

Odd knot invariants from quantum covering groups

SEAN CLARK

We show that the quantum covering group associated to $\mathfrak{osp}(1|2n)$ has an associated colored quantum knot invariant à la Reshetikhin–Turaev, which specializes to a quantum knot invariant for $\mathfrak{osp}(1|2n)$, and to the usual quantum knot invariant for $\mathfrak{so}(1+2n)$. In particular, this furnishes an “odd” variant of $\mathfrak{so}(1+2n)$ quantum invariants, even for knots labeled by spin representations. We then show that these knot invariants are essentially the same, up to a change of variables and a constant factor depending on the knot and weight.

[17B37](#), [57M27](#)

1 Introduction

1.1 Background

Quantum enveloping algebras associated to Kac–Moody Lie algebras are central objects in mathematics which have many remarkable connections to geometry, combinatorics, mathematical physics, and other areas. One such connection was produced by Reshetikhin and Turaev [23] and Turaev [27] by relating the representation theory of these quantum enveloping algebras to Laurent polynomial knot invariants, often referred to as quantum invariants, such as the (colored) Jones polynomial and the HOMFLYPT polynomial. The procedure for constructing these quantum invariants is quite general, and for instance has been generalized to construct quantum invariants from quantum enveloping superalgebras as well; see Blumen [3], Geer and Patureau-Mirand [12], Gould, Links and Zhang [13], Queffelec and Sartori [22], Sartori [25] and Zhang [30]. These super invariants are often equivalent to the nonsuper quantum invariants in some sense, but provide a novel perspective that can reveal additional features.

Many other connections have arisen from the categorification of quantum enveloping algebras and their representations; see Khovanov and Lauda [17] and Rouquier [24]. It was recently shown by Webster [28] that in fact, one can categorify all Reshetikhin–Turaev invariants using the machinery of categorified quantum enveloping algebras. This procedure generalizes Khovanov’s homological categorification of the Jones polynomial [16]. We can summarize some of these connections in the picture in

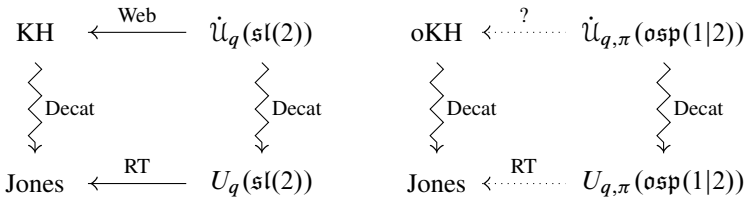


Figure 1

Figure 1 (left), where “Decat” refers to the appropriate decategorification, “RT” stands for the Reshetikhin–Turaev procedure for constructing the Jones polynomial from the standard quantum $\mathfrak{sl}(2)$ representation, and “Web” stands for Webster’s categorification of RT which produces Khovanov homology.

This beautiful picture recently developed a twist with the discovery of “odd Khovanov homology”, an alternate homological categorification of the Jones polynomial; see Ozsváth, Rasmussen and Szabó [21]. This discovery has spurred a program of “oddification”: providing analogues of (categorified) quantum groups for this odd Khovanov homology by developing “odd” analogues of standard constructions; see Ellis and Lauda [11], Ellis, Khovanov and Lauda [10] and Mikhaylov and Witten [19]. In particular, one would like an “odd (categorified) $U_q(\mathfrak{sl}(2))$ ” which could produce odd Khovanov homology in a similar way to that described in Figure 1 (left). In particular, the decategorified “odd” quantum group should produce the Jones polynomial through some analogue of the Reshetikhin–Turaev procedure. It has been proposed (see [11] and Hill and Wang [14]) that such categorifications might naturally arise through categorifying the quantum covering group $U_{q,\pi}(\mathfrak{osp}(1|2))$, in other words, producing a diagram such as in Figure 1 (right).

This proposal has some heuristic evidence. Since the work of Zhang [29] connecting the representation theories of $\mathfrak{osp}(1|2n)$ and $\mathfrak{so}(1+2n)$, it has been expected that the associated knot invariants are essentially the same [30]. This expectation was partially verified by Blumen [2; 3], who showed that there is a link invariant associated to the two-dimensional quantum representation of $\mathfrak{osp}(1|2)$ (though no relation to the Jones polynomial was claimed), and that the $\mathfrak{osp}(1|2n)$ and $\mathfrak{so}(2n+1)$ invariants which are colored by the standard $(2n+1)$ -dimensional representations are related up to some variable substitution, though the variable substitution has not been made explicit in general. However, to our knowledge, no proof of any relation between super and nonsuper colored type B knot invariants exists in the literature. The covering quantum group is a natural tool for filling this gap, as it has an explicit algebraic bridge between the super and nonsuper theories.

On the other hand, Mikhaylov and Witten [19] have produced candidates for “odd link homologies” categorifying $\mathfrak{so}(1+2n)$ -invariants via topological quantum field theories using the orthosymplectic supergroups. This suggests that the conjecture represented by Figure 1 (right) should be generalized to include colored link invariants associated to $\mathfrak{osp}(1|2n)$ for any $n \geq 1$.

1.2 Results

A quantum covering group is an algebra U that marries the quantum enveloping superalgebra of an anisotropic Kac–Moody Lie superalgebra (eg $\mathfrak{osp}(1|2n)$) with the quantum enveloping algebra of its associated Kac–Moody Lie algebra, which is obtained by forgetting the parity in the root datum (eg $\mathfrak{so}(1+2n)$). This is done by introducing a new “half-parameter” π satisfying $\pi^2 = 1$, and substituting π everywhere a sign associated to the superalgebra braiding should appear; such algebras were defined and studied in detail in the series of papers by Clark, Fan, Hill, Li and Wang [4; 5; 6; 7; 8; 9].

These quantum covering groups retain the many nice properties of usual quantum groups such as a Hopf structure, a quasi- \mathcal{R} -matrix à la Lusztig [18, Chapter 4], a category \mathcal{O} , and even canonical bases. A key feature of a quantum covering group is that by specializing $\pi = 1$ (resp. $\pi = -1$), we obtain the quantum enveloping (super)algebra associated to the Kac–Moody Lie (super)algebra. Moreover, as discovered in [5], the quantum algebra and quantum superalgebra can be identified by a twistor map; that is, an automorphism of (an extension of) the covering quantum group which sends $\pi \mapsto -\pi$ and $q \mapsto \mathbf{t}^{-1}q$, where $\mathbf{t}^2 = -1$. This construction provides an algebraic realization of the connection between $\mathfrak{osp}(1|2n)$ and $\mathfrak{so}(1+2n)$ observed in [29].

In this paper, we use the machinery of covering quantum groups to construct “quantum covering knot invariants”: knot invariants which arise from the representation theory of the finite type quantum covering groups à la Turaev [27].¹ To wit, consider the quantum covering group associated to the Lie superalgebra $\mathfrak{osp}(1|2n)$. We first associate a U -module homomorphism to each elementary tangle (cups, caps, crossings) such that a straight strand is just the identity map, along with an interpretation of combining tangles (with joining top-to-bottom being composition of the associated maps, and placing along-side being tensor products of the maps). An arbitrary tangle can then be framed and associated with a U -module homomorphism by “slicing” the diagram

¹We do not need, and will not use, the usual additional ribbon structure of [23]. While the category of representations we consider is certainly rigid, the \mathcal{R} -matrix we use does not define a braiding in that the maps it defines fail to satisfy the hexagon identity in general. This can be fixed (see Remark 2.20), but it requires additional burdensome notation and analysis that we feel detracts more than it adds, particularly for the results of Section 4.

(that is, cutting it into vertical chunks containing at most one elementary diagram alongside any number of straight strands). Each slice corresponds to a U -module homomorphism, and the tangle is sent to the composition of these maps. Note that a priori, this assignment is not unique, as many distinct slice diagrams and framings exist for an arbitrary tangle.

We then derive some identities with these maps that are versions of Turaev moves on the associated diagrams. These identities show that the map isn't dependent on the choice of slice diagram, but factors of π keep it from being an invariant of oriented framed tangles. In order to eliminate these factors, we need to expand our base ring to $\mathbb{Q}(q, \mathbf{t})^\tau$, where $\tau^2 = \pi$, and renormalize the maps corresponding to certain elementary diagrams. Finally, a normalization factor (depending on the writhe of the tangle) yields a oriented tangle invariant; see [Theorem 3.7](#).

In the rank-1 uncolored case, this invariant is simply the (unnormalized) Jones polynomial in the variable $\tau^{-1}q$; see [Example 3.10](#). This suggests that the $\pi = -1$ (ie $\tau = \mathbf{t}$) specialization of the knot invariant, viewed as a function of q , should be related to the $\pi = 1$ (ie $\tau = 1$) specialization, viewed as a function of $\mathbf{t}^{-1}q$. To make this connection precise, we further develop the theory of twistors (see [\[4; 5\]](#)) to define a general operator on tensor powers of U and compatible operators on its representations. In particular, we show that the twistors \mathfrak{X} on representations \mathbf{t} -commute with the maps S representing slices of tangles; that is, $\mathfrak{X} \circ S = \mathbf{t}^x S \circ \mathfrak{X}$ for some $x \in \mathbb{Z}$.

Once this is done, we obtain the following theorem (combining [Theorems 3.7](#) and [4.24](#)).

Main Theorem *Let K be any oriented knot and $\lambda \in X^+$ a dominant weight. There is a functor from the category $\mathcal{O}\mathcal{T}\mathcal{A}\mathcal{N}$ of oriented tangles modulo isotopy to the category \mathcal{O} of U -module representations which sends K to a constant $J_K^\lambda(q, \tau) \in \mathbb{Q}(q, \mathbf{t})^\tau$, which we call the **covering knot invariant** of K . Moreover, let ${}_{\text{so}}J_K^\lambda(q) = J_K^\lambda(q, 1)$ and ${}_{\text{osp}}J_K^\lambda(q) = J_K^\lambda(q, \mathbf{t})$ denote the specializations of the covering knot invariant to $\tau = 1$ and $\tau = \mathbf{t}$. Then*

$${}_{\text{osp}}J_K^\lambda(q) = \mathbf{t}^{\star(K, \lambda)} {}_{\text{so}}J_K^\lambda(\mathbf{t}^{-1}q)$$

for some $\star(K, \lambda) \in \mathbb{Z}$.

In particular, this shows that, after extending scalars, there is indeed a map RT as in [Figure 1](#) (right), and in fact such a map exists for all colored link invariants of any rank. In particular, this proves that the super and nonsuper colored knot invariants of type B are essentially the same. It remains to develop an analogue of the construction in [\[28\]](#) to complete the picture, though difficulties abound. For example, it is not necessarily clear how to extend the categorification to $\mathbb{Q}(q, \mathbf{t})^\tau$. Moreover, the categorification of

covering algebra representations is not yet developed enough to produce the analogous machinery to [28]. We hope that our results will help cast light on these remaining questions.

1.3 Organization

The paper is organized as follows. In [Section 2](#), we recall the definition of quantum covering $\mathfrak{osp}(1|2n)$, denoted by U , and set our conventions. We also develop some additional facts about representations of U , specifically about dual modules and (co)evaluation morphisms, and produce a universal- \mathcal{R} -matrix, which we will simply denote by \mathcal{R} , from the quasi- \mathcal{R} -matrix defined in [7]. In [Section 3](#), we interpret the maps in terms of the usual graphical calculus define an associated knot invariant. More precisely, maps are represented by a finite number of labeled, nonintersecting oriented strands such that the \mathcal{R} -matrix is a positive crossing, the (co)evaluation morphisms are various cups and caps, and orientation is determined by whether the associated module in the domain/range is the dual module or not. We show that the maps satisfy identities such that the associated graphical calculus is almost an framed oriented tangle invariant, and is indeed an oriented tangle invariant after renormalizing these elementary diagrams by an integer power of τ and a factor depending on the writhe. Finally, in [Section 4](#), we use the twistor maps introduced in [4; 5] to relate the morphisms in the $\pi = \pm 1$ cases. In particular, we develop some further details about the Hopf structure and representation theory of the enhanced quantum group \widehat{U} , and construct twistors on tensor products of simple modules and their duals. We then show that these twistors almost commute (up to an integer power of \mathfrak{t}) with the cups, caps, and crossings, allowing us to relate the \mathfrak{so} and \mathfrak{osp} knot invariants.

Acknowledgements I would like to thank David Hill for first suggesting to me this project constructing the quantum covering knot invariants, and Matt Hogancamp for helping me learn about quantum knot invariants at this project's inception. I would also like to thank Aaron Lauda, Weiqiang Wang, and Ben Webster for stimulating conversations about this project, and Sacha Blumen and several referees for very helpful comments.

2 Quantum covering $\mathfrak{osp}(1|2n)$

We begin by recalling the definition of the quantum covering algebra associated to $\mathfrak{osp}(1|2n)$ and setting our notation. We then elaborate on the representation theory of this algebra.

2.1 Root data

Let $I = I_0 \sqcup I_1$ with $I_0 = \{1, \dots, n-1\}$ and $I_1 = \{n\}$, and define the parity $p(i)$ of $i \in I$ by $i \in I_{p(i)}$. For $1 \leq r, s \leq n$, we define

$$r \cdot s = \begin{cases} 2 & \text{if } r = s = n, \\ 4 & \text{if } r = s \neq n, \\ -2 & \text{if } r = s \pm 1, \\ 0 & \text{otherwise,} \end{cases} \quad d_r = \frac{1}{2} r \cdot r,$$

and note that $p(r) = d_r \pmod 2$. Then (I, \cdot) is a bar-consistent anisotropic super Cartan datum [7]. We extend \cdot to a symmetric bilinear pairing on $\mathbb{Z}[I]$ and p to a parity function $p: \mathbb{Z}[I] \rightarrow \mathbb{Z}/2\mathbb{Z}$. Moreover, for $v = i_1 + \dots + i_t \in \mathbb{N}[I]$, we set

$$(2-1) \quad \text{ht } v = t, \quad p(v) = \sum_{1 \leq r < s \leq t} p(i_r)p(i_s), \quad \bullet(v) = \sum_{1 \leq r < s \leq t} i_r \cdot i_s.$$

Let $\Phi^+ \subset \mathbb{N}[I]$ denote the set of positive roots, and set

$$(2-2) \quad \rho = \sum_{\alpha \in \Phi^+} \alpha = \sum_{i \in I} \rho_i i \in \mathbb{N}[I].$$

Note that we have $i \cdot \rho = i \cdot i$ for all $i \in I$.

Let $Y = \mathbb{Z}[I]$ be the root lattice and $X = \text{Hom}(\mathbb{Z}[I], \mathbb{Z})$ be the weight lattice, and let $\langle \cdot, \cdot \rangle: Y \times X \rightarrow \mathbb{Z}$ be the natural pairing. We also identify $\mathbb{Z}[I]$ as a subspace of X so that $\langle r, s \rangle = 2r \cdot s / r \cdot r$. If $v = \sum_{i \in I} v_i i \in \mathbb{Z}[I]$, we set

$$(2-3) \quad \tilde{v} = \sum_{i \in I} d_i v_i i \in \mathbb{Z}[I],$$

and note $\langle \tilde{v}, \mu \rangle = v \cdot \mu$ for any $v, \mu \in \mathbb{Z}[I]$; in particular, observe that for any $i \in I$,

$$(2-4) \quad \langle \tilde{\rho}, i \rangle = i \cdot i.$$

Then $((I, \cdot), X, Y, \langle \cdot, \cdot \rangle)$ is the root datum associated to $\mathfrak{osp}(1|2n)$, and forgetting the parity on the root datum yields the root datum associated to $\mathfrak{so}(1+2n)$. As usual, we define the dominant weights to be $X^+ = \{\lambda \in X \mid \langle i, \lambda \rangle \geq 0 \text{ for all } i \in I\}$.

Example 2.1 When $n = 1$, we identify $X = \mathbb{Z}$ where $\langle 1, k \rangle = k$ for $k \in \mathbb{Z}$. Then $Y = \mathbb{Z}1$ can be identified with subset $2\mathbb{Z} \subset X$. We will freely use these identifications in later examples.

Note that the weight lattice X doesn't naturally have a parity grading compatible with that on $\mathbb{Z}[I]$. However, a parity grading on X can be defined as follows. First observe

that X carries an action of the Weyl group W of type B_n , and that in particular, $\lambda - w\lambda \in \mathbb{Z}[I]$ for any $\lambda \in X$. Let w_0 denote the longest element of B_n . If $\lambda \in X$, then $w_0\lambda = -\lambda$ hence $2\lambda = \lambda - w_0\lambda \in \mathbb{Z}[I]$. We write $2\lambda = \sum_{i \in I} (2\lambda)_i i$ and define

$$(2-5) \quad P(\lambda) = p(2\lambda) \equiv (2\lambda)_{\underline{n}} \pmod{2}.$$

This defines a parity grading on X , though it is obviously not compatible with the grading on $\mathbb{Z}[I]$ (indeed, for any $i \in I$ we have $P(i) = p(2i) = 2p(i) \equiv 0 \pmod{2}$). In particular, P is constant on cosets $X/\mathbb{Z}[I]$. This parity can be expressed explicitly in terms of the rank and weight as follows.

Lemma 2.2 *Let the notation be as above. Then $P(\lambda) \equiv n \langle \underline{n}, \lambda \rangle \pmod{2}$.*

Proof Let $1 \leq s \leq n - 1$, and for convenience, set the notation $(2\lambda)_0 = 0$. We have

$$\begin{aligned} \langle \underline{s}, \lambda \rangle &= \frac{1}{2} \langle \underline{s}, 2\lambda \rangle = \frac{1}{2} \sum_{i \in I} (2\lambda)_i \langle \underline{s}, i \rangle = (2\lambda)_{\underline{s}} - \frac{1}{2} ((2\lambda)_{\underline{s+1}} + (2\lambda)_{\underline{s-1}}), \\ \langle \underline{n}, \lambda \rangle &= (2\lambda)_{\underline{n}} - (2\lambda)_{\underline{n-1}}. \end{aligned}$$

In particular, we see that $\frac{1}{2} ((2\lambda)_{\underline{s+1}} + (2\lambda)_{\underline{s-1}}) = (2\lambda)_{\underline{s}} - \langle \underline{s}, \lambda \rangle \in \mathbb{N}$; thus $(2\lambda)_{\underline{s-1}} \equiv (2\lambda)_{\underline{s+1}} \pmod{2}$ for all $1 \leq s \leq n - 1$. Therefore, $(2\lambda)_{\underline{r}} \equiv (2\lambda)_{\underline{s}} \pmod{2}$ whenever $r \equiv s \pmod{2}$.

In particular, since $(2\lambda)_0 = 0$, we see that $(2\lambda)_{\underline{s}} \equiv 0 \pmod{2}$ for each $s \equiv 0 \pmod{2}$. If $n \equiv 0 \pmod{2}$, then $P(\lambda) \equiv (2\lambda)_{\underline{n}} \equiv 0 \pmod{2}$. If $n \equiv 1 \pmod{2}$, then $(2\lambda)_{\underline{n}} = \langle \underline{n}, \lambda \rangle - (2\lambda)_{\underline{n-1}} \equiv \langle \underline{n}, \lambda \rangle \pmod{2}$. □

Example 2.3 When $n = 1$, recall from [Example 2.1](#) that we identify $X = \mathbb{Z}$. Then $P(k) \equiv (1) \langle \underline{1}, k \rangle \equiv k \pmod{2}$ for any $k \in \mathbb{Z}$; hence our P -grading is just the natural parity grading on \mathbb{Z} .

Throughout, we will consider objects graded by $\widehat{X} = X \times (\mathbb{Z}/2\mathbb{Z})$. If M is \widehat{X} -graded and $m \in M$ is homogeneous, we let $\|m\|$ (resp. $|m|$, $p(m)$) denote its \widehat{X} -degree (resp. X -degree, $\mathbb{Z}/2\mathbb{Z}$ -degree or parity). Further, for $\zeta = (\lambda, \epsilon) \in \widehat{X}$, we will set $|\zeta| = \lambda$, $p(\zeta) = \epsilon$, and $P(\zeta) = P(\lambda)$. (Note that $P(\zeta)$ is not the same as $p(\zeta)$ in general! They are independent quantities.)

