Localization of cofibration categories
and groupoid C^*–algebras

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We prove that relative functors out of a cofibration category are essentially the same as relative functors which are only defined on the subcategory of cofibrations. As an application we give a new construction of the functor that assigns to a groupoid its groupoid C^*–algebra and thereby its topological K–theory spectrum.

Let $(\mathcal{C}, w\mathcal{C}, c\mathcal{C})$ be a *cofibration category*, ie a structure dual to a category of fibrant objects in the sense of Brown [1]. Here, $w\mathcal{C}$ and $c\mathcal{C}$ are the subcategories of weak equivalences and cofibrations, ie they have the same objects as $\mathcal{C}$ but morphisms are the weak equivalences or the cofibrations, respectively. Similarly, $wc\mathcal{C}$ will denote the subcategory of acyclic cofibrations. In addition to Brown's axioms, we will assume that $\mathcal{C}$ has *good cylinders*, which is a mild technical condition explained in Definition 9.

In this paper we will prove the following theorem. It will be formulated using the language of ∞–categories, following the notation of Lurie [11; 12]. In particular, an ordinary category $\mathcal{C}$ can be considered as an ∞–category by taking its nerve $N\mathcal{C}$.

Theorem 1 *If a cofibration category $\mathcal{C}$ has good cylinders, then the map induced by the inclusion*

$$Nc\mathcal{C}[wc^{-1}] \xrightarrow{\simeq} N\mathcal{C}[w^{-1}]$$

is an equivalence of ∞–categories.

By $N\mathcal{C}[w^{-1}]$ we denote the universal ∞–category obtained from $N\mathcal{C}$ by inverting the weak equivalences; see [12, Definition 1.3.4.1 and Remark 1.3.4.2]. By passing to opposite categories, the dual statement of Theorem 1 for fibration categories also holds.

The proof of Theorem 1 will be given at the end of the paper, but let us first establish a consequence and the application to C^*–algebras associated to groupoids.

Let $\mathcal{C}$ be a cofibration category with good cylinders and $\mathcal{M}$ a model category which is Quillen equivalent to a combinatorial model category and has functorial fibrant and cofibrant replacements, eg any of the model categories of spectra.

Proposition 2 *For any functor $F\colon c\mathcal{C} \to \mathcal{M}$ that sends acyclic cofibrations in $c\mathcal{C}$ to weak equivalences in $\mathcal{M}$ there exists a functor $\widehat{F}\colon \mathcal{C} \to \mathcal{M}$ with the following properties:*

(1) *$\widehat{F}$ sends weak equivalences in $\mathcal{C}$ to weak equivalences in $\mathcal{M}$.*

(2) *$\widehat{F}$ extends F in the sense that there exists a zigzag of natural weak equivalences between F and $\widehat{F}\mid_{c\mathcal{C}}$.*

Moreover, $\widehat{F}$ is unique in the following sense: for any other functor $\widehat{F}'\colon \mathcal{C} \to \mathcal{M}$ that satisfies (1) and (2) there exists a zigzag of natural weak equivalences between $\widehat{F}$ and $\widehat{F}'$.

Proof We denote the ∞–category $N\mathcal{M}[w^{-1}]$ associated to the model category $\mathcal{M}$ by $\mathcal{M}_\infty$. We claim that for any ordinary category $\mathcal{A}$ the canonical map

$$N\operatorname{Fun}(\mathcal{A}, \mathcal{M})[\ell^{-1}] \to \operatorname{Fun}(N\mathcal{A}, \mathcal{M}_\infty)$$

is an equivalence of ∞–categories, where ℓ is the class of levelwise weak equivalences. Here $\operatorname{Fun}(-,-)$ is used both for the ordinary category of functors between ordinary categories and the ∞–category of functors between ∞–categories; we hope that it is clear from the context which of the two is meant. If $\mathcal{M}$ is a simplicial, combinatorial model category, this is a special case of [11, Proposition 4.2.4.4], using that for a simplicial model category $\mathcal{M}$, the ∞–category $\mathcal{M}_\infty$ is equivalent to the homotopy coherent nerve of the simplicial subcategory of $\mathcal{M}$ on the fibrant and cofibrant objects; see [12, Theorem 1.3.4.20]. From the existence of functorial (co)fibrant replacements and Hovey [8, Proposition 1.3.13] it follows that a Quillen equivalence $\mathcal{M} \simeq \mathcal{M}'$ induces a Quillen equivalence $\operatorname{Fun}(\mathcal{A}, \mathcal{M}) \simeq \operatorname{Fun}(\mathcal{A}, \mathcal{M}')$. Thus the domain of the map in question is invariant under Quillen equivalences in $\mathcal{M}$. The same is true for the codomain, thus the statement that this map is an equivalence is invariant under Quillen equivalences in $\mathcal{M}$. Hence it is also true for all model categories $\mathcal{M}$ with functorial (co)fibrant replacements that are Quillen equivalent to a combinatorial, simplicial model category. Since every combinatorial model category is equivalent to a combinatorial, simplicial model category by a result of Dugger [6, Corollary 1.2], the claim holds in our generality. If $\mathcal{A}$ is a relative category, it also follows that the induced functor

