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Width of a satellite knot and its companion

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In this paper, we give a proof of a conjecture which says that $w(K) \geqslant n^2 w(J)$, where $w(\cdot)$ is the width of a knot, K is a satellite knot with J as its companion, and n is the winding number of the pattern. We also show that equality holds if K is a satellite knot with braid pattern.

57M25, 57M27

1 Introduction

Width is an important invariant of knots which was introduced by Gabai [2]. It gives rise to the notion of thin position (of knots), which is essentially used in Gabai's proof of property R [2] and Gordon and Luecke's proof of the knot complement conjecture [3], among others. We can view width as a kind of refinement of bridge number. It is an interesting question how those knot invariants behave under the operations of connected sum and taking satellites. For the bridge number, we know that $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$ and $b(K) \geqslant nb(J)$, given that K is a satellite knot with companion J and n is the wrapping number; see Schubert [6] and Schultens [7]. In the case of width, it is known that

$$w(K_1) + w(K_2) - 2 \geqslant w(K_1 \# K_2) \geqslant \max\{w(K_1), w(K_2)\},$$

and both bounds are tight. See Blair and Tomova [1], Rieck and Sedgwick [4] and Scharlemann and Schultens [5]. For width under taking satellites, it is conjectured that $w(K) \geqslant n^2 w(J)$, where n is the wrapping number, similar to the case of the bridge number. There is also a weak version conjecturing that $w(K) \geqslant n^2 w(J)$, where n is the winding number instead of the wrapping number. Zupan [8; 9] proves that $w(K) \geqslant 8n^2$, where n is the winding number, and $w(K) = q^2 w(J)$, where K is a (p,q)–cable knot with companion J and q acts as the winding number. Both of his results give partial positive answers to the weak version. In this paper, we give a complete positive answer to the weak version involving the winding number.

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Theorem 1.1 *Let K be a satellite knot with companion J, and suppose the winding number of the pattern is n. Then*

$$w(K) \geqslant n^2 w(J).$$

In Section 2, we introduce some basic concepts and construct a graph associated to the neighborhood of the companion; in Section 3, we prove that there is a simple loop in this graph, and such a loop is unique; in Section 4, we associate each knot with a word in Z_2, the free monoid of rank 2, and then use it to help calculate the width.

2 Preliminaries

First we introduce some basic definitions.

Definition 2.1 Suppose $\widehat{V}$ is a standard solid torus in S^3, and $\widehat{k}$ is a knot in $\operatorname{int}(\widehat{V})$ such that $\widehat{k}$ is not contained in any 3–ball $B \subset \widehat{V}$. Let $j \subset S^3$ be a nontrivial knot and let $V = N(j)$ be the closure of a tubular neighborhood of j in S^3. Let $f\colon \widehat{V} \to S^3$ be an embedding such that $f(\widehat{V}) = V$, and let $k = f(\widehat{k})$. Then k is called a *satellite knot* with *companion* j and *pattern* $\widehat{k}$. The *winding number* (of the pattern) is defined to be the algebraic intersection number (up to a sign) of the pattern with a meridian disk. Furthermore, if K (or J, $\widehat{K}$) is the knot type represented by k (or j, $\widehat{k}$), we could say that K is a satellite knot with companion J and pattern $\widehat{K}$ without ambiguity.

Let $\pi\colon \mathbb{R}^4 \to \mathbb{R}$ be the projection $(x_1, x_2, x_3, x_4) \mapsto x_4$, and regard S^3 as the unit sphere in $\mathbb{R}^4$. In the rest of this paper, we always assume that $h = \pi|_{S^3}$. Then h is a Morse function on S^3 with exactly two critical points. These two critical points are $h^{-1}(1)$ and $h^{-1}(-1)$, and we call them infinite (critical) points. For each $r \in (-1, 1)$, $h^{-1}(r)$ is obviously a 2–sphere, which is called a level sphere.

Definition 2.2 Let K be a knot type and let $\mathcal{K}$ be the set of all knots $k \in K$ such that

- k does not contain the two infinite points,
- $h|_k$ is Morse, and
- the critical points of $h|_k$ are in distinct levels.