For $\lambda \in X$, let $\widehat{\lambda} = (\lambda, 0) \in \widehat{X}$. The set $\{(v, p(v)) \mid v \in \mathbb{Z}[I]\} \subset \widehat{X}$ can and will be identified with $\mathbb{Z}[I]$ via $v \mapsto (v, p(v))$. In particular, if $\zeta = (\lambda, \epsilon) \in \widehat{X}$ and $v \in \mathbb{Z}[I]$, then

$$(2-6) \quad \zeta + v = (\lambda + v, \epsilon + p(v)) \in \widehat{X}.$$

With that in mind, the action of W on X generalizes naturally to \widehat{X} by setting

$$(2-7) \quad s_i(\lambda, \epsilon) = (\lambda, \epsilon) - \langle i, \lambda \rangle i = (\lambda - \langle i, \lambda \rangle i, \epsilon - \langle i, \lambda \rangle p(i)),$$

where $i \in I$ and s_i is the corresponding simple reflection.

Lastly, we have the parity swap function $\Pi: \widehat{X} \rightarrow \widehat{X}$ defined by

$$(2-8) \quad \Pi((\lambda, \epsilon)) = (\lambda, 1 - \epsilon).$$

2.2 Parameters

Let $\mathbf{t} \in \mathbb{C}$ such that $\mathbf{t}^2 = -1$. Let q be a formal parameter and let τ be an indeterminate such that

$$\tau^4 = 1.$$

For convenience, we will also define

$$\pi = \tau^2.$$

If R is a commutative ring with 1, we let

$$(2-9) \quad R^\tau = R[\tau]/(\tau^4 = 1), \quad R^\pi = R[\pi]/(\pi^2 = 1).$$

Throughout, our base ring will be $\mathbb{Q}(q, \mathbf{t})^\tau$, though occasionally we will also refer to the subring generated by $\mathbb{Q}(q)$ and π , which we identify with $\mathbb{Q}(q)^\pi$.

We denote by $\bar{\cdot}: \mathbb{Q}(q, \mathbf{t})^\tau \rightarrow \mathbb{Q}(q, \mathbf{t})^\tau$ the $\mathbb{Q}(\mathbf{t})^\tau$ -algebra automorphism satisfying $\bar{q} = \pi q^{-1}$. We also define the $\mathbb{Q}(\mathbf{t})$ -algebra automorphism \mathfrak{X} given by $\mathfrak{X}(q) = \mathbf{t}^{-1}q$ and $\mathfrak{X}(\tau) = \mathbf{t}\tau$. We caution the reader that $\bar{\cdot}$ and \mathfrak{X} will be used later to denote extensions of these algebra automorphisms which are defined on $\mathbb{Q}(q, \mathbf{t})^\tau$ -algebras and $\mathbb{Q}(q, \mathbf{t})^\tau$ -modules.

Given an $\mathbb{Q}(q, \mathbf{t})^\tau$ -module (or algebra) M and $x \in \{\pm 1, \pm \mathbf{t}\}$, the $\mathbb{Q}(q, \mathbf{t})$ -module (or algebra) $M|_{\tau=x} = \mathbb{Q}(q, \mathbf{t})_x \otimes_{\mathbb{Q}(q, \mathbf{t})^\tau} M$, where $\mathbb{Q}(q, \mathbf{t})_x = \mathbb{Q}(q, \mathbf{t})$ is viewed as a $\mathbb{Q}(q, \mathbf{t})^\tau$ -module on which τ acts as multiplication by x . We call this the *specialization of M at $\tau = x$* . Moreover, $\mathbb{Q}(q, \mathbf{t})^\tau$ has orthogonal idempotents

$$(2-10) \quad \varepsilon_{\mathbf{t}^k} = \frac{1}{4}(1 + \mathbf{t}^k \tau + (\mathbf{t}^k \tau)^2 + (\mathbf{t}^k \tau)^3), \quad 0 \leq k \leq 3,$$

such that $\mathbb{Q}(q, \mathbf{t})^\tau = \mathbb{Q}(q, \mathbf{t})_{\varepsilon_1} \oplus \mathbb{Q}(q, \mathbf{t})_{\varepsilon_{\mathbf{t}}} \oplus \mathbb{Q}(q, \mathbf{t})_{\varepsilon_{-1}} \oplus \mathbb{Q}(q, \mathbf{t})_{\varepsilon_{-\mathbf{t}}}$. In particular, since $\tau \varepsilon_x = x \varepsilon_x$, we see that for any $\mathbb{Q}(q, \mathbf{t})^\tau$ -module M ,

$$M|_{\tau=x} \cong \varepsilon_x M.$$

For $k \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}$, the (q, π) -quantum integers, along with quantum factorial and quantum binomial coefficients, are defined as follows (see [7]):

$$(2-11) \quad [n]_{q,\pi} = \frac{(\pi q)^n - q^{-n}}{\pi q - q^{-1}}, \quad [n]_{q,\pi}! = \prod_{l=1}^n [l]_{q,\pi},$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,\pi} = \frac{\prod_{l=n-k+1}^n ((\pi q)^l - q^{-l})}{\prod_{m=1}^k ((\pi q)^m - q^{-m})}.$$

If $\nu = \sum_{i \in I} \nu_i i \in \mathbb{Z}[I]$, we write

$$q_\nu = \prod_{i \in I} q^{\nu_i d_i}, \quad \tau_\nu = \prod_{i \in I} \tau^{\nu_i d_i}, \quad \pi_\nu = \prod_{i \in I} \pi^{\nu_i d_i} = \pi^{p(\nu)}, \quad \mathbf{t}_\nu = \prod_{i \in I} \mathbf{t}^{\nu_i d_i}.$$

In particular, note that $q_i = q^{d_i}$ and $\pi_i = \pi^{d_i} = \pi^{p(i)}$, and set

$$[n]_i = [n]_{q_i, \pi_i}, \quad [n]_i! = [n]_{q_i, \pi_i}!, \quad \begin{bmatrix} n \\ k \end{bmatrix}_i = \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, \pi_i}.$$

2.3 The covering quantum group

The covering quantum group associated to $\mathfrak{osp}(1|2n)$ (as well as some variants) was introduced and studied in the series of papers starting with [7]. We will recall the necessary definitions and elementary facts now.

Remark 2.4 Note that contrary to [7] and further papers in that series, we will take coefficients in the larger ring $\mathbb{Q}(q, \mathbf{t})^\tau \supset \mathbb{Q}(q)^\pi$. Nevertheless, all of the results until Section 3 are essentially statements over $\mathbb{Q}(q)^\pi$ which remain true after extending scalars to $\mathbb{Q}(q, \mathbf{t})^\tau$, so the reader may effectively ignore τ and \mathbf{t} for the present.

Definition 2.5 [7] The half-quantum covering group \mathcal{f} associated to the datum (I, \cdot) is the $\mathbb{N}[I]$ -graded $\mathbb{Q}(q, \mathbf{t})^\tau$ -algebra on the generators θ_i for $i \in I$ with $|\theta_i| = i$, satisfying the relations

$$(2-12) \quad \sum_{k=0}^{b_{ij}} (-1)^k \pi^{\binom{k}{2} p(i) + k p(i) p(j)} \begin{bmatrix} b_{ij} \\ k \end{bmatrix}_i \theta_i^{b_{ij}-k} \theta_j \theta_i^k = 0 \quad (i \neq j),$$

where $b_{ij} = 1 - \langle i, j \rangle$.

The algebra \mathcal{f} carries a nondegenerate bilinear form (\cdot, \cdot) which satisfies

$$(2-13) \quad (1, 1) = 1, \quad (\theta_i, \theta_i) = \frac{1}{1 - \pi_i q_i^{-2}}, \quad (\theta_i x, y) = (\theta_i, \theta_i)(x, i r(y)),$$

where ${}_i r: \mathfrak{f} \rightarrow \mathfrak{f}$ is the $\mathbb{Q}(q, \mathfrak{t})^\tau$ -linear map satisfying ${}_i r(1) = 0$, ${}_i r(\theta_j) = \delta_{ij}$, and ${}_i r(xy) = {}_i r(x)y + \pi^{p(i)p(x)} q^{i \cdot |x|} {}_i r(y)$. (Here, and henceforth, $\delta_{x,y}$ is set to be $\delta_{x,y} = 1$ if $x = y$ and 0 otherwise.) We define the $\mathbb{Q}(\mathfrak{t})^\tau$ -linear bar involution $\bar{\cdot}$ on \mathfrak{f} by

$$\bar{\theta}_i = \theta_i, \quad \bar{q} = \pi q^{-1}.$$

We also define the $\mathbb{Q}(q, \mathfrak{t})^\tau$ -linear anti-involution σ on \mathfrak{f} by

$$\sigma(\theta_i) = \theta_i, \quad \sigma(xy) = \sigma(y)\sigma(x),$$

and the divided powers

$$\theta_i^{(n)} = \theta_i^n / [n]_i!$$

Definition 2.6 [7] The quantum covering group U associated to the root datum $((I, \cdot), Y, X, \langle \cdot, \cdot \rangle)$ is the $\mathbb{Q}(v)^\pi$ -algebra with generators E_i, F_i, K_μ , and J_μ subject to the following relations for $i, j \in I$ and $\mu, \nu \in Y$:

$$(2-14) \quad J_\mu J_\nu = J_{\mu+\nu}, \quad K_\mu K_\nu = K_{\mu+\nu}, \quad K_0 = J_0 = J_\nu^2 = 1, \quad J_\mu K_\nu = K_\nu J_\mu,$$

$$(2-15) \quad J_\mu E_i = \pi^{\langle \mu, i \rangle} E_i J_\mu, \quad J_\mu F_i = \pi^{-\langle \mu, i \rangle} F_i J_\mu,$$

$$(2-16) \quad K_\mu E_i = q^{\langle \mu, i \rangle} E_i K_\mu, \quad K_\mu F_i = q^{-\langle \mu, i \rangle} F_i K_\mu,$$

$$(2-17) \quad E_i F_j - \pi^{p(i)p(j)} F_j E_i = \delta_{ij} \frac{J_{d_i i} K_{d_i i} - K_{-d_i i}}{\pi_i q_i - q_i^{-1}},$$

$$(2-18) \quad \sum_{k=0}^{b_{ij}} (-1)^k \pi^{\binom{k}{2} p(i) + k p(i)p(j)} \begin{bmatrix} b_{ij} \\ k \end{bmatrix}_{q_i, \pi_i} E_i^{b_{ij}-k} E_j E_i^k = 0 \quad (i \neq j),$$

$$(2-19) \quad \sum_{k=0}^{b_{ij}} (-1)^k \pi^{\binom{k}{2} p(i) + k p(i)p(j)} \begin{bmatrix} b_{ij} \\ k \end{bmatrix}_{q_i, \pi_i} F_i^{b_{ij}-k} F_j F_i^k = 0 \quad (i \neq j).$$

Since $Y = \mathbb{Z}[I]$ in this case, we note that U is actually generated by E_i, F_i, K_i, J_i for $i \in I$. For notational convenience, we set $\tilde{J}_\nu = J_\nu$ and $\tilde{K}_\nu = K_\nu$ so that (2-17) becomes

$$E_i F_j - \pi^{p(i)p(j)} F_j E_i = \delta_{ij} \frac{\tilde{J}_i \tilde{K}_i - \tilde{K}_i^{-1}}{\pi_i q_i - q_i^{-1}}.$$

We also equip U with a bar involution $\bar{\cdot}: U \rightarrow U$ extending that on $\mathbb{Q}(q, \mathfrak{t})^\tau$ by setting $\bar{E}_i = E_i$, $\bar{F}_i = F_i$, $\bar{K}_\mu = J_\mu K_{-\mu}$, and $\bar{J}_\mu = J_\mu$.

The algebras U and \mathfrak{f} are related in the following way. Let U^- be the subalgebra generated by F_i with $i \in I$, U^+ the subalgebra generated by E_i with $i \in I$, and U^0 the subalgebra generated by K_ν and J_ν for $\nu \in Y$. There is an isomorphism $\mathfrak{f} \rightarrow U^-$

(resp. $\mathfrak{f} \rightarrow U^+$) defined by $\theta_i \mapsto \theta_i^- = F_i$ (resp. $\theta_i \mapsto \theta_i^+ = E_i$). As shown in [7], there is a triangular decomposition

$$U \cong U^- \otimes U^0 \otimes U^+ \cong U^+ \otimes U^0 \otimes U^-.$$

There is also a root space decomposition

$$U = \bigoplus_{\nu \in \mathbb{Z}[I]} U_\nu, \quad U_\nu = \{x \in U \mid J_\mu K_\xi m = \pi^{(\mu, \nu)} q^{(\xi, \nu)} m\}.$$

The root space decomposition induces a parity grading via $p(u) = p(|u|)$; hence in particular, U is \widehat{X} -graded.

We say an algebra is a ‘‘Hopf covering algebra’’ if it is a $\mathbb{Z}/2\mathbb{Z}$ -graded algebra over \mathbb{Z}^π with a coproduct, antipode, and counit satisfying the usual axioms of a Hopf superalgebra, but with the braiding replaced by $x \otimes y \mapsto \pi^{p(x)p(y)} y \otimes x$. Then the algebra U is a Hopf covering algebra under the coproduct $\Delta: U \rightarrow U \otimes U$ satisfying

$$\begin{aligned} \Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, & \Delta(K_\nu) &= K_\nu \otimes K_\nu, \\ \Delta(F_i) &= F_i \otimes \tilde{K}_i^{-1} + 1 \otimes F_i, & \Delta(J_\nu) &= J_\nu \otimes J_\nu; \end{aligned}$$

the antipode $S: U \rightarrow U$ satisfying $S(xy) = \pi^{p(x)p(y)} S(y)S(x)$ for $x, y \in U$ and

$$S(E_i) = -\tilde{J}_i^{-1} \tilde{K}_i^{-1} E_i, \quad S(F_i) = -F_i \tilde{K}_i, \quad S(K_\nu) = K_\nu^{-1}, \quad S(J_\nu) = J_\nu^{-1};$$

and the counit $\epsilon: U \rightarrow \mathbb{Q}(q, \mathfrak{t})^\mp$ satisfying

$$\epsilon(E_i) = \epsilon(F_i) = 0, \quad \epsilon(K_\nu) = \epsilon(J_\nu) = 1.$$

Moreover, for $x \in \mathfrak{f}$, we have that

$$\begin{aligned} S^{\pm 1}(x^+) &= (-1)^{\text{ht } \nu} \pi^{p(\nu)} q^{\nu \cdot \nu/2} q_{\mp \nu} \tilde{J}_- \tilde{K}_- \sigma(x)^+, \\ S^{\pm 1}(x^-) &= (-1)^{\text{ht } \nu} \pi^{p(\nu)} q^{-\nu \cdot \nu/2} q_{\pm \nu} \sigma(x)^- \tilde{K}_\nu. \end{aligned} \tag{2-20}$$

2.4 U-modules

In this paper, a weight U -module is a U -module M with a \widehat{X} -grading compatible with the grading on U , such that

$$M = \bigoplus_{\lambda \in X} M_{\lambda,0} \oplus M_{\lambda,1}, \quad M_{\lambda,s} = \{m \in M \mid p(m) = s, J_\mu K_\nu m = \pi^{(\mu, \lambda)} q^{(\nu, \lambda)} m\},$$

and each $M_{\lambda,s}$ is a free $\mathbb{Q}(q, \mathfrak{t})^\mp$ -module of finite rank. For $\lambda \in X$, let $M_\lambda = M_{\lambda,0} \oplus M_{\lambda,1}$. We also define the parity-swapped module ΠM to be M as a vector space with the same action of U , but with $\Pi M_{\lambda,s} = M_{\lambda,1-s}$. We let \mathcal{O}_{fin} be the

category of weight U -modules of finite rank over $\mathbb{Q}(q, \mathbf{t})^\tau$. Henceforth, we shall always assume our U -modules are in \mathcal{O}_{fin} .

We define the (restricted) linear dual of a U -module M :

$$M^* = \bigoplus_{\lambda \in X} (M_{\lambda,0})^* \oplus (M_{\lambda,1})^*, \quad (M_{\lambda,s})^* = \text{Hom}_{\mathbb{Q}(q,\mathbf{t})^\tau}(M_{\lambda,s}, \mathbb{Q}(q, \mathbf{t})^\tau).$$

This is again a free $\mathbb{Q}(q, \mathbf{t})^\tau$ -module, which has a $\mathbb{Z}/2\mathbb{Z}$ -grading induced by that of V : namely, $p(f) = 0$ if $f(v) = 0$ for $p(v) = 1$, and vice-versa. Moreover, the Hopf superalgebra structure of U induces an action of U : for $f \in V^*$ and $x \in U$, we define $xf \in V^*$ by $xf(v) = \pi^{p(f)p(x)} f(S(x)v)$. In particular, note that V^* is a U -module with $(V^*)_{\lambda,s} = (V_{-\lambda,s})^*$. While V_λ^* is therefore ambiguous, we will always take it to denote $(V^*)_\lambda$. (In other words, our convention is that taking duals has precedence over taking weight spaces.)

We can construct the U -module $V \otimes W = V \otimes_{\mathbb{Q}(q,\mathbf{t})^\tau} W$, for any U -modules V and W , via the coproduct. In particular, we have U -modules $V^* \otimes V$ and $V \otimes V^*$, both of which contain a copy of the trivial module $V(0) = \mathbb{Q}(q, \mathbf{t})^\tau$ as a direct summand. As the following lemma shows, there are natural projection and inclusion maps to a copy of the trivial module. We borrow notation from [26].

Lemma 2.7 Fix a U -module V and recall the definition of ρ from (2-2).

- (1) Let $\text{ev}_V: V^* \otimes V \rightarrow \mathbb{Q}(q, \mathbf{t})^\tau$ be the $\mathbb{Q}(q, \mathbf{t})^\tau$ -linear map defined by $v^* \otimes w \mapsto v^*(w)$. Then ev_V is a U -module epimorphism.
- (2) Let $\text{qtr}_V: V \otimes V^* \rightarrow \mathbb{Q}(q, \mathbf{t})^\tau$ be the $\mathbb{Q}(q, \mathbf{t})^\tau$ -linear map defined by $v \otimes w^* \mapsto \pi^{p(v)p(w)} q^{-\langle \tilde{\rho}, |v| \rangle} w^*(v)$. Then qtr_V is a U -module epimorphism.
- (3) Let $\text{coev}_V: \mathbb{Q}(q, \mathbf{t})^\tau \rightarrow V^* \otimes V$ be the $\mathbb{Q}(q, \mathbf{t})^\tau$ -linear map defined by $1 \mapsto \sum_{b \in B} \pi^{p(b)} q^{\langle \tilde{\rho}, |b| \rangle} b^* \otimes b$ for some homogeneous $\mathbb{Q}(q, \mathbf{t})^\tau$ -basis B of V . Then coev_V is a U -module monomorphism.
- (4) Let $\text{coqtr}_V: \mathbb{Q}(q, \mathbf{t})^\tau \rightarrow V \otimes V^*$ be the $\mathbb{Q}(q, \mathbf{t})^\tau$ -linear map defined by $1 \mapsto \sum_{b \in B} b \otimes b^*$ for some homogeneous $\mathbb{Q}(q, \mathbf{t})^\tau$ -basis B of V . Then coqtr_V is a U -module monomorphism.

Proof This is elementary to verify. □

2.5 Simple modules and their duals

Let $\lambda \in X^+$ and recall from [7] that $V(\lambda)$ is the simple U -module of highest weight λ such that the highest weight space has even parity. Then $V(\lambda)$ has finite rank and has

the same character as the $\mathfrak{so}(2n + 1)$ module of highest weight λ . In particular, the lowest weight vector has weight $w_0\lambda = -\lambda$; hence the parity of the lowest weight vector of $V(\lambda)$ is $P(\lambda)$. Using standard arguments (for example, analogues of [15, Sections 5.3 and 5.16]), and considering the above analysis, we obtain the following lemma.

Lemma 2.8 *For each $\lambda \in X^+$, there is an isomorphism $V(\lambda)^* \cong \Pi^{P(\lambda)}V(\lambda)$ and a natural isomorphism $V(\lambda)^{**} \rightarrow V(\lambda)$.*

Example 2.9 In the case $n = 1$, the module $V = V(m)$ for $m \in \mathbb{Z}_{\geq 0}$ has basis $v_{m-2k} = F^{(k)}v_m$ with $0 \leq k \leq m$, where v_m is a choice of highest weight vector. Note that by convention, $p(v_m) = 0$, so $p(v_{m-2k}) \equiv k \pmod{2}$. The dual module $V(m)^*$ has a dual basis v_{m-2k}^* , $0 \leq k \leq m$, and the actions of E and F are given by

$$\begin{aligned}
 Ev_{m-2k}^* &= -\pi^k(\pi q)^{m-2k}[n + 1 - k]v_{m-2(k+1)}^*, \\
 Fv_{m-2k}^* &= -\pi^k(\pi q)^{m-2k+2}[k]v_{m-2(k-1)}^*.
 \end{aligned}$$

In particular, this is a simple module generated by the highest weight vector v_{-m}^* , where $|v_{-m}^*| = -|v_{-m}| = m$ and $p(v_{-m}^*) = p(v_{-m}) \equiv m \pmod{2}$; hence we have an isomorphism $V(m)^* \cong \Pi^m V(m)$.