$$N\operatorname{Fun}^w(\mathcal{A}, \mathcal{M})[\ell^{-1}] \to \operatorname{Fun}^w(N\mathcal{A}, \mathcal{M}_\infty)$$

is an equivalence, where the superscript w refers to functors that send weak equivalences in $\mathcal{A}$ to weak equivalences or equivalences in the target. This follows immediately from the nonrelative case, noting that both sides are just full subcategories of $N\operatorname{Fun}(\mathcal{A}, \mathcal{M})[\ell^{-1}]$ and $\operatorname{Fun}(N\mathcal{A}, \mathcal{M}_\infty)$. Thus in the canonical commuting square

$$N\operatorname{Fun}^w(\mathcal{C},\mathcal{M})[\ell^{-1}] \longrightarrow N\operatorname{Fun}^w(c\mathcal{C},\mathcal{M})[\ell^{-1}]$$

$$\operatorname{Fun}^w(N\mathcal{C},\mathcal{M}_\infty) \longrightarrow \operatorname{Fun}^w(Nc\mathcal{C},\mathcal{M}_\infty)$$

the vertical maps are equivalences of ∞–categories. By Theorem 1 the lower map is also an equivalence, therefore also the upper one is. Passing to homotopy categories we obtain the desired result, using that isomorphisms in homotopy categories of functor categories are represented by zigzags of natural weak equivalences. $\square$

Applications

Groupoids

We denote by Gpd the 1–category of small groupoids and by Gpd_2 the ∞–category associated to the $(2,1)$–category of groupoids in which the 2–morphisms are natural isomorphisms. The category Gpd admits a simplicial model structure in which the equivalences are equivalences of categories and the cofibrations are functors that are injective on the set of objects. In this model structure all objects are cofibrant and fibrant, compare Casacuberta, Golasiński and Tonks [2]. Furthermore, if we denote by $\operatorname{Gpd}^\omega$ the full subcategory on groupoids with at most countable many morphisms then $\operatorname{Gpd}^\omega$ inherits the structure of a cofibration category.

The following lemma is a well-known fact, but we had difficulties finding a clear reference for this so we state it as an extra lemma.

Lemma 3 *The canonical map $N\operatorname{Gpd}[w^{-1}] \to \operatorname{Gpd}_2$ is an equivalence of ∞–categories.*

Proof This follows from the description of the ∞–category associated to a simplicial model category — see [12, Theorem 1.3.4.20] — as being the homotopy coherent nerve of the simplicial category of cofibrant and fibrant objects. $\square$

Corollary 4 *Let $\mathcal{C}$ be an ∞–category. Then the canonical map $Nc\operatorname{Gpd} \to \operatorname{Grp}_2$ induces an equivalence*

$$\operatorname{Fun}(\operatorname{Gpd}_2, \mathcal{C}) \xrightarrow{\simeq} \operatorname{Fun}^w(Nc\operatorname{Gpd}, \mathcal{C}),$$

where the superscript w refers to functors that send equivalences of groupoids to equivalences in $\mathcal{C}$.

Proof Since the canonical map $N\operatorname{Gpd}[w^{-1}] \to \operatorname{Gpd}_2$ is an equivalence by Lemma 3, this is a direct application of Theorem 1. $\square$

The following corollary of Proposition 2 implies that in the approach to assembly maps discussed by Davis and Lück [4, Section 2], one can directly restrict to functors from groupoids to spectra that are only defined for maps of groupoids that are injective on objects. This resolves the issues illustrated in [4, Remark 2.3].

Corollary 5 *Let* Sp *be any of the standard model categories of spectra. Then every functor* $F \colon c\mathrm{Gpd} \to \mathrm{Sp}$ *which sends equivalences of groupoids to weak equivalences in* Sp *extends uniquely* (*in the sense of Proposition 2*) *to a functor* $\hat{F} \colon \mathrm{Gpd} \to \mathrm{Sp}$ *which also sends weak equivalences of groupoids to weak equivalences of spectra.*

Remark The statements of Corollaries 4 and 5 remain true if we replace Gpd by Gpd^ω. Furthermore, Corollary 5 does not depend on the exact choice of model category of spectra as long as it is Quillen equivalent to a combinatorial model category. Notice that this is automatically fulfilled if the model category is stable, due to the rigidity result of Schwede; see [17].