For each $k \in \mathcal{K}$, suppose all the critical values of $h|_k$ are $c_1 < c_2 < \cdots < c_m$. Choose regular values $r_1, r_2, \ldots, r_{m-1}$ such that $c_i < r_i < c_{i+1}$ for $i = 1, 2, \ldots, m-1$, and

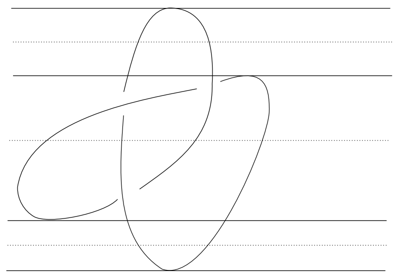

Figure 1: The width of the trefoil is 8.

let $\omega_i(k) = |k \cap h^{-1}(r_i)|$. Define

$$w(k) = \sum_{i=1}^{m-1} \omega_i(k) \quad \text{and} \quad w(K) = \min_{k \in \mathcal{K}} w(k).$$

We call $w(K)$ the *width* of the knot type K. See Figure 1 for the width of trefoil.

In this paper, we focus on "nice" solid tori defined as follows.

Definition 2.3 Let V be a solid torus in S^3. We say that V is *nice* if

- V does not contain the two infinite points,
- $h|_{\partial V}$ is also a Morse function, and
- all critical points of $h|_{\partial V}$ are in distinct levels.

If a solid torus contains a infinity point in its interior, then the first condition can be achieved by "digging them out"; ie pick an arc connecting the infinity point and a point on the boundary ∂V (disjoint from k, j), and remove a tubular neighborhood of the arc. Then we can modify the new boundary to satisfy the other two conditions and get a nice solid torus.

Definition 2.4 Let V be a nice solid torus in S^3. We construct a graph as follows. Let $c_1 < c_2 < \cdots < c_m$ be all critical values of $h|_{\partial V}$, and let $M = V - \left(\bigcup_{i=1}^m h^{-1}(c_i)\right)$. Then vertices of the graph correspond to connected components of M and edges correspond to components of $h^{-1}(c_i) \cap V$ for $i = 1, 2, \ldots, m$, which are not points. We require that two vertices v_1 and v_2 are connected by an edge if and only if the two corresponding components of M are separated by a component of $h^{-1}(c_i) \cap V$ which corresponds to the edge. We call this graph the *Reeb graph* and denote it by $\Gamma(V)$.

We need some results from [5]. In general, critical points of $h|_{\partial V}$ are classified as maximal, minimal and saddle points. Following [5], maximal (or minimal) points can be further divided into external and internal maximal (or minimal) points; saddle points can be divided into nested and unnested saddle points. We don't want to introduce the detailed definition here as they are not important for the use of this paper. Now the connectivity graph studied in [5] can be defined as follows.

Definition 2.5 Suppose $c_1', \dots, c_l'$ are all critical values of $h|_{\partial V}$ corresponding to all external maximal, external minimal and unnested saddle points, and let $M' = V - \left(\bigcup_{i=1}^{l} h^{-1}(c_i')\right)$. If we carry out the construction in Definition 2.4 using M' and c_i', then the graph we get is called the *connectivity graph*, and is denoted by $\Gamma_C(V)$.

We can see from the definition that M is obtained from M' by cutting off those critical levels containing internal maximal, internal minimal and nested saddle points. Then a component of M' is either unchanged or is cut off into a few components. For graphs, this corresponds to the fact that a vertex of $\Gamma_C(V)$ is either unchanged or replaced by some other graph which can be easily seen to be connected. Hence we have the following lemma:

Lemma 2.6 *Let V be a nice solid torus. If the connectivity graph $\Gamma_C(V)$ contains a loop, then Reeb graph $\Gamma(V)$ also has a loop.*

In this paper, we also need following results.

Lemma 2.7 [5] *For any knots K_1, K_2, we have $w(K_1 \# K_2) \geqslant \max\{w(K_1), w(K_2)\}$.*

Lemma 2.8 [5] *Let $V \subset S^3$ be a nice solid torus with boundary T. Then there is an new embedding $i\colon V \to S^3$ satisfying the following properties:*

- *$H = \overline{S^3 - i(V)}$ is a solid torus.*
- *The connectivity graph is a tree if and only if there is a meridian disk D of H such that $i^{-1}(\partial D) \subset T$ is horizontal in T under $h|_T$ (ie $i^{-1}(\partial D) \subset (h|_T)^{-1}(r)$ for some r, where r is a regular value of $h|_T$).*

Remark. Lemma 2.8 is a special case of Proposition 2.3 in [5].