For convenience, we will use the notation

$$(2-21) \quad V(-\lambda) = V(\lambda)^*, \quad \lambda \in X^+.$$

We denote the maps in Lemma 2.7 in the case $V = V(\lambda)$ with the subscript λ instead of $V(\lambda)$; for instance, $\text{ev}_\lambda = \text{ev}_{V(\lambda)}$. Note that

$$\text{ev}_\lambda \circ \text{coev}_\lambda = \sum_{v \in \mathbb{N}[I]} \text{rank}_{\mathbb{Q}(q,t)^\tau}(V_{\lambda-v})\pi^{p(v)}q^{\langle \tilde{\rho}, \lambda-v \rangle} = \pi^{P(\lambda)} \text{qtr}_\lambda \circ \text{coqtr}_\lambda.$$

Example 2.10 For $n = 1$, we have $\rho = \tilde{\rho} = \underline{1}$; hence for $\lambda = m$, $\langle \tilde{\rho}, \lambda \rangle = m$. Then

$$\text{ev}_m \circ \text{coev}_m = q^m + \pi q^{m-2} + \dots + \pi^m q^{-m} = \pi^m [m + 1] = \pi^m \text{qtr}_m \circ \text{coqtr}_m.$$

2.6 Further properties of the quasi- \mathcal{R} -matrix

Let us recall the quasi- \mathcal{R} -matrix from [7, Section 4].

Theorem 2.11 [7] Let \mathbf{B} be any $\mathbb{Q}(q, t)^\tau$ -basis of \mathbf{f} such that $\mathbf{B}_\nu = \mathbf{B} \cap \mathbf{f}_\nu$ is a basis of \mathbf{f}_ν for any $\nu \in \mathbb{N}[I]$, with $\mathbf{B}_0 = \{1\}$. Let $\mathbf{B}^* = \{b^* \mid b \in \mathbf{B}\}$ be the basis of \mathbf{f} dual to \mathbf{B} under (\cdot, \cdot) . Define

$$\Theta_\nu = (-1)^{\text{ht } \nu} \pi^{\mathbf{P}(\nu)} \pi_\nu q_\nu \sum_{b \in \mathbf{B}_\nu} b^- \otimes (b^*)^+ \in U_{-\nu}^- \otimes U_\nu^+.$$

Then if M, M' are integrable modules of U , then $\Theta = \sum_\nu \Theta_\nu$ is a well-defined operator on $M \otimes M'$ which satisfies $\Delta(u)\Theta = \Theta \overline{\Delta}(u)$ as endomorphisms of $M \otimes M'$, where $\overline{\Delta}(u) = \overline{\Delta(\overline{u})}$. Moreover, Θ is independent of the choice of basis \mathbf{B} and is invertible with inverse $\overline{\Theta}$.

Example 2.12 When $n = 1$, the quasi- \mathcal{R} -matrix Θ can be explicitly given by the formula

$$\Theta = \sum_{n \geq 0} (-1)^n (\pi q)^{-\binom{n}{2}} [n]! (\pi q - q^{-1})^n F^{(n)} \otimes E^{(n)} = 1 - (\pi q - q^{-1})F \otimes E + \dots$$

(NB there is a typo in the power of πq in [7, Example 3.1.2].)

While $\overline{\Theta}$ can be evaluated easily, it will be more convenient to have the following alternate description of $\overline{\Theta}$ using the properties of the bilinear form on \mathbf{f} ; see [7, Section 1.4].

Lemma 2.13 With the same notation as in Theorem 2.11, $\overline{\Theta} = \sum_\nu \overline{\Theta}_\nu$ is given by

$$\overline{\Theta}_\nu = \pi_\nu q^{\nu \cdot \nu / 2} \sum_{b \in \mathbf{B}_\nu} b^- \otimes \sigma(b^*)^+ \in U_{-\nu}^- \otimes U_\nu^+.$$

Proof Let $\overline{\mathbf{B}} = \{\overline{b} \mid b \in \mathbf{B}\}$, with dual basis $\overline{\mathbf{B}}^*$. Then since Θ is independent of the choice of basis, we see that $\Theta_\nu = (-1)^{\text{ht } \nu} \pi^{\mathbf{P}(\nu)} \pi_\nu q_\nu \sum_{b \in \mathbf{B}_\nu} \overline{b}^- \otimes (\overline{b}^*)^+$ for $\nu \in \mathbb{N}[I]$. We have

$$\overline{\Theta}_\nu = (-1)^{\text{ht } \nu} \pi^{\mathbf{P}(\nu)} q_{-\nu} \sum_{b \in \mathbf{B}_\nu} (\overline{b}^-) \otimes (\overline{b}^*)^+,$$

and note that $(\overline{x})^\pm = \overline{(x^\pm)}$, so $(\overline{b}^-) = b^-$.

On the other hand, recall from [7, Section 1.4] the variant bilinear form $\{-, -\}$ defined by $\{x, y\} = (\overline{x}, \overline{y})$. Note that by construction, $(\overline{b}^*, \overline{b}') = \delta_{b, b'}$. Then for any $b, b' \in \mathbf{B}$, we apply Lemma 1.4.3(b) of [loc. cit.] to deduce that

$$\delta_{b, b'} = \overline{(\overline{b}^*, \overline{b}')} = \{\overline{b}^*, b'\} = (-1)^{\text{ht } \nu} \pi^{\mathbf{P}(\nu)} \pi_\nu q^{-\nu \cdot \nu / 2} q_{-\nu} (\overline{b}^*, \sigma(b')).$$

(We note that while the power of π appears different from that in [loc. cit.], it is equivalent.) Therefore, we have

$$\overline{b^*} = (-1)^{\text{ht } \nu} \pi^{P(\nu)} q^{\nu \cdot \nu / 2} \pi_\nu q_\nu \sigma(b)^*.$$

Then the lemma follows from the observation that since $(\sigma(x), \sigma(y)) = (x, y)$, we have $\sigma(b)^* = \sigma(b^*)$. □

Now we will proceed to use Θ to define a universal map $\mathcal{R}: M \otimes N \rightarrow N \otimes M$ for any modules M and N . These constructions will be modified versions of the standard arguments in the nonsuper case; compare [15, Sections 7.3–7.6] or [18, Section 4.2 and Chapter 32].

For $1 \leq s < t \leq 3$, let $\Theta_\nu^{st} \in U \otimes U \otimes U$ be defined by

$$\Theta_\nu^{st} = (-1)^{\text{ht } \nu} \pi^{P(\nu)} \pi_\nu q_\nu \sum_{b \in \mathbf{B}_\nu} b_1 \otimes b_2 \otimes b_3,$$

where $b_s = b^-$, $b_t = (b^*)^+$, and $b_m = 1$ for $m \neq s, t$.

Proposition 2.14 *We have the following identities:*

$$\begin{aligned} (\Delta \otimes 1)(\Theta_\nu) &= \sum_{\nu' + \nu'' = \nu} \Theta_{\nu'}^{23} (1 \otimes \tilde{K}_{-\nu''} \otimes 1) \Theta_{\nu''}^{13}, \\ (\bar{\Delta} \otimes 1)(\Theta_\nu) &= \sum_{\nu' + \nu'' = \nu} \Theta_{\nu'}^{13} (1 \otimes \tilde{J}_{\nu'} \tilde{K}_{\nu''} \otimes 1) \Theta_{\nu''}^{23}, \\ (1 \otimes \Delta)(\Theta_\nu) &= \sum_{\nu' + \nu'' = \nu} \Theta_{\nu'}^{12} (1 \otimes \tilde{J}_{\nu'} \tilde{K}_{\nu''} \otimes 1) \Theta_{\nu''}^{13}, \\ (1 \otimes \bar{\Delta})(\Theta_\nu) &= \sum_{\nu' + \nu'' = \nu} \Theta_{\nu'}^{13} (1 \otimes \tilde{K}_{-\nu''} \otimes 1) \Theta_{\nu''}^{12}. \end{aligned}$$

Proof These identities are proved exactly as in [18, Section 4.2] using the analogous results in [7]. □

To construct a universal U -module homomorphism from Θ , we will need some additional maps. The first is the swap map; that is, the algebra $U \otimes U$ is equipped with an involution \mathfrak{s} defined by $\mathfrak{s}(x \otimes y) = \pi^{P(x)P(y)} y \otimes x$. This induces involutions on $U^{\otimes m}$ by applying \mathfrak{s} to sequential pairs of tensor factors; specifically, these involutions are the maps $\mathfrak{s}_{t,t+1} = 1^{\otimes t-1} \otimes \mathfrak{s} \otimes 1^{m-t-1}$, and it is not hard to see they satisfy the braid relations $\mathfrak{s}_{t-1,t} \mathfrak{s}_{t,t+1} \mathfrak{s}_{t-1,t} = \mathfrak{s}_{t,t+1} \mathfrak{s}_{t-1,t} \mathfrak{s}_{t,t+1}$. In particular, we see that to each element γ of the permutation group \mathfrak{S}_m , there is an automorphism \mathfrak{s}_γ of U^m ;

for example, $\mathfrak{s}_{(23)} = \mathfrak{s}_{2,3}$ and $\mathfrak{s}_{(123)} = \mathfrak{s}_{1,2}\mathfrak{s}_{2,3}$. Similarly, to any tensor product of modules $N = \bigotimes_{i=1}^m M_i$ and $\gamma \in \mathfrak{S}_m$, we can define $N_\gamma = \bigotimes_{i=1}^m M_{\gamma(i)}$ and a map $\mathfrak{s}_\gamma: N \rightarrow N_\gamma$ given by

$$\mathfrak{s}(v) = \pi^{p(\gamma,v)} v_\gamma,$$

where $v = v_1 \otimes \dots \otimes v_m$, $v_\gamma = v_{\gamma(1)} \otimes \dots \otimes v_{\gamma(m)}$, and

$$p(\gamma, v) = \sum_{\substack{1 \leq s < t \leq n \\ \gamma(s) > \gamma(t)}} p(v_t) p(v_s).$$

These maps are compatible in the sense that for $v = \bigotimes_{i=1}^n v_i \in N$ and $u \in U$,

$$\mathfrak{s}_\gamma(\Delta^{m-1}(u)v) = \mathfrak{s}_\gamma(\Delta^{m-1}(u))\mathfrak{s}_\gamma(v).$$

When $m = 2$, we will just write $\mathfrak{s} = \mathfrak{s}_{1,2}$.

The other ingredient is a weight-renormalization operator. This operator is induced by the weight function defined in the following lemma.

Lemma 2.15 *There exists a function $\mathfrak{f}: X \times X \rightarrow (\mathbb{Q}(q, t)^\tau)^\times$ satisfying*

$$\mathfrak{f}(\zeta + \mu', \zeta' + \nu') \mathfrak{f}(\zeta, \zeta')^{-1} = (\pi q)^{-\langle \tilde{\mu}, \zeta' \rangle} q^{-\langle \tilde{\nu}, \zeta \rangle - \mu \cdot \nu}$$

for $\zeta, \zeta' \in X$ and $\mu, \nu \in \mathbb{Z}[I]$. Moreover,

- (1) *the function $\tau(\zeta, \zeta') = \mathfrak{f}(\zeta, \zeta') \mathfrak{f}(\zeta, -\zeta')$ satisfies $\tau(\zeta + \mu, \zeta' + \nu) = \tau(\zeta, \zeta')$ for any $\mu, \nu \in \mathbb{Z}[I]$,*
- (2) *the function $\mathfrak{l}(\zeta, \zeta') = \mathfrak{f}(\zeta, \zeta') \mathfrak{f}(-\zeta, \zeta')$ satisfies $\mathfrak{l}(\zeta + \mu, \zeta' + \nu) = \mathfrak{l}(\zeta, \zeta')$ for any $\mu, \nu \in \mathbb{Z}[I]$,*
- (3) *we have $\mathfrak{f}(\zeta, \zeta') \mathfrak{f}(-\zeta, -\zeta')^{-1} = \pi^{P(\zeta)P(\zeta')}$; in particular,*

$$\mathfrak{l}(\zeta, \zeta') = \pi^{P(\zeta)P(\zeta')} \tau(\zeta, \zeta').$$

Proof It is easy to verify that such a function \mathfrak{f} exists by choosing a set of coset representatives R for $\mathbb{Z}[I]$ in X . Then (3) follows easily using (2-5) and Lemma 2.2. \square

Example 2.16 Let us consider the case $n = 1$. Then the function \mathfrak{f} is determined by the values $\mathfrak{f}(0, 0)$, $\mathfrak{f}(0, 1)$, $\mathfrak{f}(1, 0)$, and $\mathfrak{f}(1, 1)$. Then for any $\epsilon_1, \epsilon_2 \in \{0, 1\}$,

$$\mathfrak{f}(\epsilon_1 + 2s, \epsilon_2 + 2t) = \mathfrak{f}(\epsilon_1, \epsilon_2) \pi^{s\epsilon_2} q^{-t\epsilon_1 - s\epsilon_2 - 2st}.$$

By direct computation, one finds the corresponding coset functions to be

$$\begin{aligned} \tau(\epsilon_1 + 2s, \epsilon_2 + 2t) &= \mathfrak{f}(\epsilon_1, \epsilon_2)^2 q^{\epsilon_1 \epsilon_2}, \\ \mathfrak{l}(\epsilon_1 + 2s, \epsilon_2 + 2t) &= \mathfrak{f}(\epsilon_1, \epsilon_2)^2 \pi^{\epsilon_1 \epsilon_2} q^{\epsilon_1 \epsilon_2}. \end{aligned}$$

Given U -modules M and M' , define the $\mathbb{Q}(q, t)^\tau$ -linear bijection $\mathfrak{F}: M \otimes M' \rightarrow M \otimes M'$ by $\mathfrak{F}(m \otimes m') = f(|m|, |m'|)m \otimes m'$. For $1 \leq s < t \leq 3$, we define \mathfrak{F}^{st} on $M_1 \otimes M_2 \otimes M_3$ via $\mathfrak{F}^{st}(m_1 \otimes m_2 \otimes m_3) = f(|m_s|, |m_t|)m_1 \otimes m_2 \otimes m_3$. Let $\mathfrak{F}_\Theta^{st} = \Theta^{st} \circ \mathfrak{F}^{st}$. The following results are then proven entirely analogously to the classical case; see for instance, the arguments in [15, Chapter 7].

Proposition 2.17 (Yang–Baxter equation) *As operators on $M_1 \otimes M_2 \otimes M_3$,*

$$\mathfrak{F}_\Theta^{12} \circ \mathfrak{F}_\Theta^{13} \circ \mathfrak{F}_\Theta^{23} = \mathfrak{F}_\Theta^{23} \circ \mathfrak{F}_\Theta^{13} \circ \mathfrak{F}_\Theta^{12}.$$

Proposition 2.18 *Define $\mathcal{R}: M \otimes M' \rightarrow M' \otimes M$ by $\mathcal{R} = \Theta \circ \mathfrak{F} \circ s$. Then \mathcal{R} is a U -module isomorphism.*

We thus obtain the following crucial property of \mathcal{R} .

Proposition 2.19 *For any modules M_1, M_2 , and M_3 , let $\mathcal{R}_{st} = \mathfrak{F}_\Theta^{st} \circ s_{(st)}$. Then*

$$\mathcal{R}_{12}\mathcal{R}_{23}\mathcal{R}_{12} = \mathcal{R}_{23}\mathcal{R}_{12}\mathcal{R}_{23}: M_1 \otimes M_2 \otimes M_3 \rightarrow M_3 \otimes M_2 \otimes M_1.$$

Remark 2.20 In [18, Section 32], it is shown that for $\mathfrak{g} = \mathfrak{sl}(2)$, which we can view as the $\pi = 1$ (ie $\tau = \pm 1$) specialization of Example 2.12, we can extend our field $\mathbb{Q}(q)$ to $\mathbb{Q}(\sqrt{q})$ and normalize so that f is bimultiplicative; that is $f(m + a, n + b) = f(m, n)f(m, b)f(a, n)f(a, b)$. This is necessary for the maps \mathcal{R} to satisfy the Hexagon Identities and thus define a braiding on the category of finite dimensional modules.

Note that Example 2.12 shows such a renormalization is impossible in general in the $\pi = -1$ case, so in particular the maps \mathcal{R} can not be normalized to define a braiding on the category of finite dimensional weight modules. It is possible to overcome this difficulty by restricting the class of modules to those of “even” highest weight. One could also expand the definition of f to a function on $\widehat{X} \times \widehat{X}$, but as a consequence the analysis in Section 4 becomes much more complex. Since it is not essential to have the braiding, we opt to not do this here.

3 Diagrammatic calculus and knot invariants

We will now interpret the U -module homomorphisms in terms of planar diagrams. At first, these diagrams should be interpreted as slice diagrams; that is, diagrams together with vertical slices at various heights such that between consecutive slices is an elementary diagram corresponding to a U -module homomorphism. However, we will ultimately see that diagrams which can be identified by planar isotopies yield the same morphisms.

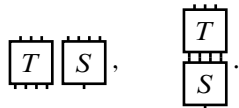
3.1 Cups, caps, and crossings

Recall that coqtr_λ , coev_λ , qtr_λ , and ev_λ are the maps defined in Lemma 2.7 where $V = V(\lambda)$. Likewise, let $\mathcal{R}_{\pm\lambda, \pm\mu}: V(\pm\lambda) \otimes V(\pm\mu) \rightarrow V(\pm\mu) \otimes V(\pm\lambda)$ be the map defined in Proposition 2.18. Furthermore, we will use the notation $1_{\pm\lambda} = 1_{V(\pm\lambda)}$.

We will now begin to represent our maps via a graphical calculus in anticipation of constructing tangle invariants. Specifically, we follow [27; 1] and interpret maps between tensor products of the modules $V(\pm\lambda)$ for various $\lambda \in X^+$ as sliced oriented tangle diagrams with X^+ -labeled strands; a concise exposition of this approach is laid out in [20, Chapter 3]. The elementary oriented tangle diagrams are interpreted as follows (note that while sideways-oriented crossings aren't considered elementary, we include them here for convenience in later arguments):

$$\begin{array}{ll}
 \downarrow_\lambda = 1_\lambda, & \downarrow_{-\lambda} = 1_{-\lambda}, \\
 \text{cup}_\lambda = \text{coqtr}_\lambda, & \text{cup}_{-\lambda} = \tau^{3P(\lambda)} \text{coev}_\lambda, \\
 \text{cap}_\lambda = \tau^{P(\lambda)} \text{qtr}_\lambda, & \text{cap}_{-\lambda} = \text{ev}_\lambda, \\
 \text{crossing}_{\lambda, \mu}^{\nearrow} = \tau^{P(\lambda)P(\mu)} \mathcal{R}_{\lambda, \mu}, & \text{crossing}_{-\lambda, -\mu}^{\searrow} = \tau^{3P(\lambda)P(\mu)} \mathcal{R}_{-\lambda, -\mu}, \\
 \text{crossing}_{\lambda, \mu}^{\searrow} = \tau^{P(\lambda)P(\mu)} \mathcal{R}_{\lambda, -\mu}, & \text{crossing}_{-\lambda, \mu}^{\nearrow} = \tau^{3P(\lambda)P(\mu)} \mathcal{R}_{-\lambda, \mu}, \\
 \text{crossing}_{\lambda, \mu}^{\nwarrow} = \tau^{3P(\lambda)P(\mu)} \mathcal{R}_{\mu, \lambda}^{-1}, & \text{crossing}_{-\lambda, -\mu}^{\nwarrow} = \tau^{P(\lambda)P(\mu)} \mathcal{R}_{-\mu, -\lambda}^{-1}, \\
 \text{crossing}_{\lambda, \mu}^{\swarrow} = \tau^{P(\lambda)P(\mu)} \mathcal{R}_{-\mu, \lambda}^{-1}, & \text{crossing}_{-\lambda, \mu}^{\swarrow} = \tau^{3P(\lambda)P(\mu)} \mathcal{R}_{\mu, -\lambda}^{-1}.
 \end{array}$$

We construct more general diagrams from these elementary ones by the following constructions. If \boxed{T} is some diagram denoting the morphism ϕ and \boxed{S} is some diagram denoting the morphism ψ , then we can combine them in two ways:



- The first is the *horizontal* composition, which denotes the tensor product $\phi \otimes \psi$.
- The second is the *vertical* composition, which denotes the composition $\phi \circ \psi$, or zero if this composition is undefined (which is to say, when the strands on top of S don't match the number and labeling of the strands on the bottom of T).

