msp
Algebraic & Geometric Topology 17 (2017) 2825–2840

Vanishing of L^2–Betti numbers and failure of acylindrical hyperbolicity of matrix groups over rings

FENG JI
SHENGKUI YE

Let R be an infinite commutative ring with identity and $n \geq 2$ an integer. We prove that for each integer $i = 0, 1, \ldots, n-2$, the L^2–Betti number $b_i^{(2)}(G)$ vanishes when G is the general linear group $\mathrm{GL}_n(R)$, the special linear group $\mathrm{SL}_n(R)$ or the group $E_n(R)$ generated by elementary matrices. When R is an infinite principal ideal domain, similar results are obtained when G is the symplectic group $\mathrm{Sp}_{2n}(R)$, the elementary symplectic group $\mathrm{ESp}_{2n}(R)$, the split orthogonal group $O(n,n)(R)$ or the elementary orthogonal group $\mathrm{EO}(n,n)(R)$. Furthermore, we prove that G is not acylindrically hyperbolic if $n \geq 4$. We also prove similar results for a class of noncommutative rings. The proofs are based on a notion of n–rigid rings.

20F65

1 Introduction

In this article, we study the s–normality of subgroups of matrix groups over rings together with two applications. Firstly, the low-dimensional L^2–Betti numbers of matrix groups are proved to be zero. Secondly, the matrix groups are proved to be not acylindrically hyperbolic in the sense of Dahmani, Guirardel and Osin [6] and Osin [17]. Let us briefly review the relevant background.

Let G be a discrete group. Denote by

$$l^2(G) = \left\{ f\colon G \to \mathbb{C} \;\middle|\; \sum_{g \in G} \|f(g)\|^2 < +\infty \right\}$$

the Hilbert space with inner product $\langle f_1, f_2 \rangle = \sum_{x \in G} f_1(x)\overline{f_2(x)}$. Let $B(l^2(G))$ be the set of all bounded linear operators on the Hilbert space $l^2(G)$. By definition, the group von Neumann algebra $\mathcal{N}G$ is the completion of the complex group ring $\mathbb{C}[G]$ in $B(l^2(G))$ with respect to the weak operator topology. There is a continuous, additive von Neumann dimension that assigns to every right $\mathcal{N}G$–module M a value $\dim_{\mathcal{N}G}(M) \in [0, \infty]$; see Definition 6.20 of Lück [14]. For a group G, let EG be the universal covering space of its classifying space BG. Denote by $C_*^{\mathrm{sing}}(EG)$ the

Published: 19 September 2017 DOI: 10.2140/agt.2017.17.2825

singular chain complex of EG with the induced $\mathbb{Z}G$–structure. The L^2–homology is the singular homology $H_i^G(EG;\mathcal{N}G)$ with coefficients in $\mathcal{N}G$, ie the homology of the $\mathcal{N}G$–chain complex $\mathcal{N}G\otimes_{\mathbb{Z}G}C_*^{\mathrm{sing}}(EG)$. The i^{th} L^2–Betti number of G is defined by

$$b_i^{(2)}(G):=\dim_{\mathcal{N}G}(H_i^G(EG;\mathcal{N}G))\in[0,\infty].$$

The L^2–homology and L^2–Betti numbers are important invariants of spaces and groups. They have many applications to geometry and K–theory. For more details, see [14].

It has been proved that the L^2–Betti numbers are (almost) zero for several classes of groups including amenable groups, Thompson's group (see [14, Theorem 7.20]), the Baumslag–Solitar group (see Bader, Furman and Sauer [1] and Dicks and Linnell [7]), the mapping class group of a closed surface with genus $g\geq 2$ except $b_{3g-3}^{(2)}$ (see Kida [12], Corollary D.15) and so on; for more information, see [14, Chapter 7]. Let R be an associative ring with identity and $n\geq 2$ an integer. The general linear group $\mathrm{GL}_n(R)$ is the group of all $n\times n$ invertible matrices with entries in R. For an element $r\in R$ and any integers i,j such that $1\leq i\neq j\leq n$, denote by $e_{ij}(r)$ the elementary $n\times n$ matrix with ones in the diagonal positions, r in the $(i,j)^{\text{th}}$ position and zeros elsewhere. The group $E_n(R)$ is generated by all such $e_{ij}(r)$, ie

$$E_n(R)=\langle e_{ij}(r)\mid 1\leq i\neq j\leq n,\ r\in R\rangle.$$

When R is commutative, we define the special linear group $\mathrm{SL}_n(R)$ as the subgroup of $\mathrm{GL}_n(R)$ consisting of matrices with determinant 1. For example, in the case $R=\mathbb{Z}$, the integers, we have that $\mathrm{SL}_n(\mathbb{Z})=E_n(R)$. The groups $\mathrm{GL}_n(R)$ and $E_n(R)$ are important in algebraic K–theory.

In this article, we prove the vanishing of lower L^2–Betti numbers for matrix groups over a large class of rings, including all infinite commutative rings. For this, we introduce the notion of n–rigid rings; for details, see Definition 3.1. Examples of n–rigid (for any $n\geq 1$) rings contain the following (see Section 3):

- infinite integral domains;
- $\mathbb{Z}$–torsion-free infinite noetherian rings (may be noncommutative);
- infinite commutative noetherian rings (moreover, any infinite commutative ring is 2–rigid);
- finite-dimensional algebras over n–rigid rings.

We prove the following results.

Theorem 1.1 *Suppose $n\geq 2$. Let R be an infinite $(n-1)$–rigid ring and $E_n(R)$ the group generated by elementary matrices. For each $i\in\{0,\dots,n-2\}$, the L^2–Betti number $b_i^{(2)}(E_n(R))$ vanishes.*

Since $b_1^{(2)}(E_2(\mathbb{Z})) \neq 0$, this result does not hold for $i = n-1$ in general.

