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Klein-four connections and the Casson invariant for nontrivial admissible $U(2)$ bundles

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Given a rank-2 hermitian bundle over a 3–manifold that is nontrivial admissible in the sense of Floer, one defines its Casson invariant as half the signed count of its projectively flat connections, suitably perturbed. We show that the 2–divisibility of this integer invariant is controlled in part by a formula involving the mod 2 cohomology ring of the 3–manifold. This formula counts flat connections on the induced adjoint bundle with Klein-four holonomy.

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1 Introduction

Let E be a $U(2)$ bundle over a closed, oriented and connected 3–manifold Y with the property that $w_2(E) \equiv c_1(E) \pmod 2$ has no torsion lifts to $H^2(Y;\mathbb{Z})$. Following Floer [4], we call such bundles *nontrivial admissible*. Floer defined the instanton homology $I_*(Y,E)$, which is an abelian group that is $\mathbb{Z}_2$–graded. Define $\lambda(Y,E)$ to be half the Euler characteristic of the instanton homology:

$$\lambda(Y,E) = \tfrac{1}{2}\chi[I_*(Y,E)].$$

This number is a signed count of suitably perturbed projectively flat connections on E. It is well known that $\lambda(Y,E)$ is an integer. Define the subset of triples

$$V_Y = \big\{\{a,b,c\} \subset H^1(Y;\mathbb{Z}_2) : a+b+c=0\big\}.$$

This set is naturally in correspondence with the set of subspaces of the $\mathbb{Z}_2$–vector space $H^1(Y;\mathbb{Z}_2)$ of dimension at most two. Write $b_1(2)$ for the $\mathbb{Z}_2$–dimension of $H_1(Y;\mathbb{Z}_2)$. Define for any given $x \in H^2(Y;\mathbb{Z}_2)$ the following nonnegative integer:

$$v_Y(x) = \Big|\big\{\{a,b,c\} \in V_Y : ab+bc+ac = x\big\}\Big|.$$

For the case in which $x = w_2(E)$ we simply write $v_Y(E)$.

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Theorem 1.1 *Suppose E is a nontrivial admissible $U(2)$ bundle over a closed, oriented, connected 3–manifold Y with $b_1(2)\geq 3$. Then $\lambda(Y,E)$ is divisible by $2^{b_1(2)-3}$. Furthermore, we have*

$$2^{3-b_1(2)}\lambda(Y,E)\equiv v_Y(E) \pmod 2. \tag{1}$$

If $b_1(2)=2$, this congruence also holds, implying that $v_Y(E)$ is even. If $b_1(2)=1$, then the integer $v_Y(E)$ is zero. In these two cases $v_Y(E) \pmod 2$ yields no information about $\lambda(Y,E)$.

Note that Y supports a nontrivial admissible bundle if and only if $b_1(Y)\geq 1$, where $b_1(Y)$ denotes the rank of $H_1(Y;\mathbb{Z})$. In general we have $b_1(2)\geq b_1(Y)$, with strict inequality if and only if $H_1(Y;\mathbb{Z})$ has 2–torsion. Theorem 1.1 and its proof are generalizations of a rather simple idea due to Ruberman and Saveliev [13]. Their result is the case of Theorem 1.1 when $H_1(Y;\mathbb{Z})$ is free abelian of rank 3, ie when Y is a homology 3–torus. To obtain their statement, one identifies $v_Y(E)$ with the triple cup product modulo 2, which for a homology 3–torus is a simple computation. (More generally, see Corollary 1.6.) Our adaptation of Ruberman and Saveliev's argument is summarized, modulo perturbations, as follows.

The invariant $\lambda(Y,E)$ is one half of a signed count of projectively flat connections on the bundle E. There is an action of $H^1(Y;\mathbb{Z}_2)$ on this set of connections, and the quotient is identified with flat connections on the adjoint SO(3) bundle induced by E. The only possible stabilizers of this action are $\{1\}$, $\mathbb{Z}_2$ and V_4, the Klein-four group isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$. Further, the connections with stabilizer V_4 are flat connections with holonomy group V_4. The number $v_Y(E)$ is the number of connections on the induced SO(3) bundle with holonomy V_4, up to gauge equivalence. The proof of Theorem 1.1 follows from counting the $H^1(Y;\mathbb{Z}_2)$–orbits with stabilizer V_4.

Vanishing conditions, and relation to Lescop's invariant The right-hand quantity $v_Y(E) \pmod 2$ of congruence (1) is often, but not always, equal to zero. The parity also turns out to be independent of our choice of nontrivial admissible bundle E. To state the result:

$$k(Y):=\dim_{\mathbb{Z}_2}\{a\in H^1(Y;\mathbb{Z}_2): a^2=0\}=\dim_{\mathbb{Z}_2}\ker(\beta^1).$$

Here β^1 is the Bockstein homomorphism defined on $H^1(Y;\mathbb{Z}_2)$ associated to the coefficient exact sequence $0\to\mathbb{Z}_2\to\mathbb{Z}_4\to\mathbb{Z}_2\to 0$. As is well known, $\beta^1(a)=a^2$. We note that if $H_1(Y;\mathbb{Z})$ is written as a direct sum of prime-power-order cyclic summands and copies of $\mathbb{Z}$, then $k(Y)$ is just the number of $\mathbb{Z}_{2^k}$ summands with $k>1$, plus the number of $\mathbb{Z}$ summands. In particular, $k(Y)\geq b_1(Y)$.

Theorem 1.2 *Let Y be a closed, oriented and connected 3–manifold with $k(Y) \geq 1$. Let $x \in H^2(Y;\mathbb{Z}_2)$ be any element that is not a cup-square. Then $v_Y(x) \pmod 2$ is independent of the choice of such x. If furthermore $k(Y) \geq 4$ then $v_Y(x) \equiv 0 \pmod 2$.*

Note that the statement holds for a larger class of elements $x \in H^2(Y;\mathbb{Z}_2)$ than those just coming from admissible bundles. The conditions are best understood through the following examples, which are surgeries on the Borromean rings; see Figure 1. These examples have $b_1(Y) = 0$ and $b_1(2) = 3$.

Example 1.3 Consider the 3–manifold Y obtained by performing $(2,2,4)$ surgery on the Borromean rings. Such a manifold has first homology group isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$. Then $k(Y) = 1$. The rank-3 vector space $H^1(Y;\mathbb{Z}_2)$ has a basis formed by a, b, c, classes that are Poincaré dual to the meridians of the surgery loops. By intersecting homology classes and using Poincaré duality we obtain

$$c^2 = 0, \quad a^2 = bc, \quad b^2 = ac,$$

where ab, bc, ac form a basis of $H^2(Y;\mathbb{Z}_2)$. Now, ab is not a square, as are not $ab + bc$, $ab + ac$ or $ab + ac + bc$. All four of these elements have $v_Y(x) = 1 \in \mathbb{Z}$. On the other hand, all other elements in $H^2(Y;\mathbb{Z}_2)$ have $v_Y(x) \in \{0,2,4\}$. This illustrates the necessity of the nonsquare condition on x.

Example 1.4 Next, consider $(2,4,4)$ surgery on the Borromean rings. The $\mathbb{Z}_2$–cohomology ring is much the same as before, except now $b^2 = 0$, and $k(Y) = 2$. All nonzero $x \in H^2(Y;\mathbb{Z}_2)$ have $v_Y(x)$ odd. In fact, if $x \neq 0$, then $v_Y(x) = 1$, while $v_Y(a^2) = 5$ and $v_Y(0) = 4$. Here a^2 is a cup-square, but does not have a different parity from the other nonzero elements.

Example 1.5 Finally, $(4,4,4)$ surgery on the Borromean rings has the same $\mathbb{Z}_2$–cohomology ring as that of the 3–torus. Here $k(Y) = 3$, and $v_Y(x) = 1$ for $x \neq 0$, all nonsquares, while $v_Y(0) = 8$.