Next we want to demonstrate how to apply these results by *functorially* constructing C^*–algebras and topological K–theory spectra associated to groupoids. This discussion is similar to the one given by Joachim [9, Section 3] but we use our main theorem to obtain full functoriality instead of an explicit construction.

Definition 6 Let $\mathcal{G}$ be a groupoid. We let $\mathbb{C}\mathcal{G}$ be the $\mathbb{C}$–linearization of the set of morphisms of $\mathcal{G}$. This is a $\mathbb{C}$–algebra by linearization of the multiplication on morphisms given by

$$f \cdot g = \begin{cases} f \circ g & \text{if } f \text{ and } g \text{ are composable,} \\ 0 & \text{otherwise.} \end{cases}$$

We remark that $\mathbb{C}\mathcal{G}$ is unital if and only if the set of objects of $\mathcal{G}$ is finite. Then we complete $\mathbb{C}\mathcal{G}$ in a universal way, like for the full group C^*–algebra, to obtain a C^*–algebra $C^*\mathcal{G}$. More precisely, the norm is given by the supremum over all norms of representations of $\mathbb{C}\mathcal{G}$ on a separable Hilbert space. This is isomorphic to the C^*–algebra associated to the maximal groupoid C^*–category of Dell'Ambrogio [5, Definition 3.16] using the construction $\mathcal{C} \mapsto A_\mathcal{C}$ of Joachim [9, Section 3].

The association $\mathcal{G} \mapsto C^*\mathcal{G}$ is functorial for cofibrations of groupoids but not for general morphisms, since it can happen that morphisms are not composable in a groupoid, but become composable after applying a functor; compare the remark in Davis and Lück [4, page 214]. We observe that the C^*–algebra $C^*\mathcal{G}$ is separable provided $\mathcal{G} \in \mathrm{Gpd}^\omega$.

Lemma 7 *Let $F\colon \mathcal{G}_1 \to \mathcal{G}_2$ be an acyclic cofibration of groupoids. Then the induced morphism*

$$C^*F\colon C^*\mathcal{G}_1 \to C^*\mathcal{G}_2$$

is a KK–equivalence.

Proof The C^*–algebra associated to a groupoid with finitely many connected components is the product of the C^*–algebras associated to each connected component. For an infinite number of components, the associated C^*–algebra is the filtered colimit of the C^*–algebras associated to finitely many connected components. Since finite products of KK–equivalences are again KK–equivalences, and the filtered colimit of these KK–equivalences is again a KK–equivalence, we may assume that $\mathcal{G}_1$ (and thus $\mathcal{G}_2$) is connected. Let $x \in \mathcal{G}$ be an object. We let $G_1 = \mathrm{End}(x)$ and $G_2 = \mathrm{End}(Fx)$ be the endomorphism groups and notice the fact that F is an equivalence implies that F induces an isomorphism $G_1 \cong G_2$. Then we consider the diagram

$$
\begin{array}{ccc}
C^*\mathcal{G}_1 & \xrightarrow{\;C^*F\;} & C^*\mathcal{G}_2 \\[2pt]
\big\uparrow & & \big\uparrow \\[2pt]
C^*G_1 & \xrightarrow{\;\cong\;} & C^*G_2
\end{array}
$$

in which the lower horizontal arrow is an isomorphism. Thus to show the lemma it suffices to prove the lemma in the special case where F is the inclusion of the endomorphisms of an object x of a connected groupoid $\mathcal{G}$.

This can be done in the abstract setting of corner algebras. For this suppose A is a C^*–algebra and $p \in A$ is a projection. It is called full if ApA is dense in A. The algebra pAp is called the corner algebra of p in A. It is called a full corner if p is a full projection. We write i_p for the inclusion $pAp \subset A$. Given a projection p the module pA is an imprimitivity $pAp - \overline{ApA}$ bimodule; see eg [15, Example 3.6]. Thus if p is full, then pA gives rise to an invertible element $[pA, i_p, 0] = \mathcal{F}(p) \in \mathrm{KK}(pAp, A)$. In this KK–group we have an equality

$$\mathcal{F}(p) = [\,pA, i_p, 0\,] + [(1-p)A, 0, 0] = [\,pA \oplus (1-p)A, i_p, 0\,] = [A, i_p, 0] = [i_p];$$

in other words, the inclusion $pAp \to A$ of a corner algebra associated to a full projection is a KK–equivalence.