Corollary 1.2 *Let R be any infinite commutative ring and $n \geq 2$. For each $i \in \{0, \dots, n-2\}$, the following L^2–Betti numbers vanish:*

$$b_i^{(2)}(\mathrm{GL}_n(R)) = b_i^{(2)}(\mathrm{SL}_n(R)) = b_i^{(2)}(E_n(R)) = 0.$$

Let $\mathrm{SL}_n(R)$ be a lattice in a semisimple Lie group, eg when $R = \mathbb{Z}$ or a subring of algebraic integers. It follows from results of Borel, which rely on global analysis on the associated symmetric space, that the L^2–Betti numbers of $\mathrm{SL}_n(R)$ vanish except possibly in the middle dimension of the symmetric space; see Borel [5] and Olbrich [16]. In particular, all the L^2–Betti numbers of $\mathrm{SL}_n(\mathbb{Z})$ $(n \geq 3)$ are zero; see Eckmann [8, Example 2.5]. For any infinite integral domain R and any $i \in \{0, \dots, n-2\}$, Bader, Furman and Sauer [1] prove that the L^2–Betti number $b_i^{(2)}(\mathrm{SL}_n(R))$ vanishes. Ershov and Jaikin-Zapirain [9] prove that the noncommutative universal lattice $E_n(\mathbb{Z}\langle x_1, \dots, x_k\rangle)$ (and therefore $E_n(R)$ for any finitely generated associative ring R) has Kazhdan's property (T) for $n \geq 3$. This implies that for any finitely generated associative ring R, the first L^2–Betti number of $E_n(R)$ vanishes; see Bekka and Valette [4].

We consider more matrix groups as follows. Let R be a commutative ring with identity. The symplectic group and the split orthogonal group are defined as

$$\mathrm{Sp}_{2n}(R) = \{A \in \mathrm{GL}_{2n}(R) \mid A^T \varphi_n A = \varphi_n\}, \quad O(n,n)(R) = \{A \in \mathrm{GL}_{2n}(R) \mid A^T \psi_n A = \psi_n\},$$

where A^T is the transpose of A and

$$\varphi_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \psi_n = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

For symplectic and orthogonal groups, we obtain the following.

Theorem 1.3 *Let R be an infinite principal ideal domain (PID), $\mathrm{Sp}_{2n}(R)$ the symplectic group with its elementary subgroup $\mathrm{ESp}_{2n}(R)$, and $O(n,n)(R)$ the orthogonal group with its elementary subgroup $\mathrm{EO}(n,n)(R)$. We have the following.*

(i) *For each $i = 0, \dots, n-2$ $(n \geq 2)$, the following L^2–Betti numbers vanish:*

$$b_i^{(2)}(\mathrm{Sp}_{2n}(R)) = b_i^{(2)}(\mathrm{ESp}_{2n}(R)) = 0.$$

(ii) *For each $i = 0, \dots, n-2$ $(n \geq 2)$, the following L^2–Betti numbers vanish:*

$$b_i^{(2)}(O(n,n)(R)) = b_i^{(2)}(\mathrm{EO}(n,n)(R)) = 0.$$

The proofs of Theorem 1.1 and Theorem 1.3 are based on a study of the notion of weak normality of particular subgroups in matrix groups, introduced in [1] and by Peterson and Thom [18]. We present another application of the weak normality of subgroups in matrix groups as follows.

Acylindrically hyperbolic groups are defined by Dahmani, Guirardel and Osin [6] and Osin [17]. Let G be a group. An isometric G–action on a metric space S is said to be acylindrical if for every $\varepsilon > 0$, there exist $R, N > 0$ such that for every two points $x, y \in S$ with $d(x, y) \geq R$, there are at most N elements $g \in G$ which satisfy $d(x, gx) \leq \varepsilon$ and $d(y, gy) \leq \varepsilon$. A G–action by isometries on a hyperbolic geodesic space S is said to be elementary if the limit set of G on the Gromov boundary ∂S contains at most two points. A group G is called *acylindrically hyperbolic* if G admits a nonelementary acylindrical action by isometries on a (Gromov-δ) hyperbolic geodesic space. The class of acylindrically hyperbolic groups includes nonelementary hyperbolic and relatively hyperbolic groups, mapping class groups of closed surface Σ_g of genus $g \geq 1$, outer automorphism groups $\mathrm{Out}(F_n)$ $(n \geq 2)$ of free groups, directly indecomposable right angled Artin groups, 1–relator groups with at least three generators, most 3–manifold groups, and many other examples.

Although there are many analogies among matrix groups, mapping class groups and outer automorphism groups of free groups, we prove that they are different on acylindrical hyperbolicity, as follows.

Theorem 1.4 *Suppose that n is an integer.*

(i) *Let R be a 2–rigid (eg commutative) ring. The group $E_n(R)$ $(n \geq 3)$ is not acylindrically hyperbolic.*

(ii) *Let R be a commutative ring. The group G is not acylindrically hyperbolic if $G = \mathrm{GL}_n(R)$ $(n \geq 3)$ the general linear group, $\mathrm{SL}_n(R)$ $(n \geq 3)$ the special linear group, $\mathrm{Sp}_{2n}(R)$ $(n \geq 2)$ the symplectic group, $\mathrm{ESp}_{2n}(R)$ $(n \geq 2)$ the elementary symplectic group, $O(n,n)(R)$ $(n \geq 4)$ the orthogonal group, or $\mathrm{EO}(n,n)(R)$ $(n \geq 4)$ the elementary orthogonal group.*

When R is commutative, the failure of acylindrical hyperbolicity of the elementary groups $E_n(R)$, $\mathrm{ESp}_{2n}(R)$ and $\mathrm{EO}(n,n)(R)$ is already known to Mimura [15] by studying property TT for weakly mixing representations. But our approach is different and Theorem 1.4 is more general, even for elementary subgroups. Explicitly, for noncommutative rings, we have the following.

Corollary 1.5 *Let R be a noncommutative $\mathbb{Z}$–torsion-free infinite noetherian ring, an integral group ring over a polycyclic-by-finite group or a finite-dimensional algebra over either. For each nonnegative integer $i \leq n-2$, we have*

$$b_i^{(2)}(E_n(R)) = 0.$$

Furthermore, the group $E_n(R)$ $(n \geq 3)$ is not acylindrically hyperbolic.