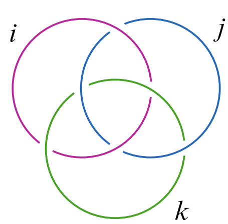

Figure 1: Surgery on the Borromean rings with framings (i,j,k) on the three components. When i, j, k are either 0 or various powers of 2, these surgeries yield nonvanishing examples of the congruence in Theorem 1.1, in which $v_Y(E) \equiv 1 \pmod 2$ and $k(Y) = 1,2,3$.

To make use of Theorem 1.1, one can replace the 4–framings in the above three examples by 0–framings, to get manifolds with the same $\mathbb{Z}_2$–cohomology rings but $b_1(Y) > 0$, ensuring that they support nontrivial admissible bundles.

In what follows, we describe how to deduce Theorem 1.2 using Theorem 1.1 and related results of Poudel [11] and Turaev [16]. By Poudel [11], the Casson invariant $\lambda(Y, E)$ may be identified with Lescop's invariant of [9], slightly modified. The proof utilizes Floer's exact triangle for instanton homology and Dehn surgery techniques à la Lescop [9]. As a result, the parity of $v_Y(E)$ is independent of E, the choice of nontrivial admissible bundle. After some substitutions, the congruences resulting from Theorem 1.1 and [11] may be summarized as follows.

Corollary 1.6 *Suppose $x \in H^2(Y;\mathbb{Z}_2)$ has no torsion lifts to $H^2(Y;\mathbb{Z})$. Then,* mod 2,

$$v_Y(x) \equiv \begin{cases} 2^{2-b_1(2)}\Delta''_Y(1) & \text{if } b_1(Y)=1, \\ 2^{3-b_1(2)}(\#(\gamma \cap F)) & \text{if } b_1(Y)=2, \\ 2^{3-b_1(2)}N\cdot(a\cup b\cup c)[Y] & \text{if } b_1(Y)=3, \\ 0 & \text{if } b_1(Y)\geq 4, \end{cases} \tag{2}$$

where N is the cardinality of Tor $H_1(Y;\mathbb{Z})$ *and other terms are defined below. In particular, if $b_1(Y)=3$ and $H_1(Y;\mathbb{Z})$ has an order-*4 *element, then $v_Y(x)\equiv 0$* (mod 2).

The right-hand sides are defined as follows. First, for $b_1(Y) = 1$, $\Delta_Y(t)$ is the Alexander polynomial of Y, normalized so that $\Delta_Y(1) = 1$ and $\Delta_Y(t) = \Delta_Y(t^{-1})$. If Y is 0–surgery on a knot K in an integral homology 3–sphere Σ, then $\Delta_Y(t)$ is just the Alexander polynomial $\Delta_{K\subset\Sigma}(t)$. Next, suppose $b_1(Y) = 2$. Take two oriented surfaces in Y that generate $H_2(Y;\mathbb{Q})$. Let γ be their intersection, and γ' the curve parallel to γ that induces the trivialization of the tubular neighborhood of γ given by the surfaces. Then $N\cdot\gamma'$ has a Seifert surface F in Y, and $\#(\gamma\cap F)$ is the count of intersection points, in general position. Finally, in the $b_1(Y) = 3$ case, the triple a, b, c generates $H^1(Y;\mathbb{Z})$ up to torsion, and $[Y]$ is the fundamental class of Y.

The vanishing implications of Corollary 1.6 look rather similar to those of Theorem 1.2, except that the role of $k(Y)$ is weakened to that of $b_1(Y)$. In other words, the role of counting summands of the form $\mathbb{Z}$ and $\mathbb{Z}_{2^k}$ for $k > 1$ is replaced by that of just counting $\mathbb{Z}$ summands. From the perspective of the $\mathbb{Z}_2$–cohomology ring, these kinds of summands are all the same. With this thought in mind, it is a rather straightforward task to establish Theorem 1.2 from Corollary 1.6 using realization results for the $\mathbb{Z}_2$–cohomology structure of 3–manifolds due to Turaev. See Section 7. We remark that, a posteriori, the divisibility properties of the quantities listed in Corollary 1.6 should imply Theorem 1.2. However, the authors prefer to mostly argue with the $\mathbb{Z}_2$–cohomology ring structure, in line with the definition of $v_Y(x)$.

Some more examples For any finitely generated abelian group H containing an element of order 4 or ∞, there is a 3–manifold Y with $H_1(Y;\mathbb{Z})$ isomorphic to H and $v_Y(x)=0$, in which x is any element that is not a cup-square. For this, just consider integer-framed surgeries on unlinks. Note also that the integer $v_Y(x)$ is stable under connect sums with $\mathbb{RP}^3$, which increases $b_1(2)$ by 1 while fixing $k(Y)$. This operation, applied to the three Borromean surgeries examples above, gives examples where $v_Y(x)\equiv 1 \pmod 2$ for any pair $b_1(2)$, $k(Y)$ such that $b_1(2)\geq 3$ and $k(Y)\in\{1,2,3\}$. In fact, it is straightforward to produce nonvanishing examples with $H_1(Y;\mathbb{Z})$ any isomorphism class of finitely generated abelian group with those same two constraints. We also have examples from Seifert-fibered spaces, with orientable base orbifold:

Proposition 1.7 *Let Y be a Seifert-fibered space with Seifert invariants given by $(g,b,(\alpha_1,\beta_1),\dots,(\alpha_r,\beta_r))$, where g is the genus of the base orbifold. Suppose that $x\in H^2(Y;\mathbb{Z}_2)$ is not a square. Then $v_Y(x)\equiv 1 \pmod 2$ if and only if $g=1$, all α_i are odd, and $b+\sum\beta_i\equiv 0 \pmod 2$.*

We note that such Seifert-fibered spaces have $b_1(Y)\in\{2,3\}$ and $b_1(2)=3$. Included in this list is of course the 3–torus. This proposition is easily proven using the description of the mod 2 cohomology ring of a Seifert-fibered space given in Aaslepp, Drawe, Hayat-Legrand, Sczesny and Zieschang [1]. See Section 8.

We mention that the Seifert-fibered spaces considered here for genus $g=0$ are double branched covers of Montesinos links. However, by Proposition 1.7 the relevant invariant $v_Y(x)$ in these cases is always even. In Section 8 we give an example of a double branched cover for which Theorem 1.1 has a nonvanishing congruence.

Discussion The integers $v_Y(x)$, and not just their parities, are interesting in the context of SO(3) gauge theory. Indeed, as is evident in the sequel, the V_4–connection classes counted by $v_Y(E)$ are persistent (unmoved) under a large class of perturbations. As such, they form a distinguished set of generators in the instanton Floer chain complex for the pair (Y,E), defined using any such perturbation. Klein-four connections also play a pivotal role in the $\mathrm{SO}(3)$ instanton homology for webs of Kronheimer and Mrowka [8] and its relation to the four-color theorem.

The authors did not see how to provide a general algebraic proof of Theorem 1.2, but we believe it can be done. Our main purpose in this article is to exhibit how the congruence in Theorem 1.1 requires hardly any work, once the picture for the relevant moduli spaces is established.

Finally, it should be mentioned that although we refer to the invariant $\lambda(Y,E)$ as a "Casson invariant", we are using the interpretation of Taubes [15] of Casson's invariant for integral homology 3–spheres, applied to nontrivial admissible bundles.

Outline In Section 2 we review the notion of nontrivial admissibility and the suitable generalization which motivates the hypotheses of Theorem 1.2. Sections 3 and 4 provide the background for the main argument of Theorem 1.1, which was sketched above and is presented concisely in Section 5. The issue of perturbations is ignored here, and then taken up in Section 6. In Section 7 we complete the proof of Theorem 1.2. Finally, in Section 8 we prove Proposition 1.7, record a connected sum formula for the parity of $v_Y(x)$, and discuss double branched covers.