To come back to our situation let us suppose $\mathcal{G}$ is a groupoid, $x \in \mathcal{G}$ is an object and let us denote its endomorphism group by $G = \mathrm{End}(x)$. We can consider the element $p = \mathrm{id}_x \in C^*\mathcal{G}$, which is clearly a projection. Its corner algebra is given by

$$p \cdot C^*\mathcal{G} \cdot p \cong C^*G.$$

If $\mathcal{G}$ is connected, it follows that every morphism in $\mathcal{G}$ may be factored through id_x and thus p is full. Hence it follows that the inclusion $C^*G \to C^*\mathcal{G}$ is an embedding of a full corner algebra. Thus, by the general theory, this inclusion is a KK–equivalence, which proves the lemma. $\square$

Let us denote by KK_∞ the ∞–category given by the localization of the category $C^*\mathrm{Alg}$ of separable C^*–algebras at the KK–equivalences; see eg [10, Definition 3.2]. In formulas we have $\mathrm{KK}_\infty := NC^*\mathrm{Alg}[w^{-1}]$, where w denotes the class of KK–equivalences. The homotopy category of KK_∞ is Kasparov's KK–category of C^*–algebras.

Corollary 8 *There exists a functor*

$$\mathrm{Gpd}_2^\omega \to \mathrm{KK}_\infty$$

which on objects sends a groupoid $\mathcal{G}$ to the full groupoid C^*–algebra $C^*\mathcal{G}$.

Remark The $(2, 1)$–category Orb^ω consisting of (countable) groups, group homomorphisms and conjugations is the full subcategory of the $(2, 1)$–category of (countable) groupoids on connected groupoids and hence along this inclusion we also obtain a functor

$$\mathrm{Orb}^\omega \to \mathrm{KK}_\infty$$

which on objects sends a group to its full group C^*–algebra. This will be used by the first two authors in [10] to compare the L–theoretic Farrell–Jones conjecture and the Baum–Connes conjecture.

Proof of Corollary 8 By Corollary 4 and the remark after Corollary 5, we have an equivalence

$$\mathrm{Fun}^w(Nc\mathrm{Gpd}^\omega, \mathrm{KK}_\infty) \simeq \mathrm{Fun}(\mathrm{Gpd}_2^\omega, \mathrm{KK}_\infty),$$

and thus it suffices to construct a functor

$$c\mathrm{Gpd}^\omega \to C^*\mathrm{Alg}$$

which has the *property* that it sends equivalences of groupoids to KK–equivalences. We have established in Lemma 7 that the functor of Definition 6 satisfies this property. $\square$

Remark In [10, Proposition 3.7] it is shown that the topological K–theory functor

$$K \colon NC^*\mathrm{Alg} \to \mathrm{Sp}$$

factors over KK_∞, in fact becomes corepresentable there. It thus follows from Corollary 8 that there is a functor sending a groupoid to the topological K–theory spectrum of its C^*–algebra.

The proof of Theorem 1

In this section we will prove Theorem 1. Recall that we consider a cofibration category $(\mathcal{C}, w\mathcal{C}, c\mathcal{C})$ and aim to compare the ∞–categories associated to the relative categories $(\mathcal{C}, w\mathcal{C})$ and $(c\mathcal{C}, wc\mathcal{C})$. As our model of the homotopy theory of $(\infty, 1)$–categories we will use the *complete Segal spaces* of Rezk; see [16]. This homotopy theory is modelled by the Rezk model structure on the category of bisimplicial sets in which fibrant objects are the complete Segal spaces. The model structure is constructed as a Bousfield localization of the Reedy model structure and hence every levelwise weak equivalence of bisimplicial sets is a Rezk equivalence, ie an equivalence of ∞–categories.

The ∞–category associated to a relative category $(\mathcal{D}, w\mathcal{D})$ is modelled by the *classification diagram* $N^R\mathcal{D}$ of Rezk, which is given by

$$(N^R\mathcal{D})_k \mapsto Nw(\mathcal{D}^{[k]}),$$

where the weak equivalences in $\mathcal{D}^{[k]}$ are levelwise weak equivalences; compare [16, Section 3.3; 13, Theorem 3.8]. See also Cisinski's response in [3]. Here, again, the notation N refers to the nerve of a category, which is a simplicial set, and here it should be thought of as a homotopy type as opposed to an ∞–category. The classification diagram is not fibrant in the Rezk model structure, but it is levelwise equivalent to a fibrant object if $\mathcal{D}$ is a cofibration category.

