\, P). \tag{13}$$

The equivariance here is understood with respect to a lift of $\tau\colon W \to W$ to the bundle $\mathrm{ad}\, P$ which has second order on the fibers over the fixed point set. The connection A is equivariant with respect to this lift, hence it splits the coefficient bundle $\mathrm{ad}\, P$ into a sum of three real line bundles corresponding to $\mathrm{Ad}\, k = \mathrm{diag}(-1, -1, 1)$. Accordingly, the complex (13) splits into a sum of three elliptic complexes, one with the trivial real coefficients and two with the twisted coefficients. Application of [32, Proposition 4.1] to the former complex and of [32, Corollary 4.2] to the latter two reduces the index problem to computing the singular cohomology

$$H^k(W; \mathrm{ad}\, \pi^*\rho) = H^k(W; \mathbb{R}) \oplus H^k(W; \mathbb{R}_-) \oplus H^k(W; \mathbb{R}_-) \quad \text{for } k = 0, 1, 2,$$

where $\mathbb{R}_-$ stands for the real line coefficients on which $\mathbb{Z}/2$ acts as multiplication by -1, and their equivariant versions.

The zeroth equivariant cohomology of the complex (13) vanishes since $H^0(W;\mathbb{R}_-)=0$ and the lift of τ acts as minus identity on the remaining group $H^0(W;\mathbb{R})=\mathbb{R}$. This vanishing result could also be derived directly from the irreducibility of the singular connection A'. The next two subsections are dedicated to computing the first and second cohomology of (13).

5.3 Trivial coefficients

Our computation will be based on the Mayer–Vietoris exact sequence applied twice, first to compute cohomology of W_1° and W_2°, and then to compute cohomology of $W = W_1^\circ \cup W_2^\circ$.

The cohomology groups of W_1° and $W_1 = W_1^\circ \cup (I \times D^3)$ are related by the following Mayer–Vietoris exact sequence:

$$\begin{aligned} 0 \longrightarrow H^1(W_1;\mathbb{R}) &\longrightarrow H^1(W_1^\circ;\mathbb{R}) \longrightarrow 0 \\ \longrightarrow H^2(W_1;\mathbb{R}) &\longrightarrow H^2(W_1^\circ;\mathbb{R}) \longrightarrow H^2(I\times S^2;\mathbb{R}) \\ \xrightarrow{\delta} H^3(W_1;\mathbb{R}) &\longrightarrow H^3(W_1^\circ;\mathbb{R}) \longrightarrow 0 \end{aligned}$$

Since W_1 and therefore W_1° are simply connected, both $H^1(W_1;\mathbb{R})$ and $H^1(W_1^\circ;\mathbb{R})$ vanish. Applying the Poincaré–Lefschetz duality to the manifold W_1 and using the long exact sequence of the pair $(W_1,\partial W_1)$, we obtain

$$H^3(W_1;\mathbb{R}) = H_1(W_1,\partial W_1;\mathbb{R}) = \widetilde{H}_0(\partial W_1;\mathbb{R}) = \mathbb{R}.$$

Similarly, viewing W_1° as a manifold whose boundary is a connected sum of the two boundary components of W_1, we obtain

$$H^3(W_1^\circ;\mathbb{R}) = H_1(W_1^\circ,\partial W_1^\circ;\mathbb{R}) = \widetilde{H}_0(\partial W_1^\circ;\mathbb{R}) = 0.$$

Therefore, the connecting homomorphism δ in the above exact sequence must be an isomorphism, which leads to the isomorphisms

$$H^2(W_1^\circ;\mathbb{R}) = H^2(W_1;\mathbb{R}) = H^2(V;\mathbb{R}).$$

A similar long exact sequence relates the cohomology of W_2° and $W_2 = W_2^\circ \cup (I\times D^3)$, implying that

$$H^2(W_2^\circ;\mathbb{R}) = H^2(W_2;\mathbb{R}) = H^2(\mathbb{R}\mathrm{P}^3;\mathbb{R}) = 0.$$

Since $\pi_1 W_2 = \pi_1 W_2^\circ = \mathbb{Z}/2$, both $H^1(W_2;\mathbb{R})$ and $H^1(W_2^\circ;\mathbb{R})$ vanish. The Mayer–Vietoris exact sequence of the splitting $W = W_1^\circ \cup W_2^\circ$,

$$\begin{aligned} 0 \longrightarrow H^1(W;\mathbb{R}) &\longrightarrow H^1(W_1^\circ;\mathbb{R}) \oplus H^1(W_2^\circ;\mathbb{R}) \longrightarrow 0 \\ \longrightarrow H^2(W;\mathbb{R}) &\longrightarrow H^2(W_1^\circ;\mathbb{R}) \oplus H^2(W_2^\circ;\mathbb{R}) \to H^2(I \times S^2;\mathbb{R}) \\ \longrightarrow H^3(W;\mathbb{R}) &\longrightarrow H^3(W_1^\circ;\mathbb{R}) \oplus H^3(W_2^\circ;\mathbb{R}) \longrightarrow 0 \end{aligned}$$

together with the isomorphisms $H^3(W;\mathbb{R}) = H_1(W,\partial W;\mathbb{R}) = \widetilde{H}_0(\partial W;\mathbb{R}) = \mathbb{R}$ and $\pi_1 W = \mathbb{Z}/2$, implies that

$$H^1(W;\mathbb{R}) = 0 \quad \text{and} \quad H^2(W;\mathbb{R}) = H^2(V;\mathbb{R}).$$

5.4 Twisted coefficients

We will now do a similar calculation using the Mayer–Vietoris sequence of $W = W_1^\circ \cup W_2^\circ$ with twisted coefficients. Since W_1° is simply connected, the twisted coefficients $\mathbb{R}_-$ pull back to the trivial $\mathbb{R}$–coefficients over W_1° and the cohomology calculations from the previous section are unchanged. A direct calculation using homotopy equivalences $W_2 \simeq \mathbb{R}\mathrm{P}^3$ and $W_2^\circ \simeq \mathbb{R}\mathrm{P}^2$ shows that

$$H^1(W_2^\circ;\mathbb{R}_-) = 0 \quad \text{and} \quad H^2(W_2^\circ;\mathbb{R}_-) = \mathbb{R}.$$

The latter isomorphism is induced by the inclusion $I \times S^2 \to W_2^\circ$, which can be easily seen from the Mayer–Vietoris exact sequence of $W_2 = W_2^\circ \cup (I \times D^3)$. Now, consider the Mayer–Vietoris exact sequence of the splitting $W = W_1^\circ \cup W_2^\circ$ with twisted $\mathbb{R}$–coefficients:

$$\begin{aligned} 0 \longrightarrow H^1(W;\mathbb{R}_-) &\longrightarrow H^1(W_1^\circ;\mathbb{R}) \oplus H^1(W_2^\circ;\mathbb{R}_-) \longrightarrow 0 \\ \longrightarrow H^2(W;\mathbb{R}_-) &\longrightarrow H^2(W_1^\circ;\mathbb{R}) \oplus H^2(W_2^\circ;\mathbb{R}_-) \to H^2(I \times S^2;\mathbb{R}) \\ \longrightarrow H^3(W;\mathbb{R}_-) &\longrightarrow H^3(W_1^\circ;\mathbb{R}) \oplus H^3(W_2^\circ;\mathbb{R}_-) \longrightarrow 0 \end{aligned}$$

Keeping in mind that the map $H^2(W_1^\circ;\mathbb{R}) \to H^2(I \times S^2;\mathbb{R})$ in this sequence is zero and the map $H^2(W_2^\circ;\mathbb{R}_-) \to H^2(I \times S^2;\mathbb{R})$ is an isomorphism $\mathbb{R} \to \mathbb{R}$, we conclude that

$$H^1(W;\mathbb{R}_-) = 0 \quad \text{and} \quad H^2(W;\mathbb{R}_-) = H^2(V;\mathbb{R}).$$

5.5 Equivariant cohomology

Combining results of the previous two sections, we obtain $H^1(W;\mathrm{ad}\,P) = 0$ and $H^2(W;\mathrm{ad}\,P) = H^2(V;\mathbb{R}^3)$. The action of τ is compatible with these isomorphisms, from which we immediately conclude that

$$H^1_\tau(W;\operatorname{ad}P)=0$$

and $H^2_\tau(W;\operatorname{ad}P)$ is the fixed point set of the map $H^2(V;\mathbb{R}^3)\to H^2(V;\mathbb{R}^3)$ obtained by twisting $\tau^*\colon H^2(V;\mathbb{R})\to H^2(V;\mathbb{R})$ by the action on the coefficients $\mathbb{R}^3\to\mathbb{R}^3$. The involution τ^* is minus the identity, which follows from the usual transfer argument applied to the covering $V\to D^4$, while the action on the coefficients is given by an SO(3) operator of second order. Such an operator must have a single eigenvalue 1 and a double eigenvalue -1, which leads us to the conclusion that $\operatorname{rk}H^2_\tau(W;\operatorname{ad}P)=2\cdot b_2(V)$. Similarly,

$$\operatorname{rk}H^2_{\tau,+}(W;\operatorname{ad}P)=2\cdot b_2^+(V).$$

5.6 Proof of Theorem 5.1

It follows from the discussion in Section 5.2 and the calculation in Section 5.5 that

$$\operatorname{ind}\mathcal{D}_{A'}(\alpha_u,\alpha)=\operatorname{rk}H^1_\tau(W;\operatorname{ad}P)-\operatorname{rk}H^2_{+,\tau}(W;\operatorname{ad}P)=-2\cdot b_2^+(V).$$

Taking into account (10) and (11), we obtain the formula

$$\operatorname{gr}(\alpha)=2\cdot b_2^+(V)-\chi(F')+1\pmod 4.$$

To simplify it, let us compute $\chi(V)$ in two different ways: $\chi(V)=1+b_2^+(V)+b_2^-(V)$ by definition, and $\chi(V)=2\chi(D^4)-\chi(F')=2-\chi(F')$ using the fact that V is a double branched cover of D^4 with branch set F'. Combining these formulas with the knot signature formula of Viro [41], we obtain the desired result (remember that $\operatorname{sign}k$ is always even):

$$\operatorname{gr}(\alpha)=-\operatorname{sign}V=-\operatorname{sign}k=\operatorname{sign}k\pmod 4.$$

6 Knot homology: gradings of other generators

Proposition 4.1 identified the critical points of the Chern–Simons functional with the fibers of the map $\psi\colon\mathcal{PR}_c(K^\natural,\mathrm{SU}(2))\to\mathcal{R}_0(K,\mathrm{SO}(3))$. Assuming that the space $\mathcal{R}_0(K,\mathrm{SO}(3))$ is nondegenerate, all of these fibers (with the exception of the special generator α) are Morse–Bott circles. In this section, we will compute their Floer gradings using the equivariant index theory of Section 3.5. The actual generators of the chain complex $IC^\natural(k)$ are then obtained by perturbing each Morse–Bott circle of index μ into two points of indices μ and $\mu+1$, as in [20]. Our index calculation will depend on whether an irreducible trace-free representation $\rho_0\colon\pi_1K\to\mathrm{SO}(3)$ giving rise to the Morse–Bott circle $C(\rho_0)$ is dihedral or not. The two cases will be considered separately, starting with the case when ρ_0 is dihedral. If $\mathcal{R}_0(K, \mathrm{SO}(3))$ fails to be nondegenerate, similar results hold after additional perturbations.