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2 Nontrivial admissible bundles

Here we briefly discuss Floer's nontrivial admissibility condition. A good reference for this material is [2]. As in the introduction, we let Y be a closed, oriented and connected 3–manifold. An SO(3) bundle over Y is *nontrivial admissible* if its second Stiefel–Whitney class $x \in H^2(Y;\mathbb{Z}_2)$ satisfies the following three equivalent conditions; see [2, Lemma 1.1]:

- The image of x under $h\colon H^2(Y;\mathbb{Z}_2) \to \mathrm{Hom}(H_2(Y;\mathbb{Z}),\mathbb{Z}_2)$ is nonzero.
- There is an *orientable* surface $\Sigma \subset Y$ such that $\langle x,[\Sigma]\rangle \not\equiv 0$.
- The element $x \in H^2(Y;\mathbb{Z}_2)$ has no torsion lifts to $H^2(Y;\mathbb{Z})$.

One then defines a $U(2)$ bundle to be nontrivial admissible if its induced adjoint SO(3) bundle is nontrivial admissible. The definition is motivated by the fact that a nontrivial admissible $U(2)$ bundle admits no reducible flat connections. This avoids complications in instanton Floer theory. Using that h is surjective, and the fact that SO(3) bundles over a 3–manifold are characterized by the second Stiefel–Whitney class, we count the number of nontrivial admissible SO(3) bundles:

$$(2^{b_1(Y)}-1)2^{b_1(2)-b_1(Y)}.$$

According to Theorem 1.1 and Poudel's result mentioned in the introduction, the parity of $v_Y(E)$ is the same for all nontrivial admissible bundles E. However, Theorem 1.2 indicates that the parity of $v_Y(E)$ is invariant under a larger collection of bundles. Such bundles are characterized by having a second Stiefel–Whitney class $x \in H^2(Y;\mathbb{Z}_2)$ that satisfies the following equivalent conditions:

- The image of x under $g\colon H^2(Y;\mathbb{Z}_2) \to \mathrm{Hom}(\mathrm{PD}(\ker(\beta^1)),\mathbb{Z}_2)$ is nonzero.
- There is a surface $\Sigma \subset Y$ such that $\langle x,[\Sigma]\rangle \not\equiv 0$ and $\Sigma\cdot\Sigma \equiv 0 \in H_1(Y;\mathbb{Z}_2)$.
- The element $x \in H^2(Y;\mathbb{Z}_2)$ has no order-2 lifts to $H^2(Y;\mathbb{Z})$.
- The element $x \in H^2(Y;\mathbb{Z}_2)$ is not the cup-square of an element from $H^1(Y;\mathbb{Z}_2)$.

Note that here Σ is not necessarily orientable, and PD: $H^1(Y;\mathbb{Z}_2) \to H_2(Y;\mathbb{Z}_2)$ is the Poincaré duality isomorphism. We briefly remark on the equivalence of these conditions, leaving the details to the reader. The first two bullets are equivalent because $\mathrm{PD}(\ker(\beta^1)) \subset H_2(Y;\mathbb{Z}_2)$ is spanned by the classes $[\Sigma]$ with the stated conditions. The equivalence of the third and fourth conditions follow from understanding the Bockstein homomorphisms in this setting — see eg [5, Section 3.E] — and the remaining equivalences make use of the nondegeneracy of Poincaré duality. These conditions are the natural extensions of the prior three conditions when one wants to treat $\mathbb{Z}$ summands and $\mathbb{Z}_{2^k}$ summands for $k > 1$ the same. We note that the ring $H^*(Y;\mathbb{Z}_2)$ cannot see the difference between such summands. Since g is surjective, the number of SO(3) bundles of this more general type is

$$(2^{k(Y)} - 1)2^{b_1(2)-k(Y)}.$$

The most basic example of such a bundle that is not nontrivial admissible is the nontrivial SO(3) bundle over the lens space $L(4,1)$.

3 Configuration spaces and stabilizers

Fix a connection A_0 on $\det(E)$, and let $\mathcal{C}_E$ be the space of connections A on E with determinant connection $\mathrm{Tr}(A) = A_0$. Let $\mathcal{G}_E$ be the gauge transformation group consisting of smooth unitary automorphisms of E that are determinant 1. The configuration space is the quotient $\mathcal{B}_E = \mathcal{C}_E/\mathcal{G}_E$. The nontrivial admissibility of E implies that all projectively flat points in $\mathcal{B}_E$ are irreducible, meaning that the $\mathcal{G}_E$–stabilizer of every such connection $A \in \mathcal{C}_E$ is as small as possible:

$$\mathrm{Stab}_{\mathcal{G}_E}(A) = \{\pm 1\}.$$

The $U(2)$ bundle E induces an SO(3) bundle $\mathfrak{su}(E)$, which may be defined as the subbundle of $\mathrm{End}(E)$ consisting of trace-free, skew-hermitian endomorphisms. We let $\mathcal{G}_{\mathfrak{su}(E)}$ denote the full SO(3) gauge transformation group of $\mathfrak{su}(E)$. Any $A \in \mathcal{C}_E$ induces a connection $A_{\mathrm{ad}} \in \mathcal{C}_{\mathfrak{su}(E)}$, and this induces a bijection between $\mathcal{C}_E$ and $\mathcal{C}_{\mathfrak{su}(E)}$. Indeed, any $U(2)$ connection A on E is uniquely determined by $\mathrm{Tr}(A)$ on $\det(E)$ and A_{ad} on $\mathfrak{su}(E)$. The condition that A be projectively flat is equivalent to A_{ad} being flat. In contrast to the $U(2)$ case, however, when A_{ad} is flat we have

$$\mathrm{Stab}_{\mathcal{G}_{\mathfrak{su}(E)}}(A_{\mathrm{ad}}) \in \big\{\{1\}, \mathbb{Z}_2, V_4\big\}.$$

Indeed, the difference between the determinant-1 unitary gauge group and the SO(3) gauge group is described by an action of $H^1(Y;\mathbb{Z}_2)$ on $\mathcal{B}_E$ that gives $\mathcal{B}_{\mathfrak{su}(E)}$ as its quotient space. The action is as follows: $H^1(Y;\mathbb{Z}_2)$ parametrizes the isomorphism classes of flat complex line bundles (with connection) χ with holonomy $\{\pm 1\}$. Then

$[\chi]$ acts on $[A] \in \mathcal{B}_E$ by tensoring the bundle-with-connection (E, A) with χ. See eg [3, Section 5.6]. We then have the more precise statement that $\mathrm{Stab}_{\mathcal{G}_{\mathfrak{su}(E)}}(A_{\mathrm{ad}})$ is naturally a subspace of $H^1(Y;\mathbb{Z}_2)$, with the constraint that

$$\dim_{\mathbb{Z}_2} \mathrm{Stab}_{\mathcal{G}_{\mathfrak{su}(E)}}(A_{\mathrm{ad}}) \in \{0, 1, 2\}.$$

In summary, we see that even though any projectively flat connection in $\mathcal{B}_E$ is irreducible, its image in $\mathcal{B}_{\mathfrak{su}(E)}$ may not be irreducible. Flat connections on $\mathfrak{su}(E)$ with $\mathcal{G}_{\mathfrak{su}(E)}$–stabilizer isomorphic to $\mathbb{Z}_2$ are exactly those whose holonomy is contained in $O(2)$, but not in an SO(2) or Klein-four subgroup. Equivalently, these are flat connections that are compatible with a splitting

$$\mathfrak{su}(E) = \lambda \oplus L,$$

where λ is a nontrivial real line bundle and L is an unoriented real 2–plane bundle, and for which the connection on L is irreducible. Connections with stabilizer V_4 are those whose holonomy is also isomorphic to V_4. Equivalently, these are flat connections compatible with a splitting

$$\mathfrak{su}(E) = \lambda_1 \oplus \lambda_2 \oplus \lambda_3$$

into a sum of three nontrivial real line bundles. We write $\mathcal{B}^{V_4}_{\mathfrak{su}(E)} \subset \mathcal{B}_{\mathfrak{su}(E)}$ for the subset of flat connections on $\mathfrak{su}(E)$ with V_4–stabilizer, which we henceforth call *Klein-four connections*.