We will say two diagrams are equal if the corresponding morphisms agree. Note that, by construction and by [Proposition 2.19](#), the following diagrams are equal for any choice of orientation and labeling of strands:

$$(3-2) \quad \boxed{T} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}, \quad \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array},$$

$$(3-3) \quad \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}, \quad \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}.$$

In (3-2), the symbols \boxed{T} and \boxed{S} stand for arbitrary subdiagrams with an arbitrary number of strands protruding from the top and bottom.

3.2 Graphical identities

Now we shall prove some more substantial diagrammatic identities.

Lemma 3.1 *We have an equality of diagrams*

$$\begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}$$

for any choice of orientation or labeling of the strand.

Proof This follows by choosing a homogeneous basis for the module and applying the definitions. □

Lemma 3.2 *Let $\lambda \in X^+$, and define $c_\lambda = \mathfrak{f}(\lambda, \lambda)\pi^{P(\lambda)}q^{-\langle \bar{\rho}, \lambda \rangle}$. Then we have*

$$(a) \quad \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = c_\lambda \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array},$$

$$(b) \quad \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = c_\lambda^{-1} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array},$$

$$(c) \quad \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = c_\lambda \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array},$$

$$(d) \quad \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = c_\lambda^{-1} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}.$$

Next, we apply $\mathcal{R}_{-\lambda, -\mu}^{-1} = \mathfrak{s} \circ \mathfrak{F}^{-1} \circ \bar{\Theta}$ to the middle tensor factors, and thus setting $y = \mathcal{R}_{-\lambda, -\mu}^{-1}(w_0^* \otimes v^*)$, we see that

$$y = \sum_{\nu} \mathfrak{f}(-\xi - \nu, -|\nu| + \nu)^{-1} \pi^{p(w_0)p(\nu)+p(\nu)p(\nu)} q^{\nu \cdot \nu/2} \sum_{b \in \mathbf{B}_{\nu}} \sigma(b^*)^+ v^* \otimes b^- w_0^*.$$

Finally, we apply the quantum trace to the two tensor factors on the left; hence we need to compute $\text{qtr}(v_0 \otimes \sigma(b^*)^+ v^*)$. Since $x^*(y) = 0$ unless $\|x\| = \|y\|$ (that is, unless x and y have the same weight and parity), we can assume $|\nu| = \kappa + \nu$ and $p(\nu) = p(v_0) + p(\nu)$. Then we have

$$\begin{aligned} \text{qtr}_{\lambda}(v_0 \otimes \sigma(b^*)^+ v^*) &= \pi^{p(v_0)} q^{-\langle \tilde{\rho}, |v_0| \rangle} (\sigma(b^*)^+ v^*)(v_0) \\ &= (-1)^{\text{ht } \nu} \pi^{p(\nu)+p(v_0)+p(\nu)p(v_0)+p(\nu)} q^{-\nu \cdot \nu/2 - \langle \tilde{\rho}, \kappa \rangle} (\pi q)^{-\langle \tilde{\nu}, \kappa \rangle} q_{-\nu} v^*((b^*)^+ v_0). \end{aligned}$$

Putting these computations together, we see that

$$\begin{aligned} \phi(v_0 \otimes w_0^*) &= \sum_{\nu \in \mathbf{B}(\lambda)} \sum_{\nu} \sum_{b \in \mathbf{B}_{\nu}} \pi^{p(v_0)+p(\nu)} q^{\langle \tilde{\rho}, \kappa + \nu \rangle} \\ &\quad \times \mathfrak{f}(-\xi - \nu, -\kappa)^{-1} \pi^{p(w_0)p(v_0)+p(w_0)p(\nu)+p(\nu)p(v_0)+p(\nu)} q^{\nu \cdot \nu/2} \\ &\quad \times (-1)^{\text{ht } \nu} \pi^{p(\nu)+p(v_0)+p(\nu)p(v_0)+p(\nu)} q^{-\nu \cdot \nu/2 - \langle \tilde{\rho}, \kappa \rangle} (\pi q)^{-\langle \tilde{\nu}, \kappa \rangle} q_{-\nu} \\ &\quad \times v^*((b^*)^+ v_0) b^- w_0^* \otimes v \\ &= \sum_{\nu} (-1)^{\text{ht } \nu} \mathfrak{f}(-\xi - \nu, -\kappa)^{-1} (\pi q)^{-\langle \tilde{\nu}, \kappa \rangle} \pi_{\nu} q^{\langle \tilde{\rho}, \nu \rangle} q_{-\nu} \pi^{p(v_0)p(w_0)+p(\nu)} \\ &\quad \times \sum_{b \in \mathbf{B}_{\nu}} \pi^{p(w_0)p(\nu)} b^- w_0^* \otimes \left(\sum_{\nu \in \mathbf{B}(\lambda)} v^*((b^*)^+ v_0) v \right). \end{aligned}$$

But note that we have the following identities:

$$\begin{aligned} \mathfrak{f}(-\xi - \nu, -\kappa) (\pi q)^{\langle \tilde{\nu}, \kappa \rangle} &= \mathfrak{f}(-\xi, -\kappa), \\ q^{\langle \tilde{\rho}, \nu \rangle} &= q_{\nu}^2, \\ \sum_{\nu \in \mathbf{B}(\lambda)} v^*((b^*)^+ v_0) v &= (b^*)^+ v_0. \end{aligned}$$

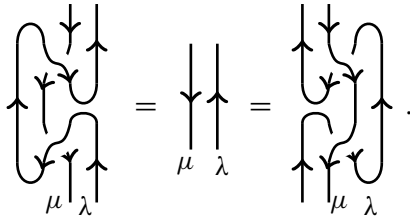
Applying these identities to the computation of $\phi(v_0 \otimes w_0^*)$, we find

$$\phi(v_0 \otimes w_0^*) = \mathfrak{r}(-\xi, \kappa)^{-1} R_{\lambda, -\mu}(v_0 \otimes w_0^*).$$

Finally, since $-\xi \in \mu + \mathbb{Z}[I]$ and $\kappa \in \lambda + \mathbb{Z}[I]$, we can apply Lemma 2.15(1) to conclude that $\phi = \mathfrak{r}(\mu, \lambda)^{-1} R_{\lambda, -\mu}$. A similar computation shows that $\psi = \mathfrak{l}(\mu, \lambda)^{-1} \mathcal{R}_{\lambda, -\mu}$, and the result then follows from Lemma 2.15. \square

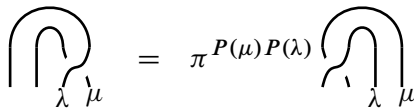
Note that since the maps in Lemma 3.3(a) and (b) are mutually inverse, we obtain the following corollary.

Corollary 3.4 We have an equality of diagrams



Finally, we show a somewhat more involved identity, leading us to our final result.

Lemma 3.5 We have an equality of diagrams



for any choice of orientation.

Proof In order to prove the identity without referring to a particular orientation, it will be convenient to introduce the following notation. Suppose $m \in V(\zeta)$ and $n \in V(-\zeta)$ for some $\zeta \in X^+$. Let us denote by (n, m) (resp. (m, n)) the evaluation $\text{ev}_\zeta(n \otimes m)$ (resp. the quantum trace $\text{qtr}_\zeta(m \otimes n)$). In particular, one may think of $(-, -)$ as a pairing on $V(\zeta) \oplus V(-\zeta)$ satisfying, for $v, w \in V(\zeta)$,

$$(3-5) \quad \begin{aligned} (v, w) &= (v^*, w^*) = 0, & (v, w^*) &= \pi^{p(v)p(w)} q^{-\langle \tilde{\rho}, |v| \rangle} (w^*, v), \\ (uv, w^*) &= \pi^{p(u)p(v)} (v, S(u)w^*), & (uw^*, v) &= \pi^{p(u)p(w)} (w^*, S(u)v). \end{aligned}$$

Indeed, all the statements of (3-5) are obvious except $(uv, w^*) = \pi^{p(u)p(v)} (v, S(u)w^*)$, which follows from a simple calculation on the generators: for example,

$$(E_i v, w^*) = \pi^{p(v)p(w)} q^{-\langle \tilde{\rho}, |v| \rangle} q_i^{-2} (-E_i \tilde{J}_i^{-1} \tilde{K}_i^{-1} w^*)(v) = \pi^{p(v)p(i)} (v, S(E_i)w^*).$$

In this proof we will use the notation $(-, -)$ as shorthand for ev_ζ and qtr_ζ for both $\zeta = \lambda, \mu$ with the intended map (and highest weight) being clear from context. Using this notation, the diagram equality is equivalent to showing that the maps

$$\begin{aligned} \psi &= (-, -) \circ (1_{s\mu} \otimes (-, -) \otimes 1_{-s\mu}) \circ (\mathcal{R}_{s\lambda, t\mu} \otimes 1_{-s\lambda} \otimes 1_{-t\mu}), \\ \phi &= (-, -) \circ (1_{t\lambda} \otimes (-, -) \otimes 1_{-t\lambda}) \circ (1_{s\lambda} \otimes 1_{t\mu} \otimes \mathcal{R}_{-s\lambda, -t\mu}) \end{aligned}$$

are $\pi^{P(\mu)P(\lambda)}$ multiples of each other for any choice of $s, t \in \{1, -1\}$.

Let $w \in V(s\lambda)$, $x \in V(t\mu)$, $y \in V(-s\lambda)$, and $z \in V(-t\mu)$, where $V(-\xi) = V(\xi)^*$ for $\xi \in X^+$. Then on one hand,

$$\psi(w \otimes x \otimes y \otimes z) = \sum_v \sum_{b \in \mathbf{B}_v} \pi^{p(x)p(w)} \mathfrak{f}(|x|, |w|) (-1)^{\text{ht } v} \pi^{p(v)} \pi_v q_v \pi^{p(v)p(x)} (b^-x, z) ((b^*)^+ w, y).$$

On the other hand, using the representation of Θ in the basis $\sigma(\mathbf{B})$,

$$\phi(w \otimes x \otimes y \otimes z) = \sum_v \sum_{b \in \mathbf{B}_v} \pi^{p(y)p(z)} \mathfrak{f}(|z|, |y|) (-1)^{\text{ht } v} \pi^{p(v)} \pi_v q_v \pi^{p(v)p(z)} (x, \sigma(b)^-z) (w, \sigma(b^*)^+ y).$$

Thus to see that $\psi(w \otimes x \otimes y \otimes z) = \pi^{P(\mu)P(\lambda)} \phi(w \otimes x \otimes y \otimes z)$, and hence that $\psi = \pi^{P(\mu)P(\lambda)} \phi$ since w, x, y, z are arbitrary, it is enough to show that $l = \pi^{P(\mu)P(\lambda)} r$, where

$$l = \pi^{p(y)p(z)+p(v)p(z)} \mathfrak{f}(|z|, |y|) (x, \sigma(b)^-z) (w, \sigma(b^*)^+ y),$$

$$r = \pi^{p(w)p(x)+p(v)p(x)} \mathfrak{f}(|x|, |w|) (b^-x, z) ((b^*)^+ w, y).$$

Using the properties of $(-, -)$ (see (3-5)) and S (see (2-20)), we see that

$$(x, \sigma(b)^-z) (w, \sigma(b^*)^+ y) = \pi^{p(x)p(v)+p(w)p(v)} q^{-v \cdot v + \langle \tilde{v}, |x| \rangle} (\pi q)^{-\langle \tilde{v}, |w| \rangle} (b^-x, z) ((b^*)^+ w, y).$$

Note that l and r are both zero unless $-\|x\| = \|z\| - v$ and $-\|w\| = \|y\| + v$. In particular, l and r are both zero unless $p(y) = p(w) + p(v)$ and $p(z) = p(x) + p(v)$, in which case

$$p(y)p(z) + p(v)p(z) + p(x)p(v) + p(w)p(v) \equiv p(w)p(x) + p(w)p(v) \pmod{2}.$$

Likewise, l and r are both zero unless $-|y| = |w| + v$ and $-|z| = |x| - v$, in which case

$$\mathfrak{f}(|z|, |y|) q^{-v \cdot v + \langle \tilde{v}, |x| - |w| \rangle} = \mathfrak{f}(-|x|, -|w|).$$

Finally, note that $\mathfrak{f}(-|x|, -|w|) = \pi^{P(-|x|)P(-|w|)} \mathfrak{f}(|x|, |w|)$. Putting these observations together,

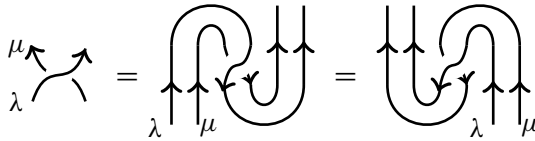
$$l = \pi^{p(w)p(x)+p(v)p(x)+P(-|x|)P(-|w|)} \mathfrak{f}(|x|, |w|) (b^-x, z) ((b^*)^+ w, y)$$

$$= \pi^{P(-|x|)P(-|w|)} r.$$

Since parity in X only depends on the $X/\mathbb{Z}[I]$ cosets and we have $-|x| \in \mu + \mathbb{Z}[I]$ and $-|w| \in \lambda + \mathbb{Z}[I]$, the result follows. □

Lastly, note that Lemmas 3.5 and 3.1 immediately imply the following corollary.

Corollary 3.6 We have an equality of diagrams



for any $\lambda, \mu \in X^+$.

3.3 Defining the tangle invariant

Recall that the *writhe* $wr(T)$ of an oriented tangle T is defined by forgetting the orientation and setting

$$wr(\begin{smallmatrix} \nearrow \\ \searrow \end{smallmatrix}) = 1, \quad wr(\begin{smallmatrix} \searrow \\ \nearrow \end{smallmatrix}) = -1, \quad wr(T) = \sum_{\lambda} wr(X),$$

where the sum is over all crossings X in T .

Theorem 3.7 Let T be an oriented tangle, and $\lambda \in X^+$ be a dominant weight. For any slice diagram $S(T)$ of T , let $S(T)_\lambda$ be the associated map defined by the diagrammatic calculus with strands colored by λ . Then $S(T)_\lambda$ is independent of the choice of slice diagram, and $T_\lambda = S(T)_\lambda$ is an isotopy invariant of oriented framed tangles. Moreover, if $J_T^\lambda = (\pi^{p(\lambda)} \mathfrak{f}(\lambda, \lambda)^{-1} q^{(\tilde{\rho}, \lambda)})^{wr(T)} T_\lambda$, then J_T^λ is independent of the framing, hence is an invariant of T .

Proof To prove the theorem, it suffices to show that the maps $S(T)_\lambda$ (resp. J_T^λ) are invariant under the Turaev moves for framed (resp. unframed) oriented tangles [27, Theorem 3.2; 20, Theorem 3.3, Equations (3.9)–(3.16)]. This is straightforward to check. □

We note that the proof of [Theorem 3.7](#) actually implies a more general result, though we first need to recall some notions. The category of X^+ -colored oriented tangles is the strict monoidal category whose objects are finite sequences of pairs (λ, s) where $\lambda \in X^+$ and $s \in \{\pm 1\}$, and whose morphisms from $(\lambda_a, s_a)_{1 \leq a \leq b}$ to $(\mu_c, s_c)_{1 \leq c \leq d}$ are tangle diagrams where the labeling and orientation of the r^{th} strand from the left at the lower (resp. upper) boundary corresponds to (λ_r, s_r) (resp. (μ_c, s_c)); see [27; 1] for more details. In particular, morphisms in this category (and thus colored tangles) are generated from the elementary morphisms

$$\curvearrowright_\lambda, \quad \curvearrowleft_\lambda, \quad \cup_\lambda, \quad \cap_\lambda, \quad (\otimes)_{\lambda, \mu}^\pm, \quad (\boxtimes)_{\lambda, \mu}^\pm,$$

subject to relations which are simply colored versions of the Turaev moves.

We can extend [Theorem 3.7](#) to framed multicolored tangles with the same proof. To obtain the unframed invariant, we must similarly renormalize but now by the factor $\prod_{\lambda \in X^+} (\pi^{P(\lambda)} \mathfrak{f}(\lambda, \lambda)^{-1} q^{\langle \tilde{\rho}, \lambda \rangle})^{\text{wr}_\lambda(T)}$, where wr_λ is defined to be the writhe where we exclude from the sum any crossings where there is a strand not labeled by λ . Therefore, we obtain the following corollary.

Corollary 3.8 *There exists a covariant functor J from the category of X^+ -colored oriented tangles modulo isotopy to \mathcal{O}_{fin} satisfying*

$$\begin{aligned} \curvearrowright_\lambda &\mapsto \tau^{P(\lambda)} \text{ev}_\lambda, & \curvearrowleft_\lambda &\mapsto \text{qtr}_\lambda, & \cup_\lambda &\mapsto \text{coqtr}_\lambda, & \cap_\lambda &\mapsto \tau^{-P(\lambda)} \text{coev}_\lambda, \\ (\otimes)_{\lambda, \mu}^\pm &\mapsto (\pi^{P(\lambda)} \mathfrak{f}(\lambda, \lambda)^{-1} q^{\langle \tilde{\rho}, \lambda \rangle})^{\pm \delta_{\lambda, \mu}} \tau^{\pm P(\lambda)} P(\mu) \mathcal{R}_{\lambda, \mu}^\pm, \\ (\otimes)_{\lambda, \mu}^\pm &\mapsto (\pi^{P(\lambda)} \mathfrak{f}(\lambda, \lambda)^{-1} q^{\langle \tilde{\rho}, \lambda \rangle})^{\pm \delta_{\lambda, \mu}} \tau^{\mp P(\lambda)} P(\mu) \mathcal{R}_{-\lambda, -\mu}^\pm. \end{aligned}$$

In particular, if L is an oriented colored link, then $J(L) \in \mathbb{Q}(q, \mathbf{t})^\tau$ is the associated quantum covering $\mathfrak{osp}(1|2n)$ colored link invariant.

Remark 3.9 Recall from [Remark 2.4](#) that all the module homomorphisms of [Section 2](#) are defined over the subring $\mathbb{Q}(q)^\pi$ of $\mathbb{Q}(q, \mathbf{t})^\tau$. We observe that whenever $P(\lambda) = 0$, the maps represented by the diagrams are defined over $\mathbb{Q}(q)^\pi$. By [Lemma 2.2](#), this holds whenever λ is an even weight (ie $\langle \underline{n}, \lambda \rangle \in 2\mathbb{N}$) or n is even. In particular, this means that the functor J can be defined to the category \mathcal{O} of the quantum group over $\mathbb{Q}(q)^\pi$ provided all weights are even, or n is even.

Example 3.10 Let's take $\mathfrak{g} = \mathfrak{osp}(1|2)$ and $\lambda = 1$. Fix $\mathfrak{f}(1, 1) = 1$, and note that $\langle \tilde{\rho}, \lambda \rangle = 1$ and $p(\lambda) = 1$. We can explicitly compute the maps represented by our diagrams on $V(1) \otimes V(1)$. Let v_1, v_{-1} be the basis of $V(1)$ from [Example 2.9](#). Then with respect to the ordered basis $\{v_1 \otimes v_1, v_1 \otimes v_{-1}, v_{-1} \otimes v_1, v_{-1} \otimes v_{-1}\}$ of $V(1) \otimes V(1)$, we have

$$\Theta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q^{-1} - \pi q & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tilde{\mathfrak{F}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & \pi q & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}, \quad \mathfrak{s} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix},$$

and thus

$$\begin{aligned} \curvearrowright &= \begin{bmatrix} \tau & 0 & 0 & 0 \\ 0 & 0 & \tau q & 0 \\ 0 & \tau^3 q & \tau - \tau^3 q^2 & 0 \\ 0 & 0 & 0 & \tau \end{bmatrix}, & \curvearrowleft &= \begin{bmatrix} \tau^3 & 0 & 0 & 0 \\ 0 & \tau^3 - \tau q^{-2} & \tau q^{-1} & 0 \\ 0 & \tau^3 q^{-1} & 0 & 0 \\ 0 & 0 & 0 & \tau^3 \end{bmatrix}. \end{aligned}$$

Note that $\pi^{p(\lambda)} q^{(\tilde{p}, \lambda)} = \pi q$. Then it is easy to verify directly that

$$\overbrace{\nearrow \nearrow} - q^2 \overbrace{\searrow \searrow} = (\tau - \tau^3 q^2) \uparrow \uparrow$$

Now let T be an unframed oriented link with all strands colored by λ , and fix a subdiagram which consists of two strands with either no crossing or a single crossing. Since $T^\#$ is isotopy invariant and independent of framing, we may assume that the strands are directed upward. Let T_+ , T_0 and T_- be T with the subdiagram replaced by

$$\overbrace{\nearrow \nearrow}, \uparrow \uparrow \text{ and } \overbrace{\searrow \searrow},$$

respectively. Then using the above relation and the definition in [Theorem 3.7](#),

$$(\pi q^{-1}) J_{T_+}^1 - (\pi q^3) J_{T_-}^1 = (\tau - \tau^3 q^2) J_{T_0}^1,$$

hence

$$(\pi q^2)^{-1} J_{T_+}^1 - \pi q^2 J_{T_-}^1 = (\tau q^{-1} - \tau^3 q) J_{T_0}^1.$$

Moreover, if T is the unknot, then for either orientation we have $J_T^1 = \tau^3 q + \tau q^{-1}$. In particular, for any link K , we see that J_K^1 is simply a multiple of the Jones polynomial of T in the variable $\tau^3 q = \tau^{-1} q$. In particular, note that using the specialization $\tau = t$, which corresponds to $\pi = -1$, this shows the uncolored $U_q(\mathfrak{osp}(1|2))$ link invariant is equal to the $U_{t^{-1}q}(\mathfrak{sl}_2)$ link invariant.