6.1 Dihedral representations

Let $\rho_0\colon \pi_1 K \to \mathrm{SO}(3)$ be an irreducible trace-free dihedral representation. The pullback via $\pi\colon M \to \Sigma$ identifies the Morse–Bott circle $C(\rho_0)$ with the circle of the conjugacy classes of equivariant representations of the form $\beta * \gamma\colon \pi_1 Y * \mathbb{Z}/2 \to \mathrm{SO}(3)$, where β is a nontrivial reducible representation of $\pi_1 Y$ and γ is the representation of $\mathbb{Z}/2$ sending the generator to $\mathrm{Ad}\, k$. These representations are equivariant in that $\tau^*(\beta * \gamma) = u \cdot (\beta * \gamma) \cdot u^{-1}$ with $u = \mathrm{Ad}\, i$; see Remark 4.3.

We wish to compute the equivariant index $\mathrm{ind}\, \mathcal{D}^\tau_A(\beta * \gamma, \theta * \gamma)$, where A is any equivariant connection on the cylinder $\mathbb{R} \times (Y \# \mathbb{R}\mathrm{P}^3)$ whose limits are the flat connections $\beta * \gamma$ and $\theta * \gamma$ over the negative and positive ends, respectively. The Morse–Bott index of the circle corresponding to $\beta * \gamma$ will then equal

$$\mu = \mathrm{ind}\, \mathcal{D}^\tau_A(\beta * \gamma, \theta * \gamma) + \mathrm{sign}\, k \pmod 4. \tag{14}$$

Proposition 6.1 *Let $\beta\colon \pi_1 Y \to \mathrm{SO}(3)$ be a nontrivial equivariant reducible representation. Then for any equivariant connection B on the cylinder $\mathbb{R} \times Y$ whose limits are the flat connections β and θ over the negative and positive ends,*

$$\mathrm{ind}\, \mathcal{D}^\tau_A(\beta * \gamma, \theta * \gamma) = \mathrm{ind}\, \mathcal{D}^\tau_B(\beta, \theta) \pmod 4.$$

Proof To compute the index on the left-hand side of this formula, we will apply the formula of Proposition 3.11 to the manifold $X = \mathbb{R} \times (Y \# \mathbb{R}\mathrm{P}^3)$ with two product ends. Since the metric on X is a product metric, the terms $p_1(TX)$ and $e(TX)$ in the integrand

$$\widehat{A}(X)\, \mathrm{ch}(S^+)\, \mathrm{ch}(\mathrm{ad}\, P)_{\mathbb{C}} = -2 p_1(A) - \tfrac{1}{2} p_1(TX) - \tfrac{3}{2} e(TX)$$

will vanish, as will the topological terms $\chi(F)$ and $F \cdot F$, leading to the formula

$$\begin{aligned} \mathrm{ind}\, \mathcal{D}^\tau_A(\beta * \gamma, \theta * \gamma) = -\int_X p_1(A) &- \tfrac{1}{4}(h_{\theta*\gamma} - \rho_{\theta*\gamma}) - \tfrac{1}{4}(h_{\beta*\gamma} + \rho_{\beta*\gamma}) \\ &- \tfrac{1}{4}(h^\tau_{\theta*\gamma} - \rho^\tau_{\theta*\gamma}) - \tfrac{1}{4}(h^\tau_{\beta*\gamma} + \rho^\tau_{\beta*\gamma}), \end{aligned}$$

where $\rho_{\beta*\gamma} = \eta_{\beta*\gamma}(0) - \eta_\theta(0)$ and $\rho^\tau_{\beta*\gamma} = \eta^\tau_{\beta*\gamma}(0) - \eta^\tau_\theta(0)$ are ρ–invariants of the manifold $Y \# \mathbb{R}\mathrm{P}^3$.

The connection A in this formula is any equivariant connection whose limits are the flat connections $\beta * \gamma$ and $\theta * \gamma$ at the two ends of X, hence we are free to choose A

to equal γ over $\mathbb{R}\times(\mathbb{R}\mathrm{P}^3-D^3)$ and to be trivial in the gluing region. This determines the integral term in the above formula as follows:

$$\int_X p_1(A)=\int_{\mathbb{R}\times Y} p_1(A).$$

To evaluate the ρ–invariants, build a cobordism W from the disjoint union $Y\cup\mathbb{R}\mathrm{P}^3$ to the connected sum $Y\#\mathbb{R}\mathrm{P}^3$ by attaching a 1–handle to $[0,1]\times(Y\cup\mathbb{R}\mathrm{P}^3)$. The flat connection $\beta*\gamma$ extends to W making it into a flat cobordism from $(Y,\beta)\cup(\mathbb{R}\mathrm{P}^3,\gamma)$ to $(Y\#\mathbb{R}\mathrm{P}^3,\beta*\gamma)$. It then follows from [5, Theorem 2.4] that

$$\rho_{\beta*\gamma}-\rho_\beta-\rho_\gamma=\operatorname{sign}_{\beta*\gamma}W-3\operatorname{sign}W,$$

where ρ_β and ρ_γ are ρ–invariants of the manifolds Y and $\mathbb{R}\mathrm{P}^3$, respectively. One can easily see from the description of W that both signature terms in the above formula vanish, implying that $\rho_{\beta*\gamma}=\rho_\beta+\rho_\gamma$. Since the involution τ extends to W, a similar argument using the index theorem of Donnelly [12] instead of [5, Theorem 2.4] shows that $\rho^\tau_{\beta*\gamma}=\rho^\tau_\beta+\rho^\tau_\gamma$. Similar formulas also hold with $\theta*\gamma$ in place of $\beta*\gamma$.

Plugging all of this back into the above index formula and keeping in mind that $\rho_\theta=\rho^\tau_\theta=0$, we obtain

$$\begin{aligned}\operatorname{ind}\mathcal{D}^\tau_A(\beta*\gamma,\theta*\gamma)=-\int_{\mathbb{R}\times Y}&p_1(A)\\&-\tfrac14(h_{\beta*\gamma}+\rho_\beta)-\tfrac14 h_{\theta*\gamma}-\tfrac14(h^\tau_{\beta*\gamma}+\rho^\tau_\beta)-\tfrac14 h^\tau_{\theta*\gamma}.\end{aligned}$$

On the other hand, one can apply the formula of Proposition 3.11 to the manifold $X=\mathbb{R}\times Y$ to obtain

$$\operatorname{ind}\mathcal{D}^\tau_A(\beta,\theta)=-\int_{\mathbb{R}\times Y}p_1(A)-\tfrac14(h_\beta+\rho_\beta)-\tfrac14 h_\theta-\tfrac14(h^\tau_\beta+\rho^\tau_\beta)-\tfrac14 h^\tau_\theta.$$

Therefore,

$$\begin{aligned}\operatorname{ind}\mathcal{D}^\tau_A(\beta*\gamma,\theta*\gamma)-\operatorname{ind}\mathcal{D}^\tau_A(\beta,\theta)&=-\tfrac14(h_{\beta*\gamma}-h_\beta)-\tfrac14(h_{\theta*\gamma}-h_\theta)\\&\quad-\tfrac14(h^\tau_{\beta*\gamma}-h^\tau_\beta)-\tfrac14(h^\tau_{\theta*\gamma}-h^\tau_\theta),\end{aligned}$$

and the proof of the proposition reduces to a calculation with twisted cohomology.

Since Y is a rational homology sphere, $H^1(Y;\operatorname{ad}\theta)=0$, which implies that

$$h_\theta=\dim H^0(Y;\operatorname{ad}\theta)=3\quad\text{and}\quad h^\tau_\theta=\operatorname{tr}(\operatorname{Ad}u)=-1.$$

It follows from a calculation in Section 5 that $H^1(Y\#\mathbb{R}\mathrm{P}^3;\operatorname{ad}(\theta*\gamma))=0$. Therefore, $h_{\theta*\gamma}=\dim H^0(Y;\operatorname{ad}(\theta*\gamma))=1$ because $H^0(Y;\operatorname{ad}(\theta*\gamma))$ is the $(+1)$–eigenspace of $\operatorname{Ad}(k)\colon \mathfrak{so}(3) \to \mathfrak{so}(3)$. The operator $\operatorname{Ad} i$ acts as minus identity on the $(+1)$–eigenspace of $\operatorname{Ad} k$, making $h^{\tau}_{\theta*\gamma} = -1$.