Remark 3.1 If the assumption of nontrivial admissibility is removed, three other kinds of stabilizers in the SO(3)–gauge group can occur: SO(2), $O(2)$ and SO(3).

4 Klein-four connections

The subset of Klein-four connections in $\mathcal{B}_{\mathfrak{su}(E)}$ is a finite, discrete set. As the elements are characterized by having holonomy V_4, a finite group, they must all be flat, as a simple continuity argument shows. Alternatively, each splitting $\mathfrak{su}(E) = \lambda_1 \oplus \lambda_2 \oplus \lambda_3$ into nontrivial real line bundles supports a unique compatible connection, which of course must be flat. Let us consider the larger set

$$\mathcal{B}^{\geq V_4} = \{\text{connections over } Y \text{ on any SO(3) bundle with holonomy inside a } V_4\}/\text{gauge}.$$

Then $\mathcal{B}^{\geq V_4}$ is parametrized by SO(3) bundles of the form $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ over Y. Noting that $w_1(\mathfrak{su}(E)) = 0$, sending such a bundle to the triple $\{w_1(\lambda_1), w_1(\lambda_2), w_1(\lambda_3)\}$ sets up a bijection

$$\mathcal{B}^{\geq V_4} \overset{1:1}{\longleftrightarrow} \bigl\{\{a, b, c\} \subset H^1(Y;\mathbb{Z}_2) \text{ with } a + b + c = 0\bigr\} =: V_Y.$$

Yet another description of $\mathcal{B}^{\geq V_4}$ is as the set of homomorphisms $\mathrm{Hom}(\pi_1(Y), V_4)$ modulo the action of $S_3 = \mathrm{Aut}(V_4)$. A simple counting argument shows that $\mathcal{B}^{\geq V_4}$ has cardinality

$$2^{b_1(2)-1} + \tfrac{1}{6}(4^{b_1(2)} + 2).$$

Now, the elements of $\mathcal{B}^{\geq V_4}$ that live on $\mathfrak{su}(E)$ are the ones with

$$w_2(E) = w_2(\lambda_1 \oplus \lambda_2 \oplus \lambda_3) = a_1a_2 + a_2a_3 + a_1a_3, \quad a_i = w_1(\lambda_i).$$

Thus we have the following bijection describing Klein-four connections on $\mathfrak{su}(E)$:

$$\mathcal{B}^{V_4}_{\mathfrak{su}(E)} \overset{1:1}{\longleftrightarrow} \bigl\{\{a,b,c\} \in V_Y \text{ with } ab + bc + ac = w_2(E)\bigr\}.$$

We see now that $v_Y(E) = |\mathcal{B}^{V_4}_{\mathfrak{su}(E)}|$, and the statement of Theorem 1.1 is the congruence

$$\lambda(Y,E) \equiv 2^{b_1(2)-3} \cdot |\mathcal{B}^{V_4}_{\mathfrak{su}(E)}| \pmod{2^{b_1(2)-2}}. \tag{3}$$

5 The argument modulo perturbations

We now prove Theorem 1.1 under the assumption that all moduli spaces to follow are nondegenerate, so that no perturbations are needed. The argument uses the most basic information we have from the $H^1(Y;\mathbb{Z}_2)$–action. Consider the moduli space of projectively flat connections on E:

$$\mathcal{M}_E := \bigl\{[A] \in \mathcal{B}_E : F_A = \tfrac{1}{2}F_{A_0} \cdot \mathrm{id}_E\bigr\}.$$

This is a finite set, and each of its points is irreducible. This moduli space is invariant under the $H^1(Y;\mathbb{Z}_2)$–action, and its quotient is the space of flat connections on $\mathfrak{su}(E)$:

$$\mathcal{M}_{\mathfrak{su}(E)} := \{[B] \in \mathcal{B}_{\mathfrak{su}(E)} : F_B = 0\}.$$

We need the following observation. An element $w \in H^1(Y;\mathbb{Z}_2)$ affects the relative mod 8 Floer grading $\mathrm{gr}[A]$ of $[A] \in \mathcal{M}_E$ by (see [2, Propositions 1.9 and 1.13])

$$\mathrm{gr}(w \cdot [A]) - \mathrm{gr}[A] \equiv 4(w_2(E)w + w^3)[Y] \pmod 8,$$

so the $H^1(Y;\mathbb{Z}_2)$–action preserves the $\mathbb{Z}_2$–gradings. Here $[Y]$ is the fundamental class of Y. In particular, each $H^1(Y;\mathbb{Z}_2)$–orbit lies in a single $\mathbb{Z}_2$–grading. The proof is now completed by counting orbit sizes. Each connection in $\mathcal{M}_{\mathfrak{su}(E)}$ with stabilizer at most $\mathbb{Z}_2$ gives an orbit of size either

$$|H^1(Y;\mathbb{Z}_2)| = 2^{b_1(2)} \quad \text{or} \quad |H^1(Y;\mathbb{Z}_2)/\mathbb{Z}_2| = 2^{b_1(2)-1},$$

lying upstairs in $\mathcal{M}_E$. Thus $2^{b_1(2)-1}$ divides the *signed* count of $\mathcal{M}_E$, with the prior observation about gradings in mind. The remaining connections downstairs in $\mathcal{M}_{\mathfrak{su}(E)}$

are Klein-four connections, and so in fact are given by the set $\mathcal{B}^{V_4}_{\mathfrak{su}(E)}$. Each point in this set contributes an orbit of size

$$|H^1(Y;\mathbb{Z}_2)/V_4| = 2^{b_1(2)-2}$$

upstairs in $\mathcal{M}_E$. Recalling that $\lambda(Y,E)$ is *half* the signed count of points in $\mathcal{M}_E$, we recover the congruence (3), proving Theorem 1.1 under the assumption of nondegeneracy.

6 Including holonomy perturbations

In general, the moduli space $\mathcal{M}_E$ is degenerate and we need to perturb the projectively flat equation to achieve the transversality we want. Henceforth we assume that our 3–manifold Y is equipped with a Riemannian metric. The standard class of perturbations used are known as *holonomy perturbations* [6; 13]. The input for such a perturbation is an embedding $\Gamma = \{\gamma_k\}_{k=1}^m$ into Y of solid tori $\gamma_k\colon S^1\times D^2\to Y$. We require that the embedded tori γ_k have a common normal disk, meaning that the image of $\{1\}\times D^2$ under γ_k is the same for all k. We also require that the images of the core loops $S^1\times\{0\}$ are disjoint away from the normal disk. Fix a trivialization of $\det(E)$ over the image of Γ, which is homotopically a wedge (bouquet) of circles. This allows us to consider the holonomy around the γ_k as living in $\mathrm{SU}(2)$. Let $f\colon \mathrm{SU}(2)^m\to\mathbb{R}$ be a conjugation invariant function, ie

$$f(ga_1g^{-1},\dots,ga_mg^{-1}) = f(a_1,\dots,a_m) \quad \text{for all } g\in\mathrm{SU}(2).$$

We also choose a smooth 2–form μ on D^2 with compact support in the interior and integral 1. From this data one constructs a holonomy perturbation h, given as follows:

$$h(A) = \int_{D^2} f(\mathrm{Hol}_{\gamma_{1,z}}(A),\dots,\mathrm{Hol}_{\gamma_{m,z}}(A))\,\mu(z).$$

Here $\gamma_{k,z}$ is the loop $t\mapsto\gamma_k(t,z)$ in Y. Fixing only the data Γ, we define $\mathcal{H}_\Gamma$ to be the space of perturbations constructed as above. Each $h\in\mathcal{H}_\Gamma$ yields a well-defined function $h\colon \mathcal{B}_E\to\mathbb{R}$.