2 s–normality

Recall from [1] that the n–step s–normality is defined as follows.

Definition 2.1 Let $n \geq 1$ be an integer. A subgroup H of a group G is called n–step s–normal if for any $(n+1)$–tuple $\omega = (g_0, g_1, \dots, g_n) \in G^{n+1}$, the intersection

$$H^{\omega} := \bigcap_{i=0}^{n} g_i H g_i^{-1}$$

is infinite. A 1–step s–normal group is simply called s–normal.

The following result is proved by Bader, Furman and Sauer; see [1, Theorem 1.3].

Lemma 2.2 *Let H be a subgroup of G. Assume that*

$$b_i^{(2)}(H^{\omega}) = 0$$

for all integers $i, k \geq 0$ with $i + k \leq n$ and every $\omega \in G^{k+1}$. In particular, H is an n–step s–normal subgroup of G. Then for every $i \in \{0, \dots, n\}$,

$$b_i^{(2)}(G) = 0.$$

The following result is important for our later arguments; see [14, Theorem 7.2, (1-2), page 294].

Lemma 2.3 *Let n be any nonnegative integer. Then:*

(i) *For any infinite amenable group G, the L^2–Betti numbers $b_n^{(2)}(G)$ vanish.*

(ii) *Let H be a normal subgroup of a group G with vanishing $b_i^{(2)}(H)$ for each $i \in \{0, 1, \dots, n\}$. Then for each $i \in \{0, 1, \dots, n\}$, we have $b_i^{(2)}(G) = 0$.*

We will also need the following fact; see [17, Corollaries 1.5, 7.3].

Lemma 2.4 *The class of acylindrically hyperbolic groups is closed under taking s–normal subgroups. Furthermore, acylindrically hyperbolic groups have finite center.*

3 Rigidity of rings

We introduce the notion of n–rigidity of rings. For a ring R, all R–modules are right modules and homomorphisms are right R–module homomorphisms.

Definition 3.1 For a positive integer n, an infinite ring R is called *n–rigid* if every R–homomorphism $R^n \to R^{n-1}$ of free modules has an infinite kernel.

A related concept is the strong rank condition: a ring R satisfies the strong rank condition if there is no injection $R^n \to R^{n-1}$ for any n; see Lam [13, page 12]. Clearly, n–rigidity for any n implies the strong rank condition for a ring. Fixing the standard basis of both R^n and R^{n-1}, the kernel of an R–homomorphism $\phi\colon R^n \to R^{n-1}$ corresponds to a system S of $n-1$ linear equations with n unknowns over R:

$$S: \quad \sum_{1\le i\le n} a_{ij}x_i = 0, \quad 1 \le j \le n-1,$$

with $a_{ij} \in R$, $1 \le i \le n$, $1 \le j \le n-1$. Therefore, the strong rank condition asserts that the system S has nontrivial solutions over R, while the n–rigidity property requires that S has infinitely many solutions.

Many rings are n–rigid. For example, infinite integral rings are n–rigid for any n by considering the dimensions over quotient fields. Moreover, let A be a ring satisfying the strong rank condition (eg a noetherian ring, see Theorem 3.15 of [13]). Suppose that A is a torsion-free $\mathbb{Z}$–module, where $\mathbb{Z}$ acts on A via $\mathbb{Z}\cdot 1_A$. Since the kernel $A^n \to A^{n-1}$ is a nontrivial $\mathbb{Z}$–module, the ring A is n–rigid for any n.

We present several basic facts on n–rigid rings as follows.

Lemma 3.2 *n–rigidity implies $(n-1)$–rigidity.*

Proof For any R–homomorphism $f\colon R^{n-1} \to R^{n-2}$, we could add a copy of R as direct summand to get a map $f \oplus \mathrm{id}\colon R^{n-1} \oplus R \to R^{n-2} \oplus R$. The two maps have the same kernel. □

Lemma 3.3 *Let R be an n–rigid ring for any $n \ge 1$. Suppose that an associative ring A is a finite-dimensional R–algebra (ie A is a free R–module of finite rank with compatible multiplications in A and R). Then A is n–rigid for any $n \ge 1$.*

Proof Let $f\colon A^n \to A^{n-1}$ be an A–homomorphism. If we view A as a finite-dimensional R–module, we see that f is also an R–homomorphism. Embed the target A^{n-1} into $R^{n\cdot \mathrm{rank}_R(A)-1}$. The kernel $\ker f$ is infinite by the assumption that R is $n\cdot \mathrm{rank}_R(A)$–rigid. □

Proposition 3.4 *Let R be an n–rigid ring and let $u_1, u_2, \dots, u_{n-1} \in R^m$ $(m \ge n)$ be arbitrary $n-1$ elements. Then the set*

$$\{\phi \in \mathrm{Hom}_R(R^m, R) \mid \phi(u_i) = 0,\ i = 1, 2, \dots, n-1\}$$

is infinite.

Proof When $m = n$, we define an R–homomorphism

$$\mathrm{Hom}_R(R^n, R) \to R^{n-1}, \quad f \mapsto (f(u_1), f(u_2), \dots, f(u_{n-1})).$$

Since $\mathrm{Hom}_R(R^n, R)$ is isomorphic to R^n, such an R–homomorphism has an infinite kernel. When $m > n$, we may project R^m to its last n–components and apply a similar proof. □

Lemma 3.5 *An infinite commutative ring R is 2–rigid.*

Proof Let $f\colon R^2 \to R$ be any R–homomorphism, and let

$$I = \langle xR + yR \mid (x, y) \in \ker f \rangle \trianglelefteq R.$$

Suppose that $\ker f$ is finite. When $(x, y) \in \ker f$, the set xR and yR are also finite. Thus I is finite. Let $a = f((1,0))$ and $b = f((0,1))$. Note that $(-b, a) \in \ker f$. For any $(x, y) \in R^2$, we have $ax + by \in I$. Since the set of right cosets R/I is infinite, we may choose (x, x) and (y, y) with x, y from distinct cosets such that

$$ax + bx = ay + by.$$

However, $(x - y, x - y) \in \ker f$, and thus $x - y \in I$. This is a contradiction. □

To state our result in the most general form, we introduce the following notion.