The calculation with $\beta * \gamma$ will rely on the Mayer–Vietoris exact sequence of the splitting $Y \,\#\, \mathbb{R}\mathrm{P}^3 = Y_0 \cup \mathbb{R}\mathrm{P}^3_0$ with twisted coefficients:

$$0 \to H^0(Y \,\#\, \mathbb{R}\mathrm{P}^3; \operatorname{ad}(\beta * \gamma)) \to H^0(Y; \operatorname{ad}\beta) \oplus H^0(\mathbb{R}\mathrm{P}^3; \operatorname{ad}\gamma) \\ \to H^0(S^2; \operatorname{ad}\theta) \to H^1(Y \,\#\, \mathbb{R}\mathrm{P}^3; \operatorname{ad}(\beta * \gamma)) \\ \to H^1(Y; \operatorname{ad}\beta) \oplus H^1(\mathbb{R}\mathrm{P}^3; \operatorname{ad}\gamma) \to 0.$$

Since β is reducible but nontrivial, $H^0(Y; \operatorname{ad}\beta) = \mathbb{R}$. Therefore, keeping in mind that $H^0(S^2; \operatorname{ad}\theta) = \mathbb{R}^3$, $H^0(\mathbb{R}\mathrm{P}^3; \operatorname{ad}\gamma) = \mathbb{R}$, and $H^1(\mathbb{R}\mathrm{P}^3; \operatorname{ad}\gamma) = 0$, we obtain

$$h_{\beta*\gamma} - h_{\beta} = 2 \cdot \dim H^0(Y \,\#\, \mathbb{R}\mathrm{P}^3; \operatorname{ad}(\beta * \gamma)).$$

The involution τ induces involutions $\tilde{\tau}^*$ on each of the groups in the Mayer–Vietoris exact sequence comprising a chain map. Keeping in mind that the traces of $\tilde{\tau}^*$ are equal to -1 on both $H^0(S^2; \operatorname{ad}\theta) = \mathbb{R}^3$ and $H^0(\mathbb{R}\mathrm{P}^3; \operatorname{ad}\gamma) = \mathbb{R}$, we obtain

$$h^{\tau}_{\beta*\gamma} - h^{\tau}_{\beta} = 2\operatorname{tr}\bigl(\tilde{\tau}^* \,|\, H^0(Y \,\#\, \mathbb{R}\mathrm{P}^3; \operatorname{ad}(\beta * \gamma))\bigr) - 2\operatorname{tr}\bigl(\tilde{\tau}^* \,|\, H^0(Y; \operatorname{ad}\beta)\bigr).$$

Even though both β and γ are reducible, the representation $\beta * \gamma$ may be either reducible or irreducible. In the former case, $H^0(Y \,\#\, \mathbb{R}\mathrm{P}^3; \operatorname{ad}(\beta * \gamma)) = \mathbb{R}$ is the $(+1)$–eigenspace of the operator $\operatorname{Ad} k\colon \mathfrak{so}(3) \to \mathfrak{so}(3)$ on which $\tilde{\tau}^*$ acts as minus identity, therefore

$$h_{\beta*\gamma} - h_{\beta} = 2 \quad \text{and} \quad h^{\tau}_{\beta*\gamma} - h^{\tau}_{\beta} = 0.$$

In the latter case, $H^0(Y \,\#\, \mathbb{R}\mathrm{P}^3; \operatorname{ad}(\beta * \gamma)) = 0$, therefore

$$h_{\beta*\gamma} - h_{\beta} = 0 \quad \text{and} \quad h^{\tau}_{\beta*\gamma} - h^{\tau}_{\beta} = 2.$$

In both cases, we conclude that

$$\operatorname{ind} \mathcal{D}^{\tau}_{A}(\beta * \gamma, \theta * \gamma) = \operatorname{ind} \mathcal{D}^{\tau}_{A}(\beta, \theta).$$

The result now follows from the fact that $\operatorname{ind} \mathcal{D}^{\tau}_{A}(\beta, \theta) = \operatorname{ind} \mathcal{D}^{\tau}_{B}(\beta, \theta) \pmod 4$ for any choice of connections A and B on the cylinder $\mathbb{R} \times Y$ whose limits are β and θ over the negative and positive ends. □

Remark 6.2 The formula of Proposition 6.1 holds as well for equivariant irreducible representations β, the proof requiring just minor adjustments.

Combining Proposition 6.1 with formula (14), we obtain the following formula for the Floer grading.

Corollary 6.3 *Let $\beta\colon \pi_1 Y \to \mathrm{SO}(3)$ be a nontrivial equivariant reducible representation. Then the Floer grading of the Morse–Bott circle arising from $\beta * \gamma$ is given by*

$$\mu = \operatorname{ind} \mathcal{D}^{\tau}_{B}(\beta,\theta) + \operatorname{sign} k \pmod 4,$$

where B is an arbitrary equivariant connection on the infinite cylinder $\mathbb{R}\times Y$ whose limits are β and θ over the negative and positive ends.

The index $\operatorname{ind}\mathcal{D}^{\tau}_{B}(\beta,\theta)$ in the above corollary can be computed using the formula

$$\operatorname{ind}\mathcal{D}^{\tau}_{B}(\beta,\theta) = \tfrac{1}{2}\operatorname{ind}\mathcal{D}_{B}(\beta,\theta) + \tfrac{1}{2}\operatorname{ind}(\tau,\mathcal{D}_{B})(\beta,\theta). \tag{15}$$

According to Donnelly [12],

$$\operatorname{ind}(\tau,\mathcal{D}_{B})(\beta,\theta) = \frac{1}{2}\int_F (e(TF)+e(NF)) - \tfrac{1}{2}(h^{\tau}_{\theta} - \eta^{\tau}_{\theta}(0))(Y) - \tfrac{1}{2}(h^{\tau}_{\beta} + \eta^{\tau}_{\beta}(0))(Y),$$

where the integral term vanishes and $h^{\tau}_{\beta} = h^{\tau}_{\theta} = -1$ as in the proof of Proposition 6.1. Therefore,

$$\operatorname{ind}(\tau,\mathcal{D}_{B})(\beta,\theta) = 1 - \tfrac{1}{2}\cdot\rho^{\tau}_{\beta}(Y). \tag{16}$$

The ρ–invariants in this formula are difficult to compute in general but they can be shown to vanish in several special cases, for example for two-bridge knots, as discussed in Section 7.1.

6.2 Nondihedral representations

Let $\rho_0\colon \pi_1 K \to \mathrm{SO}(3)$ be an irreducible trace-free representation which is not dihedral, and assume that it is nondegenerate. Proposition 4.1(c) then tells us that the fiber $C(\rho_0)$ consists of two circles which are permuted by the involution χ_k.

Lemma 6.4 *The involution χ_k permuting the two circles in $C(\rho_0)$ has degree zero mod 4.*

Proof This follows as in Lemma 2.5 whose proof in Section 3.7 needs to be amended to allow for the 1–dimensional critical point sets $C(\rho_0)$. This is easily accomplished by replacing $\mathrm{gr}(\chi_1\cdot\rho,\rho)$ with $\mathrm{gr}(\chi_1\cdot\rho,\rho)+1$ in the first two displayed formulas. □

Therefore, the two circles in $C(\rho_0)$ have the same Morse–Bott index μ. Perturbing both of them, we obtain four generators, two of grading μ and two of grading $\mu+1$. The calculation of the previous section leading up to the formula of Corollary 6.3 can be easily amended to work in the current situation, producing the following result.

Proposition 6.5 *Let $\beta\colon \pi_1 Y \to \mathrm{SO}(3)$ be an irreducible representation. Then the Floer grading of the two Morse–Bott circles arising from $\beta * \gamma$ is*

$$\mu = \operatorname{ind} \mathcal{D}^{\tau}_{\boldsymbol{B}}(\beta,\theta) + \operatorname{sign} k \pmod 4,$$

where B is an arbitrary equivariant connection on the infinite cylinder $\mathbb{R}\times Y$ whose limits are β and θ over the negative and positive ends.

The index $\operatorname{ind}\mathcal{D}^{\tau}_{\boldsymbol{B}}(\beta,\theta)$ in this proposition can be computed using the formula (15). Since h^{τ}_{β} now vanishes, the formula (16) takes the form

$$\operatorname{ind}(\tau,\mathcal{D}_{\boldsymbol{B}})(\beta,\theta) = \tfrac{1}{2} - \tfrac{1}{2}\cdot \rho^{\tau}_{\beta}(Y). \tag{17}$$

Remark 6.6 Let $\Delta(t)$ be the Alexander polynomial of a knot $k \subset S^3$ normalized so that $\Delta(t)=\Delta(t^{-1})$ and $\Delta(1)=1$. The knots k with $\Delta(-1)=1$ are precisely the knots whose double branched covers Y are integral homology spheres, and which are known to have no dihedral representations in $\mathcal{R}_0(K,\mathrm{SO}(3))$; see [23, Theorem 10] or [11, Proposition 3.4]. Therefore, all the generators in $IC^{\natural}(k)$ are of the nondihedral type studied in this section. In addition, $\operatorname{sign} k = 0 \pmod 8$ because $1=\Delta(-1)=\det(i\cdot Q)$, where Q is the (even) quadratic form of the knot.

7 Knot homology: explicit calculations

The equivariant techniques work particularly well for Montesinos knots, including two-bridge and pretzel knots, as we will demonstrate in this section. We begin with two-bridge knots, then discuss the Montesinos knots whose double branched covers are integral homology spheres, and then move on to the general Montesinos knots. We finish with a short section on torus knots.

7.1 Two-bridge knots

Let p be an odd positive integer and k a two-bridge knot of type $-p/q$ in the 3–sphere. Its double branched cover Y is the lens space $L(p,q)$ oriented as the $(-p/q)$–surgery on an unknot in S^3. One can use Proposition 3.1 to show that all representations $\beta\colon \pi_1 Y \to \mathrm{SO}(3)$ are equivariant. The invariant $\rho^{\tau}_{\beta}(Y)$ of formula (16) has been shown to vanish in [36, Proposition 27]. Therefore, $\operatorname{ind}(\tau,\mathcal{D}_{\boldsymbol{B}})(\beta,\theta)=1$ and formula (15) reduces to

$$\operatorname{ind}\mathcal{D}^{\tau}_{\boldsymbol{B}}(\beta,\theta) = \tfrac{1}{2}(\operatorname{ind}\mathcal{D}_{\boldsymbol{B}}(\beta,\theta)+1) \pmod 4.$$

Let $\beta\colon \pi_1 Y \to \mathrm{SO}(3)$ be a representation sending the canonical generator of $\pi_1 Y$ to the adjoint of $\exp(2\pi i \ell/p)$. The quantity $\operatorname{ind} \mathcal{D}_B(\beta,\theta) + 1 \pmod 8$ was shown by Sasahira [34, Corollary 4.3] (see also Austin [7]) to equal

$$2N_1(k_1,k_2) + N_2(k_1,k_2) \pmod 8,$$

where the integers $0 < k_1 < p$ and $0 < k_2 < p$ are uniquely determined by the equations

$$k_1 = \ell \pmod p, \quad k_2 = -r\ell \pmod p, \quad qr = 1 \pmod p,$$

and

$$N_1(k_1,k_2) = \#\bigl\{(i,j) \in \mathbb{Z}^2 \mid i + qj = 0 \pmod p, |i| < k_1, |j| < k_2\bigr\},$$
$$N_2(k_1,k_2) = \#\bigl\{(i,j) \in \mathbb{Z}^2 \mid i + qj = 0 \pmod p \text{ and } \text{either } |i| = k_1, |j| < k_2 \text{ or } |i| < k_1, |j| = k_2\bigr\}.$$

Example 7.1 The figure-eight knot k is the two-bridge knot of type $-\frac{5}{3}$. Its double branched cover is the lens space $L(5,3)$, whose fundamental group has no irreducible representations and has two nontrivial reducible representations, up to conjugacy. For these two representations, ℓ equals 1 and 2 and, by Sasahira's formula, $\operatorname{ind} \mathcal{D}_B(\beta,\theta)+1$ equals 2 and 4 mod 8, respectively. Since $\operatorname{sign} k = 0$, the corresponding Morse–Bott circles have indices $\mu = 1$ and 2 mod 4 by Corollary 6.3. After perturbation, they contribute the generators of Floer indices 1, 2 and 2, 3 mod 4, respectively. The ranks of the chain groups $IC^\natural(k)$ are then equal to $(1,0,0,0) + (0,1,1,0) + (0,0,1,1) = (1,1,2,1)$. This equals the Khovanov homology of (the mirror image of) k, hence we conclude from the Kronheimer–Mrowka spectral sequence that the ranks of $I^\natural(k)$ also equal $(1,1,2,1)$.