4 Relating $\mathfrak{so}(2n + 1)$ and $\mathfrak{osp}(1|2n)$ invariants

The results of [Example 3.10](#) suggest a connection between the specializations of the tangle invariants in [Theorem 3.7](#). We now make this precise by extending the constructions in [\[5; 4\]](#). We begin by recalling the definition of the twistor maps.

4.1 Definition of twistors

An *enhancer* ϕ is a function $\phi: \mathbb{Z}[I] \times X \rightarrow \mathbb{Z}$ satisfying

$$(4-1) \quad \begin{aligned} \phi(v, \lambda + \mu) &\equiv \phi(v, \mu) + \phi(v, \lambda) \pmod{4} && \text{for } v, \mu \in \mathbb{Z}[I], \\ \phi(v + \mu, \lambda) &\equiv \phi(v, \lambda) + \phi(\mu, \lambda) \pmod{4} && \text{for } v, \mu \in \mathbb{Z}[I], \\ \phi(i, i) &= d_i \quad \text{and} \quad \phi(i, j) \in 2\mathbb{Z} && \text{for } i \neq j \in I, \\ \phi(i, j) - \phi(j, i) &\equiv i \cdot j + 2p(i)p(j) \pmod{4} && \text{for } i, j \in I. \end{aligned}$$

Note that $\phi(i, i) - \phi(i, i) = 0 \equiv i \cdot i + 2p(i)p(i) \pmod 4$ since $i \cdot i = 2d_i$ and $2p(i)p(i) = 2p(i) = 2d_i$. In particular, note that these congruences imply that

$$(4-2) \quad \begin{aligned} \phi_4: \mathbb{Z}[I] \times \mathbb{Z}[I] &\rightarrow \mathbb{Z}/4\mathbb{Z}, & (\mu, \nu) &\mapsto \phi(\mu, \nu) \pmod 4 \text{ is } \mathbb{Z}\text{-bilinear,} \\ \phi(\mu, \nu) &\equiv \phi(\nu, \mu) + \mu \cdot \nu + 2p(\mu)p(\nu) \pmod 4 & \text{ for } \mu, \nu \in \mathbb{Z}[I]. \end{aligned}$$

Note that an enhancer can always be defined on $\mathbb{Z}[I] \times \mathbb{Z}[I]$ by defining it for I and extending in \mathbb{Z} -bilinearly, and then it can be extended to $\mathbb{Z}[I] \times X$ by translation along a transversal of $X/\mathbb{Z}[I]$.

When I has a unique odd element, as in the present case, the enhancer is closely related to the usual pairing.

Lemma 4.1 *Let ϕ be an enhancer. Then $\phi(\mu, \nu) + \phi(\nu, \mu) \equiv \mu \cdot \nu \pmod 4$.*

Proof It is straightforward using the fact that $|I_1| = 1$. □

The ϕ -enhanced quantum covering group \widehat{U} associated to U and the enhancer ϕ is the semidirect product of U with the algebra $\mathbb{Q}(q, \mathfrak{t})^\tau[T_\mu, \Upsilon_\mu \mid \mu \in \mathbb{Z}[I]]$ subject to the relations

$$(4-3) \quad T_\mu T_\nu = T_{\mu+\nu}, \quad \Upsilon_\mu \Upsilon_\nu = \Upsilon_{\mu+\nu}, \quad T_0 = \Upsilon_0 = T_\nu^4 = \Upsilon_\nu^4 = 1, \quad T_\mu \Upsilon_\nu = \Upsilon_\nu T_\mu,$$

$$(4-4) \quad T_\mu u = \mathfrak{t}^{(\mu, |u|)} u T_\mu, \quad u \in U, \quad \mu \in \mathbb{Z}[I],$$

$$(4-5) \quad \Upsilon_\mu u = \mathfrak{t}^{\phi(\mu, |u|)} u \Upsilon_\mu, \quad u \in U, \quad \mu \in \mathbb{Z}[I].$$

See [5; 4] for a more formal definition. The enhanced quantum covering group has a useful $\mathbb{Q}(\mathfrak{t})$ -linear automorphism called a *twistor*. There are several ways to define such a twistor; we will need the following.

Proposition 4.2 [5, Theorems 4.3, 4.12] *Define a product $*$ on \mathfrak{f} by the following rule: if x and y are homogeneous elements of \mathfrak{f} , let $x * y = \mathfrak{t}^{\phi(|x|, |y|)} xy$. Let $(\mathfrak{f}, *)$ denote \mathfrak{f} with this multiplication. Finally, let $\mathfrak{X}: \mathbb{Q}(q, \mathfrak{t})^\tau \rightarrow \mathbb{Q}(q, \mathfrak{t})^\tau$ be the $\mathbb{Q}(\mathfrak{t})$ -linear automorphism satisfying $\mathfrak{X}(q) = \mathfrak{t}^{-1}q$ and $\mathfrak{X}(\tau) = \mathfrak{t}^{-1}\tau$.*

(1) *There is a $\mathbb{Q}(\mathfrak{t})$ -linear algebra isomorphism $\mathfrak{X}: \mathfrak{f} \rightarrow (\mathfrak{f}, *)$ defined by*

$$\mathfrak{X}(\theta_i) = \theta_i, \quad \mathfrak{X}(q) = \mathfrak{t}^{-1}q, \quad \mathfrak{X}(\tau) = \mathfrak{t}^{-1}\tau.$$

(2) *Let \mathcal{B} be the canonical basis of \mathfrak{f} [8]. Then \mathfrak{X} on \mathfrak{f} satisfies $\mathfrak{X}(b) = \mathfrak{t}^{\ell(b)}b$ for all $b \in \mathcal{B}$, where $\ell(b)$ is some integer depending on b .*

(3) There is a $\mathbb{Q}(\mathbf{t})$ -algebra automorphism $\mathfrak{X}: \widehat{U} \rightarrow \widehat{U}$ defined by

$$\begin{aligned} \mathfrak{X}(E_i) &= \mathbf{t}_i^{-1} \widetilde{T}_i \Upsilon_i E_i, & \mathfrak{X}(F_i) &= F_i \Upsilon_{-i}, & \mathfrak{X}(K_\mu) &= T_{-\mu} K_\mu, & \mathfrak{X}(J_\mu) &= T_{2\mu} J_\mu, \\ \mathfrak{X}(T_\mu) &= T_\mu, & \mathfrak{X}(\Upsilon_\mu) &= \Upsilon_\mu, & \mathfrak{X}(q) &= \mathbf{t}^{-1} q, & \mathfrak{X}(\tau) &= \mathbf{t}^{-1} \tau, \end{aligned}$$

where $\widetilde{T}_\mu = \prod_{i \in I} T_{\mu_i d_i}$ for $\mu = \sum_{i \in I} \mu_i i$.

(4) For $x \in \mathfrak{f}[\mathbf{t}]$, we have

- (a) $\mathfrak{X}(x^+) = \mathbf{t}_v^2 \mathbf{t}^{\bullet(|x|)} \mathfrak{X}(x)^+ \widetilde{T}_{|x|} \Upsilon_{|x|}$,
- (b) $\mathfrak{X}(x^-) = \mathfrak{X}(x)^- \Upsilon_{-|x|}$.

Later on, we will need some alternate versions of the results in Proposition 4.2 which we shall prove now. First, we note the following analogue of Proposition 4.2(2) for the dual canonical basis.

Lemma 4.3 Let $(-, -)$ be the bilinear form on \mathfrak{f} defined in (2-13). Then

$$\mathfrak{X}^{-1}((\mathfrak{X}(x), \mathfrak{X}(y))) = (-1)^{\mathbf{p}(|x|)}(x, y).$$

In particular, $\mathfrak{X}(b^*) = (-1)^{\mathbf{p}(b)} \mathbf{t}^{-\ell(b)} b^*$ for any $b \in \mathcal{B}$.

Proof Let $(x, y)^{\mathfrak{X}} = (-1)^{\mathbf{p}(|x|)} \mathfrak{X}^{-1}((\mathfrak{X}(x), \mathfrak{X}(y)))$ and observe that this is a $\mathbb{Q}(q, \mathbf{t})^{\mathfrak{r}}$ -bilinear form on $\mathfrak{f}[\mathbf{t}]$. It suffices to show that this bilinear form satisfies the defining properties of $(-, -)$, which is elementary to verify. □

Remark 4.4 Though Lemma 4.3 as stated requires $|I_1| = 1$, a version of it also holds for arbitrary enhanced quantum covering algebras. Indeed, if $|I_1| > 1$, then $\mathfrak{X}^{-1}((\mathfrak{X}(x), \mathfrak{X}(y))) = \mathbf{t}^{\binom{v}{2}}(x, y)$, where $|x| = v = \sum_{i \in I} v_i i$ and $\binom{v}{2} = \sum_{i \in I} \binom{v_i}{2} d_i$.

It will also be more convenient to have the following variant of Proposition 4.2(4b).

Lemma 4.5 We have

$$\mathfrak{X}(x^+) = \mathbf{t}_{|x|}^{-1} \widetilde{T}_{|x|} \Upsilon_{|x|} \mathfrak{X}(x)^+.$$

Proof This is true if $x = \theta_i$. It suffices to show if it is true for x , then it is true for $\theta_i x$. We have

$$\begin{aligned} \mathfrak{X}(\theta_i x^+) &= \mathfrak{X}(\theta_i^+) \mathfrak{X}(x^+) = \mathbf{t}_i^{-1} \widetilde{T}_i \Upsilon_i E_i \mathbf{t}_v^{-1} T_v \Upsilon_v \mathfrak{X}(x)^+ \\ &= \mathbf{t}_{i+v}^{-1} \mathbf{t}^{-i \cdot v - \phi(v, i)} \widetilde{T}_i \Upsilon_i \widetilde{T}_v \Upsilon_v E_i \mathfrak{X}(x)^+ \\ &= \mathbf{t}_{|\theta_i x|}^{-1} \mathbf{t}^{-i \cdot v - \phi(v, i) - \phi(i, v)} \widetilde{T}_{i+v} \Upsilon_{i+v} \mathfrak{X}(\theta_i x)^+. \end{aligned}$$

But then by Lemma 4.1,

$$-i \cdot v - \phi(v, i) - \phi(i, v) \equiv_4 -2i \cdot v \equiv_4 0. \quad \square$$

4.2 \widehat{U} -modules and Hopf structure

Let M be a U -weight module. Then M is naturally a \widehat{U} -module by defining

$$(4-6) \quad T_\mu m = \mathbf{t}^{(\mu,\lambda)} m, \quad \mu \in \mathbb{Z}[I], m \in M_\lambda;$$

$$(4-7) \quad \Upsilon_\mu m = \mathbf{t}^{\phi(\mu,\lambda)} m, \quad \mu \in \mathbb{Z}[I], m \in M_\lambda.$$

To that end, we will call any \widehat{U} -module which restricts to a U -weight module and satisfies (4-6) a \widehat{U} -weight module. If it additionally satisfies (4-7), we shall call it an *untwisted \widehat{U} -weight module*.

In particular, any tensor product of \widehat{U} -modules can be given an untwisted \widehat{U} -weight module structure. However, such a procedure forgets the action of the Υ elements on the factors due to the lack of additivity in the second component of ϕ .

Example 4.6 Consider the case $n = 1$. Then \widehat{U} has the untwisted weight module $\widehat{V}(1) = \mathbb{Q}(q, \mathbf{t})^\tau v_1 \oplus \mathbb{Q}(q, \mathbf{t})^\tau v_{-1}$ which is isomorphic to $V(1)$ as a U -module and satisfies $T_i v_1 = \mathbf{t} v_1$ and $\Upsilon_{\underline{1}} v_1 = \mathbf{t}^{\phi(\underline{1},1)} v_1$. Then $\widehat{V}(1) \otimes \widehat{V}(1)$ is a U -weight module hence has a untwisted \widehat{U} -module structure, but note that

$$\Upsilon_{\underline{1}} v_1 \otimes v_1 = \mathbf{t}^{\phi(\underline{1},2)} v_1 \otimes v_1,$$

and by the definition of ϕ , we have $\phi(\underline{1}, 2) = \phi(\underline{1}, \underline{1}) = 1 \neq \phi(\underline{1}, 1) + \phi(\underline{1}, 1)$.

In particular, untwisted module structures will be too naive for our purposes. Instead, we will introduce Hopf structure which will inform our classes of weight modules.

Proposition 4.7 *The algebra \widehat{U} has a Hopf covering algebra structure given by the following:*

- (1) A coassociative coproduct $\Delta: \widehat{U} \rightarrow \widehat{U} \otimes_{\mathbb{Q}(q,\mathbf{t})^\tau} \widehat{U}$ extending $\Delta: U \rightarrow U \otimes_{\mathbb{Q}(q,\mathbf{t})^\tau} U$ such that $\Delta(T_\mu) = T_\mu \otimes T_\mu$ and $\Delta(\Upsilon_\mu) = \Upsilon_\mu \otimes \Upsilon_\mu$ for $\mu \in \mathbb{Z}[I]$. In particular, we inductively define $\Delta^t = (\Delta \otimes 1^{t-1}) \circ \Delta^{t-1}: \widehat{U} \rightarrow \widehat{U}^{\otimes(t+1)}$ for any integer $t > 1$.
- (2) An antipode $S: \widehat{U} \rightarrow \widehat{U}$ extending $S: U \rightarrow U$ such that $S(T_\mu) = T_{-\mu}$ and $S(\Upsilon_\mu) = \Upsilon_{-\mu}$ for $\mu \in \mathbb{Z}[I]$.
- (3) A counit map $\epsilon: \widehat{U} \rightarrow \mathbb{Q}(q, \mathbf{t})^\tau$ extending $\epsilon: U \rightarrow \mathbb{Q}(q, \mathbf{t})^\tau$ such that $\epsilon(T_\mu) = \epsilon(\Upsilon_\mu) = 1$ for $\mu \in \mathbb{Z}[I]$.

Proof To show that these maps define a Hopf structure, we need only check that these morphisms respect (4-3)–(4-5). This is obvious for (4-3), and can be quickly verified

for (4-4) and (4-5) by checking it for the generators of U . Finally, the coassociativity of Δ on \widehat{U} follows immediately from the coassociativity of Δ on U and the fact that T_μ and Υ_μ are grouplike elements. \square

The coproduct gives us another way to define an action of \widehat{U} on tensor products of untwisted \widehat{U} -weight modules. Henceforth, given \widehat{U} -weight modules M and N , we let $M \widehat{\otimes} N$ denote the space $M \otimes_{\mathbb{Q}(q,t)^\pi} N$ with the \widehat{U} -weight module structure induced by the coproduct on \widehat{U} . (Note that in general, the module $M \widehat{\otimes} N$ is *not* untwisted!)

Example 4.8 Continuing the previous example, the action of $\Upsilon_{\underline{1}}$ on $\widehat{V}(1) \widehat{\otimes} \widehat{V}(1)$ is given by

$$\Delta(\Upsilon_{\underline{1}})v_1 \otimes v_1 = t^{2\phi(\underline{1},1)}v_1 \otimes v_1.$$

Another natural module to consider is the following. Given an untwisted \widehat{U} -weight module M , we can construct the restricted linear dual M^* . This space is naturally a U -weight module as in Section 2.4, hence has an untwisted \widehat{U} structure. On the other hand, let M^\natural denote the space M^* with the action of \widehat{U} defined by $(uf)(x) = \pi^{p(f)p(u)} f(S(u)x)$. Note that M^\natural is not untwisted: if $f \in (M_\lambda, s)^*$, then $|f| = -\lambda$, but nevertheless,

$$\Upsilon_\mu f = t^{-\phi(\mu,\lambda)} f.$$

Since modules with these unorthodox actions of the Υ_μ will be important, we give the following definitions.

Definition 4.9 We say that a \widehat{U} -weight module M is $*$ -twisted if $\Upsilon_\mu m = t^{-\phi(\mu,-\lambda)} m$ for all $m \in M_\lambda$. More generally, we say that M is a *mixed* weight module if there exists an integer $t \geq 1$ and a sequence $c = (c_1, \dots, c_t) \in \{\pm 1\}^t$ such that $M_\lambda = \bigoplus_{(\lambda_s) \in (X^t)_\lambda} M_{(\lambda_s)}$, where

$$(X^t)_\lambda = \{(\lambda_1, \dots, \lambda_t) \in X^t \mid \lambda = \lambda_1 + \dots + \lambda_s\},$$

$$M_{(\lambda_s)} = \{m \in M \mid \Upsilon_\mu m = t^{\sum_{1 \leq s \leq t} c_s \phi(\mu, c_s \lambda_s)} m \text{ for all } \mu \in \mathbb{Z}[I]\}.$$

We say c is the *signature* of M , and denote it by $\text{sig}(M) = c$.

Remark 4.10 We note that just as any weight U -module can be given a untwisted \widehat{U} -module structure, it can also be given an $*$ -twisted \widehat{U} -module structure. Indeed, suppose M is a weight U -module and define $T_i m = t^{(i,|m|)} m$ and $\Upsilon_i m = t^{-\phi(i,-|m|)}$. Then this defines an action of \widehat{U} , since for any $i \in I$ and $u \in U$, $\Upsilon_i u m = t^{-\phi(i,-(|m|+|u|))} u m = t^{\phi(i,|u|)} u \Upsilon_i m$.

In addition to classifying modules by the action of the Υ elements, another property of \widehat{U} -weight modules which will be important to us is their interaction with the twistor map $\mathfrak{X}: \widehat{U} \rightarrow \widehat{U}$.

Definition 4.11 Let M be a \widehat{U} -weight module. We say M carries a twistor \mathfrak{X} (or \mathfrak{X} is a twistor on M) if there exists a homogeneous $\mathbb{Q}(t)$ -linear bijection $\mathfrak{X}: M \rightarrow M$ such that $\mathfrak{X}(um) = \mathfrak{X}(u)\mathfrak{X}(m)$.

Modules which carry twistors are not hard to find. Indeed, the simple U -modules $V(\lambda)$ are themselves examples when given untwisted (or $*$ -twisted) actions of \widehat{U} .

Lemma 4.12 [4, Lemma 6.9] Let $\lambda \in X^+$, and let $\widehat{V}(\lambda)$ be the space $V(\lambda)$ with the untwisted action of \widehat{U} . There is a $\mathbb{Q}(t)$ -linear map $\mathfrak{X}: \widehat{V}(\lambda) \rightarrow \widehat{V}(\lambda)$ which satisfies $\mathfrak{X}(v_\lambda) = v_\lambda$ and $\mathfrak{X}_\lambda(um) = \mathfrak{X}(u)\mathfrak{X}(m)$ for all $u \in \widehat{U}$ and $m \in \widehat{V}(\lambda)$.

In light of Lemma 2.8, it follows that the U -module $V(\lambda)^*$, viewed as an untwisted \widehat{U} -module, also carries a twistor. A similar argument to [4, Lemma 6.9] can be used to construct a twistor on $V(\lambda)$ with a $*$ -twisted action of U ; hence the \widehat{U} -module $\widehat{V}(\lambda)^\natural$ carries a twistor. However, this construction is not very compatible with the dual basis, since it relies on an isomorphism $V(\lambda) \rightarrow \Pi^{P(\lambda)}V(\lambda)$ and is defined by descent from the highest weight vector. To obtain a convenient definition of a twistor on the dual modules, we will define a map directly on $\widehat{V}(\lambda)^\natural$.