Let X be an object of a cofibration category $\mathcal{C}$. Recall that a *cylinder* on X is a factorization of the canonical morphism $X \sqcup X \to X$ via a cofibration $X \sqcup X \to IX$ and a weak equivalence $IX \to X$. A *cylinder functor* on $\mathcal{C}$ is a functor $I\colon \mathcal{C} \to \mathcal{C}$ equipped with natural transformations that provide such factorizations for all objects of $\mathcal{C}$. In the introduction we stated Theorem 1 under the following assumption on $\mathcal{C}$.

Definition 9 A cofibration category $\mathcal{C}$ has *good cylinders* if it has a cylinder functor I such that for every cofibration $X \rightarrowtail Y$ the induced morphism $IX \sqcup_{X \sqcup X} (Y \sqcup Y) \to IY$ is a cofibration.

For example, any cofibration category arising from a monoidal model category (or a model category enriched over a monoidal model category) has good cylinders, since they are given by tensoring with a chosen interval object. In particular the cofibration category underlying the model category of groupoids we discussed has good cylinders.

Theorem 10 *If $\mathcal{C}$ has good cylinders, then the inclusion $c\mathcal{C} \to \mathcal{C}$ induces a levelwise weak equivalence of the classification diagrams $N^R c\mathcal{C} \to N^R \mathcal{C}$.*

For the proof we will need a series of auxiliary definitions and lemmas. Let us first fix some notation. If J is a category, then $\widehat{J}$ denotes J considered as a relative category with all morphisms as weak equivalences. If J is any relative category, then $\mathcal{C}^J$ stands for the cofibration category of all relative diagrams $J \to \mathcal{C}$ with levelwise weak equivalences and cofibrations. If J is any relative direct category, then $\mathcal{C}^J_R$ stands for the cofibration category of all relative Reedy cofibrant diagrams $J \to \mathcal{C}$ with levelwise weak equivalences and Reedy cofibrations. See [14, Theorem 9.3.8] for the construction of these cofibration categories and [14, Sections 9.1 and 9.2] for definitions of (relative) direct categories and Reedy cofibrations. (Note that Radulescu-Banu [14] uses the word "restricted" instead of "relative".) For our purposes we only need the direct category $J = [k]$, so we will recall the definitions just in this case. A diagram over $[k]$ is Reedy cofibrant if all its structure maps are cofibrations. A morphism $X \to Y$ of such diagrams is a Reedy cofibration if all the induced morphisms $X_{i+1} \sqcup_{X_i} Y_i \to Y_{i+1}$ are cofibrations. In [14] cofibration categories are assumed to have certain infinite colimits that are necessary for these results to hold for arbitrary J. However, as mentioned above, we will only use finite categories $J = [k]$, in which case the cited theorem is valid with Brown's original definition, which asserts only existence of an initial object and pushouts along cofibrations.

Definition 11 A subcategory $g\mathcal{C}$ of a cofibration category $\mathcal{C}$ is said to be *good* if

- all cofibrations are in $g\mathcal{C}$;
- the morphisms of $g\mathcal{C}$ are stable under pushouts along cofibrations;
- $\mathcal{C}$ has functorial factorizations that preserve $g\mathcal{C}$, in the sense that if

$$
\begin{array}{ccc}
A_0 & \longrightarrow & B_0 \\
\downarrow & & \downarrow \\
A_1 & \longrightarrow & B_1
\end{array}
$$

  is a square in $\mathcal{C}$ such that both vertical morphisms are in $g\mathcal{C}$ and

$$
\begin{array}{ccccc}
A_0 & \rightarrowtail & \widetilde{B}_0 & \overset{\sim}{\longrightarrow} & B_0 \\
\downarrow & & \downarrow & & \downarrow \\
A_1 & \rightarrowtail & \widetilde{B}_1 & \overset{\sim}{\longrightarrow} & B_1
\end{array}
$$

  is the resulting factorization, then the induced morphism $A_1 \sqcup_{A_0} \widetilde{B}_0 \to \widetilde{B}_1$ is also in $g\mathcal{C}$. In particular, so is $\widetilde{B}_0 \to \widetilde{B}_1$ by the second condition and most of the time only this conclusion will be used. However, the stronger property that $A_1 \sqcup_{A_0} \widetilde{B}_0 \to \widetilde{B}_1$ is in $g\mathcal{C}$ is necessary for the inductive argument in the proof of Lemma 15(3).