7.2 Special Montesinos knots

Let p, q, and r be pairwise relatively prime positive integers, and view the Brieskorn homology sphere $\Sigma(p,q,r)$ as the link of the singularity at zero of the complex polynomial $x^p + y^q + z^r$. The involution τ induced by complex conjugation on the link makes $\Sigma(p,q,r)$ into a double branched cover of S^3 with branch set a Montesinos knot which will be called $k(p,q,r)$; see for instance [36, Section 7].

Since $\Sigma(p,q,r)$ is an integral homology sphere, apart from the trivial one, all representations $\beta\colon \pi_1(\Sigma(p,q,r)) \to \mathrm{SO}(3)$ are irreducible. Fintushel and Stern [15] showed that all irreducible representations β are nondegenerate and, up to conjugation, there are $-2\lambda(\Sigma(p,q,r))$ of them, where $\lambda(\Sigma(p,q,r))$ is the Casson invariant of $\Sigma(p,q,r)$. The representations β are also equivariant (see [36, Proposition 8]), hence each conjugacy class of them contributes four generators to the chain complex $IC^\natural(k(p,q,r))$, two of grading $\mu(\beta)$ and two of grading $\mu(\beta)+1$.

Theorem 7.2 *The ranks of the chain groups* $IC^\natural(k(p,q,r))$ *are* $(1+b, b, b, b)$, *where* $b = -2\lambda(\Sigma(p,q,r))$.

Proof Our proof will use the flat cobordism of Fintushel and Stern [15], which is constructed as follows. The mapping torus of the Seifert fibration $\Sigma(p,q,r) \to S^2$ is an orbifold with three singular points whose neighborhoods are open cones over lens spaces. The compact manifold obtained from W by excising these cones is an equivariant flat cobordism W_0 between $\Sigma(p,q,r)$ and the lens spaces. One can easily see that the intersection form on $H^2(W_0;\mathbb{R}) = \mathbb{R}$ is negative definite.

An equivariant version of [5, Theorem 2.4] together with the vanishing of the ρ^τ–invariants of lens spaces [36, Proposition 27] imply that

$$\rho^\tau_\beta(\Sigma(p,q,r)) = \operatorname{sign}_\beta(\tau, W_0) - \operatorname{sign}_\theta(\tau, W_0),$$

where

$$\operatorname{sign}_\beta(\tau, W_0) = \operatorname{tr}(\tilde{\tau}^* \,|\, H^2_+(W_0;\operatorname{ad}\beta)) - \operatorname{tr}(\tilde{\tau}^* \,|\, H^2_-(W_0;\operatorname{ad}\beta)),$$

and similarly for $\operatorname{sign}_\theta(\tau, W_0)$. It follows from [15, Proposition 2.5 and Lemma 2.6] that $H^2(W_0;\operatorname{ad}\beta) = 0$, hence $\rho^\tau_\beta(\Sigma(p,q,r)) = \operatorname{tr}(\operatorname{Ad} u) = -1$ and $\operatorname{ind}(\tau, \mathcal{D}_{\boldsymbol{B}})(\beta,\theta) = 1$ by formula (17). Proposition 6.5 and formula (15) now imply that

$$\mu(\beta) = \tfrac{1}{2}(\operatorname{ind}\mathcal{D}_{\boldsymbol{B}}(\beta,\theta) + 1).$$

The index $\operatorname{ind}\mathcal{D}_{\boldsymbol{B}}(\beta,\theta)$ in this formula can be computed explicitly using either [15] or Corollary 7.7, however, this alone will not lead us to the closed-form formula of Theorem 7.2.

Instead, we will use the 4–periodicity in the instanton Floer homology due to Frøyshov [17, Theorem 2]. In the case at hand, the Floer homology of $\Sigma(p,q,r)$ equals its Floer chain complex, whose generators are the conjugacy classes of irreducible representations β, hence the 4–periodicity simply means that there is a (noncanonical) free involution of degree 4 on these generators. For any pair of generators β_1 and β_2,

$$\mu(\beta_2) - \mu(\beta_1) = \tfrac{1}{2}(\operatorname{ind}\mathcal{D}_{\boldsymbol{B}}(\beta_2,\theta) - \operatorname{ind}\mathcal{D}_{\boldsymbol{B}}(\beta_1,\theta)) \pmod 4,$$

which is exactly half the relative grading of the generators β_1 and β_2 in the Floer chain complex of $\Sigma(p,q,r)$. For any involutive pair (β_1,β_2), we have

$$\mu(\beta_2) - \mu(\beta_1) = 2 \pmod 4,$$

therefore, each such pair contributes $(2,2,2,2)$ to the chain complex $IC^\natural(k(p,q,r))$. The special generator α resides in degree zero so the result follows. □

Example 7.3 $\Sigma(2,3,7)$ is a double branched cover of S^3 whose branch set $k(2,3,7)$ is the pretzel knot $P(-2,3,7)$. Since $\lambda(\Sigma(2,3,7)) = -1$, we conclude that the ranks of the chain groups $IC^\natural(P(-2,3,7))$ are $(3,2,2,2)$. This is consistent with the calculation in [18, Section 5].

We expect that the formula of Theorem 7.2 can be proved for all Seifert fibered homology spheres $\Sigma(a_1,\dots,a_n)$ and the corresponding Montesinos knots $k(a_1,\dots,a_n)$ using τ–equivariant perturbations of [38] modeled after the perturbations of Kirk and Klassen [22]. Note that the action of $H^1(K;\mathbb{Z}/2)$ on the conjugacy classes of projective representations is free hence it causes no equivariant transversality issues.

7.3 General Montesinos knots

Let $(a_1,b_1),\dots,(a_n,b_n)$ be pairs of integers such that, for each i, the integers a_i and b_i are relatively prime and a_i is positive. Burde and Zieschang [9, Chapter 7] associated with these pairs a Montesinos link $K((a_1,b_1),\dots,(a_n,b_n))$ and showed that its double branched cover is a Seifert fibered manifold Y with unnormalized Seifert invariants $(a_1,b_1),\dots,(a_n,b_n)$. In particular,

$$\pi_1 Y = \langle x_1,\dots x_n, h \mid h \text{ central},\ x_i^{a_i} = h^{-b_i},\ x_1\cdots x_n = 1\rangle,$$

with the covering translation $\tau\colon Y \to Y$ acting on the fundamental group by the rule

$$\tau_*(h) = h^{-1}, \qquad \tau_*(x_i) = x_1\cdots x_{i-1}x_i^{-1}x_{i-1}^{-1}\cdots x_1^{-1} \quad \text{for } i = 1,\dots,n;$$

see Burde and Zieschang [9, Proposition 12.30]. The knots $k(a_1,\dots,a_n)$ of the previous section are of the type $K((a_1,b_1),\dots,(a_n,b_n))$; we omitted the parameters $(b_1,\dots,b_n)$ from the notation because they can be uniquely recovered from the pairwise relatively prime $a_1,\dots,a_n$ up to isotopy of the knot. All two-bridge and pretzel knots and links are special cases of Montesinos knots and links. In this section, we will only be interested in Montesinos knots; the case of Montesinos links of two components will be addressed in Section 8.3.

Let k be a Montesinos knot $K((a_1,b_1),\dots,(a_n,b_n))$ and Y the double branched cover of S^3 with branch set k. The manifold Y need not be an integral homology sphere; in fact, one can easily see that its first homology is a finite abelian group of order

$$|H_1(Y;\mathbb{Z})| = \left(\sum_{i=1}^n b_i/a_i\right)\cdot a_1\cdots a_n.$$

Note that this integer is always odd because Y is a $\mathbb{Z}/2$ homology sphere.

All reducible representations $\beta\colon \pi_1 Y \to \mathrm{SO}(3)$ are equivariant because the involution $\tau_*\colon H_1(Y) \to H_1(Y)$ acts as multiplication by -1; see Proposition 3.1. There are no irreducible representations for $n \le 2$. If $n = 3$, all irreducible representations are nondegenerate and equivariant, which can be shown using a minor modification of the arguments of [15, Proposition 2.5] and [36, Proposition 30]. For $n \ge 4$, one encounters positive-dimensional manifolds of representations; the action of τ^* on these manifolds was described in [38], together with equivariant perturbations making them nondegenerate. This discussion, together with Propositions 4.1 and 4.2, identifies the generators of the chain complex $IC^\natural(k)$ for all Montesinos knots in terms of representations for Seifert fibered manifolds, which are well known. An independent calculation of the generators of $IC^\natural(k)$ for pretzel knots k with $n = 3$ can be found in Zentner [43].

Let W_0 be the mapping cylinder of the Seifert fibration $Y \to S^2$ with excised open cones around its singular points. Then W_0 is a cobordism from a disjoint union of the lens spaces $L(a_i, -b_i)$ to Y.