Define the *dual twistor* on \widehat{U} to be the map $\mathfrak{X}^\natural(u) = S \circ \mathfrak{X} \circ S^{-1}(u)$. This map is clearly a bijection, and for any $u, v \in \widehat{U}$ we have

$$\begin{aligned} \mathfrak{X}^\natural(uv) &= S(\mathfrak{X}(S^{-1}(uv))) \\ &= t^{2p(u)p(v)} S(\mathfrak{X}(S^{-1}(u)))S(\mathfrak{X}(S^{-1}(v))) \\ &= t^{2p(u)p(v)} \mathfrak{X}^\natural(u)\mathfrak{X}^\natural(v). \end{aligned}$$

Therefore, it is determined by the images of the generators, which are

$$\begin{aligned} \mathfrak{X}^\natural(E_i) &= t_i E_i \Upsilon_{-i}, & \mathfrak{X}^\natural(F_i) &= \Upsilon_i F_i \widetilde{T}_i, & \mathfrak{X}^\natural(K_\mu) &= T_{-\mu} K_\mu, & \mathfrak{X}^\natural(J_\mu) &= T_{2\mu} J_\mu, \\ \mathfrak{X}^\natural(q) &= t^{-1}q, & \mathfrak{X}^\natural(\tau) &= t\tau. \end{aligned}$$

In particular, note that

$$(4-8) \quad \mathfrak{X}^\natural(x^-) = \Upsilon_v \mathfrak{X}(x)^- \widetilde{T}_v.$$

While \mathfrak{X}^\natural is not an algebra automorphism of \widehat{U} , it shares many properties with \mathfrak{X} . In particular, we have a version of Lemma 4.12.

Lemma 4.13 *Let $\lambda \in X^+$. There is a $\mathbb{Q}(\mathfrak{t})$ -linear map $\mathfrak{X}^{\natural}: \widehat{V}(\lambda) \rightarrow \widehat{V}(\lambda)$ which satisfies $\mathfrak{X}^{\natural}(v_{\lambda}) = v_{\lambda}$ and $\mathfrak{X}^{\natural}(um) = \mathfrak{t}^{2p(u)p(m)} \mathfrak{X}^{\natural}(u)\mathfrak{X}^{\natural}(m)$ for all $u \in \widehat{U}$ and $m \in \widehat{V}(\lambda)$.*

Proof This follows from more or less the same proof as [4, Lemmas 6.8 and 6.9]. To wit, we can identify the Verma module of highest weight λ for U with \mathfrak{f} (see [loc. cit.] for details), and in particular this is naturally an untwisted \widehat{U} -module. Then we define a map $\mathfrak{X}_{\lambda}^{\natural}: \mathfrak{f} \rightarrow \mathfrak{f}$ via $\mathfrak{X}_{\lambda}^{\natural}(x) = \mathfrak{t}^{\langle \tilde{v}, \lambda \rangle + \phi(v, \lambda - v)} \mathfrak{X}(x)$. Then it is straightforward to verify that $\mathfrak{X}_{\lambda}^{\natural}(ux) = \mathfrak{t}^{2p(u)p(x)} \mathfrak{X}_{\lambda}^{\natural}(u)\mathfrak{X}_{\lambda}^{\natural}(x)$ for $x \in \mathfrak{f}$ and $u = F_i, T_{\mu}, J_{\mu}, K_{\mu}, \Upsilon_{\mu}$. From the calculations in [loc. cit.] and the definition, we see that

$$\mathfrak{X}_{\lambda}^{\natural}(E_i x) = \mathfrak{t}^{\star} E_i \mathfrak{X}_{\lambda}^{\natural}(x),$$

where

$$\star = \langle \tilde{v} - \tilde{i}, \lambda \rangle - \langle \tilde{v}, \lambda \rangle + \phi(v - i, \lambda - v + i) - \phi(v, \lambda - v) - d_i + \langle \tilde{i}, \lambda - v + i \rangle - \phi(i, v - i).$$

Now we can simplify \star and apply (4-2) to see that

$$\star \equiv \phi(v, i) - \phi(i, v) - \phi(i, \lambda - v) + i \cdot v + d_i = 2p(v)p(i) - \phi(i, \lambda - v) + d_i \pmod{4},$$

and thus

$$\mathfrak{X}_{\lambda}^{\natural}(E_i x) = \mathfrak{t}^{p(v)p(i)} \mathfrak{t}_i E_i \Upsilon_{-i} \mathfrak{X}_{\lambda}^{\natural}(x) = \mathfrak{t}^{p(v)p(i)} \mathfrak{X}^{\natural}(E_i) \mathfrak{X}_{\lambda}^{\natural}(x).$$

Finally, we note that the kernel of the projection $\mathfrak{f} \rightarrow \widehat{V}(\lambda)$ is trivially preserved by $\mathfrak{X}_{\lambda}^{\natural}$; hence it descends to a map on $\widehat{V}(\lambda)$. □

The dual twistor \mathfrak{X}^{\natural} is what will allow us to define a convenient twistor map on dual modules, as follows. Recall that $V(-\lambda)$ denotes the U -module $V(\lambda)^*$. We will adapt this notation to $\widehat{V}(\lambda)^{\natural}$.

Lemma 4.14 *For $\lambda \in X^+$, let $\widehat{V}(-\lambda) = \widehat{V}(\lambda)^{\natural}$; that is, the space $V(\lambda)^*$ with the action of \widehat{U} induced by the antipode $S: \widehat{U} \rightarrow \widehat{U}$. Define a map \mathfrak{X} on $\widehat{V}(-\lambda)$ by $\mathfrak{X}(f)(x) = \mathfrak{t}^{2p(f)p(x)} \mathfrak{X}(f((\mathfrak{X}^{\natural})^{-1}(x)))$ for homogeneous $x \in \widehat{V}(\lambda)$ and $f \in \widehat{V}(-\lambda)$. Then $\mathfrak{X}(uf) = \mathfrak{X}(u)\mathfrak{X}(f)$ for all $u \in \widehat{U}$ and $f \in \widehat{V}(-\lambda)$.*

Proof Let $f \in \widehat{V}(-\lambda)$ and $x \in \widehat{V}(\lambda)$ be homogeneous. First, observe that since \mathfrak{X}^{\natural} preserves the \widehat{X} -grading, $\mathfrak{X}(f)(x) = 0$ unless $\|x\| = \|f\|$. Moreover, if $a \in \mathbb{Q}(q, \mathfrak{t})^{\tau}$,

$$\begin{aligned} \mathfrak{X}(f)(ax) &= \mathfrak{t}^{2p(f)p(x)} \mathfrak{X}(f((\mathfrak{X}^{\natural})^{-1}(ax))) \\ &= \mathfrak{t}^{2p(f)p(x)} \mathfrak{X}(\mathfrak{X}^{-1}(a)f((\mathfrak{X}^{\natural})^{-1}(x))) = a\mathfrak{X}(f)(x), \end{aligned}$$

so $\mathfrak{X}(f)$ is indeed an element of $\widehat{V}(-\lambda)$.

Now suppose $u \in \widehat{U}$. We compute that

$$\begin{aligned} \mathfrak{X}(uf)(x) &= \mathbf{t}^{2p(uf)p(x)} \mathfrak{X}((uf)((\mathfrak{X}^{\natural})^{-1}(x))) \\ &= \mathbf{t}^{2p(u)p(x)+2p(f)p(x)} \mathfrak{X}(\pi^{p(u)p(f)} f(S(u)(\mathfrak{X}^{\natural})^{-1}(x))) \end{aligned}$$

and

$$\begin{aligned} \mathfrak{X}(u)\mathfrak{X}(f)(x) &= \mathbf{t}^{2p(f)p(ux)} \pi^{p(u)p(f)} \mathfrak{X}(f((\mathfrak{X}^{\natural})^{-1}(S(\mathfrak{X}(u))x))) \\ &= \mathbf{t}^{2p(f)p(u)+2p(f)p(x)+2p(u)p(x)} \pi^{p(u)p(f)} \mathfrak{X}(f((\mathfrak{X}^{\natural})^{-1}(S(\mathfrak{X}(u))\mathfrak{X}'^{-1}(x)))) \\ &= \mathbf{t}^{2p(f)p(x)+2p(u)p(x)} \mathfrak{X}(\pi^{p(u)p(f)} f(S(u)(\mathfrak{X}^{\natural})^{-1}(x))). \end{aligned}$$

Therefore, $\mathfrak{X}(uf) = \mathfrak{X}(u)\mathfrak{X}(f)$. □

4.3 Twistor on tensor products

Now let us return to the question of relating the $\mathfrak{osp}(1|2)$ and $\mathfrak{sl}(2)$ link invariants. Since the invariants arise from maps between tensor products of simple modules and their duals, we shall also need variants of the twistor maps on the corresponding \widehat{U} -modules. In the following, we shall define a number of versions of \mathfrak{X} in different settings. However, they will all be compatible in natural ways, so rather than label these maps differently, we shall treat them en suite as an operator on \widehat{U} and its modules.

The following proposition takes the first step in this direction by showing that there is a natural extension of the twistor maps to tensor powers of U .

Proposition 4.15 *For each positive integer t , there exists a $\mathbb{Q}(\mathbf{t})$ -algebra isomorphism $\mathfrak{X}: \widehat{U}^{\otimes t+1} \rightarrow \widehat{U}^{\otimes t+1}$ which satisfies*

$$\mathfrak{X}(x \otimes y) = \mathfrak{X}(x)\Delta^s(\Upsilon_{|y|}) \otimes \Delta^{s'}(\widetilde{T}_{|x|}\Upsilon_{|x|})\mathfrak{X}(y)$$

for any positive integers s, s' satisfying $s + s' = t + 1$, $x \in \widehat{U}^{\otimes s}$, and $y \in \widehat{U}^{\otimes s'}$. Moreover, $\Delta^t(\mathfrak{X}(x)) = \mathfrak{X}(\Delta^t(x))$ for any $x \in U$.

Proof Define $\mathfrak{X}': \widehat{U}^{\otimes t+1} \rightarrow \widehat{U}^{\otimes t+1}$ as follows: for $x = \bigotimes_{s=1}^{t+1} x_s \in \widehat{U}^{\otimes t+1}$, let $\mathfrak{X}(x) = \bigotimes_{s=1}^{t+1} \mathfrak{X}(x)_s$, where

$$(4-9) \quad \mathfrak{X}(x)_s = \widetilde{T}_{|x_1|+\dots+|x_{s-1}|}\Upsilon_{|x_1|+\dots+|x_{s-1}|}\mathfrak{X}(x_s)\Upsilon_{|x_{s+1}|+\dots+|x_{t+1}|}.$$

It is elementary to check that

$$\mathfrak{X}(x \otimes y) = \mathfrak{X}(x)\Delta^s(\Upsilon_{|y|}) \otimes \Delta^{s'}\widetilde{T}_{|x|}\Upsilon_{|x|}\mathfrak{X}(y)$$

for any positive integers s, s' satisfying $s + s' = t + 1$, $x \in \widehat{U}^{\otimes s}$, and $y \in \widehat{U}^{\otimes s'}$. Moreover, since \mathfrak{X} on \widehat{U} is a bijection, it is easy to see that so is \mathfrak{X} on $\widehat{U}^{\otimes t+1}$.

We will prove that \mathfrak{X} is an algebra homomorphism, and hence isomorphism, by induction. Since \mathfrak{X} on \widehat{U} is an isomorphism, let us assume \mathfrak{X} on $\widehat{U}^{\otimes t}$ is an isomorphism. Then for $x, w \in \widehat{U}^{\otimes t}$ and $y, z \in \widehat{U}$,

$$\begin{aligned} &\mathfrak{X}(x \otimes y)\mathfrak{X}(w \otimes z) \\ &= (\mathfrak{X}(x)\Upsilon_{|y|} \otimes \widetilde{T}_{|x|}\Upsilon_{|x|}\mathfrak{X}(y))(\mathfrak{X}(w)\Upsilon_{|z|} \otimes \widetilde{T}_{|w|}\Upsilon_{|w|}\mathfrak{X}(z)) \\ &= \pi^{p(y)p(w)}\mathfrak{X}(x)\Upsilon_{|y|}\mathfrak{X}(w)\Upsilon_{|z|} \otimes \widetilde{T}_{|x|}\Upsilon_{|x|}\mathfrak{X}(y)\widetilde{T}_{|w|}\Upsilon_{|w|}\mathfrak{X}(z) \\ &= \pi^{p(y)p(w)}\mathfrak{t}^{\phi(|y|,|w|)-\phi(|w|,|y|)-|w|\cdot|y|}\mathfrak{X}(xw)\Upsilon_{|yz|} \otimes \widetilde{T}_{|xw|}\Upsilon_{|xw|}\mathfrak{X}(yz) \\ &= \pi^{p(y)p(w)}\mathfrak{t}^{2p(y)p(z)}\mathfrak{X}(xw \otimes yz) = \mathfrak{X}(\pi^{p(y)p(w)}xw \otimes yz) \\ &= \mathfrak{X}((x \otimes y)(w \otimes z)). \end{aligned}$$

This completes the induction showing \mathfrak{X} on $\widehat{U}^{\otimes t+1}$ is an isomorphism as claimed. Finally, showing that \mathfrak{X} commutes with Δ^t is straightforward using (4-9) and checking on the generators. □

Now that we have a viable twistor map on tensor powers of \widehat{U} , we need an analogue on the tensor powers of modules. In particular, suppose we have a collection of \widehat{U} modules which are untwisted or $*$ -twisted, and which carry twistors. We will produce a twistor on the tensor product of these modules.

As might be suggested by (4-9), this is not as simple as taking the tensor power of the twistors. A version of such a twistor is produced in [4, Proposition 6.11] by rescaling the tensor product of twistors by a power of \mathfrak{t} given by a function of the weights of the tensor factors. We will do something similar, but it turns out that we will need functions which depend not only on the weights of tensor factors but also their parities, as well as the signature of the tensor product.

Lemma 4.16 *Let $c = (c_1, c_2)$ where $c_1, c_2 \in \{1, -1\}$. There exists a function $\kappa_c: \widehat{X}^2 \rightarrow \mathbb{Z}$ satisfying $\kappa((0, 0), \zeta) \equiv \kappa(\zeta, (0, 0)) \equiv 0 \pmod{4}$ and*

$$\begin{aligned} &\kappa_c(\zeta + \mu, \zeta' + \nu) - \kappa_c(\zeta, \zeta') \\ &\equiv \langle \widetilde{\mu}, |\zeta'| \rangle + c_2\phi(\mu, c_2|\zeta'|) + 2p(\zeta)p(\nu) + c_1\phi(\nu, c_1|\zeta|) + \mu \cdot \nu + \phi(\mu, \nu) \pmod{4} \end{aligned}$$

for all $\zeta, \zeta' \in \widehat{X}$ and $\mu, \nu \in \mathbb{Z}[I]$.

Proof Fix $c = (c_1, c_2)$ where $c_1, c_2 \in \{1, -1\}$. Note that it suffices to show such a function $\kappa = \kappa_c$ exists on each coset of $\mathbb{Z}[I] \times \mathbb{Z}[I]$ (where as in (2-6), we view $\mathbb{Z}[I]$

as a subset of \widehat{X}), so fix a set of representatives C of $\widehat{X}/\mathbb{Z}[I]$. For $\zeta_0, \zeta_1 \in C$, set $\kappa(\zeta_0 + \mu, \zeta_1 + \nu)$

$$= \langle \tilde{\mu}, |\zeta_1| \rangle + c_2 \phi(\mu, c_2 |\zeta_1|) + 2p(\zeta_0)p(\nu) + c_1 \phi(\nu, c_1 |\zeta_0|) + \mu \cdot \nu + \phi(\mu, \nu).$$

It is elementary to verify that this has the desired properties. □

We henceforth suppose we have fixed choices of κ_c for each $c \in \{1, -1\}^2$. We can extend κ naturally to larger powers of \widehat{X} . Let $t > 1$ be a positive integer and fix a sequence $c = (c_s) \in \{\pm 1\}^t$. Let $\kappa_c: \widehat{X}^t \rightarrow \mathbb{Z}$ be the function defined by

$$\kappa_c(\zeta) = \sum_{1 \leq r < s \leq t} \kappa_{(c_r, c_s)}(\zeta_r, \zeta_s), \quad \zeta = (\zeta_s) \in \widehat{X}^t.$$

Then if we have $\zeta = (\zeta_s)$ and $\zeta' = (\zeta'_s)$ in \widehat{X}^t with $\zeta'_s = \zeta_s + \delta_{r,s}i$ for some $1 \leq r \leq t$, then

$$\kappa(\zeta') - \kappa(\zeta) \equiv \sum_{r < s \leq t} (\langle \tilde{i}, |\zeta_s| \rangle + c_s \phi(i, c_s |\zeta_s|)) + \sum_{1 \leq s' < r} (2p(\zeta_{s'})p(i) + c_{s'} \phi(i, c_{s'} |\zeta_{s'}|))$$

modulo 4.

We can observe some convenient properties of the maps κ_c .

Lemma 4.17 *Let $c = (c_s) \in \{\pm 1\}^t$, and let $\zeta = (\zeta_s)$ and $\zeta' = (\zeta'_s)$ in \widehat{X}^t .*

- (1) *Let $1 \leq r \leq t$, and define $c_{\leq r} = (c_1, \dots, c_r)$ and $c_{> r} = (c_{r+1}, \dots, c_t)$. Likewise, define $\zeta''_{\leq r} = (\zeta''_1, \dots, \zeta''_r)$ and $\zeta''_{> r} = (\zeta''_{r+1}, \dots, \zeta''_t)$ for any $\zeta'' = (\zeta''_s) \in \widehat{X}^t$. Then*

$$\kappa_c(\zeta, \zeta') = \kappa_{c_{\leq r}}(\zeta_{\leq r}, \zeta'_{\leq r}) + \kappa_{c_{> r}}(\zeta_{> r}, \zeta'_{> r}) + \sum_{1 \leq s < r < s' \leq t} \kappa_{(c_s, c_{s'})}(\zeta_s, \zeta'_{s'}).$$

- (2) *Suppose that there exists $1 \leq r < t$ such that $\zeta_r = \zeta'_r + \nu$, $\zeta_{r+1} = \zeta'_{r+1} - \nu$, and $\zeta_s = \zeta'_s$ for $s \neq r, r + 1$ and some $\nu \in \mathbb{Z}[I]$. Then*

$$\kappa_c(\zeta) - \kappa_c(\zeta') = \langle \tilde{\nu}, \zeta_{r+1} \rangle + c_{r+1} \phi(\nu, c_{r+1} \zeta_{r+1}) + 2p(\nu)p(\zeta_r) - c_r \phi(\nu, c_r \zeta_r) - \nu \cdot \nu - \phi(\nu, \nu).$$

- (3) *For any $\zeta \in \widehat{X}$ and $c_1 = \pm 1$, we have*

$$\begin{aligned} \kappa_{c_1, \pm 1, \mp 1}(\zeta + \widehat{\nu}, (\pm\lambda, 0), (\mp\lambda, 0)) &= \kappa_{c_1, \pm 1, \mp 1}(\zeta, (\pm\lambda, 0), (\mp\lambda, 0)), \\ \kappa_{\pm 1, \mp 1, c_1}((\pm\lambda, 0), (\mp\lambda, 0), \zeta + \widehat{\nu}) &= \kappa_{\pm 1, \mp 1, c_1}((\pm\lambda, 0), (\mp\lambda, 0), \zeta). \end{aligned}$$

Proof We note that (1) is an immediate consequence of the definition of κ_c . On the other hand, (2) and (3) both follow from direct computations and the definition. □

The functions κ_c allows us to define a twistor on tensor product modules as follows.

Proposition 4.18 *Let M_1, M_2, \dots, M_t be untwisted or $*$ -twisted \widehat{U} -modules carrying twistors and let $M = M_1 \widehat{\otimes} M_2 \otimes \dots \widehat{\otimes} M_t$ be the $\widehat{U}^{\otimes t}$ -module (and hence a mixed \widehat{U} -module via Δ^{t-1}) with the natural action. Set $c = \text{sig}(M) = (c_1, \dots, c_t)$. Then the automorphism*

$$\mathfrak{X}(m_1 \otimes \dots \otimes m_t) = t^{\kappa_c(\|m\|_i)} \mathfrak{X}(m_1) \otimes \dots \otimes \mathfrak{X}(m_t)$$

satisfies

$$\mathfrak{X}((x_1 \otimes \dots \otimes x_t)(m_1 \otimes \dots \otimes m_t)) = \mathfrak{X}(x_1 \otimes \dots \otimes x_t) \mathfrak{X}(m_1 \otimes \dots \otimes m_t).$$

In particular, $\mathfrak{X}(um) = \mathfrak{X}(u)\mathfrak{X}(m)$ for $u \in \widehat{U}$ and $m \in M$.