Now suppose that $\mathcal{C}$ is a cofibration category with a good subcategory $g\mathcal{C}$. We let $W\mathcal{C}$ be the bisimplicial set whose (m, n)–bisimplices are all diagrams in $\mathcal{C}$ of the form

ie relative diagrams $\widehat{[m]} \times \widehat{[n]} \to \mathcal{C}$ where all horizontal morphisms are cofibrations and all vertical morphisms are in $g\mathcal{C}$. In other words, $W\mathcal{C}$ is the nerve of a double category with the same objects as $\mathcal{C}$, whose horizontal morphisms are acyclic cofibrations, vertical morphisms are weak equivalences in $g\mathcal{C}$, and double morphisms are just commutative squares.

Lemma 12 *The bisimplicial set $W\mathcal{C}$ is vertically homotopically constant, ie every simplicial operator $[n] \to [n']$ induces a weak homotopy equivalence $(W\mathcal{C})_{*,n'} \to (W\mathcal{C})_{*,n}$.*

Proof Note that $(W\mathcal{C})_{*,n} = N\widetilde{\mathcal{C}}_n$, where $\widetilde{\mathcal{C}}_n$ is a category whose objects are diagrams $\widehat{[n]} \to c\mathcal{C}$ and whose morphisms are weak equivalences with all components in $g\mathcal{C}$. It is enough to consider the case $n' = 0$, ie to show that the constant functor const: $\widetilde{\mathcal{C}}_0 \to \widetilde{\mathcal{C}}_n$ is a homotopy equivalence. The evaluation at n functor $\mathrm{ev}_n \colon \widetilde{\mathcal{C}}_n \to \widetilde{\mathcal{C}}_0$ satisfies $\mathrm{ev}_n \, \mathrm{const} = \mathrm{id}_{\widetilde{\mathcal{C}}_0}$. Moreover, the structure maps of every diagram $X \in \widetilde{\mathcal{C}}_n$ form a natural weak equivalence $X \to \mathrm{const} \, \mathrm{ev}_n \, X$ since every cofibration is in $g\mathcal{C}$. □

Lemma 13 *The bisimplicial set $W\mathcal{C}$ is horizontally homotopically constant, ie every simplicial operator $[m] \to [m']$ induces a weak homotopy equivalence $(W\mathcal{C})_{m',*} \to (W\mathcal{C})_{m,*}$.*

Proof Note that $(W\mathcal{C})_{m,*} = N\bar{\mathcal{C}}_m$, where $\bar{\mathcal{C}}_m$ is a category whose objects are diagrams $\widehat{[m]} \to g\mathcal{C}$ and whose morphisms are acyclic levelwise cofibrations. Again, it is enough to consider the case $m' = 0$ and to show that the constant functor const: $\bar{\mathcal{C}}_0 \to \bar{\mathcal{C}}_m$ and the evaluation at m functor $\mathrm{ev}_m \colon \bar{\mathcal{C}}_n \to \bar{\mathcal{C}}_0$ form a homotopy equivalence.

We have $\mathrm{ev}_m \, \mathrm{const} = \mathrm{id}_{\bar{\mathcal{C}}_0}$. Moreover, given any object $X \in \bar{\mathcal{C}}_m$ and $i \in [m]$ we consider the composite weak equivalence $X_i \xrightarrow{\sim} X_m$. We combine it with the identity $X_m \to X_m$ and factor functorially the resulting morphism $X_i \sqcup X_m \to X_m$ as $X_i \sqcup X_m \rightarrowtail \widetilde{X}_i \xrightarrow{\sim} X_m$. In the square

$$
\begin{array}{ccc}
X_m \sqcup X_i & \longrightarrow & X_m \\
\downarrow & & \downarrow \\
X_m \sqcup X_{i+1} & \longrightarrow & X_m
\end{array}
$$

both vertical morphisms are in $g\mathcal{C}$ (since $g\mathcal{C}$ is closed under pushouts). Thus the induced morphism $\widetilde{X}_i \to \widetilde{X}_{i+1}$ is in $g\mathcal{C}$. Moreover, we obtain acyclic cofibrations $X_i \overset{\sim}{\rightarrowtail} \widetilde{X}_i$ and $X_m \overset{\sim}{\rightarrowtail} \widetilde{X}_i$ that constitute a zigzag of natural weak equivalences connecting $\mathrm{const} \, \mathrm{ev}_m$ and $\mathrm{id}_{\bar{\mathcal{C}}_m}$. $\qquad\square$

Lemma 14 *The inclusion $Nwc\mathcal{C} \to Nwg\mathcal{C}$ is a weak homotopy equivalence.*