Lemma 7.4 *W_0 is a flat cobordism provided $a_1 \cdots a_n = \mathrm{lcm}(a_1, \dots, a_n) \cdot |H_1(Y;\mathbb{Z})|$.*

Proof The fundamental group $\pi_1 W_0$ is obtained from $\pi_1 Y$ by setting the homotopy class $h \in \pi_1 Y$ of the circle fiber equal to one. Since h is a central element in $\pi_1 Y$, every irreducible representation $\beta\colon \pi_1 Y \to \mathrm{SO}(3)$ has the property that $\beta(h) = 1$. This property need not hold for reducible representations but it does if $h = 1$ in the first homology group $H_1(Y)$. The algebraic condition of the lemma ensures exactly that; see Lee and Raymond [26, page 331]. □

To avoid dealing with perturbations, we will assume from now on that our knot k is a Montesinos knot of type $K((a_1, b_1)(a_2, b_2), (a_3, b_3))$ and that W_0 is a flat cobordism. We wish to calculate Floer gradings of the generators in the chain complex $IC^\natural(k)$. Recall that every conjugacy class of nontrivial reducible representations $\beta\colon \pi_1 Y \to \mathrm{SO}(3)$ gives rise to two generators of gradings $\mu(\beta)$ and $\mu(\beta)+1$, and every conjugacy class of irreducible representations to four generators, two of grading $\mu(\beta)$ and two of grading $\mu(\beta) + 1$. The trivial representation as usual gives rise to just one generator α of grading $\operatorname{sign} k$.

Lemma 7.5 *For any nontrivial representation $\beta\colon \pi_1 Y \to \mathrm{SO}(3)$, we have*

$$\mu(\beta) = \operatorname{sign} k + \tfrac{1}{2}(\operatorname{ind} \mathcal{D}_B(\beta, \theta) + 1) \pmod 4.$$

Proof This formula holds for all irreducible representations β by the same argument as in the proof of Theorem 7.2. That argument can be easily amended for nontrivial reducible representations $\beta\colon \pi_1 Y \to \mathrm{SO}(3)$ by using (16) in place of (17). The ρ–invariant in formula (16) is given by the formula

$$\rho^{\tau}_{\beta}(Y) = \operatorname{sign}_{\beta}(\tau, W_0) - \operatorname{sign}_{\theta}(\tau, W_0),$$

with $\operatorname{sign}_{\theta}(\tau, W_0) = 1$. To compute the cohomology of W_0 with coefficients in $\operatorname{ad}\beta$, write $\operatorname{ad} P = \mathbb{R} \oplus L$, where L is a line bundle with a nontrivial flat connection. Then $H^2(W_0; L) = 0$ by the argument of [15, Lemma 2.6] and $H^2(W_0; \mathbb{R}) = \mathbb{R}$. Since the manifold W_0 is negative definite, we easily conclude that $\operatorname{sign}_{\beta}(\tau, W_0) = 1$. Therefore, $\rho^{\tau}_{\beta}(Y) = 0$, and the result follows. □

To complete the calculation of Floer gradings, we only need to compute the index $\operatorname{ind}\mathcal{D}_{\boldsymbol{B}}(\beta, \theta)$. This can be done by extending the formulas of Fintushel and Stern [15] from integral homology spheres to the more general situation at hand. We will restrict ourselves to the relatively easy case of odd a_i and leave the case of even a_i open because it would require passing to a double branched cover as in the proof of [15, Theorem 3.7].

Given a flat cobordism W_0, any representation $\beta\colon \pi_1(Y) \to \mathrm{SO}(3)$ gives rise to a representation $\pi_1(W_0) \to \mathrm{SO}(3)$ and to representations $\beta_i\colon \pi_1(L(a_i, -b_i)) \to \mathrm{SO}(3)$. Let us assume that a_i are odd and $\beta_i \neq \theta$ for $i = 1, \dots, m$, and that $\beta_i = \theta$ for $i = m+1, \dots, 3$. Applying the excision principle for the ASD operator twice, first to $\mathbb{R} \times L(a_i, -b_i)$ with $i = 1, \dots, m$, and then to W_0 with the attached product ends, we obtain

$$-3 = \operatorname{ind}\mathcal{D}_{\boldsymbol{B}}(\theta, \theta) = \operatorname{ind}\mathcal{D}_{\boldsymbol{B}}(\theta, \beta_i) + 1 + \operatorname{ind}\mathcal{D}_{\boldsymbol{B}}(\beta_i, \theta)$$

$$-3 = \operatorname{ind}\mathcal{D}_{\boldsymbol{B}}(W_0, \theta, \theta) = \sum_{i=1}^{m} (\operatorname{ind}\mathcal{D}_{\boldsymbol{B}}(\theta, \beta_i) + 1) + \operatorname{ind}\mathcal{D}_{\boldsymbol{B}}(W_0) + 1 + \operatorname{ind}\mathcal{D}_{\boldsymbol{B}}(\beta, \theta),$$

where $\mathcal{D}_{\boldsymbol{B}}(W_0)$ stands for the ASD operator on W_0 twisted by a flat connection B whose holonomy is the representation $\pi_1(W_0) \to \mathrm{SO}(3)$. A similar argument with even a_i does not work because representations β_i and θ may end up living in different SO(3)–bundles.

Lemma 7.6 *Let $\beta\colon \pi_1(Y) \to \mathrm{SO}(3)$ be a nontrivial representation. Then $\operatorname{ind}\mathcal{D}_{\boldsymbol{B}}(W_0) = -1$ if β is reducible, and $\operatorname{ind}\mathcal{D}_{\boldsymbol{B}}(W_0) = 0$ if β is irreducible.*

Proof The proof of [15, Proposition 3.3] implies the formula for irreducible β immediately, and for reducible β after a minor modification. To be precise, let us assume that β is reducible. The index at hand equals $h^1 - h^0 - h^2$, where h^0, h^1, and h^2 are the Betti numbers of the elliptic complex

$$0 \to \Omega^0(W_0, \operatorname{ad} P) \xrightarrow{-d_B} \Omega^1(W_0, \operatorname{ad} P) \xrightarrow{d_B^+} \Omega^2_+(W_0, \operatorname{ad} P).$$

Since B has 1–dimensional stabilizer we immediately conclude that $h^0 = 1$. To compute the remaining Betti numbers, write $\operatorname{ad} P = \mathbb{R} \oplus L$, where L is a line bundle with a nontrivial flat connection. The argument of [15, Lemma 2.6] can be used to show that the homomorphisms $H^1(W_0; L) \to H^1(Y; L)$ and $H^2(W_0; L) \to H^2(Y; L)$ induced by the inclusion $Y \to W_0$ are injective. Both $H^1(W_0; \mathbb{R})$ and $H^1(Y; \mathbb{R})$ vanish, and the long exact sequence of the pair (W_0, Y) shows that the kernel of the map $H^2(W_0; \mathbb{R}) \to H^2(Y; \mathbb{R})$ is 1–dimensional. Keeping in mind that the manifold W_0 is negative definite, we conclude as in the proof of [15, Proposition 3.3] that $h^1 = h^2 = 0$. □

Corollary 7.7 *Let $\beta\colon \pi_1(Y) \to \mathrm{SO}(3)$ be a nontrivial representation such that a_i is odd and $\beta_i \neq \theta$ for $i = 1, \dots, m$, and $\beta_i = \theta$ for $i = m+1, \dots, 3$. Then*

$$\mu(\beta) = \operatorname{sign} k - 1 + \frac{1}{2} \sum_{i=1}^{m} (\operatorname{ind} \mathcal{D}_B(\beta_i, \theta) + 3) \pmod 4,$$

where the index $\operatorname{ind} \mathcal{D}_B(\beta_i, \theta)$ on the infinite cylinder $\mathbb{R} \times L(a_i, -b_i)$ can be computed as in Section 7.1.

Example 7.8 Let us view the pretzel knot $k = P(-2, 3, 3)$ as the Montesinos knot $K((2, -1), (3, 1), (3, 1))$. It obviously satisfies the condition of Lemma 7.4. Its double branched cover is a Seifert fibered manifold Y whose fundamental group has presentation

$$\langle x_1, x_2, x_3, x_4, h \mid h \text{ central},\ x_1^2 = h,\ x_2^3 = h^{-1},\ x_3^3 = h^{-1},\ x_1 x_2 x_3 = 1 \rangle.$$

This group admits one nontrivial reducible representation β with $\beta(x_1) = 1$, $\beta(x_2) = \operatorname{Ad}(\exp(2\pi i/3))$ and $\beta(x_3) = \operatorname{Ad}(\exp(-2\pi i/3))$ contributing generators of gradings μ and $\mu + 1$ to the chain complex $IC^{\natural}(k)$. To compute μ, we apply the formulas of Section 7.1 to the lens space $L(3, -1) = L(3, 2)$ twice to obtain $\operatorname{ind} \mathcal{D}_B(\beta_2, \theta) = \operatorname{ind} \mathcal{D}_B(\beta_3, \theta) = 1 \pmod 8$. Since $\operatorname{sign} k = 2 \pmod 4$, it follows from Corollary 7.7 that $\mu = 1 \pmod 4$ hence the contribution of β to the chain complex is $(0, 1, 1, 0)$. The special generator α contributes $(0, 0, 1, 0)$.

The group $\pi_1 Y$ also admits one irreducible representation β such that all of the induced representations $\beta_1\colon \pi_1(L(2, 1)) \to \mathrm{SO}(3)$ and $\beta_2, \beta_3\colon \pi_1(L(3, 2)) \to \mathrm{SO}(3)$

are nontrivial. Corollary 7.7 no longer applies, hence we can only conclude that the contribution of this representation to $IC^{\natural}(k)$ is $(2,2,0,0)$ up to cyclic permutation.

This information can be combined with the fact that the Kronheimer–Mrowka spectral sequence of the knot $k = P(-2,3,3)$ is trivial and that the Khovanov homology groups of k have ranks $(2,1,1,1)$; see Lobb and Zentner [28]. It then follows that the ranks of the chain groups $IC^{\natural}(k)$ must be $(2,1,2,2)$, with the contribution of the irreducible being $(2,0,0,2)$, and that the boundary operator $IC_2^{\natural}(k) \to IC_3^{\natural}(k)$ must be nontrivial.

A similar calculation can be done for all Montesinos knots $K((a_1,b_1),\dots,(a_n,b_n))$ satisfying the condition of Lemma 7.4 with the help of the equivariant perturbations of [38].