One way to guarantee that the perturbation h is $H^1(Y;\mathbb{Z}_2)$–equivariant is to require that each loop $\mathrm{im}(\gamma_k)$ is zero as a class in $H_1(Y;\mathbb{Z}_2)$. We call such Γ *mod-2 trivial*, following [13], where this condition is introduced. We record their observation:

Lemma 6.1 *If Γ is mod-2 trivial, then each $h\in\mathcal{H}_\Gamma$ is $H^1(Y;\mathbb{Z}_2)$–equivariant.*

Now, the perturbed $U(2)$ moduli space $\mathcal{M}_E^h$ is the set of critical points of the perturbed Chern–Simons functional $\mathrm{CS}+h$. Specifically, for a suitable normalization of CS, we obtain

$$\mathcal{M}_E^h = \big\{[A]\in\mathcal{B}_E : F_A - \tfrac{1}{2}F_{A_0}\cdot\mathrm{id}_E + \star\,\nabla h(A) = 0\big\}.$$

If Γ is mod-2 trivial, this perturbed moduli space inherits the $H^1(Y;\mathbb{Z}_2)$–action from $\mathcal{B}_E$, and its quotient space is the perturbed SO(3) moduli space for $\mathfrak{su}(E)$. We also record the following:

Lemma 6.2 *Suppose Γ is mod-2 trivial. For any $h \in \mathcal{H}_\Gamma$, Klein-four connections are unmoved in the SO(3) moduli space. More precisely, we always have the relation*

$$\mathcal{M}^h_{\mathfrak{su}(E)} \cap \mathcal{B}^{V_4}_{\mathfrak{su}(E)} = \mathcal{B}^{V_4}_{\mathfrak{su}(E)}.$$

As such perturbations are $H^1(Y;\mathbb{Z}_2)$–equivariant, a similar statement holds for the connections in the $U(2)$ moduli space $\mathcal{M}^h_E$ lying above Klein-four connections. In fact, the lemma clearly follows from this latter case, which is justified as follows. First, the $H^1(Y;\mathbb{Z}_2)$–equivariance of our perturbations imply that Klein-four connections in $\mathcal{B}_E$ are always perturbed to Klein-four connections. Second, we recall that the space of Klein-four connection classes is a finite discrete set. Important here is our earlier observation that any connection with Klein-four stabilizer is in fact flat. In particular, the gradient of our perturbation is a Klein-four invariant vector $v \in T\mathcal{B}_E$, which must be the 0 vector by discreteness of the set of Klein-four connections.

Our goal is to find a mod-2 trivial Γ such that for small, generic $h \in \mathcal{H}_\Gamma$ the moduli space $\mathcal{M}^h_E$ is nondegenerate. Section 5 of [13] shows that this can be achieved if Γ is *abundant* at each projectively flat $[A] \in \mathcal{M}_E$. We need to slightly generalize the definition of abundancy given in [13], which only considers stabilizers isomorphic to $\{1\}$ and $\mathbb{Z}_2$. To begin, note that $H^1(Y;A_{\text{ad}})$, the Zariski tangent space to $[A]$ in $\mathcal{M}_E$, carries an action by the stabilizer, denoted

$$S_A := \text{Stab}_{H^1(Y;\mathbb{Z}_2)}[A] = \text{Stab}_{\mathcal{G}_{\mathfrak{su}(E)}}(A_{\text{ad}}). \tag{4}$$

We remark that the second equality in (4) is not true in general, and is contingent upon the nontrivial admissibility of E. Recall that S_A is one of $\{1\}$, $\mathbb{Z}_2$ or V_4. Now, decompose the tangent space into its S_A–invariant subspace V_A, and the S_A–equivariant orthogonal complement to V_A:

$$H^1(Y;A_{\text{ad}}) = V_A \oplus V_A^\perp.$$

The space V_A is the Zariski tangent space of $[A]$ internal to the stratum of $\mathcal{M}_E$ consisting of connection classes with stabilizer isomorphic to S_A. The complement $V_A^\perp$ is the Zariski normal bundle fiber in $\mathcal{M}_E$ at $[A]$ relative to the aforementioned stratum. For a vector space W we write $\text{Sym}(W)$ for the space of symmetric bilinear forms on W. If W has a linear G–action by some group G, we write $\text{Sym}(W)^G$ for the forms that are G–invariant.

Definition 6.3 A mod-2 trivial Γ is *abundant* at a projectively flat $[A] \in \mathcal{M}_E$ if there exist perturbations $\{h_i\}_{i=1}^n \subset \mathcal{H}_\Gamma$ and some k such that $Dh_i(A) = 0$ for $k+1 \le i \le n$, and such that the following map that is defined from $\mathbb{R}^n$ to $\mathrm{Hom}(V_A, \mathbb{R}) \oplus \mathrm{Sym}(V_A^\perp)^{S_A}$ is surjective:

$$(x_1, \dots, x_n) \mapsto \left(\sum_{i=1}^k x_i \, Dh_i(A), \sum_{i=k+1}^n x_i \operatorname{Hess} h_i(A) \right). \tag{5}$$

Note that if S_A is trivial, then V_A accounts for the entire tangent space, and in particular $V_A^\perp = 0$. Thus only the left-hand factor of the map (5) is relevant. This is the condition of "first-order abundancy", and is sufficient to achieve nondegeneracy for small, generic perturbations when there are no other (lower) strata to consider. At the other extreme, when S_A is isomorphic to V_4, we have $V_A = 0$. In this case (5) reduces to a condition purely of "second-order abundancy".

If S_A is isomorphic to $\mathbb{Z}_2$, then V_A and $V_A^\perp$ are the $+1$ and -1 eigenspaces of the $\mathbb{Z}_2$–action, respectively, and are V_+ and V_- in the notation of [13]. In this case $\mathrm{Sym}(V_A^\perp)^{S_A}$ is the same as $\mathrm{Sym}(V_-)$. Our choice of $\mathrm{Sym}(V_A^\perp)^{S_A}$ in Definition 6.3 is sufficient for the arguments of Section 5 in [13] to go through in part because a generic element therein is nondegenerate; see the proof of Proposition 5.4 in [13]. When S_A is isomorphic to $\{1\}$ or $\mathbb{Z}_2$, our definition agrees with that of [13].

We are left with producing a mod-2 trivial Γ which is abundant for all $[A] \in \mathcal{M}_E$. To this end, the work of Ruberman and Saveliev implies the following:

Lemma 6.4 [13, Proposition 5.2] *There exists a mod-2 trivial Γ that is abundant for all connections in $\mathcal{M}_E$ that do not descend to $\mathrm{SO}(3)$ Klein-four connections.*

This allows us to focus on the situations in which S_A is isomorphic to V_4, the case in which A_{ad} is a Klein-four connection. We have the following facts, used in [13, Section 5.5], stated informally:

- If Γ is abundant, and Γ' is close to Γ, then Γ' is abundant.
- If Γ is abundant and $\Gamma \subset \Gamma'$, then Γ' is abundant.

In these situations, we are assuming that Γ and Γ' have the same fixed normal disk with basepoint. Now suppose we can show, for each A with S_A isomorphic to V_4, the existence of a mod-2 trivial Γ abundant at $[A]$. Then it is straightforward to conclude, using these two facts and Lemma 6.4, that there exists a mod-2 trivial Γ' abundant at all $[A] \in \mathcal{M}_E$. Thus the following lemma completes the proof of Theorem 1.1:

Lemma 6.5 *There is an abundant mod-2 trivial Γ for any $[A] \in \mathcal{M}_E$ that descends to an $\mathrm{SO}(3)$ Klein-four connection.*

Proof We follow the method used in [13] of passing to a finite cover. Let A be a projectively flat connection on E with stabilizer S_A isomorphic to V_4. The SO(3) connection A_{ad} is compatible with a splitting $\mathfrak{su}(E) = \lambda_1 \oplus \lambda_2 \oplus \lambda_3$ in which the λ_i are nontrivial and distinct real line bundles. The stabilizer S_A is given explicitly by

$$S_A = \{0, a_1, a_2, a_3\} \subset H^1(Y; \mathbb{Z}_2), \quad a_i = w_1(\lambda_i).$$

Here, a_i corresponds to the gauge transformation of $\mathfrak{su}(E)$ that simultaneously reflects λ_{i+1} and λ_{i+2}, while fixing λ_i, where indices are taken mod 3. Define a homomorphism $\pi_1(Y) \to S_A$ by

$$\gamma \mapsto a_1(\gamma)a_1 + a_2(\gamma)a_2 + a_3(\gamma)a_3.$$

Let $p\colon Y' \to Y$ be the covering space corresponding to this homomorphism. Under this covering A_{ad} pulls back to a trivial connection, denoted A'_{ad}; see [13, Lemma 5.6]. In particular, each of λ_i pulls back under p to a trivial real line bundle λ'_i. Note that the covering transformation group of $Y' \to Y$ is the Klein-four group S_A.