Definition 3.6 A ring R is called *size-balanced* if any finite right ideal of R generates a finite two-sided ideal of R.

It is immediate that any commutative ring is size-balanced.

Proposition 3.7 *A size-balanced infinite noetherian ring is n–rigid for any n.*

Proof Let $f\colon R^n \to R^{n-1}$ be any R–homomorphism. Let $A = (a_{ij})_{(n-1)\times n}$ be the matrix representation of f with respect to the standard basis, and let

$$I' = \langle x_1 R + x_2 R + \cdots + x_n R \mid (x_1, x_2, \dots, x_n) \in \ker f \rangle \trianglelefteq R.$$

First we notice that I' is nontrivial by the strong rank condition of noetherian rings; see Theorem 3.15 of [13]. Suppose that $\ker f$ is finite. For any

$$(x_1, x_2, \dots, x_n) \in \ker f$$

and $r \in R$, each $(x_1 r, x_2 r, \dots, x_n r) \in \ker f$. As $\ker f$ is finite, each right ideal $x_i R$ is finite, and hence so is I'. Let I be the two-sided ideal generated by the finite right ideal I'. It is finite as R is assumed to be size-balanced. Therefore, the quotient ring R/I is infinite and noetherian.

Let $\overline{f}\colon R/I \to R/I$ be the R/I–homomorphism induced by the matrix $\overline{A} = (\overline{a}_{ij})$, where $\overline{a}_{ij}$ is the image of a_{ij}. If $(\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n) \in \ker \overline{f}$ and x_i is any preimage of $\overline{x}_i$, we have

$$A(x_1, x_2, \dots, x_n)^T \in I^{n-1}.$$

As I is finite, so is I^{n-1}. If $\ker \overline{f}$ is infinite, there are two distinct elements in $\ker \overline{f}$ with preimage $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ in R^n such that

$$A(x_1, x_2, \dots, x_n)^T = A(y_1, y_2, \dots, y_n)^T \in I^{n-1}.$$

However, this implies that

$$(x_1, x_2, \dots, x_n) - (y_1, y_2, \dots, y_n) \in \ker f.$$

We have a contradiction as $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ are distinct in $(R/I)^n$. Therefore, $\ker \overline{f}$ is finite. Moreover, $\ker \overline{f}$ is nontrivial by the strong rank condition of noetherian rings. Let I_1' be the preimage of the right ideal generated by components of elements in $\ker \overline{f}$ in R, which is a finite right ideal by a similar argument as above. It generates a finite two-sided ideal I_1 of R, and it properly contains I.

Repeating the argument, we get an infinite ascending sequence

$$I \subsetneqq I_1 \subsetneqq I_2 \subsetneqq \cdots$$

of finite ideals of R. This is a contradiction to the assumption that R is noetherian. □

Corollary 3.8 *Any commutative ring R containing an infinite noetherian subring is n–rigid for each n.*

Proof Let R_0 be an infinite noetherian subring of R. Let

$$S: \quad \sum_{1 \le i \le n} a_{ij} x_i = 0, \quad 1 \le j \le m$$

be a system of linear equations with $a_{ij} \in R$. Form the infinite commutative subring $R' = R_0[a_{ij}, 1 \le i \le n, 1 \le j \le m]$ of R. By the Hilbert basis theorem, R' is infinite noetherian. Proposition 3.7 asserts that the system S has infinitely many solutions in R', and hence in R. □

Example 3.9 Let G be a polycyclic-by-finite group and $R = \mathbb{Z}[G]$ be its integral group ring. It is known that R is infinite noetherian [11]. Moreover, R is size-balanced by the trivial reason that there are no nontrivial finite right ideals. According to Proposition 3.7, the ring R is n–rigid for any n.

Example 3.10 Let F be a nonabelian free group and $\mathbb{Z}[F]$ the group ring. Since $\mathbb{Z}[F]$ does not satisfy the strong rank condition [13, Exercise 29, page 21], the ring $\mathbb{Z}[F]$ is not n–rigid for any $n \geq 2$.

4 Proofs

Let

$$Q = \left\{ \begin{pmatrix} 1 & x \\ 0 & A \end{pmatrix} \,\middle|\, x \in R^{n-1},\ A \in \mathrm{GL}_{n-1}(R),\ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in E_n \right\}.$$

It is straightforward that Q contains the normal subgroup

$$S = \left\{ \begin{pmatrix} 1 & x \\ 0 & I_{n-1} \end{pmatrix} \,\middle|\, x \in R^{n-1} \right\},$$

an abelian group. Therefore, all the L^2–Betti numbers of S and Q are zero when the ring R is infinite.

Lemma 4.1 *Let $k < n$ $(n \geq 3)$ be two positive integers. Suppose that R is an infinite k–rigid ring. Then the subgroup Q is $(k-1)$–step s–normal in $E_n(R)$. In particular, Q is s–normal if R is infinite 2–rigid.*

Proof Without loss of generality, we assume $k = n-1$. Let $g_1, g_2, \dots, g_{n-2}$ be any $n-2$ elements in $E_n(R)$. We will show that the intersection $Q \cap \bigcap_{i=1}^{n-2} g_i Q g_i^{-1}$ is infinite, which implies the $(k-1)$–step s–normality of H. Let $\{e_i\}_{i=1}^n$ be the standard basis of R^n. Denote by $U = R^{n-1}$ the R–submodule spanned by $\{e_i\}_{i=2}^n$ and $p\colon R^n \to U$ the natural projection.