7.4 Torus knots

Let p and q be positive integers which are odd and relatively prime. The double branched cover of the right-handed (p,q)–torus knot $T_{p,q}$ is the Brieskorn homology sphere $\Sigma(2,p,q)$. According to Fintushel and Stern [15], all irreducible SO(3) representations of the fundamental group of $\Sigma(2,p,q)$ are nondegenerate and, up to conjugacy, there are $-\operatorname{sign}(T_{p,q})/4$ of them. All of these representations are equivariant [11, Section 4.2], hence each of them contributes four generators to the chain complex of $I^{\natural}(T_{p,q})$, two of index μ and two of index $\mu+1$. Calculating μ would require equivariant index theory on the double branched cover of $T_{p,q}$ which is currently not sufficiently well developed. We know that the special generator resides in degree zero because $\operatorname{sign} T_{p,q} = 0 \pmod 8$, and we conjecture that the ranks of the chain groups $IC^{\natural}(T_{p,q})$ are

$$(1+a,a,a,a), \quad \text{where } a = -\operatorname{sign}(T_{p,q})/4.$$

This conjecture is consistent with the calculations for torus knots by Hedden, Herald and Kirk [20].

Let us now assume that p and q are relatively prime positive integers such that p is odd and $q = 2r$ is even. The double branched cover Y, which is no longer an integral homology sphere, is the link of the singularity at zero of the complex polynomial $x^2 + y^p + z^{2r} = 0$, with the covering translation given by the formula $\tau(x,y,z) = (-x,y,z)$. Neumann and Raymond [30] showed that Y admits a fixed-point-free circle action making it into a Seifert fibration over S^2 with the Seifert invariants

$$\{(a_1,b_1),\dots,(a_n,b_n)\} = \{(1,b_1),(p,b_2),(p,b_2),(r,b_3)\},$$

where $b_1 \cdot pr + 2b_2 \cdot r + b_3 \cdot p = 1$. In principle, this allows for calculation of the generators in the Floer chain complex $IC^{\natural}(T_{p,q})$.

Example 7.9 Let us consider the torus knot $T_{3,4}$. The Seifert invariants of the manifold Y are $\{(1,-1),(2,1),(3,1),(3,1)\}$, while those of the manifold in Example 7.8 are $\{(2,-1),(3,1),(3,1)\}$. These match for the good reason that $P(-2,3,3)$ and $T_{3,4}$ are the same knot. The calculation of Example 7.8 then tells us that the ranks of the chain groups $IC^{\natural}(T_{3,4})$ are $(2,1,2,2)$, with a nontrivial boundary operator $IC_2^{\natural}(T_{3,4}) \to IC_3^{\natural}(T_{3,4})$. This is consistent with [20].

8 Link homology of general two-component links

This section deals with general two-component links $\mathcal{L} = \ell_1 \cup \ell_2$ and not just the links $\mathcal{L} = k^{\natural}$ used in the definition of the knot Floer homology $I^{\natural}(k)$. After computing the Euler characteristic of $I_*(\Sigma, \mathcal{L})$, we explicitly compute the Floer chain groups for some links $\mathcal{L}$ with particularly simple double branched covers.

8.1 Euler characteristic

Let $\mathcal{L} = \ell_1 \cup \ell_2$ be a two-component link in an integral homology sphere Σ. The linking number $\ell k(\ell_1, \ell_2)$ is well defined up to a sign for nonoriented links $\mathcal{L}$.

Theorem 8.1 *The Euler characteristic of the Floer homology $I_*(\Sigma, \mathcal{L})$ of a two-component link $\mathcal{L} = \ell_1 \cup \ell_2$ equals $\pm \ell k(\ell_1, \ell_2)$.*

Proof The Floer excision principle can be used as in [24] to establish an isomorphism between $I_*(\Sigma, \mathcal{L})$ and the sutured Floer homology of $\mathcal{L}$. The latter is the Floer homology of the 3–manifold X_φ obtained by identifying the two boundary components of $S^3 - \operatorname{int} N(\mathcal{L})$ via an orientation-reversing homeomorphism $\varphi\colon T^2 \to T^2$. According to [19, Lemma 2.1], the homeomorphism φ can be chosen so that X_φ has integral homology of $S^1 \times S^2$. The result then follows from [19, Theorem 2.3], which asserts that the Euler characteristic of the sutured Floer homology of $\mathcal{L}$ equals $\pm \ell k(\ell_1, \ell_2)$. □

Theorem 8.1 implies in particular that the Euler characteristic of $I^{\natural}(k)$ equals ± 1, which is the linking number of the two components of the link $k^{\natural}$. This also follows from the fact that the critical point set of the orbifold Chern–Simons functional used to define $I^{\natural}(k)$ consists of an isolated point and finitely many isolated circles, possibly after a perturbation. An absolute grading on $I^{\natural}(k)$ was fixed in [24] so that the grading of the isolated point is even; this is consistent with our Theorem 5.1 because $\operatorname{sign} k$ is always even. The Euler characteristic of $I^{\natural}(k)$ then equals $+1$. We do not know how to fix an absolute grading on $I_*(\Sigma, \mathcal{L})$ for a general two-component link $\mathcal{L}$.

8.2 Pretzel link $P(2,-3,-6)$

This is the two-component link $\mathcal{L}$ whose double branched cover is the Seifert fibered manifold M with unnormalized Seifert invariants $(2,1)$, $(3,-1)$, and $(6,-1)$; see for instance [37, Section 4]. In particular,

$$\pi_1 M = \langle x, y, z, h \mid h \text{ central},\ x^2 = h^{-1},\ y^3 = h,\ z^6 = h,\ xyz = 1\rangle,$$

with the covering translation $\tau\colon M \to M$ acting on the fundamental group by the rule

$$\tau_*(h) = h^{-1}, \quad \tau_*(x) = x^{-1}, \quad \tau_*(y) = xy^{-1}x^{-1}, \quad \tau_*(z) = xyz^{-1}y^{-1}x^{-1};$$

see Burde and Zieschang [9, Proposition 12.30]. The manifold M has integral homology of $S^1 \times S^2$. In fact, it can be obtained by 0–surgery on the right-handed trefoil, so that $\pi_1 M = \pi_1 K/\langle\lambda\rangle$, where K is the exterior of the trefoil and λ is its longitude. The relation $\lambda = 1$ shows up as the relation $z^6 = h$ in the above presentation of $\pi_1 M$.

We will use this surgery presentation of M to describe representations $\pi_1 M \to \mathrm{SO}(3)$ with nontrivial $w_2 \in H^2(M;\mathbb{Z}/2) = \mathbb{Z}/2$. According to Example 2.2, the conjugacy classes of such representations are in one-to-two correspondence with the conjugacy classes of representations $\rho\colon \pi_1 K \to \mathrm{SU}(2)$ such that $\rho(\lambda) = -1$. In the terminology of Section 2.2, these ρ are projective representations $\rho\colon \pi_1 M \to \mathrm{SU}(2)$, and the group $H^1(M;\mathbb{Z}/2) = \mathbb{Z}/2$ acts on them freely, providing the claimed one-to-two correspondence. Therefore, we wish to find all the $\mathrm{SU}(2)$ matrices $\rho(h)$, $\rho(x)$, $\rho(y)$, and $\rho(z)$ such that

$$\rho(x)^2 = \rho(h)^{-1}, \quad \rho(y)^3 = \rho(h), \quad \rho(z)^6 = -\rho(h), \quad \rho(x)\rho(y)\rho(z) = 1$$

and such that $\rho(h)$ commutes with $\rho(x)$, $\rho(y)$, and $\rho(z)$. Since ρ is irreducible, we conclude as in Fintushel and Stern [15, Section 2] that $\rho(h) = -1$ and that $\rho(x)$ is conjugate to i, $\rho(y)$ is conjugate to $e^{\pi i/3}$, and $\rho(z)$ is conjugate to either $e^{\pi i/3}$ or $e^{2\pi i/3}$. These give rise to two conjugacy classes of projective representations $\rho\colon \pi_1 M \to \mathrm{SU}(2)$ corresponding to a single conjugacy class of representations $\mathrm{Ad}\,\rho\colon \pi_1 M \to \mathrm{SO}(3)$.

The arguments of [15, Proposition 2.5] and [36, Proposition 8] can be easily adapted to conclude that the representation $\mathrm{Ad}\,\rho$ is nondegenerate and equivariant. It gives rise to a single $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ orbit of generators in $IC_*(S^3, \mathcal{L})$. Since the linking number between the components of $\mathcal{L}$ is even, Lemma 2.5 tells us that the (relative) Floer indices of these four generators are $0, 0, 2, 2 \pmod 4$. The boundary operators then must vanish, and we conclude that the Floer homology groups $I_k(S^3, \mathcal{L})$ are free abelian groups of ranks $(2, 0, 2, 0)$, up to cyclic permutation.

Remark 8.2 The same result can be obtained independently using the isomorphism between $I_*(S^3, \mathcal{L})$ and the sutured Floer homology of $\mathcal{L}$ defined in [25]. The latter is the Floer homology of the manifold X_φ obtained by identifying the two boundary components of $X = S^3 - \operatorname{int} N(\mathcal{L})$ via an orientation-reversing homeomorphism $\varphi\colon T^2 \to T^2$. A surgery description of X_φ can be found in [19]; computing its Floer homology is then an exercise in applying the Floer exact triangle to this surgery description.

8.3 Montesinos links

Let $(a_1, b_1), \dots, (a_n, b_n)$ be pairs of integers such that, for each i, the integers a_i and b_i are relatively prime and a_i is positive. Associated with these pairs is the Montesinos link $K((a_1, b_1), \dots, (a_n, b_n))$, whose definition can be found for instance in [9, Chapter 7]. All two-bridge and pretzel links are Montesinos links; for example, the link $P(2, -3, -6)$ considered in the previous section is the Montesinos link with the parameters $(2, 1)$, $(3, -1)$ and $(6, -1)$. The double branched covers M of Montesinos links were described in Section 7.3. In this section, we will only be interested in Montesinos links whose double branched covers have integral homology of $S^1 \times S^2$, a condition that is easily checked by abelianizing $\pi_1 M$. This condition guarantees that the unique SO(3)–bundle $P \to M$ with nontrivial $w_2(P) \in H^2(M; \mathbb{Z}/2) = \mathbb{Z}/2$ does not carry any reducible connections.

The generators of Floer chain complex of the link $K((a_1, b_1), \dots, (a_n, b_n))$ and their gradings can be computed explicitly using the equivariant theory developed in this paper; here is a brief outline.