Proof First, observe it is enough to show

$$\mathfrak{X}((1^{s-1} \otimes x_s \otimes 1^{t-s})(m_1 \otimes \dots \otimes m_t)) = \mathfrak{X}(1^{s-1} \otimes x_s \otimes 1^{t-s}) \mathfrak{X}(m_1 \otimes \dots \otimes m_t),$$

where $1 \leq s \leq t$ and x_s is a generator of \widehat{U} . This is trivial when x_s is K_μ, J_μ, T_μ and Υ_μ for some $\mu \in \mathbb{Z}[I]$ so it suffices to check the case $x_s = E_i, F_i$ for $i \in I$. To do this, let us make our equations more compact with the following notation: for $m_1 \otimes \dots \otimes m_t \in M$, let

$$\begin{aligned} m_{<s} &= m_1 \otimes \dots \otimes m_{s-1}, & m_{>s} &= m_{s+1} \otimes \dots \otimes m_t, \\ \mathfrak{X}(m)_{<s} &= \mathfrak{X}(m_1) \otimes \dots \otimes \mathfrak{X}(m_{s-1}), & \mathfrak{X}(m)_{>s} &= \mathfrak{X}(m_{s+1}) \otimes \dots \otimes \mathfrak{X}(m_t), \\ \|m\|_{<s} &= (\|m_1\|, \dots, \|m_{s-1}\|), & \|m\|_{>s} &= (\|m_{s+1}\|, \dots, \|m_t\|), \\ \phi'(i, m_{<s}) &= \sum_{1 \leq r < s} c_r \phi(i, c_r | m_r), & \phi''(i, m_{>s}) &= \sum_{s < r \leq t} c_r \phi(i, c_r | m_r). \end{aligned}$$

Using this notation, we compute that

$$\begin{aligned} &\mathfrak{X}((1^{s-1} \otimes E_i \otimes 1^{t-s})(m_{<s} \otimes m_s \otimes m_{>s})) \\ &= \mathfrak{X}(\pi_i^{p(m_{<s})} m_{<s} \otimes E_i m_s \otimes m_{>s}) \\ &= t^{2p(i)p(m_{<s}) + \kappa_c(\|m\|_{<s}, \|m_s\| + \widehat{i}, \|m\|_{>s})} \pi^{p(i)p(m_{<s})} \mathfrak{X}(m)_{<s} \otimes \mathfrak{X}(E_i m_s) \otimes \mathfrak{X}(m_{>s}) \\ &= t^{\kappa(\|m_{<s}\|, \|m_s\|, \|m_{>s}\|) + \phi'(i, m_{<s}) + \phi''(i, m_{>s}) + (\widehat{i}, |m_{>s}|)} \\ &\quad \times \pi^{p(i)p(m_{<s})} \mathfrak{X}(m)_{<s} \otimes \mathfrak{X}(E_i) \mathfrak{X}(m_s) \otimes \mathfrak{X}(m_{>s}) \\ &= t^{\kappa(\|m_{<s}\|, \|m_s\|, \|m_{>s}\|)} \pi^{p(i)p(m_{<s})} \\ &\quad \times (\Upsilon_i^{\otimes(s-1)} \mathfrak{X}(m)_{<s}) \otimes (\mathfrak{X}(E_i) \mathfrak{X}(m_s)) \otimes ((\Upsilon_i \widetilde{T}_i)^{\otimes(t-s)} \mathfrak{X}(m_{>s})) \\ &= \mathfrak{X}(1^{s-1} \otimes E_i \otimes 1^{t-s}) \mathfrak{X}(m_{<s} \otimes m_s \otimes m_{>s}). \end{aligned}$$

The case $x_s = F_i$ proceeds similarly. □

We now have defined a family of compatible twistor maps on untwisted modules, $*$ -twisted modules and their tensor products. Moreover, the twistor maps on tensor products of modules are compatible with one another in the following sense. Let $M_1, \dots, M_t, c_1, \dots, c_s$ and M be as in Proposition 4.18. Fix $1 \leq r \leq t$ and set $m_{\leq r} = m_1 \otimes \dots \otimes m_r$ and $m_{>r} = m_{r+1} \otimes \dots \otimes m_t$. Then by Lemma 4.17(1),

$$(4-10) \quad \mathfrak{X}(m_{\leq r} \otimes m_{>r}) = \left(\prod_{1 \leq s \leq r < s' \leq t} \mathbf{t}^{\kappa_{c_s, c_{s'}}(\|m_s\|, \|m_{s'}\|)} \right) \mathfrak{X}(m_{\leq r}) \otimes \mathfrak{X}(m_{>r}).$$

4.4 Twisting the crossings, caps, and cups

We have now laid the groundwork for studying the atomic maps in our graphical calculus from Section 3 under the twistor functor. Specifically, we will show that the twistor almost commutes with cups, caps, and crossings up to a factor of an integral power of \mathbf{t} , where the power depends on the map. We begin by considering the cups and caps on their domains of definition.

Proposition 4.19 *Let $\lambda \in X^+$. Then the map ev_λ (resp. qtr_λ , coev_λ , and coqtr_λ) viewed as a function $\widehat{V}(-\lambda) \widehat{\otimes} \widehat{V}(\lambda) \rightarrow \mathbb{Q}(q, \mathbf{t})^\tau$ (resp. $\widehat{V}(\lambda) \widehat{\otimes} \widehat{V}(-\lambda) \rightarrow \mathbb{Q}(q, \mathbf{t})^\tau$, $\mathbb{Q}(q, \mathbf{t})^\tau \rightarrow \widehat{V}(-\lambda) \widehat{\otimes} \widehat{V}(\lambda)$, and $\mathbb{Q}(q, \mathbf{t})^\tau \rightarrow \widehat{V}(\lambda) \widehat{\otimes} \widehat{V}(-\lambda)$) is a \widehat{U} -module homomorphism. Moreover, we have*

- (1) $\text{ev}_\lambda \mathfrak{X} = \mathbf{t}^{\kappa(-1,1)((-\lambda,0),(\lambda,0))} \mathfrak{X} \text{ev}_\lambda$;
- (2) $\text{qtr}_\lambda \mathfrak{X} = \mathbf{t}^{\kappa(1,-1)((\lambda,0),(-\lambda,0)) - \langle \tilde{\rho}, \lambda \rangle} \mathfrak{X} \text{qtr}_\lambda$;
- (3) $\text{coev}_\lambda \mathfrak{X} = \mathbf{t}^{-\kappa(-1,1)((-\lambda,0),(\lambda,0)) + \langle \tilde{\rho}, \lambda \rangle} \mathfrak{X} \text{coev}_\lambda$;
- (4) $\text{coqtr}_\lambda \mathfrak{X} = \mathbf{t}^{-\kappa(1,-1)((\lambda,0),(-\lambda,0))} \mathfrak{X} \text{coqtr}_\lambda$.

Proof First, observe that since these maps are U -module homomorphisms, they preserve weight spaces hence preserve the action of T_i for $i \in I$. Therefore, it only remains to check that they commute with the action of Υ_i for $i \in I$. As the arguments are all similar, let us show this for ev_λ . Let $f \in \widehat{V}(-\lambda)$ and $x \in V(\lambda)$. Then

$$\Upsilon_i \text{ev}_\lambda(f \otimes x) = \mathbf{t}^{\phi(i,0)} \text{ev}_\lambda(f \otimes x) = f(x).$$

On the other hand, $\Upsilon_i(f \otimes x) = (\Upsilon_i f) \otimes (\Upsilon_i x) = \mathbf{t}^{-\phi(i,-|f|) + \phi(i,|x|)} f \otimes x$, hence

$$\text{ev}_\lambda(\Upsilon_i(f \otimes x)) = \mathbf{t}^{\phi(i,|x|) - \phi(i,-|f|)} f(x).$$

However, since $f(x) = 0$ if $|f| \neq -|x|$, we see that

$$\text{ev}_\lambda(\Upsilon_i(f \otimes x)) = \mathbf{t}^{\phi(i,|x|) - \phi(i,|x|)} f(x) = f(x) = \Upsilon_i \text{ev}_\lambda(f \otimes x).$$

In order to verify (1)–(4), it suffices to compute the images $\mathfrak{X}(b^-v_\lambda \otimes (b^-v_\lambda)^*)$ and $\mathfrak{X}((b^-v_\lambda)^* \otimes b^-v_\lambda)$ for $b \in \mathcal{B}_v = \mathcal{B} \cap \mathfrak{f}_v$. We compute directly that

$$\begin{aligned} \mathfrak{X}(b^-v_\lambda) &= \mathfrak{t}^{\ell(b)-\phi(v,\lambda)} b^-v_\lambda, \\ \mathfrak{X}^\natural(b^-v_\lambda) &= \mathfrak{t}^{\ell(b)+\langle v,\lambda \rangle + \phi(v,\lambda-v)} b^-v_\lambda. \end{aligned}$$

This implies that for any $b, b' \in \mathcal{B}_v$,

$$\begin{aligned} \mathfrak{X}((b^-v_\lambda)^*)(b'^-v_\lambda) &= \mathfrak{t}^{2p(v)} \mathfrak{X}((b^-v_\lambda)^* ((\mathfrak{X}^\natural)^{-1}(b'^-v_\lambda))) \\ &= \mathfrak{t}^{2p(v)-\ell(b)-\langle v,\lambda \rangle - \phi(v,\lambda-v)} \delta_{b,b'}, \end{aligned}$$

and hence $\mathfrak{X}((b^-v_\lambda)^*) = \mathfrak{t}^{2p(v)-\ell(b)-\langle v,\lambda \rangle - \phi(v,\lambda-v)} (b^-v_\lambda)^*$. In particular, for $c = (1, -1)$, observe that

$$\begin{aligned} \mathfrak{X}((b^-v_\lambda) \otimes (b^-v_\lambda)^*) &= \mathfrak{t}^{\kappa_c((\lambda,0)-\widehat{v},(-\lambda,0)+\widehat{v})+\ell(b)-\phi(v,\lambda)+2p(v)-\ell(b)-\langle v,\lambda \rangle - \phi(v,\lambda-v)} (b^-v_\lambda) \otimes (b^-v_\lambda)^* \\ &= \mathfrak{t}^{\kappa_c((\lambda,0),(-\lambda,0))+2p(v)-v \cdot v} (b^-v_\lambda) \otimes (b^-v_\lambda)^*. \end{aligned}$$

It is easy to verify that $\frac{1}{2}v \cdot v = p(v) \pmod 2$ by induction; hence we see that

$$\mathfrak{X}((b^-v_\lambda) \otimes (b^-v_\lambda)^*) = \mathfrak{t}^{\kappa_{(1,-1)}((\lambda,0),(-\lambda,0))} (b^-v_\lambda) \otimes (b^-v_\lambda)^*.$$

A similar computation shows that

$$\mathfrak{X}((b^-v_\lambda)^* \otimes (b^-v_\lambda)) = \mathfrak{t}^{\kappa_{(-1,1)}((-\lambda,0),(\lambda,0))} (b^-v_\lambda)^* \otimes (b^-v_\lambda).$$

Note that in either case, the power of \mathfrak{t} is independent of $b \in \mathcal{B}$, and applying this to the definition of the maps proves (1) and (4). For (2) and (3), also note that $\pi^{p(v)} q^\pm(\tilde{\rho},\lambda-v) = \pi_v q_v^{\mp 2} q^\pm(\tilde{\rho},\lambda)$, and we compute that $\mathfrak{X}(\pi_v q_v^{\mp 2} q^\pm(\tilde{\rho},\lambda)) = \mathfrak{t}^{\mp \langle \tilde{\rho},\lambda \rangle} \pi_v q_v^{\mp 2} q^\pm(\tilde{\rho},\lambda)$; the result follows. \square

Example 4.20 Consider the case $n = 1$ and $\lambda = m$. As noted in [Example 2.10](#), $\langle \tilde{\rho}, \lambda \rangle = m$ and $ev_m \circ coev_m = \pi^m[m + 1]$. Then we have $ev_m \circ coev_m \circ \mathfrak{X}(1) = \pi^m[m + 1]$, and

$$\mathfrak{X} \circ ev_m \circ coev_m(1) = \mathfrak{X}(\pi^m[m + 1]) = \mathfrak{t}^{-m} \pi^m[m + 1] = \mathfrak{t}^{-m} ev_m \circ coev_m \circ \mathfrak{X}(1).$$

Note that this is consistent with [Proposition 4.19](#), as we see that

$$ev_m \circ coev_m \circ \mathfrak{X} = \mathfrak{t}^{-\kappa_{(-1,1)}((-\lambda,0),(\lambda,0))+\langle \tilde{\rho},\lambda \rangle} ev_m \circ \mathfrak{X} \circ coev_m = \mathfrak{t}^m \mathfrak{X} \circ ev_m \circ coev_m.$$

The last elementary diagram to consider is the crossing, which is to say the automorphism $R = \Theta \mathfrak{F}_s$ of a tensor product of two modules. In order to have a concrete comparison of $R\mathfrak{X}$ and $\mathfrak{X}R$ on tensor products carrying twistors, it will be necessary

to have a precise description of $\mathfrak{X}(\mathfrak{f}(\zeta, \eta))$ for any $\zeta, \eta \in X$. To that end, let us once and for all fix a transversal T of $X/\mathbb{Z}[I]$ and note that $\widehat{T} = \{(\zeta, 0), (\zeta, 1) \mid \zeta \in T\}$ is a transversal of $\widehat{X}/\mathbb{Z}[I]$. Then for $\zeta_0, \zeta_1 \in T$, we shall henceforth require that

$$(4-11) \quad \mathfrak{f}(|\zeta_0|, |\zeta_1|) = 1.$$

Then we have the following proposition.

Proposition 4.21 *Let $\lambda, \lambda' \in X^+ \cup -X^+$. Let $\widehat{\zeta}, \widehat{\zeta}' \in \widehat{T}$ be the corresponding coset representatives of $(\lambda, 0)$ and $(\lambda', 0)$ in $\widehat{X}/\mathbb{Z}[I]$, and let $(c_1, c_2) = \text{sig}(\widehat{V}(\lambda) \widehat{\otimes} \widehat{V}(\lambda'))$. Let $\mathcal{R}: \widehat{V}(\lambda) \widehat{\otimes} \widehat{V}(\lambda') \rightarrow \widehat{V}(\lambda') \widehat{\otimes} \widehat{V}(\lambda)$ be the map described in Proposition 2.18. Then \mathcal{R} is a \widehat{U} -module homomorphism. Moreover, as maps on $\widehat{V}(\lambda) \widehat{\otimes} \widehat{V}(\lambda')$, we have*

$$\mathfrak{X}\mathcal{R} = \mathbf{t}^{\kappa(c_2, c_1)(\widehat{\zeta}', \widehat{\zeta}) - \kappa(c_1, c_2)(\widehat{\zeta}, \widehat{\zeta}') + 2p(\widehat{\zeta})p(\widehat{\zeta}')}\mathcal{R}\mathfrak{X}.$$

Proof Recall that $\mathcal{R} = \Theta\mathfrak{f}\mathfrak{s}$ by definition. It is easy to see that \mathcal{R} is a \widehat{U} -module homomorphism: indeed, since \mathcal{R} preserve weight spaces, it commutes with the action of the T_i for $i \in I$; moreover, $\mathfrak{f}\mathfrak{s}$ obviously commutes with the diagonal action of Υ_i , and it is easy to check directly that $\Theta_\nu \Delta(\Upsilon_i) = \Delta(\Upsilon_i)\Theta_\nu$. We will prove the remainder of the proposition in two steps.

First we shall show that $\mathfrak{X}(\Theta_\nu) = \Theta_\nu$ for any $\nu \in \mathbb{Z}_{\geq 0}[I]$, and thus $\mathfrak{X}\Theta = \Theta\mathfrak{X}$ as maps on $V(\lambda) \otimes V(\lambda')$. This is straightforward: applying Lemmas 4.5, 4.3 and Proposition 4.15, we compute that

$$\begin{aligned} \mathfrak{X}(\Theta_\nu) &= (-1)^{\text{ht}\nu} \mathbf{t}^{2p(\nu)} \pi^{p(\nu)} \mathbf{t}_\nu^2 \pi_\nu \mathbf{t}_\nu^{-1} q_\nu \sum_{b \in \mathcal{B}_\nu} \mathfrak{X}(b^-) \Upsilon_\nu \otimes \widetilde{T}_{-\nu} \Upsilon_{-\nu} \mathfrak{X}((b^*)^+) \\ &= (-1)^{\text{ht}\nu + p(\nu)} \pi^{p(\nu)} \mathbf{t}_\nu \pi_\nu q_\nu \sum_{b \in \mathcal{B}_\nu} (\mathfrak{X}(b)^- \Upsilon_{-\nu}) \Upsilon_\nu \otimes \widetilde{T}_{-\nu} \Upsilon_{-\nu} (\mathbf{t}_\nu^{-1} \widetilde{T}_\nu \Upsilon_\nu \mathfrak{X}(b^*)^+) \\ &= (-1)^{\text{ht}\nu + p(\nu)} \pi^{p(\nu)} \mathbf{t}_\nu \pi_\nu q_\nu \sum_{b \in \mathcal{B}_\nu} (\mathbf{t}^{\ell(b)} b^-) \otimes (\mathbf{t}_\nu^{-1} \mathbf{t}^{-\ell(b)} (-1)^{p(\nu)} (b^*)^+) \\ &= (-1)^{\text{ht}\nu} \pi^{p(\nu)} \pi_\nu q_\nu \sum_{b \in \mathcal{B}_\nu} b^- \otimes (b^*)^+ = \Theta_\nu. \end{aligned}$$

Now it remains to show that we have $\mathfrak{X}\mathfrak{F}\mathfrak{s} = \mathbf{t}^{\kappa(c_2, c_1)(\widehat{\zeta}', \widehat{\zeta}) - \kappa(c_1, c_2)(\widehat{\zeta}, \widehat{\zeta}') + 2p(\widehat{\zeta})p(\widehat{\zeta}')}\mathfrak{F}\mathfrak{s}\mathfrak{X}$ as maps on $V(\lambda) \otimes V(\lambda')$. Set $c = (c_1, c_2)$, and $\widetilde{c} = (c_2, c_1)$. Let $m \in V(\lambda)$ and $n \in V(\lambda')$. Then we see directly that

$$\begin{aligned} \mathfrak{X}\mathfrak{F}\mathfrak{s}(m \otimes n) &= \mathbf{t}^{2p(m)p(n) + \kappa_{\widetilde{c}}(\|n\|, \|m\|)} \mathfrak{X}(\mathfrak{f}(|n|, |m|)) \pi^{p(m)p(n)} \mathfrak{X}(n) \otimes \mathfrak{X}(m), \\ \mathfrak{F}\mathfrak{s}\mathfrak{X}(m \otimes n) &= \mathbf{t}^{\kappa_c(\|m\|, \|n\|)} \mathfrak{f}(|n|, |m|) \pi^{p(m)p(n)} \mathfrak{X}(n) \otimes \mathfrak{X}(m). \end{aligned}$$

The proposition then follows by verifying that

$$\begin{aligned} \mathbf{t}^{2p(m)p(n)+\kappa_{\tilde{c}}(\|n\|,\|m\|)} \mathfrak{X}(\mathfrak{f}(|n|, |m|)) \\ = \mathbf{t}^{\kappa_{\tilde{c}}(\widehat{\xi}', \widehat{\xi}) - \kappa_c(\widehat{\xi}, \widehat{\xi}') + 2p(\widehat{\xi})p(\widehat{\xi}') + \kappa_c(\|m\|, \|n\|)} \mathfrak{f}(|n|, |m|). \end{aligned}$$

Note that $\widehat{\xi} = \|m\| + \mu$ and $\widehat{\xi}' = \|n\| + \nu$ for some $\mu, \nu \in \mathbb{Z}[I]$. Let $\zeta = |\widehat{\xi}| \in X$ and $\zeta' = |\widehat{\xi}'| \in X$. Then in particular, (4-11) implies

$$\mathfrak{f}(|n|, |m|) = (\pi q)^{\langle \tilde{\nu}, \zeta \rangle} q^{\langle \tilde{\mu}, \zeta' \rangle - \mu \cdot \nu},$$

so $\mathfrak{X}(\mathfrak{f}(|n|, |m|)) = \mathbf{t}^{\langle \tilde{\nu}, \zeta \rangle - \langle \tilde{\mu}, \zeta' \rangle + \mu \cdot \nu} \mathfrak{f}(|n|, |m|)$. Therefore, we are reduced to showing that $\ell \equiv r \pmod 4$, where

$$\begin{aligned} \ell &= 2p(m)p(n) + \langle \tilde{\nu}, \zeta \rangle - \langle \tilde{\mu}, \zeta' \rangle + \mu \cdot \nu + \kappa_{\tilde{c}}(\|n\|, \|m\|), \\ r &= 2p(\widehat{\xi})p(\widehat{\xi}') + \kappa_{\tilde{c}}(\widehat{\xi}', \widehat{\xi}) - \kappa_c(\widehat{\xi}, \widehat{\xi}') + \kappa_c(\|m\|, \|n\|) \pmod 4. \end{aligned}$$

We compute directly that

$$\begin{aligned} \kappa_{\tilde{c}}(\|n\|, \|m\|) - \kappa_{\tilde{c}}(\widehat{\xi}', \widehat{\xi}) + \kappa_c(\widehat{\xi}, \widehat{\xi}') - \kappa_c(\|m\|, \|n\|) \\ = \kappa_{\tilde{c}}(\widehat{\xi}' - \nu, \widehat{\xi} - \mu) - \kappa_{\tilde{c}}(\widehat{\xi}', \widehat{\xi}) + \kappa_c(\widehat{\xi}, \widehat{\xi}') - \kappa_c(\widehat{\xi} - \mu, \widehat{\xi}' - \nu) \\ \equiv_4 -\langle \tilde{\nu}, \zeta \rangle - c_1\phi(\nu, c_1\zeta) + 2p(\widehat{\xi})p(\mu) - c_2\phi(\mu, c_2\zeta) + \mu \cdot \nu + \phi(\mu, \nu) \\ \quad + \langle \tilde{\mu}, \zeta \rangle + c_1\phi(\mu, c_1\zeta) - 2p(\widehat{\lambda})p(\nu) + \phi(\nu, \lambda) - \mu \cdot \nu - \phi(\nu, \mu) \\ \equiv_4 2p(\widehat{\lambda})p(\nu) + 2p(\widehat{\xi})p(\mu) + 2p(\mu)p(\nu) - \langle \tilde{\nu}, \lambda \rangle + \langle \tilde{\mu}, \zeta \rangle + \mu \cdot \nu \\ \equiv_4 2p(m)p(n) - 2p(\widehat{\lambda})p(\widehat{\xi}) - \langle \tilde{\nu}, \lambda \rangle + \langle \tilde{\mu}, \zeta \rangle + \mu \cdot \nu, \end{aligned}$$

where here \equiv_4 denotes equivalence modulo 4. This finishes the proof. □

We have seen that the twistor map commutes (up to an integral power of \mathbf{t}) with the elementary functions in our graphical calculus. However, note that in Theorem 3.7, the typical composition factor of a tangle invariant is not just one of these maps, but in fact is a tensor product of these maps with various identities. It is important to note that a consequence of Proposition 4.18 is that the twistor maps on tensor products are not local, since the power of \mathbf{t} in the construction depends on the weight and signature of each tensor factor. Nevertheless, we can extend Propositions 4.19 and 4.21 to this more general setting.