For each g_i $(i = 1, 2, \dots, n-2)$, suppose that

$$g_i e_1 = x_i e_1 + u_i$$

for $x_i \in R$ and $u_i \in U$. Let

$$\Phi = \{\phi \in \mathrm{Hom}_R(U, R) \mid \phi(u_i) = 0,\ i = 1, 2, \dots, n-2\}.$$

For any $\phi \in \Phi$, define $T_\phi\colon R^n \to R^n$ by $T_\phi(v) = v + \phi \circ p(v) e_1$. It is obvious that

$$g_i^{-1} T_\phi g_i(e_1) = e_1$$

for each $i = 1, 2, \dots, n-2$. Note that Q is the stabilizer of e_1. This shows that for each $\phi \in \Phi$, the transformation T_ϕ lies in $Q \cap \bigcap_{i=1}^{n-2} g_i Q g_i^{-1}$. Denote by T the subgroup

$$T = \{T_\phi \mid \phi \in \Phi\}. \tag{1}$$

By Proposition 3.4, Φ is infinite, and thus T is infinite. The proof is finished. □

Lemma 4.2 *The subgroup T from* (1) *is normal in* $Q \cap \bigcap_{i=1}^{n-2} g_i Q g_i^{-1}$.

Proof For any ϕ, write $e_\phi = (\phi(e_2), \dots, \phi(e_n))$. With respect to the standard basis, the representation matrix of the transformation T_ϕ is $\begin{pmatrix} 1 & e_\phi \\ 0 & I_{n-1} \end{pmatrix}$. For any $\begin{pmatrix} 1 & x \\ 0 & A \end{pmatrix} \in Q \cap \bigcap_{i=1}^{n-2} g_i Q g_i^{-1}$, the conjugate of the representation matrix has the following form:

$$\begin{pmatrix} 1 & x \\ 0 & A \end{pmatrix}^{-1} \begin{pmatrix} 1 & e_\phi \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & A \end{pmatrix} = \begin{pmatrix} 1 & e_\phi A \\ 0 & I_{n-1} \end{pmatrix}.$$

Define $\psi\colon U = R^{n-1} \to R$ by $\psi(x) = e_\phi A x$. For each $i = 1, \dots, n-2$, we have that $\begin{pmatrix} 1 & x \\ 0 & A \end{pmatrix} = g_i q_i g_i^{-1}$ for some $q_i \in Q$. Therefore,

$$\begin{pmatrix} 1 & x \\ 0 & A \end{pmatrix} g_i e_1 = g_i q_i e_1,$$

and $A u_i = u_i$. This implies that $\psi(u_i) = e_\phi u_i = 0$ for each i, and thus $\psi \in \Phi$. Therefore, the conjugate $\begin{pmatrix} 1 & e_\phi A \\ 0 & I_{n-1} \end{pmatrix}$ lies in T, which proves that T is normal. □

Proof of Theorem 1.1 By Lemma 4.2, any intersection $Q \cap \bigcap_{i=1}^{n-2} g_i Q g_i^{-1}$ contains an infinite normal amenable subgroup T. Therefore, all the L^2–Betti numbers of any intersection $Q \cap \bigcap_{i=1}^{k} g_i Q g_i^{-1}$ vanish for $k \le n-2$ considering Lemma 3.2. We have that $b_i^{(2)}(E_n(R)) = 0$ for any $0 \le i \le n-2$ by Lemma 2.2. □

Proof of Corollary 1.2 When $n = 2$, it is clear that both $\mathrm{GL}_n(R)$ and $\mathrm{SL}_n(R)$ are infinite, since $E_2(R)$ is an infinite subgroup. Thus $b_0^{(2)}(\mathrm{GL}_2(R)) = b_0^{(2)}(\mathrm{SL}_2(R)) = 0$. We have already proved that $b_i^{(2)}(E_n(S)) = 0$ for infinite commutative noetherian ring S and $0 \le i \le n-2$, since the ring S would be k–rigid for any integer k by Proposition 3.7. If S is a finite subring of R, the group $E_n(S)$ is also finite. Therefore, we still have $b_i^{(2)}(E_n(S)) = 0$ for $1 \le i \le n-2$. Note that every commutative ring R is the directed colimit of its subrings S that are finitely generated as $\mathbb{Z}$–algebras (noetherian rings by the Hilbert basis theorem). Since the group $E_n(R)$ is the union of the directed system of subgroups $E_n(S)$, we get that

$$b_i^{(2)}(E_n(R)) = 0$$

for $0 \le i \le n-2$; see [14, Theorem 7.2(3)] and its proof. When R is commutative and $n \ge 3$, a result of Suslin [19] says that the group $E_n(R)$ is a normal subgroup of $\mathrm{GL}_n(R)$ and $\mathrm{SL}_n(R)$. Lemma 2.3 implies that $b_i^{(2)}(\mathrm{GL}_n(R)) = b_i^{(2)}(\mathrm{SL}_n(R)) = 0$ for each $i \in \{0, \dots, n-2\}$. □

We follow [2] to define the elementary subgroups of symplectic groups and orthogonal groups. Let E_{ij} denote the $n \times n$ matrix with 1 in the $(i,j)^{\text{th}}$ position and zeros elsewhere. Then for $i \neq j$, the matrix $e_{ij}(a) = I_n + aE_{ij}$ is an elementary matrix, where I_n is the identity matrix of size n. With n fixed, for any integer $1 \le k \le 2n$, set $\sigma k = k + n$ if $k \le n$ and $\sigma k = k - n$ if $k > n$. For $a \in R$ and $1 \le i \neq j \le 2n$, we define the elementary unitary matrices $\rho_{i,\sigma i}(a)$ and $\rho_{ij}(a)$ with $j \neq \sigma i$ as follows:

- $\rho_{i,\sigma i}(a) = I_{2n} + aE_{i,\sigma i}$ with $a \in R$.
- Fix $\varepsilon = \pm 1$. We define $\rho_{ij}(a) = \rho_{\sigma j,\sigma i}(-a') = I_{2n} + aE_{ij} - a'E_{\sigma j,\sigma i}$ with $a' = a$ when $i, j \le n$; $a' = \varepsilon a$ when $i \le n < j$; $a' = a\varepsilon$ when $j \le n < i$; and $a' = a$ when $n + 1 \le i, j$.