It is known [6, Proposition 67 and Lemma 58] that there is some Γ', a collection of embedded solid tori in Y', that is abundant at the trivial connection A'_{ad} in the following sense: there exist perturbations $\{h_i\}_{i=1}^n \subset \mathcal{H}_{\Gamma'}$ such that the map from $\mathbb{R}^n$ to $\mathrm{Sym}(H^1(Y'; A'_{\mathrm{ad}}))^{\mathrm{SO}(3)}$ given by

$$(x_1, \dots, x_n) \mapsto \sum_{i=1}^n x_i \,\mathrm{Hess}\, h_i(A'_{\mathrm{ad}}) \tag{6}$$

is surjective. The appearance of the SO(3) here is the gauge stabilizer of the connection A'_{ad}. Let Γ be the image of Γ' under p, slightly perturbed in Y so that it is of the form described at the beginning of this section. By construction, Γ is mod-2 trivial. Consider the following map:

$$\mathrm{Sym}(H^1(Y'; A'_{\mathrm{ad}}))^{\mathrm{SO}(3)} \to \mathrm{Sym}(H^1(Y; A_{\mathrm{ad}}))^{V_4}. \tag{7}$$

Here the V_4 refers to S_A. The map (5) is the composition of (6) with (7). Thus, to show abundancy of Γ at A, it suffices to show that (7) is surjective. The map (7) is induced by the pull-back map:

$$V_4 \circlearrowright H^1(Y; A_{\mathrm{ad}}) \xrightarrow{\;p^*\;} H^1(Y'; A'_{\mathrm{ad}}) \circlearrowleft \mathrm{SO}(3) \tag{8}$$

This map is equivariant with respect to the indicated gauge stabilizer actions, upon considering V_4 as a subgroup of $\mathrm{SO}(3)$. More precisely, V_4 refers to the $\mathcal{G}_{\mathfrak{su}(E)}$–stabilizer of A_{ad}, while $\mathrm{SO}(3)$ refers to the $\mathcal{G}_{p^*\mathfrak{su}(E)}$–stabilizer of A'_{ad}.

To show that (7) is surjective, consider the following two decompositions:

$$H^1(Y;A_{\mathrm{ad}}) = \bigoplus_{i=1}^{3} H^1(Y;\lambda_i), \quad H^1(Y';A'_{\mathrm{ad}}) = H^1(Y';\mathbb{R})\otimes\mathbb{R}^3. \tag{9}$$

Implicit here is a trivialization for each λ_i', and the $\mathbb{R}^3$ should be thought of as coming from the induced trivialization of $\lambda_1'\oplus\lambda_2'\oplus\lambda_3'$. The map (8) respects these decompositions. In the left-hand decomposition of (9), the V_4 action is as follows: a_i acts as -1 on $H^1(Y;\lambda_{i+1})\oplus H^1(Y;\lambda_{i+2})$, and $+1$ on $H^1(Y;\lambda_i)$. In the tensor product appearing in (9), the SO(3)–action on $\mathbb{R}^3$ is standard, and is trivial on $H^1(Y';\mathbb{R})$. From these descriptions, it is straightforward to verify that these decompositions induce identifications between the domain and codomain of (7) with $\mathrm{Sym}(H^1(Y';\mathbb{R}))$ and $\bigoplus_{i=1}^3 \mathrm{Sym}(H^1(Y;\lambda_i))$, respectively. The map (7) can then be seen as the map

$$\mathrm{Sym}(H^1(Y';\mathbb{R})) \to \bigoplus_{i=1}^{3} \mathrm{Sym}(H^1(Y;\lambda_i)), \tag{10}$$

in which each of the three components is the map induced by pull-back, after trivializing λ_i'. Now, (10) is surjective because the three relevant pull-back maps are injective, and their three images pairwise intersect at 0. This is evident from the decomposition

$$H^1(Y';\mathbb{R}) = H^1(Y;\mathbb{R})\oplus H^1(Y;\lambda_1)\oplus H^1(Y;\lambda_2)\oplus H^1(Y;\lambda_3),$$

which is induced by the covering transformation group S_A acting on $H^1(Y';\mathbb{R})$. This action should not to be confused with the gauge stabilizer action of S_A on $H^1(Y;A_{\mathrm{ad}})$ which was used above. The summand $H^1(Y;\mathbb{R})$ is the invariant subspace under this action, while $H^1(Y;\lambda_i)$ is the complement of $H^1(Y;\mathbb{R})$ inside the invariant subspace for the subgroup $\{0,a_i\}$. □

Remark 6.6 For a discussion of some of the technical assumptions used here, see Section 5.6 of [13]. For a detailed study of the abundancy of holonomy perturbations in the context of the equivariant Kuranishi method, see [7].

7 Establishing the vanishing result

Here we complete the proof of Theorem 1.2. The remaining step is to use a realization result for the $\mathbb{Z}_2$–cohomology ring due to Turaev in conjunction with Corollary 1.6. Recall that for a closed, oriented and connected 3–manifold we have the triple cup product form

$$u_Y\colon H^1(Y;\mathbb{Z}_2)\otimes H^1(Y;\mathbb{Z}_2)\otimes H^1(Y;\mathbb{Z}_2)\to\mathbb{Z}_2, \quad u_Y(a,b,c) = (a\cup b\cup c)[Y].$$

The trilinear form u_Y determines the $\mathbb{Z}_2$–cohomology ring of Y. It was originally proven by Postnikov that any symmetric trilinear form satisfying $u(a,a,b) = u(b,b,a)$ is realized by a closed, oriented and connected 3–manifold. Recall also that we have the linking form

$$L_Y\colon \operatorname{Tor} H_1(Y;\mathbb{Z}) \otimes \operatorname{Tor} H_1(Y;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z},$$

which is a nondegenerate symmetric bilinear form. The linking form interacts with the $\mathbb{Z}_2$–cohomology ring in the following way. Let $\psi\colon \mathbb{Z}_2 \to \mathbb{Q}/\mathbb{Z}$ be the injection defined by $\psi(k \pmod 2) = k/2$. Then for all $a, b \in H^1(Y;\mathbb{Z}_2)$ we have the relation

$$\psi(u_Y(a,a,b)) = L_Y(a^\dagger, b^\dagger), \tag{11}$$

where for any $a \in H^1(Y;\mathbb{Z}_2)$ the element $a^\dagger \in \operatorname{Tor} H_1(Y;\mathbb{Z})$ is defined by the condition that $L_Y(a^\dagger, c) = \psi(a(c))$ for all $c \in \operatorname{Tor} H_1(Y;\mathbb{Z})$. Here we are of course identifying $H^1(Y;\mathbb{Z}_2)$ with $\operatorname{Hom}(H_1(Y;\mathbb{Z}),\mathbb{Z}_2)$. An implication of Turaev's work is the following result (see also [10; 14] for related results):

Theorem 7.1 [16] *Let H be a finitely generated abelian group, and let*

$$u\colon \operatorname{Hom}(H,\mathbb{Z}_2)^{\otimes 3} \to \mathbb{Z}_2$$

be a symmetric trilinear form. There exists a closed, orientable and connected 3–manifold Y such that the pair (H,u) is equivalent to $(H_1(Y;\mathbb{Z}), u_Y)$ if and only if there exists a nondegenerate symmetric bilinear form $L\colon \operatorname{Tor} H^{\otimes 2} \to \mathbb{Q}/\mathbb{Z}$ such that (11) *holds with $u_Y = u$ and $L_Y = L$.*