Proof Observe that the 0th row and the 0th column of $W\mathcal{C}$ are $Nwg\mathcal{C}$ and $Nwc\mathcal{C}$, respectively. Since $W\mathcal{C}$ is homotopically constant in both directions, it follows from [7, Proposition IV.1.7] that we have weak equivalences

$$
Nwg\mathcal{C} \xrightarrow{\sim} \mathrm{diag} \, W\mathcal{C} \xleftarrow{\sim} Nwc\mathcal{C}.
$$

Moreover, the restrictions along the diagonal inclusions $[m] \to [m] \times [m]$ induce a simplicial map $\mathrm{diag} \, W\mathcal{C} \to Nwg\mathcal{C}$ whose composites with the two maps above are the identity on $Nwg\mathcal{C}$ and the inclusion $Nwc\mathcal{C} \to Nwg\mathcal{C}$. Hence the latter is a weak equivalence by 2-out-of-3. $\qquad\square$

Next we establish that under specific circumstances certain subcategories of $\mathcal{C}$ are good.

Lemma 15 *Let $\mathcal{C}$ be a cofibration category.*

(1) *If $\mathcal{C}$ has functorial factorizations, then $\mathcal{C}$ itself is a good subcategory.*

(2) *If $\mathcal{C}$ has good cylinders, then $c\mathcal{C}$ is a good subcategory of $\mathcal{C}$.*

(3) *If $c\mathcal{C}$ is a good subcategory of $\mathcal{C}$, then the subcategory of levelwise cofibrations is a good subcategory of $\mathcal{C}_{\mathrm{R}}^{[k]}$ for all k.*

Proof (1) This is vacuously true.

(2) We will show that the standard mapping cylinder factorization makes $c\mathcal{C}$ into a good subcategory. Let

$$
\begin{array}{ccc}
A_0 & \longrightarrow & B_0 \\
\downarrow & & \downarrow \\
A_1 & \longrightarrow & B_1
\end{array}
$$

be a square where both vertical morphisms are cofibrations. The mapping cylinder of $A_i \to B_i$ is constructed as $IA_i \sqcup_{A_i \sqcup A_i} (A_i \sqcup B_i)$. We need to show that the morphism induced by the square

$$
\begin{array}{ccc}
A_0 & \longrightarrow & IA_0 \sqcup_{A_0 \sqcup A_0} (A_0 \sqcup B_0) \\
\downarrow & & \downarrow \\
A_1 & \longrightarrow & IA_1 \sqcup_{A_1 \sqcup A_1} (A_1 \sqcup B_1)
\end{array}
$$

is a cofibration. This morphism coincides with

$$
IA_0 \sqcup_{A_0 \sqcup A_0} (A_1 \sqcup B_0) \to IA_1 \sqcup_{A_1 \sqcup A_1} (A_1 \sqcup B_1),
$$

which factors as

$$
IA_0 \sqcup_{A_0 \sqcup A_0} (A_1 \sqcup B_0) \to IA_0 \sqcup_{A_0 \sqcup A_0} (A_1 \sqcup B_1) \to IA_1 \sqcup_{A_1 \sqcup A_1} (A_1 \sqcup B_1).
$$

The first morphism is a pushout of $A_1 \sqcup B_0 \to A_1 \sqcup B_1$, which is a cofibration since $B_0 \to B_1$ is. Comparing the pushouts of rows and columns in the diagram

$$
\begin{array}{ccccc}
IA_0 & \longleftarrow & IA_0 & \longrightarrow & IA_1 \\
\uparrow & & \uparrow & & \uparrow \\
A_0 \sqcup A_0 & \longleftarrow & A_0 \sqcup A_0 & \longrightarrow & A_1 \sqcup A_1 \\
\downarrow & & \downarrow & & \downarrow \\
A_1 \sqcup B_1 & \longleftarrow & A_1 \sqcup A_1 & \longrightarrow & A_1 \sqcup A_1
\end{array}
$$

shows that the second morphism above is a pushout of $IA_0 \sqcup_{A_0 \sqcup A_0} (A_1 \sqcup A_1) \to IA_1$, which is a cofibration since $A_0 \to A_1$ is and $\mathcal{C}$ has good cylinders.