When $\varepsilon = -1$, we have the elementary symplectic group

$$\mathrm{ESp}_{2n}(R) = \langle \rho_{i,\sigma i}(a),\ \rho_{ij}(a) \mid a \in R,\ i \neq j,\ i \neq \sigma j \rangle.$$

When $\varepsilon = 1$, we have the elementary orthogonal group

$$\mathrm{EO}(n,n)(R) = \langle \rho_{ij}(a) \mid a \in R,\ i \neq j,\ i \neq \sigma j \rangle.$$

Note that for the orthogonal group, each matrix $\rho_{i,\sigma i}(a)$ is not in $\mathrm{EO}(n,n)(R)$.

There is an obvious embedding

$$\mathrm{Sp}_{2n}(R) \to \mathrm{Sp}_{2n+2}(R), \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & 1 & 0 \\ 0 & \gamma & 0 & \delta \end{pmatrix}.$$

Denote the image of $A \in \mathrm{Sp}_{2n}(R)$ by $I \oplus A \in \mathrm{Sp}_{2n+2}(R)$. Let

$$Q_1 = \left\langle (I \oplus A) \cdot \prod_{i=1}^{2n} \rho_{1i}(a_i) \;\middle|\; a_i \in R,\ A \in \mathrm{Sp}_{2n-2}(R),\ I \oplus A \in \mathrm{ESp}_{2n}(R) \right\rangle$$

and

$$S_1 = \left\langle \prod_{i=1}^{2n} \rho_{1i}(a_i) \;\middle|\; a_i \in R \right\rangle.$$

Similarly, we can define

$$Q_2 = \left\langle (I \oplus A) \prod_{\substack{1 \le i \le 2n \\ i \neq n+1}} \rho_{1i}(a_i) \;\middle|\; a_i \in R,\ A \in O(2n-2, 2n-2)(R),\ I \oplus A \in \mathrm{EO}(n,n)(R) \right\rangle$$

and

$$S_2 = \left\langle \prod_{\substack{1 \le i \le 2n \\ i \neq n+1}} \rho_{1i}(a_i) \;\middle|\; a_i \in R \right\rangle.$$

Since S_i is abelian and normal in Q_i, all the L^2–Betti numbers of Q_i vanish for $i = 1, 2$.

Proof of Theorem 1.3 We prove the theorem by induction on n. When $n=2$, both $\mathrm{Sp}_{2n}(R)$ and $O(n,n)(R)$ are infinite, and therefore we have

$$b_0^{(2)}(\mathrm{Sp}_4(R)) = b_0^{(2)}(O(4,4)(R)) = 0.$$

The subgroup $\mathrm{ESp}_{2n}(R)$ is normal in $\mathrm{Sp}_{2n}(R)$ when $n \geq 2$, and the subgroup $\mathrm{EO}(n,n)(R)$ is normal in $O(n,n)(R)$ when $n \geq 3$; see [3, Corollary 3.10]. It suffices to prove the vanishing of Betti numbers for $G = \mathrm{ESp}_{2n}(R)$ and $\mathrm{EO}(n,n)(R)$.

We check the condition of Lemma 2.2 for $Q = Q_1$ (resp. Q_2) as follows. Note that

$$Q = \{g \in G \mid ge_1 = e_1\}.$$

Let $g_1, g_2, \dots, g_k$ ($g_0 = I_{2n}$, $k \leq n-2$) be any k elements in G and

$$K = \langle g_0e_1, g_1e_1, \dots, g_ke_1\rangle$$

the submodule in R^{2n} generated by all g_ie_1. Recall that the symplectic (resp. orthogonal) form $\langle -,-\rangle\colon R^{2n} \times R^{2n} \to R$ is defined by $\langle x,y\rangle = x^T\varphi_n y$ (resp. $\langle x,y\rangle = x^T\psi_n y$). Let

$$C := \{v \in R^{2n} \mid \langle v, g_ie_1\rangle = 0 \text{ for each } i = 0,\dots,k-1\}.$$

Let $\varepsilon = -1$ for $\mathrm{ESp}_{2n}(R)$ and $\varepsilon = 1$ for $\mathrm{EO}(n,n)(R)$. For each $r \in R$, set $\delta_\varepsilon^r = r$ if $\varepsilon = -1$ and $\delta_\varepsilon^r = 0$ if $\varepsilon = 1$. For each $u, v \in C$ with $\langle u,u\rangle = \langle u,v\rangle = \langle v,v\rangle = 0$, define the transvections in G (see [20, page 287], Eichler transformations in [10, pages 214, 223–224])

$$\tau(u,v)\colon R^{2n} \to R^{2n}, \quad x \mapsto x + \varepsilon u\langle v,x\rangle - v\langle u,x\rangle,$$
$$\tau_{v,r}\colon R^{2n} \to R^{2n}, \quad x \mapsto x - \delta_\varepsilon^r v\langle v,x\rangle.$$

Note that $\tau_{v,r}$ is nonidentity only in $\mathrm{ESp}_{2n}(R)$. We have

$$\tau(u,v)(g_ie_1) = \tau_{v,r}(g_ie_1) = g_ie_1$$

for each i. Therefore, the transvections $\tau(u,v), \tau_{v,r} \in \bigcap_{i=0}^k g_iQg_i^{-1}$. Let

$$T = \langle \tau(u,v), \tau_{v,r} \mid u,v \in C,\ \langle u,u\rangle = \langle u,v\rangle = \langle v,v\rangle = 0,\ r \in R\rangle$$

be the subgroup generated by the transvections in G. For any $g \in \bigcap_{i=0}^k g_iQg_i^{-1}$, we have $gg_ie_1 = g_ie_1$, and thus

$$\langle gu, g_ie_1\rangle = \langle gu, gg_ie_1\rangle = \langle u, g_ie_1\rangle = 0.$$

This implies that $g\tau(u,v)g^{-1} = \tau(gu,gv) \in T$ and $g\tau_{v,r}g^{-1} = \tau_{gv,r} \in T$. Therefore, the subgroup T is a normal subgroup in $\bigcap_{i=0}^k g_iQg_i^{-1}$.