Proof of Theorem 1.2 Let Y be such that $k(Y) \geq 4$, and suppose that x is not a cup-square. Equivalently, x has no order-2 lift to $H^2(Y;\mathbb{Z})$. Our goal is show that $v_Y(x) \equiv 0 \pmod 2$. We choose an isomorphism

$$H_1(Y;\mathbb{Z}) \simeq \bigoplus_{i=1}^{4} A_i \oplus B,$$

where A_i is an abelian group of the form $\mathbb{Z}_{2^k}$ for $k > 1$ or a copy of $\mathbb{Z}$. Make these choices so that x has a lift to $H^2(Y;\mathbb{Z})$ with support in A_1, not of order 2, which can be done by our assumption on x. Recall that $\operatorname{Tor} H_1(Y;\mathbb{Z})$ is the torsion of $H^2(Y;\mathbb{Z})$ by the universal coefficients theorem. Now define H by replacing the A_i summands with copies of $\mathbb{Z}$:

$$H := \bigoplus_{i=1}^{4} A_i' \oplus B, \quad A_i' := \mathbb{Z}$$

With our identifications we have a natural isomorphism between $H^1(Y;\mathbb{Z}_2)$ and $\operatorname{Hom}(H,\mathbb{Z}_2)$, and with this understood we set $u := u_Y$. Also, noting that $\operatorname{Tor} H$

is simply Tor $H_1(Y;\mathbb{Z})$ with some summands possibly thrown away, we define L to be the restriction of L_Y. With our identifications, the terms appearing in (11) are unchanged. Thus Theorem 7.1 implies the existence of a closed, oriented and connected 3–manifold Z with first homology and triple cup product form given by (H,u). By our choices, x has no torsion lifts, and is thus equal to $w_2(E)$ for a nontrivial admissible $U(2)$ bundle E over Z. Now Poudel's result in the guise of Corollary 1.6 says $v_Z(x)\equiv 0 \pmod 2$, since $b_1(Z)\geq 4$. Since the $\mathbb{Z}_2$–cohomology rings of Y and Z are the same, we then get $v_Y(x)\equiv 0 \pmod 2$. The independence of x as a choice having no order-2 lift to $H^2(Y;\mathbb{Z})$ is established in much the same way as the vanishing. □

8 Examples and properties

In this section we prove Proposition 1.7, which yields examples of $v_Y(x) \pmod 2$ for Seifert-fibered spaces. We then produce a connected sum formula for the parity of $v_Y(x)$. Finally, we illustrate how to compute $v_Y(x)$ for double branched covers of links.

Seifert-fibered spaces Let Y be a Seifert-fibered 3–manifold over an oriented base orbifold, with Seifert invariants $(g,b,(\alpha_1,\beta_1),\dots,(\alpha_r,\beta_r))$. Here g is the genus of the base orbifold. The mod 2 cohomology ring of Y is completely described in [1].

Lemma 8.1 *Suppose $x\in H^2(Y;\mathbb{Z}_2)$ is not a square. If any of the α_i are even, or if all α_i are odd and $b+\sum\beta_i\equiv 1 \pmod 2$, then $v_Y(x)=0$.*

Proof We begin with the following easily verified observation. In general, we have

$$\{a^2 : a\in H^1(Y;\mathbb{Z}_2)\}\subset\{ab : a,b\in H^1(Y;\mathbb{Z}_2)\}. \tag{12}$$

When these sets are equal, then $v_Y(x)=0$. For if the triple $\{a,b,a+b\}\in V_Y$ had $a^2+b^2+ab=x$, then x would in fact be a square, a contradiction. Now we appeal to [1, Theorem 2.9]. When there is some even α_i ("case $n=0$" in [1]), we easily check that these two sets in (12) are equal. This is particularly immediate when there is an α_i divisible by 4, and the mod 2 cohomology ring of Y is isomorphic to that of a connect sum of some copies of $\mathbb{RP}^3$ and some copies of $S^1\times S^2$. Finally, if all α_i are odd and $b+\sum\beta_i\equiv 1 \pmod 2$, then the ring is isomorphic to that of a connect sum of $2g$ copies of $S^1\times S^2$, whence by the same reasoning $v_Y(x)=0$. □

Proof of Proposition 1.7 First, since $b_1(Y)$ is equal to either $2g$ or $2g+1$, the integer $v_Y(x)$ is even by Corollary 1.6 unless $g=1$. By the above lemma, it remains to check that $v_Y(x) \equiv 1 \pmod 2$ when $g=1$ and all α_i are odd and $b+\sum \beta_i \equiv 0 \pmod 2$. One can conclude from [1] that $H^1(Y;\mathbb{Z}_2)$ has a basis a, b, c with $a^2=b^2=0$ and nonzero products ab, bc, ac, the three of which provide a basis for $H^2(Y;\mathbb{Z}_2)$. Depending on some divisibility conditions on the β_i, either $c^2=0$ or $c^2=ab$. The element ac, for one, is never a square, so we set $x=ac$. In either case we compute $v_Y(x)=1$. □

Connected sums Now let x be any element of $H^2(Y;\mathbb{Z}_2)$. Recall that V_Y may be viewed as $\mathrm{Hom}(\pi_1(Y),V_4)$ modulo the action of $S_3=\mathrm{Aut}(V_4)$. As such, it makes sense to keep track of the S_3–stabilizers of the orbits. For a set X with S_3–action we define the triple $\check{v}(X)=(\check{v}_1,\check{v}_2,\check{v}_3)$ where $\check{v}_1$, $\check{v}_2$, $\check{v}_3$ are the numbers of orbits with stabilizers of orders 1, 2, 6, respectively. For two such sets X_1 and X_2 with S_3–actions we have

$$\check{v}(X_1\times_{S_3}X_2)=\check{v}(X_1)\times\check{v}(X_2),$$

where we define the product $\times$ on triples as follows:

$$\check{v}\times\check{u} := (6\check{v}_1\check{u}_1+3\check{v}_1\check{u}_2+3\check{v}_2\check{u}_1+\check{v}_1\check{u}_3+\check{v}_3\check{u}_1+\check{v}_2\check{u}_2,\ \check{v}_2\check{u}_2+\check{v}_2\check{u}_3+\check{v}_3\check{u}_2,\ \check{v}_3\check{u}_3).$$

Define the norm of a triple to be the L^1–norm: $|\check{v}|=\check{v}_1+\check{v}_2+\check{v}_3$. Write $\check{v}_Y(x)$ for the triple $\check{v}(X)$, with X the subset of $\mathrm{Hom}(\pi_1(Y),V_4)$ that lives on an SO(3) bundle E with $x=w_2(E)$. Thus X/S_3 is the subset of $\{a,b,c\}\in V_Y$ such that $ab+bc+ac=x$. With our new notation, we have

$$v_Y(x)=|\check{v}_Y(x)|.$$

Now, given $x_i\in H^2(Y_i;\mathbb{Z}_2)$ it is easy to verify the connect sum relation

$$v_{Y_1\#Y_2}(x_1+x_2)=|\check{v}_{Y_1}(x_1)\times\check{v}_{Y_2}(x_2)|.$$

Note also that if x is not a cup-square, then $\check{v}_Y(x)$ has the form

$$\check{v}_Y(x)=(\check{v}_1,0,0).$$

In general, the third entry $\check{v}_3$ is equal to 1 if and only if $x=0$, and is otherwise 0. Also, the second entry $\check{v}_2$ is the number of nontrivial cup-square-roots of x:

$$\check{v}_2=|\{a\in H^1(Y;\mathbb{Z}_2): a\neq 0, a^2=x\}|, \quad \text{where } \check{v}_Y(x)=(\check{v}_1,\check{v}_2,\check{v}_3).$$