Since M is Seifert fibered, the representations $\pi_1 M \to \mathrm{SO}(3)$ with nontrivial w_2 can be described in terms of their rotation numbers using a slight modification of the Fintushel–Stern algorithm [15]; complete details can be found in [35]. If $n = 3$, there are finitely many conjugacy classes of such representations, all of which are nondegenerate and equivariant with the conjugating element of order 2. If $n \ge 4$, the same conclusion holds after using τ–equivariant perturbations similar to those described in [38]. Note that no equivariant transversality issues are caused by the action of $H^1(M; \mathbb{Z}/2)$ or $H^1(X; \mathbb{Z}/2)$ because both actions are free. In what follows, we will restrict ourselves to the case when $n = 3$; however, we expect that the same results will hold for all n.

The relative indices of the operator $\mathcal{D}_A$ on $\mathbb{R} \times M$ were computed explicitly in [35] and shown to be even. The relative Floer gradings of the generators in the Floer chain complex of the link $K((a_1, b_1), (a_2, b_2), (a_3, b_3))$ are equal to one half times those indices, by the argument of [36, Section 5.2] modified to take into account the nontriviality of the bundle $P \to M$.

The final outcome of this calculation can be stated in terms of the Floer homology groups $I_*(M,P)$ of the unique admissible bundle $P \to M$ as follows. The groups $I_*(M,P)$ are free abelian of ranks (n_0,n_1,n_2,n_3), up to cyclic permutation, with either $n_0 = n_2 = 0$ or $n_1 = n_3 = 0$. Assume for the sake of concreteness that $n_0 = n_2 = 0$. Then the Floer chain groups of $K((a_1,b_1),\dots,(a_n,b_n))$, up to cyclic permutation, have the ranks

$$(2n_1, 2n_3, 2n_1, 2n_3). \tag{18}$$

Example 8.3 The double branched cover M of the Montesinos link

$$\mathcal{L} = K((2,1),(5,-2),(10,-1))$$

can be obtained by 0–surgery on the right-handed torus knot $T_{2,5}$. Applying the Floer exact triangle to this surgery, we see that $I_*(M,P) \oplus I_{*+4}(M,P) = I_*(\Sigma(2,15,11))$, where we use the mod 8 grading in both groups. Fintushel and Stern [15] showed[3] that the groups $I_k(\Sigma(2,5,11))$ are free abelian of the ranks $(0,1,0,2,0,1,0,2)$. Therefore $n_1 = 1$, $n_3 = 2$, and the Floer chain groups of the link $\mathcal{L}$ have the ranks $(2,4,2,4)$.

In fact, the integers n_1 and n_3 in the formula (18) can be computed much more easily in terms of classical knot invariants without any reference to the Floer homology. They are known to satisfy the equations

$$-n_1 - n_3 = \lambda'(M) \quad \text{and} \quad -n_1 + n_3 = \bar{\mu}'(M),$$

where $\lambda'(M)$ is the Casson invariant of M and $\bar{\mu}'(M)$ its Neumann invariant [29]. The former equation follows from the Casson surgery formula and the latter from [37]. The Casson and Neumann invariants can then be computed explicitly using the formulas

$$\lambda'(M) = -\tfrac{1}{2}\cdot\Delta''_M(1) \quad \text{and} \quad \bar{\mu}'(M) = \pm \ell k(\ell_1,\ell_2),$$

where $\Delta_M(t)$ is the Alexander polynomial of M normalized so that $\Delta_M(1) = 1$ and $\Delta(t) = \Delta(t^{-1})$, and $\ell k(\ell_1,\ell_2)$ is the linking number between the components of the link $\mathcal{L}$. Note that there is no need to fix the sign in the above formula because switching that sign preserves the answer (18) up to cyclic permutation.

Appendix: Homology of double branched covers

This section contains a proof of Proposition 3.2 which was postponed until later in Section 3.1.

[3] We adjusted the formulas of [15] to take into account that Fintushel and Stern work with SD rather than ASD equations.

A.1 Computing $H_*(M;\mathbb{Z}/2)$

In this section, we will compute the groups $H_*(M;\mathbb{Z}/2)$ using the transfer homomorphism approach of [27].

The transfer homomorphisms can be defined in the following two equivalent ways; see for instance [14, Section 3]. For each singular simplex $\sigma\colon \Delta \to \Sigma$, choose a lift $\widetilde{\sigma}\colon \Delta \to M$ and define the chain map $\pi_!\colon C_*(\Sigma) \to C_*(M)$ by the formula $\pi_!(\sigma) = \widetilde{\sigma} + \tau \circ \widetilde{\sigma}$. This map is obviously independent of the choice of $\widetilde{\sigma}$, and it induces homomorphisms $\pi_!\colon H_*(\Sigma) \to H_*(M)$ and $\pi^!\colon H^*(M) \to H^*(\Sigma)$ in homology and cohomology with arbitrary coefficients, called transfer homomorphisms. Another way to define $\pi_!$ is as the map that makes the diagram

$$\begin{array}{ccc} H_*(M) & \xleftarrow{\ \text{PD}\ } & H^*(M) \\ \uparrow{\scriptstyle \pi_!} & & \uparrow{\scriptstyle \pi^*} \\ H_*(\Sigma) & \xleftarrow{\ \text{PD}\ } & H^*(\Sigma) \end{array}$$

commute, where PD stands for the Poincaré duality isomorphism, and similarly for $\pi^!$.

From now on, all chain complexes and (co)homology will be assumed to have $\mathbb{Z}/2$ coefficients. It is then immediate from the definition of $\pi_!\colon C_*(\Sigma) \to C_*(M)$ that $\ker \pi_! = C_*(\mathcal{L})$ and that we have a short exact sequence of chain complexes

$$0 \longrightarrow C_*(\Sigma,\mathcal{L}) \xrightarrow{\ \pi_!\ } C_*(M) \xrightarrow{\ \pi_*\ } C_*(\Sigma) \longrightarrow 0.$$

This exact sequence induces long exact sequences in homology

$$\begin{aligned} 0 \longrightarrow H_3(\Sigma,\mathcal{L}) &\xrightarrow{\ \pi_!\ } H_3(M) \longrightarrow H_3(\Sigma) \\ \longrightarrow H_2(\Sigma,\mathcal{L}) &\xrightarrow{\ \pi_!\ } H_2(M) \longrightarrow H_2(\Sigma) \\ \longrightarrow H_1(\Sigma,\mathcal{L}) &\xrightarrow{\ \pi_!\ } H_1(M) \longrightarrow H_1(\Sigma) \longrightarrow 0 \end{aligned}$$

and in cohomology

$$\begin{aligned} 0 \longrightarrow H^1(\Sigma) &\longrightarrow H^1(M) \xrightarrow{\ \pi^!\ } H^1(\Sigma,\mathcal{L}) \\ \longrightarrow H^2(\Sigma) &\longrightarrow H^2(M) \xrightarrow{\ \pi^!\ } H^2(\Sigma,\mathcal{L}) \\ \longrightarrow H^3(\Sigma) &\longrightarrow H^3(M) \xrightarrow{\ \pi^!\ } H^3(\Sigma,\mathcal{L}) \to 0. \end{aligned}$$

Combining these with the long exact sequence of the pair $(\Sigma,\mathcal{L})$, we obtain the following result.

Proposition A.1 *Let $\pi\colon M \to \Sigma$ be a double branched cover over an integral homology sphere Σ with branching set a two-component link $\mathcal{L}$. Then $H_i(M;\mathbb{Z}/2) = H^i(M;\mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}/2$ if $i = 0, 1, 2, 3$, and is zero otherwise.*

A.2 The cup product on $H^*(M;\mathbb{Z}/2)$

This section is devoted to the proof of the following result. We continue working with $\mathbb{Z}/2$ coefficients.

Proposition A.2 *The cup product $H^1(M) \times H^1(M) \to H^2(M)$ is the bilinear form $\mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2$ with the matrix $\ell k(\ell_1, \ell_2) \pmod 2$.*

Proof We will reduce the cup product calculation to intersection theory using the commutative diagram

$$\begin{array}{ccc} H_2(M) \times H_2(M) & \xrightarrow{\;\cdot\;} & H_1(M) \\ \uparrow{\scriptstyle \mathrm{PD}} & & \uparrow{\scriptstyle \mathrm{PD}} \\ H^1(M) \times H^1(M) & \xrightarrow{\;\cup\;} & H^2(M) \end{array}$$

where PD stands for the Poincaré duality isomorphisms and $\cdot$ for the intersection product. The transfer homomorphism $\pi_!\colon H_*(\Sigma, \mathcal{L}) \to H_*(M)$ will give us explicit generators of $H_1(M)$ and $H_2(M)$ that we need to proceed with this approach.

We begin with the group $H_1(M)$. Note that $H_1(\Sigma, \mathcal{L}) = \mathbb{Z}/2$ is generated by the homology class $[w]$ of any embedded arc $w \subset \Sigma$ whose endpoints belong to two different components of $\mathcal{L}$. The transfer homomorphism $\pi_!\colon H_1(\Sigma, \mathcal{L}) \to H_1(M)$ maps the homology class of w to that of the circle $\pi^{-1}(w)$. Since $\pi_!$ is an isomorphism, we conclude that the circle $\pi^{-1}(w)$ represents a generator of $H_1(M)$.

To describe a generator of $H_2(M)$, observe that $H_2(\Sigma, \mathcal{L}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is generated by the homology classes of Seifert surfaces S_1 and S_2 of the knots ℓ_1 and ℓ_2. We will assume that S_1 and S_2 intersect transversely in a finite number of circles and arcs, and note that $S_1 \cap S_2$ is homologous to $\ell k(\ell_1, \ell_2) \cdot w$. We claim that the closed orientable surfaces $\pi^{-1}(S_1)$ and $\pi^{-1}(S_2)$, representing the homology classes $\pi_!([S_1])$ and $\pi_!([S_2])$, are homologous to each other and generate $H_2(M)$. To see this, we will appeal to Theorem 2 of [27], which supplies us with the commutative diagram with an exact row

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_3(\Sigma) & \xrightarrow{\;d_*\;} & H_2(\Sigma, \mathcal{L}) & \xrightarrow{\;\pi_!\;} & H_2(M) & \longrightarrow & 0 \\ & & & \searrow^{f} & \downarrow{\scriptstyle \partial_*} & & & & \\ & & & & H_1(\mathcal{L}) & & & & \end{array}$$

where $f([\Sigma]) = [\ell_1] + [\ell_2]$ and ∂_* is the connecting homomorphism in the long exact sequence of the pair $(\Sigma, \mathcal{L})$. One can easily see that ∂_* is an isomorphism. Since $\partial_*([S_1] + [S_2]) = [\ell_1] + [\ell_2] = f([\Sigma])$, we conclude that $[S_1] + [S_2] \in \operatorname{im} d_* = \ker \pi_!$ and hence $\pi_!([S_1]) = \pi_!([S_2])$ is a generator of $H_2(M)$.