When R is a PID, the submodule K and the complement C are free of smaller ranks.

Case (i) $\boldsymbol{K \cap C = 0}$ Since $R^{2n} = K \oplus C$ (note that each $g_i e_1$ is unimodular), the symplectic (resp. orthogonal) form on R^{2n} restricts to a nondegenerate symplectic (resp. orthogonal) form on C. Let $T < G$ be as defined before. It is known that the transvections generate the elementary subgroups [10, pages 223–224], and thus $T \cong \mathrm{ESp}_{2m}(R)$ (resp. $\mathrm{EO}(m,m)(R)$) for $m = \mathrm{rank}(C) \le n-2$. Since $k \le n-2$, we have $m \ge 4$. By induction,

$$b_s^{(2)}\Bigl(\bigcap_{i=0}^{k} g_i Q g_i^{-1}\Bigr) = b_s^{(2)}(T) = 0$$

for $s \le \frac{1}{2}\mathrm{rank}(C) - 2$. When $s + k \le n-2$, we have that $s \le \frac{1}{2}\mathrm{rank}(C) - 2$ since $\mathrm{rank}(C) \ge 2n-(k+1)$. Therefore, $b_s^{(2)}\bigl(\bigcap_{i=0}^{k} g_i Q_1 g_i^{-1}\bigr) = 0$, and Lemma 2.2 implies that for any $i \le n-2$,

$$b_i^{(2)}(G) = 0.$$

Case (ii) $\boldsymbol{K \cap C \neq 0}$ For any $u, v \in K \cap C$ and any $g \in \bigcap_{i=0}^{k} g_i Q g_i^{-1}$, we have that $gu = u$, $gv = v$ and

$$g\tau(u,v)g^{-1} = \tau(gu, gv) = \tau(u,v).$$

This implies that $\tau(u,v)$ lies in the center of $\bigcap_{i=0}^{k} g_i Q g_i^{-1}$. Note that when $G = \mathrm{ESp}_{2n}(R)$, the transvection $\tau(u,u)$ is not trivial for any $u \in K \cap C$. When $G = \mathrm{EO}(n,n)(R)$ and $\mathrm{rank}(K \cap C) \ge 2$, the transvection $\tau(u,v)$ is not trivial for any linearly independent $u, v \in K \cap C$. Moreover, for two elements r, s with $r^2 \neq s^2$, we have $\tau(ru, rv) \neq \tau(su, sv)$ when $\tau(u,v) \neq I_{2n}$ (take note that for $G = \mathrm{ESp}_{2n}(R)$, we can just let $u = v$ from above). The infinite PID R contains infinitely many square elements. In summary, as $K \cap C$ is a free R–module, the subgroup

$$T' = \langle \tau(u,v) \mid v \in K \cap C \rangle < G$$

is an infinite abelian normal subgroup of $\bigcap_{i=0}^{k} g_i Q g_i^{-1}$. Therefore,

$$b_s^{(2)}\Bigl(\bigcap_{i=0}^{k} g_i Q g_i^{-1}\Bigr) = b_s^{(2)}(T') = 0$$

for each integer $s \ge 0$. Therefore, for any $i \le n-2$, we have that $b_i^{(2)}(G) = 0$ by Lemma 2.2.

The remaining situation is that $G = \mathrm{EO}(n,n)(R)$ and $\mathrm{rank}(K \cap C) = 1$. Choose the decomposition $C = (K \cap C) \oplus C_1$. The orthogonal form restricts to a nondegenerate orthogonal form on C_1. (Suppose that for some $x \in C_1$, we have $\langle x, y\rangle = 0$ for any $y \in C_1$. Since $\langle x, k\rangle = 0$ for any $k \in K$, we know that $\langle x, y\rangle = 0$ for any $y \in C$. This implies $x \in K$, which gives $x = 0$.) Since $k \le n-2$, the even number $\operatorname{rank}(C_1) \ge 4$. A similar argument as in case (i) finishes the proof. □

Remark 4.3 Let T be the normal subgroup of $\bigcap_{i=0}^{k} g_i Q g_i^{-1}$ constructed in the proof of Theorem 1.3. We do not know whether the L^2–Betti numbers $b_i^{(2)}(T)$ vanish for a general infinite $(2n-1)$–rigid commutative ring R when $i \le n-2-k$. If so, Theorem 1.3 would hold for any general infinite commutative ring by a similar argument as in the proof of Corollary 1.2.

Proof of Theorem 1.4 Note that when R is commutative, the elementary subgroups $E_n(R), \mathrm{ESp}_{2n}(R)$ and $\mathrm{EO}(n,n)(R)$ are normal in $\mathrm{SL}_n(R), \mathrm{Sp}_{2n}(R)$ and $O(n,n)(R)$, respectively; see [19; 3, Corollary 3.10]. Therefore, it is enough to prove the failure of acylindrically hyperbolicity for elementary subgroups. We prove (i) first. If R is finite, all the groups will be finite and thus not acylindrically hyperbolic. If R is infinite, then it is 2–rigid, and the subgroup Q is s–normal by Lemma 4.1. Suppose that $E_n(R)$ is acylindrically hyperbolic. Lemma 2.4 implies that both Q and S are acylindrically hyperbolic. However, the subgroup S is infinite abelian, which is a contradiction to the second part of Lemma 2.4.