Proposition 4.22 *Let M_1, \dots, M_t be \widehat{U} -modules such that for each $1 \leq s \leq t$, $M_s = \widehat{V}(\mu_s)$ for some $\mu_s \in X^+ \cup -X^+$. Let $M = M_1 \widehat{\otimes} \dots \widehat{\otimes} M_t$ and let $c = (c_1, \dots, c_t) = \text{sig}(M)$. For any $\lambda \in X^+$ and $0 \leq r \leq t$, we define $M_{\leq r} = M_1 \widehat{\otimes} \dots \widehat{\otimes} M_r$, $M_{>r} = M_{r+1} \widehat{\otimes} \dots \widehat{\otimes} M_t$, and*

$$M(r, \pm\lambda) = M_{\leq r} \widehat{\otimes} \widehat{V}(\pm\lambda) \widehat{\otimes} \widehat{V}(\mp\lambda) \widehat{\otimes} M_{>r}.$$

- (1) Let $\mathcal{R}_s = 1_{M_{\leq s-1}} \otimes \mathcal{R} \otimes 1_{M_{> s+1}} : M \rightarrow M$ for some $1 \leq s \leq t-1$. Then as maps on M , we have that $\mathfrak{X}\mathcal{R}_s$ and $\mathcal{R}_s\mathfrak{X}$ are proportional up to an integral power of \mathfrak{t} .
- (2) Let $\text{ev}(M, r, \lambda) = 1_{M_{\leq r}} \otimes \text{ev}_\lambda \otimes 1_{M_{> r}}$ for some $1 \leq r \leq t$. Then as maps on $M(r, -\lambda)$, we have that $\mathfrak{X}\text{ev}(M, r, \lambda)$ and $\text{ev}(M, r, \lambda)\mathfrak{X}$ are proportional up to an integral power of \mathfrak{t} .
- (3) Let $\text{qtr}(M, r, \lambda) = 1_{M_{\leq r}} \otimes \text{qtr}_\lambda \otimes 1_{M_{> r}}$ for some $1 \leq r \leq t$. Then as maps on $M(r, \lambda)$, we have that $\mathfrak{X}\text{qtr}(M, r, \lambda)$ and $\text{qtr}(M, r, \lambda)\mathfrak{X}$ are proportional up to an integral power of \mathfrak{t} .
- (4) Let $\text{coev}(M, r, \lambda) = 1_{M_{\leq r}} \otimes \text{coev}_\lambda \otimes 1_{M_{> r}}$ for some $1 \leq r \leq t$. Then as maps on M , we have that $\mathfrak{X}\text{coev}(M, r, \lambda)$ and $\text{coev}(M, r, \lambda)\mathfrak{X}$ are proportional up to an integral power of \mathfrak{t} .
- (5) Let $\text{coqtr}(M, r, \lambda) = 1_{M_{\leq r}} \otimes \text{coqtr}_\lambda \otimes 1_{M_{> r}}$ for some $1 \leq r \leq t$. Then as maps on M , we have that $\mathfrak{X}\text{coqtr}(M, r, \lambda)$ and $\text{coqtr}(M, r, \lambda)\mathfrak{X}$ are proportional up to an integral power of \mathfrak{t} .

Remark 4.23 The precise constants of proportionality can be determined directly as in Propositions 4.19 and 4.21 (and can be worked out from the following proof), but we leave them out of the statement of Proposition 4.22 because they are not particularly illuminating, and are not necessary for Theorem 4.24

Proof As the proofs of (2)–(5) are similar, we shall only prove (1) and (2) here in detail.

We will begin with the proof of (1), which is essentially the same as the proof of Proposition 4.21. To wit, we first observe that for any $a, b \geq 0$ and $\nu \in \mathbb{N}[I]$,

$$\mathfrak{X}(1^{\otimes a} \otimes \Theta_\nu \otimes 1^{\otimes b}) = (\Upsilon_{|\Theta_\nu|})^{\otimes a} \otimes \mathfrak{X}(\Theta_\nu) \otimes (\Upsilon_{|\Theta_\nu|} \tilde{T}_{|\Theta_\nu|})^{\otimes b},$$

and the result follows from the observation that $|\Theta_\nu| = \nu - \nu = 0$. Then $\mathfrak{X}\mathcal{R}_s = (1^{\otimes s-1} \otimes \Theta \otimes 1^{t-s-1})\mathfrak{X}\mathfrak{F}_s\mathfrak{s}_s$. Then we verify directly that

$$\mathfrak{X}\mathfrak{F}_s\mathfrak{s}_s = \mathfrak{t}^{\kappa(c_s+1, c_s)(\hat{\zeta}', \hat{\xi}) - \kappa(c_s, c_{s+1})(\hat{\zeta}, \hat{\xi}') + 2p(\hat{\xi})p(\hat{\zeta}')} \mathfrak{F}_s\mathfrak{s}_s\mathfrak{X},$$

where $\hat{\zeta}$ (resp. $\hat{\zeta}'$) is the coset representative for $(\mu_s, 0)$ (resp. $(\mu_{s+1}, 0)$).

Now, we shall prove (2). Note that an arbitrary element of $M(r, -\lambda)$ is a linear combination of simple tensors of the form $x = m_{\leq r} \otimes (b^- v_\lambda)^* \otimes (b'^- v_\lambda) \otimes m_{> r}$, where $b, b' \in \mathcal{B}$, $m_{\leq r} = m_1 \otimes \dots \otimes m_r \in M_{\leq r}$ and $m_{> r} = m_{r+1} \otimes \dots \otimes m_t \in M_{> r}$; hence we need only prove (1) holds when evaluating both sides at such elements. Since

$ev_\lambda((b^-v_\lambda)^* \otimes (b'^-v_\lambda)) = \delta_{b,b'}$, note that (1) is trivially true when $b \neq b'$, so let's assume $b = b' \in \mathcal{B}_v$. Then

$$ev(M, r, \lambda)\mathfrak{X}(x) = \mathbf{t}^{\diamond(m_1, \dots, m_t) + \clubsuit} \mathfrak{X}(m_{\leq r}) \otimes ev_\lambda \mathfrak{X}((b^-v_\lambda)^* \otimes (b^-v_\lambda)) \otimes \mathfrak{X}(m_{> r}),$$

where we set

$$\begin{aligned} \diamond(m_1, \dots, m_t) &= \sum_{s < r < s'} \kappa_{(c_s, c'_s)}(\|m_s\|, \|m'_s\|), \\ \clubsuit &= \sum_{s < r} (\kappa_{(c_s, -1)}(\|m_s\|, (-\lambda, 0) + \widehat{v}) + \kappa_{(c_s, 1)}(\|m_s\|, (\lambda, 0) - \widehat{v})) \\ &\quad + \sum_{s > r} (\kappa_{(-1, c_s)}((-\lambda, 0) + \widehat{v}, \|m_s\|) + \kappa_{(1, c_s)}((\lambda, 0) - \widehat{v}, \|m_s\|)). \end{aligned}$$

Now \clubsuit can be simplified to

$$\begin{aligned} \clubsuit &= \sum_{s < r} (\kappa_{(c_s, -1)}((\mu_s, 0), (-\lambda, 0)) + \kappa_{(c_s, 1)}((\mu_s, 0), (\lambda, 0))) \\ &\quad + \sum_{s > r} (\kappa_{(-1, c_s)}((-\lambda, 0), (\mu_s, 0)) + \kappa_{(1, c_s)}((\lambda, 0), (\mu_s, 0))). \end{aligned}$$

Note that \clubsuit is independent of x . Then

$$\begin{aligned} ev(M, r, \lambda)\mathfrak{X}(x) &= \mathbf{t}^{\diamond(m_1, \dots, m_t) + \clubsuit} \mathfrak{X}(m_{\leq r}) \otimes ev_\lambda \mathfrak{X}((b^-v_\lambda)^* \otimes (b^-v_\lambda)) \otimes \mathfrak{X}(m_{> r}) \\ &= \mathbf{t}^{\diamond(m_1, \dots, m_t) + \clubsuit + \kappa_{(-1, 1)}((-\lambda, 0), (\lambda, 0))} \mathfrak{X}(m_{\leq r}) \otimes \mathfrak{X}(m_{> r}) \\ &= \mathbf{t}^{\clubsuit + \kappa_{(-1, 1)}((-\lambda, 0), (\lambda, 0))} \mathfrak{X}(m_{\leq r} \otimes m_{> r}). \end{aligned}$$

Since $\mathfrak{X}(m_{\leq r} \otimes m_{> r}) = \mathfrak{X}(ev_\lambda(x))$ and the exponent of \mathbf{t} is independent of x , this completes the proof of (2). □

Theorem 4.24 *Let K be any oriented knot, and let $J_K^\lambda(q, \tau) \in \mathbb{Q}(q, \mathbf{t})^\tau$ be the λ -colored knot invariant defined in Theorem 3.7. Let ${}_{so}J_K^\lambda(q) = J_K^\lambda(q, 1)$ and ${}_{osp}J_K^\lambda(q) = J_K^\lambda(q, \mathbf{t})$. Then*

$${}_{osp}J_K^\lambda(q) = \mathbf{t}^{\star(K, \lambda)} {}_{so}J_K^\lambda(\mathbf{t}^{-1}q)$$

for some $\star(K, \lambda) \in \mathbb{Z}$.

Proof Let $J = J_K^\lambda(q, \tau)$. First, observe that J can be thought of as a function $\mathbb{Q}(q, \mathbf{t})^\tau \rightarrow \mathbb{Q}(q, \mathbf{t})^\tau$, and in that spirit $\mathfrak{X}(J)$ is $\mathfrak{X} \circ J(1)$. On the other hand, $J = W_K \circ S$, where $W_K = (f(\lambda, \lambda)^{-1} \pi^{P(\lambda)} q^{(\rho, \lambda)})^{\text{wr}(K)}$ (interpreted as a function $\mathbb{Q}(q, \mathbf{t})^\tau \rightarrow \mathbb{Q}(q, \mathbf{t})^\tau$) and S is a slice diagram of K interpreted as a composition of morphisms as described in Section 3 (with strands colored by λ). In particular, observe that $\mathfrak{X}(f(\lambda, \lambda)) = \mathbf{t}^x f(\lambda, \lambda)$ for some $x \in \mathbb{Z}$ depending on the coset representative of λ in $X/\mathbb{Z}[I]$, and that $\mathfrak{X}(\pi^{P(\lambda)} q^{(\rho, \lambda)}) = \pi^{P(\lambda)} q^{(\rho, \lambda)}$. Then $\mathfrak{X}W = \mathbf{t}^{-x \text{wr}(K)} W \mathfrak{X}$.

Likewise, note that S can be written as a composition of maps of the form $\text{ev}(M, r, \lambda)$, $\text{coev}(M, r, \lambda)$, $\text{qtr}(M, r, \lambda)$, $\text{coqtr}(M, r, \lambda)$, and $R_s: M \rightarrow M$ for various $r, s \in \mathbb{N}$ with the notation being the same as in [Proposition 4.22](#). In particular, we see that $\mathfrak{X} \circ S = \mathbf{t}^y S \circ \mathfrak{X}$ for some $y \in \mathbb{Z}$, and thus

$$\mathfrak{X}(J) = \mathfrak{X} \circ W_K \circ S(1) = \mathbf{t}^{-x\text{wr}(K)+y} W_K \circ S \circ \mathfrak{X}(1) = \mathbf{t}^{-x\text{wr}(K)+y} J.$$

On the other hand, observe that $\mathfrak{X}(J_K^\lambda(q, \tau)) = J_K^\lambda(\mathbf{t}^{-1}q, \mathbf{t}^{-1}\tau)$, and so

$$\mathbf{t}^{y-x\text{wr}(K)} J_K^\lambda(\mathbf{t}^{-1}q, \mathbf{t}^{-1}\tau) = J_K^\lambda(q, \tau).$$

The theorem follows from specializing $\tau = \mathbf{t}$. □

Remark 4.25 Note that since ${}_{\text{so}}J_K^\lambda(q) \in \mathbb{Z}[q, q^{-1}]$, [Theorem 4.24](#) implies that (after a renormalization) ${}_{\text{osp}}J_K^\lambda(q) = {}_{\text{so}}J_K^\lambda(v) \in \mathbb{Z}[v, v^{-1}]$ where $v = q\mathbf{t}^{-1}$. Furthermore, note that when n or $\langle n, \lambda \rangle$ is even, ${}_{\text{osp}}J_K^\lambda(q) \in \mathbb{Q}(q)$ (see [Remark 2.4](#)); thus in this case, ${}_{\text{osp}}J_K^\lambda(q) \equiv {}_{\text{so}}J_K^\lambda(q) \pmod{2}$.

References

- [1] **Y Akutsu, T Deguchi, T Ohtsuki**, *Invariants of colored links*, J. Knot Theory Ramifications 1 (1992) 161–184 [MR](#)
- [2] **S C Blumen**, *The Birman–Wenzl–Murakami algebra, Hecke algebra and representations of $U_q(\text{osp}(1|2n))$* , preprint (2006) [arXiv](#)
- [3] **S C Blumen**, *On the $U_q(\text{osp}(1|2n))$ and $U_{-q}(\text{so}(2n+1))$ uncolored quantum link invariants*, J. Knot Theory Ramifications 19 (2010) 335–353 [MR](#)
- [4] **S Clark**, *Quantum supergroups, IV: The modified form*, Math. Z. 278 (2014) 493–528 [MR](#)
- [5] **S Clark, Z Fan, Y Li, W Wang**, *Quantum supergroups, III: Twistors*, Comm. Math. Phys. 332 (2014) 415–436 [MR](#)
- [6] **S Clark, D Hill**, *Quantum supergroups, V: Braid group action*, Comm. Math. Phys. 344 (2016) 25–65 [MR](#)
- [7] **S Clark, D Hill, W Wang**, *Quantum supergroups, I: Foundations*, Transform. Groups 18 (2013) 1019–1053 [MR](#)
- [8] **S Clark, D Hill, W Wang**, *Quantum supergroups, II: Canonical basis*, Represent. Theory 18 (2014) 278–309 [MR](#)
- [9] **S Clark, W Wang**, *Canonical basis for quantum $\text{osp}(1|2)$* , Lett. Math. Phys. 103 (2013) 207–231 [MR](#)
- [10] **A P Ellis, M Khovanov, A D Lauda**, *The odd nilHecke algebra and its diagrammatics*, Int. Math. Res. Not. 2014 (2014) 991–1062 [MR](#)

- [11] **A P Ellis, A D Lauda**, *An odd categorification of $U_q(\mathfrak{sl}_2)$* , *Quantum Topol.* 7 (2016) 329–433 [MR](#)
- [12] **N Geer, B Patureau-Mirand**, *Multivariable link invariants arising from Lie superalgebras of type I*, *J. Knot Theory Ramifications* 19 (2010) 93–115 [MR](#)
- [13] **MD Gould, JR Links, Y-Z Zhang**, *Type-I quantum superalgebras, q -supertrace, and two-variable link polynomials*, *J. Math. Phys.* 37 (1996) 987–1003 [MR](#)
- [14] **D Hill, W Wang**, *Categorification of quantum Kac–Moody superalgebras*, *Trans. Amer. Math. Soc.* 367 (2015) 1183–1216 [MR](#)
- [15] **J C Jantzen**, *Lectures on quantum groups*, *Graduate Studies in Mathematics* 6, Amer. Math. Soc., Providence, RI (1996) [MR](#)
- [16] **M Khovanov**, *A categorification of the Jones polynomial*, *Duke Math. J.* 101 (2000) 359–426 [MR](#)
- [17] **M Khovanov, A D Lauda**, *A categorification of quantum $\mathfrak{sl}(n)$* , *Quantum Topol.* 1 (2010) 1–92 [MR](#)
- [18] **G Lusztig**, *Introduction to quantum groups*, *Progress in Mathematics* 110, Birkhäuser, Boston (1993) [MR](#)
- [19] **V Mikhaylov, E Witten**, *Branes and supergroups*, *Comm. Math. Phys.* 340 (2015) 699–832 [MR](#)
- [20] **T Ohtsuki**, *Quantum invariants: a study of knots, 3–manifolds, and their sets*, *Series on Knots and Everything* 29, World Scientific, River Edge, NJ (2002) [MR](#)
- [21] **PS Ozsváth, J Rasmussen, Z Szabó**, *Odd Khovanov homology*, *Algebr. Geom. Topol.* 13 (2013) 1465–1488 [MR](#)
- [22] **H Queffelec, A Sartori**, *Mixed quantum skew Howe duality and link invariants of type A*, preprint (2015) [arXiv](#)
- [23] **NY Reshetikhin, V G Turaev**, *Ribbon graphs and their invariants derived from quantum groups*, *Comm. Math. Phys.* 127 (1990) 1–26 [MR](#)
- [24] **R Rouquier**, *2–Kac–Moody algebras*, preprint (2008) [arXiv](#)
- [25] **A Sartori**, *The Alexander polynomial as quantum invariant of links*, *Ark. Mat.* 53 (2015) 177–202 [MR](#)
- [26] **P Tingley**, *A minus sign that used to annoy me but now I know why it is there*, preprint (2010) [arXiv](#)
- [27] **V G Turaev**, *Operator invariants of tangles, and R -matrices*, *Izv. Akad. Nauk SSSR Ser. Mat.* 53 (1989) 1073–1107 [MR](#) In Russian; translated in *Math. of the USSR–Izvestiya* 35 (1990) 411–444
- [28] **B Webster**, *Knot invariants and higher representation theory*, preprint (2013) [arXiv](#)
To appear in *Mem. Amer. Math. Soc.*

- [29] **RB Zhang**, *Finite-dimensional representations of $U_q(\mathfrak{osp}(1/2n))$ and its connection with quantum $\mathfrak{so}(2n + 1)$* , Lett. Math. Phys. 25 (1992) 317–325 [MR](#)
- [30] **RB Zhang**, *Quantum supergroups and topological invariants of three-manifolds*, Rev. Math. Phys. 7 (1995) 809–831 [MR](#)

Department of Mathematics, Northeastern University
Boston, MA, United States

se.clark@northeastern.edu

<https://web.northeastern.edu/sclark/>

Received: 18 October 2016 Revised: 28 February 2017