(3) Clearly, every Reedy cofibration is a levelwise cofibration and levelwise cofibrations are stable under pushouts. Consider a diagram

$$
\begin{array}{ccccc}
A_0 & \rightarrowtail & \widetilde{B}_0 & \xrightarrow{\sim} & B_0 \\
\downarrow & & \downarrow & & \downarrow \\
A_1 & \rightarrowtail & \widetilde{B}_1 & \xrightarrow{\sim} & B_1
\end{array}
$$

in $\mathcal{C}_{\mathrm{R}}^J$, where $\widetilde{B}_0$ and $\widetilde{B}_1$ are obtained by the standard Reedy factorization (see [14, Proof of Theorem 9.2.4(v)]) induced by the given functorial factorization in $\mathcal{C}$. Assuming that $A_0 \to A_1$ and $B_0 \to B_1$ are levelwise cofibrations, we need to check that $A_{1,i} \sqcup_{A_{0,i}} \widetilde{B}_{0,i} \to \widetilde{B}_{1,i}$ is a cofibration for every $i \in [m]$.

For $i = 0$, this follows directly from the assumption that $c\mathcal{C}$ is a good subcategory of $\mathcal{C}$. The Reedy factorization is constructed by induction over $[m]$, so assume that the conclusion is already known for $i < m$. The factorization at level $i + 1$ arises as

$$
\begin{array}{ccccc}
A_{0,i+1} \sqcup_{A_{0,i}} \widetilde{B}_{0,i} & \rightarrowtail & \widetilde{B}_{0,i+1} & \xrightarrow{\sim} & B_{0,i+1} \\
\downarrow & & \downarrow & & \downarrow \\
A_{1,i+1} \sqcup_{A_{1,i}} \widetilde{B}_{1,i} & \rightarrowtail & \widetilde{B}_{1,i+1} & \xrightarrow{\sim} & B_{1,i+1}
\end{array}
$$

where the left square comes from the diagram
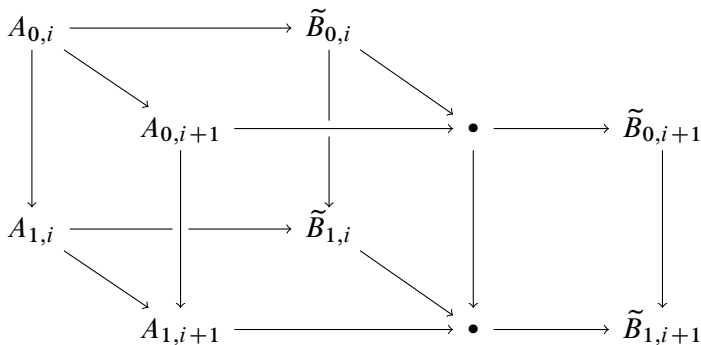
where the bullets stand for the pushouts above. The conclusion we need to obtain amounts to the composite of the two squares in the front being a Reedy cofibration when seen as a morphism from left to right. The right square is a Reedy cofibration since $c\mathcal{C}$ is a good subcategory of $\mathcal{C}$ and so is the left one since it is a pushout of the back square, which is a Reedy cofibration by the inductive hypothesis. $\square$

Lemma 16 *The inclusion $Nw(\mathcal{C}_{\mathrm{R}}^{[k]}) \to Nw(\mathcal{C}^{[k]})$ is a weak homotopy equivalence.*

Proof Functorial factorization induces a functor in the opposite direction as well as natural weak equivalences connecting both composites with identities. $\square$

Proof of Theorem 10 Recall that we want to show that $Nw((c\mathcal{C})^{[k]}) \to Nw(\mathcal{C}^{[k]})$ is a weak equivalence for all k. In the diagram

$$\begin{array}{ccc}
& Nwc(\mathcal{C}_{\mathrm{R}}^{[k]}) & \xrightarrow{\ (1)\ } & Nw(\mathcal{C}_{\mathrm{R}}^{[k]}) \\
{\scriptstyle(3)}\ \swarrow & \downarrow & & \downarrow\ {\scriptstyle(4)} \\
Nw((c\mathcal{C})^{[k]}) \longrightarrow & Nwc(\mathcal{C}^{[k]}) & \xrightarrow{\ (2)\ } & Nw(\mathcal{C}^{[k]})
\end{array}$$

the labelled maps are weak equivalences. The map (1) is a weak equivalence by Lemma 14 applied to $\mathcal{C}_{\mathrm{R}}^{[k]}$ with itself as a good subcategory and so is the map (2) by the same argument applied to $\mathcal{C}^{[k]}$. The map (3) is a weak equivalence by Lemma 14 applied to $\mathcal{C}_{\mathrm{R}}^{[k]}$ with the good subcategory of levelwise cofibrations, which is indeed good by Lemma 15. Finally, the map (4) is a weak equivalence by Lemma 16. Hence by 2-out-of-3, the bottom composite is also a weak equivalence as required.   □

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