For (ii), we may also assume that R is infinite since any finite group is not acylindrically hyperbolic. It suffices to prove that Q_1 (resp. Q_2) is s–normal in $\mathrm{ESp}_{2n}(R)$ (resp. $\mathrm{EO}(n,n)(R)$). (Note that Q_1 and Q_2 contain the infinite normal subgroups S_1 and S_2, respectively. If G is acylindrically hyperbolic, the infinite abelian subgroup S_1 or S_2 would be acylindrically hyperbolic. This is a contradiction to the second part of Lemma 2.4.) By definition, this is equivalent to proving that for any $g \in G$, the intersection $Q \cap g^{-1}Qg$ is infinite for $Q = Q_1$ and $Q = Q_2$. Let $ge_1 = (x_1, \dots, x_n, y_1, \dots, y_n)^T$. Let

$$t_A = \prod_{1 \le i < j \le n} \rho_{i,n+j}(a_{ij}) = \begin{pmatrix} I_n & A \\ 0 & I_n \end{pmatrix} \in G,$$

where $A = (a_{ij})$ is an $n \times n$ matrices with entries in R. Note that $a_{ji} = a_{ij}$ if $G = \mathrm{ESp}_{2n}(R)$ and $a_{ij} = -a_{ij}$ if $G = \mathrm{EO}(n,n)(R)$. Moreover, we have $\rho_{i,n+i}(a) \notin \mathrm{EO}(n,n)(R)$ and $\rho_{i,n+i}(a) \in \mathrm{ESp}_{2n}(R)$ for any $a \in R$. Direct calculation shows that $t_A(ge_1) - ge_1 = ((y_1, \dots, y_n)A^T, 0, \dots, 0)^T$. When $n \ge 4$ and $G = \mathrm{EO}(n,n)(R)$, the map $f\colon R^{n(n-1)/2} \to R^n$ defined by

$$(a_{ij})_{1 \le i < j \le n} \mapsto A(y_1, \dots, y_n)^T$$

has an infinite kernel $\ker f$ by 2–rigidity of infinite commutative rings. This implies that $\langle t_A \mid (a_{ij})_{1\le i<j\le n} \in \ker f \rangle < Q \cap g^{-1}Qg$ is infinite. When $n \ge 2$ and $G = \mathrm{ESp}_{2n}(R)$, the map $f\colon R^{n(n+1)/2} \to R^n$ defined by $(a_{ij})_{1\le i\le j\le n} \mapsto A(y_1,\dots,y_n)^T$ has an infinite kernel $\ker f$, and $Q \cap g^{-1}Qg$ is infinite by a similar argument. □

Proof of Corollary 1.5 By Proposition 3.7, Example 3.9 and Lemma 3.3, all these rings are n–rigid for any $n \ge 1$. The corollary follows Theorems 1.1 and 1.4. □

Acknowledgements The authors would like to thank the referee for pointing out a gap in a previous version of this paper. Ye is supported by Jiangsu Natural Science Foundation (No. BK20140402) and NSFC (No. 11501459).

References

[1] **U Bader**, **A Furman**, **R Sauer**, *Weak notions of normality and vanishing up to rank in L^2–cohomology*, Int. Math. Res. Not. 2014 (2014) 3177–3189 MR

[2] **A Bak**, *K–theory of forms*, Annals of Mathematics Studies 98, Princeton Univ. Press (1981) MR

[3] **A Bak**, **N Vavilov**, *Normality for elementary subgroup functors*, Math. Proc. Cambridge Philos. Soc. 118 (1995) 35–47 MR

[4] **M E B Bekka**, **A Valette**, *Group cohomology, harmonic functions and the first L^2–Betti number*, Potential Anal. 6 (1997) 313–326 MR

[5] **A Borel**, *The L^2–cohomology of negatively curved Riemannian symmetric spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. 10 (1985) 95–105 MR

[6] **F Dahmani**, **V Guirardel**, **D Osin**, *Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces*, Mem. Amer. Math. Soc. 1156, Amer. Math. Soc., Providence, RI (2017) MR

[7] **W Dicks**, **P A Linnell**, *L^2–Betti numbers of one-relator groups*, Math. Ann. 337 (2007) 855–874 MR

[8] **B Eckmann**, *Lattices, l_2–Betti numbers, deficiency, and knot groups*, Enseign. Math. 50 (2004) 123–137 MR

[9] **M Ershov**, **A Jaikin-Zapirain**, *Property (T) for noncommutative universal lattices*, Invent. Math. 179 (2010) 303–347 MR

[10] **A J Hahn**, **O T O'Meara**, *The classical groups and K–theory*, Grundl. Math. Wissen. 291, Springer (1989) MR

[11] **A V Jategaonkar**, *Integral group rings of polycyclic-by-finite groups*, J. Pure Appl. Algebra 4 (1974) 337–343 MR

[12] **Y Kida**, *The mapping class group from the viewpoint of measure equivalence theory*, Mem. Amer. Math. Soc. 916, Amer. Math. Soc., Providence, RI (2008) MR

[13] **T Y Lam**, *Lectures on modules and rings*, Graduate Texts in Mathematics 189, Springer (1999) MR

[14] **W Lück**, *L^2–invariants: theory and applications to geometry and K–theory*, Ergeb. Math. Grenzgeb. 44, Springer (2002) MR

[15] **M Mimura**, *Superrigidity from Chevalley groups into acylindrically hyperbolic groups via quasi-cocycles*, preprint (2015) arXiv to appear in J. Eur. Math. Soc.

[16] **M Olbrich**, *L^2–invariants of locally symmetric spaces*, Doc. Math. 7 (2002) 219–237 MR

[17] **D Osin**, *Acylindrically hyperbolic groups*, Trans. Amer. Math. Soc. 368 (2016) 851–888 MR

[18] **J Peterson**, **A Thom**, *Group cocycles and the ring of affiliated operators*, Invent. Math. 185 (2011) 561–592 MR

[19] **A A Suslin**, *The structure of the special linear group over rings of polynomials*, Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977) 235–252 MR In Russian; translated in Math. USSR-Izv. 11 (1977) 221–238

[20] **L N Vaseršteĭn**, *Stabilization for the classical groups over rings*, Mat. Sb. 93(135) (1974) 268–295 MR In Russian; translated in Math. USSR-Sb. 22 (1974) 271–303

Infinitus, Nanyang Technological University
Singapore

Department of Mathematical Sciences, Xi'an Jiaotong-Liverpool University
Jiangsu, China

jifeng@ntu.edu.sg, shengkui.ye@xjtlu.edu.cn

https://yeshengkui.wordpress.com/

Received: 15 August 2016 Revised: 18 February 2017