In particular, the sum $\check{v}_2+\check{v}_3$ is either zero or the cardinality of the kernel of the Bockstein map $H^1(Y;\mathbb{Z}_2)\to H^2(Y;\mathbb{Z}_2)$, which is by definition $2^{k(Y)}$. Putting these observations together, and using our freedom to choose x that is not a square (below choose $x_2=0$), we compute the following:

Proposition 8.2 *Suppose $x_i \in H^2(Y_i;\mathbb{Z}_2)$ and that x_1 is not a cup-square. Then*

$$v_{Y_1\#Y_2}(x_1+x_2) \equiv \begin{cases} v_{Y_1}(x_1) \pmod 2 & \text{if } k(Y_2)=0,\\ 0 \pmod 2 & \text{otherwise.}\end{cases}$$

In particular, we recover the fact (mod 2) that $v_Y(x)$ is stable under connect summing with $\mathbb{RP}^3$. More generally, these statements clearly hold when the decompositions are only algebraic, instead of geometric: for example, if there is a decomposition $H^1(Y;\mathbb{Z}_2) = A \oplus B$ where $A \cup B = 0$ and B has an element of order 4 or ∞, then $v_Y(x) \equiv 0 \pmod 2$ for any x not a cup-square.

Double branched covers The above Seifert-fibered examples for genus $g=0$ are double branched covers of Montesinos links, but in all of those cases $v_Y(x)$ vanishes (mod 2) for nonsquares x. Here we compute a nonvanishing example in which Y is a double branched cover $\Sigma(L)$ of a link L in S^3. First, we describe the $\mathbb{Z}_2$ cohomology rings of such manifolds. Let L be a link with components $L_1,\dots,L_n$, and let S_i be a Seifert surface for L_i. Then S_i lifts to a closed surface F_i in the branched cover $\Sigma(L)$. Write $a_i \in H^1(\Sigma(L);\mathbb{Z}_2)$ for the Poincaré dual of $[F_i]$.

Proposition 8.3 *Let L be an n–component link. The vector space $H^1(\Sigma(L);\mathbb{Z}_2)$ has dimension $n-1$, and it is generated by the n classes a_i subject to the one relation*

$$a_1+\cdots+a_n = 0. \tag{13}$$

The triple cup product form on $H^1(\Sigma(L);\mathbb{Z}_2)$ is determined by the values

$$(a_i \cup a_j \cup a_k)[\Sigma(L)] \equiv \begin{cases} \sum_{\ell\neq i} \mathrm{lk}(L_i,L_\ell) \pmod 2 & \text{for } i=j=k,\\ \mathrm{lk}(L_i,L_k) \pmod 2 & \text{for } i=j\neq k,\\ 0 \pmod 2 & \text{for } i,j,k \text{ distinct.}\end{cases}$$

This proposition is proved for two-component links in [12, Proposition 9.2], and the proof easily generalizes. We sketch the argument. To begin, we mention that $H_1(\Sigma(L);\mathbb{Z}_2)$ is in bijection with the subsets of $\{1,\dots,n\}$ of *even* cardinality:

$$H_1(\Sigma(L);\mathbb{Z}_2) \overset{1:1}{\longleftrightarrow} \big\{S \subset \{1,\dots,n\} : |S| \equiv 0 \pmod 2\big\}. \tag{14}$$

The bijection goes as follows. Given such a subset, pair off elements. For the pair $\{i,j\}$, draw an arc in S^3 between components L_i and L_j, otherwise missing L. Lift the arcs to a union of loops in $\Sigma(L)$ to obtain a class in $H_1(\Sigma(L);\mathbb{Z}_2)$. Now, assume the F_i are transverse to one another. Then it is not hard to see, when $i\neq j$, that $F_i \cap F_j$ is mod 2 homologous to

$$\mathrm{lk}(L_i,L_j)\cdot\{i,j\},$$

where we view $\{i,j\}$ as an element of $H_1(\Sigma(L);\mathbb{Z}_2)$ via the above bijection. Upon

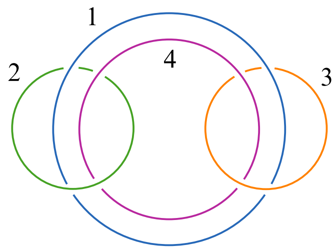

Figure 2: The link $L = \text{L8n8}$ with its four components labeled by $\{1,2,3,4\}$. This link has determinant zero and thus its branched double cover supports nontrivial admissible bundles.

taking Poincaré duals, this yields the proposition. We note that addition on the subsets appearing on the right side of (14) is the symmetric difference of sets.

Let $Y = \Sigma(L)$, and let f be the function from V_Y to $H_1(Y;\mathbb{Z}_2)$ that sends a flat Klein-four connection class to the Poincaré dual of its second Stiefel–Whitney class:

$$f\{a,b,c\} = \mathrm{PD}(ab+bc+ac).$$

Let L be the four-component link L8n8 depicted in Figure 2, and let a_i be the classes described in Proposition 8.3 for L, so that a_i is dual to the lifted Seifert surface of L_i. In particular, a_1, a_2, a_3 form a basis for $H^1(Y;\mathbb{Z}_2)$. For illustration, using Proposition 8.3 we compute

$$\mathrm{PD}(a_1^2) = \mathrm{PD}\,(a_1(a_2+a_3+a_4)) = \sum_{i=2}^{4} \mathrm{lk}(L_1,L_i)\cdot\{1,i\} = \{1,2\}+\{1,3\} = \{2,3\}.$$

The bijection (14) is implicit in our notation, aligning subsets of $\{1,2,3,4\}$ of even size with elements of $H_1(Y;\mathbb{Z}_2)$. We then compute f on all fifteen of the Klein-four connection classes in V_Y:

$$\begin{aligned}
f\{0,0,0\} &= 0, & f\{a_1,a_2,a_1{+}a_2\} &= \underline{\{3,4\}},\\
f\{a_1,a_1,0\} &= \{2,3\}, & f\{a_1,a_3,a_1{+}a_3\} &= \underline{\{2,4\}},\\
f\{a_2,a_2,0\} &= \{1,4\}, & f\{a_2,a_3,a_2{+}a_3\} &= 0,\\
f\{a_3,a_3,0\} &= \{1,4\}, & f\{a_1,a_2{+}a_3,a_1{+}a_2{+}a_3\} &= 0,\\
f\{a_1{+}a_2,a_1{+}a_2,0\} &= \{1,2,3,4\}, & f\{a_2,a_1{+}a_3,a_1{+}a_2{+}a_3\} &= \underline{\{1,3\}},\\
f\{a_1{+}a_3,a_1{+}a_3,0\} &= \{1,2,3,4\}, & f\{a_3,a_1{+}a_2,a_1{+}a_2{+}a_3\} &= \underline{\{1,2\}},\\
f\{a_2{+}a_3,a_2{+}a_3,0\} &= 0, & f\{a_1{+}a_2,a_1{+}a_3,a_2{+}a_3\} &= 0,\\
f\{a_1{+}a_2{+}a_3,a_1{+}a_2{+}a_3,0\} &= \{2,3\}.
\end{aligned}$$

We find that the cup-squares form a 2–dimensional subspace of $H^2(Y;\mathbb{Z}_2)$, appearing as the outputs of the left-hand column. Thus $k(Y)=1$. We have four nonsquares, appearing as the nonzero (underlined) entries in the right-hand column. Each has one Klein-four class, and so $v_Y(x)\equiv 1 \pmod 2$ when x is not a cup-square. The link L has determinant zero, ie $b_1(Y)>0$, so Y has a nontrivial admissible $U(2)$ bundle E. By Theorem 1.1 we conclude

$$\lambda(Y,E)\equiv 1 \pmod 2.$$

Proposition 8.3 similarly computes the parity of $2^{4-n}\lambda(Y,E)$, when $\det(L)=0$, from only knowing the mod 2 linking matrix of L.

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