The calculation of the intersection form $H_2(M) \times H_2(M) \to H_1(M)$ is now completed as follows:

$$\begin{aligned}[\pi^{-1}(S_1)] \cdot [\pi^{-1}(S_2)] &= [\pi^{-1}(S_1) \cap \pi^{-1}(S_2)] \\ &= [\pi^{-1}(S_1 \cap S_2)] = \ell k(\ell_1, \ell_2) \cdot [\pi^{-1}(w)]. \end{aligned} \qquad \square$$

Remark A.3 Let $\beta \in H^1(M) = \mathbb{Z}/2$ be a generator and assume that $\ell k(\ell_1, \ell_2)$ is odd. Proposition A.2 implies that $\beta \cup \beta \in H^2(M)$ is nontrivial, and a straightforward argument with Poincaré duality shows that $\beta \cup \beta \cup \beta$ generates $H^3(M)$. If $\ell k(\ell_1, \ell_2)$ is even then $\beta \cup \beta = 0$, and the cup product of β with a generator of $H^2(M)$ generates $H^3(M)$. This gives a complete description of the cohomology ring $H^*(M)$.

Example A.4 The real projective space $\mathbb{R}P^3$ is a double branched cover over the Hopf link in S^3 with linking number ± 1. Choose Seifert surfaces S_1 and S_2 to be the obvious disks intersecting in a single interval w. Then $\pi^{-1}(S_1)$ and $\pi^{-1}(S_2)$ are two copies of $\mathbb{R}P^2$, each represented as a double branched cover of a disk with branching set a disjoint union of a circle and a point. These two copies of $\mathbb{R}P^2$ intersect in the circle $\pi^{-1}(w)$, thereby recovering the familiar cup product structure on $H^*(\mathbb{R}P^3; \mathbb{Z}/2)$.

References

[1] **S Akbulut**, **R Kirby**, *Branched covers of surfaces in 4–manifolds*, Math. Ann. 252 (1979/80) 111–131 MR

[2] **N Anvari**, *Equivariant rho-invariants and instanton homology of torus knots*, preprint (2016) arXiv To appear in Comm. Anal. Geom.

[3] **M F Atiyah**, **N J Hitchin**, **I M Singer**, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A 362 (1978) 425–461 MR

[4] **M F Atiyah**, **V K Patodi**, **I M Singer**, *Spectral asymmetry and Riemannian geometry, I*, Math. Proc. Cambridge Philos. Soc. 77 (1975) 43–69 MR

[5] **M F Atiyah**, **V K Patodi**, **I M Singer**, *Spectral asymmetry and Riemannian geometry, II*, Math. Proc. Cambridge Philos. Soc. 78 (1975) 405–432 MR

[6] **M F Atiyah**, **I M Singer**, *The index of elliptic operators, III*, Ann. of Math. 87 (1968) 546–604 MR

[7] **D M Austin**, $\mathrm{SO}(3)$*–instantons on* $L(p,q)\times\mathbf{R}$, J. Differential Geom. 32 (1990) 383–413 MR

[8] **P J Braam**, **S K Donaldson**, *Floer's work on instanton homology, knots and surgery*, from "The Floer memorial volume" (H Hofer, C H Taubes, A Weinstein, E Zehnder, editors), Progr. Math. 133, Birkhäuser, Basel (1995) 195–256 MR

[9] **G Burde**, **H Zieschang**, *Knots*, de Gruyter Studies in Mathematics 5, de Gruyter, Berlin (1985) MR

[10] **W Chen**, **Y Ruan**, *Orbifold Gromov–Witten theory*, from "Orbifolds in mathematics and physics" (A Adem, J Morava, Y Ruan, editors), Contemp. Math. 310, Amer. Math. Soc., Providence, RI (2002) 25–85 MR

[11] **O Collin**, **N Saveliev**, *Equivariant Casson invariants via gauge theory*, J. Reine Angew. Math. 541 (2001) 143–169 MR

[12] **H Donnelly**, *Eta invariants for* G*–spaces*, Indiana Univ. Math. J. 27 (1978) 889–918 MR

[13] **S Dostoglou**, **D Salamon**, *Instanton homology and symplectic fixed points*, from "Symplectic geometry" (D Salamon, editor), London Math. Soc. Lecture Note Ser. 192, Cambridge Univ. Press (1993) 57–93 MR

[14] **A Durfee**, **L Kauffman**, *Periodicity of branched cyclic covers*, Math. Ann. 218 (1975) 157–174 MR

[15] **R Fintushel**, **R J Stern**, *Instanton homology of Seifert fibred homology three spheres*, Proc. London Math. Soc. 61 (1990) 109–137 MR

[16] **A Floer**, *Instanton homology and Dehn surgery*, from "The Floer memorial volume" (H Hofer, C H Taubes, A Weinstein, E Zehnder, editors), Progr. Math. 133, Birkhäuser, Basel (1995) 77–97 MR

[17] **K A Frøyshov**, *Equivariant aspects of Yang–Mills Floer theory*, Topology 41 (2002) 525–552 MR

[18] **Y Fukumoto**, **P Kirk**, **J Pinzón-Caicedo**, *Traceless* $SU(2)$ *representations of* 2*–stranded tangles*, Math. Proc. Cambridge Philos. Soc. 162 (2017) 101–129 MR

[19] **E Harper**, **N Saveliev**, *Instanton Floer homology for two-component links*, J. Knot Theory Ramifications 21 (2012) art. id. 1250054, 8 pages MR

[20] **M Hedden**, **C M Herald**, **P Kirk**, *The pillowcase and perturbations of traceless representations of knot groups*, Geom. Topol. 18 (2014) 211–287 MR

[21] **A Kawauchi**, *A survey of knot theory*, Birkhäuser, Basel (1996) MR

[22] **P A Kirk**, **E P Klassen**, *Representation spaces of Seifert fibered homology spheres*, Topology 30 (1991) 77–95 MR

[23] **E P Klassen**, *Representations of knot groups in* $\mathrm{SU}(2)$, Trans. Amer. Math. Soc. 326 (1991) 795–828 MR

[24] **P B Kronheimer**, **T S Mrowka**, *Khovanov homology is an unknot-detector*, Publ. Math. Inst. Hautes Études Sci. 113 (2011) 97–208 MR

[25] **P B Kronheimer**, **T S Mrowka**, *Knot homology groups from instantons*, J. Topol. 4 (2011) 835–918 MR

[26] **K B Lee**, **F Raymond**, *Seifert fiberings*, Mathematical Surveys and Monographs 166, Amer. Math. Soc., Providence, RI (2010) MR

[27] **R Lee**, **S H Weintraub**, *On the homology of double branched covers*, Proc. Amer. Math. Soc. 123 (1995) 1263–1266 MR

[28] **A Lobb**, **R Zentner**, *On spectral sequences from Khovanov homology*, preprint (2013) arXiv

[29] **W D Neumann**, *An invariant of plumbed homology spheres*, from "Topology Symposium Siegen 1979" (U Koschorke, W D Neumann, editors), Lecture Notes in Math. 788, Springer (1980) 125–144 MR

[30] **W D Neumann**, **F Raymond**, *Seifert manifolds, plumbing, μ–invariant and orientation reversing maps*, from "Algebraic and geometric topology" (K C Millett, editor), Lecture Notes in Math. 664, Springer (1978) 163–196 MR

[31] **D Ruberman**, *Doubly slice knots and the Casson–Gordon invariants*, Trans. Amer. Math. Soc. 279 (1983) 569–588 MR

[32] **D Ruberman**, *Rational homology cobordisms of rational space forms*, Topology 27 (1988) 401–414 MR

[33] **D Ruberman**, **N Saveliev**, *Rohlin's invariant and gauge theory, I: Homology 3–tori*, Comment. Math. Helv. 79 (2004) 618–646 MR

[34] **H Sasahira**, *Instanton Floer homology for lens spaces*, Math. Z. 273 (2013) 237–281 MR

[35] **N N Savel'ev**, *Adding relations to instanton homology groups of Seifert fibered homology spheres*, Mat. Sb. 183 (1992) 125–140 MR In Russian; translated in Russian Acad. Sci. Sb. Math 77 (1994) 497–510

[36] **N Saveliev**, *Floer homology of Brieskorn homology spheres*, J. Differential Geom. 53 (1999) 15–87 MR

[37] **N Saveliev**, *A surgery formula for the $\overline{\mu}$–invariant*, Topology Appl. 106 (2000) 91–102 MR

[38] **N Saveliev**, *Representation spaces of Seifert fibered homology spheres*, Topology Appl. 126 (2002) 49–61 MR

[39] **P Shanahan**, *The Atiyah–Singer index theorem*, Lecture Notes in Math. 638, Springer (1978) MR

[40] **C H Taubes**, *Gauge theory on asymptotically periodic 4–manifolds*, J. Differential Geom. 25 (1987) 363–430 MR

[41] **O J Viro**, *Branched coverings of manifolds with boundary and link invariants, I*, Izv. Akad. Nauk SSSR Ser. Mat. 37 (1973) 1241–1258 MR In Russian; translated in Math. USSR-Izv. 7 (1973) 1239–1256

[42] **S Wang**, *Moduli spaces over manifolds with involutions*, Math. Ann. 296 (1993) 119–138 MR

[43] **R Zentner**, *Representation spaces of pretzel knots*, Algebr. Geom. Topol. 11 (2011) 2941–2970 MR

Department of Mathematics & Statistics, McMaster University
Hamilton, ON, Canada

Department of Mathematics, University of Miami
Coral Gables, FL, United States

ppoudel@math.mcmaster.ca, saveliev@math.miami.edu

http://www.ms.mcmaster.ca/~ppoudel,
http://www.math.miami.edu/~saveliev

Received: 15 June 2015