We will also use the word "vertex" to refer to its corresponding connected component of M. It is not hard to see that each vertex has a product structure $P \times (c_i, c_{i+1})$,

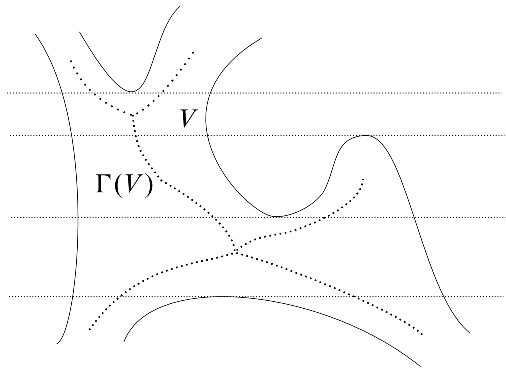

Figure 2: Embedding of $\Gamma(V)$ into V

where P is a horizontal planar surface (what we call a horizontal piece in Definition 3.5), and c_i, c_{i+1} are the two critical values that bound the vertex from below and above. We could embed $\Gamma(V)$ into $\operatorname{int}(V)$ as follows:

(1) Pick one point in the interior of each component of M.

(2) For two adjacent vertices, connecting the two points in the two vertices by a monotone decreasing arc in the interior of V.

Then $\Gamma(V)$ can be thought of as a 1–dimensional complex in $\mathring{V}$ as in Figure 2. The solid curves are the boundary of the solid torus V, and the interior of V is bounded by them. The dashed curves indicate the embedding of $\Gamma(V)$ into $\mathring{V}$.

3 $\Gamma(V)$ contains a unique simple loop

In this section, we always assume that $V \subset S^3$ is a nice solid torus and $\Gamma(V)$ is the graph constructed as in Definition 2.4. We need some preliminary results before proving that there is a unique loop in $\Gamma(V)$.

Lemma 3.1 *Suppose V is knotted in S^3, $T = \partial V$, and r is a regular value of $h|_T$ such that one component of $(h|_T)^{-1}(r)$ is an essential curve on T. Then at least one component X of $h^{-1}(r) \cap V$ is a surface with boundary such that exactly one boundary component δ is essential on T. Furthermore, δ is a meridian on T.*

Proof Since r is a regular value of $h|_T$, we have that $(h|_T)^{-1}(r)$ is a disjoint union of some simple closed curves. Note that $h^{-1}(r)$ is a sphere in S^3, so each component α of $(h|_T)^{-1}(r)$, which is essential in T, bounds two disks on $h^{-1}(r)$. We say α is *innermost* if α bounds a disk in $h^{-1}(r)$ which does not contain any other essential curve of $(h|_T)^{-1}(r)$. By the innermost arguments, we can find a component P of $h^{-1}(r) - T$ such that only one component of ∂P is essential in T.

Claim $$P \subset V.$$

Proof of claim Suppose P is not contained in V. Then $P \cap \mathring{V} = \varnothing$. Let δ be the component of ∂P which is essential in T. Then any other component of P bounds a disk in T; hence δ bounds a singular disk in $\overline{S^3 - V}$. By Dehn's lemma, δ bounds an embedded disk in $\overline{S^3 - V}$, and thus T is compressible in $\overline{S^3 - V}$, which contradicts to the assumption that V is knotted. □

By similar argument as above, we can see that δ bounds an embedded disk in V; hence δ is a meridian of T. □

Corollary 3.2 *Suppose V is knotted and r is a regular value of $h|_T$. Suppose δ is a component of $(h|_T)^{-1}(r)$ which is essential in T. Then δ is a meridian of T.*

Corollary 3.3 *The graph $\Gamma(V)$ is not a tree if V is a knotted solid torus.*

Proof Suppose, on the contrary, that $\Gamma(V)$ is a tree. By Lemma 2.6, the connectivity graph defined in [5] for V is also a tree. Let $i\colon V \to S^3$ be the embedding as in Lemma 2.8. Then there is a meridian disk D of solid torus $H = \overline{S^3 - i(V)}$ such that $i^{-1}(\partial D) \subset (h|_T)^{-1}(r)$ for some r. Obviously, $i^{-1}(\partial D)$ is an essential curve in T, so by Corollary 3.2, $i^{-1}(\partial D)$ is a meridian of V; hence ∂D bounds a disk in $i(V)$, which contradicts that $S^3 = i(V) \cup H$. □

Definition 3.4 Let l be a simple loop of Γ. A vertex of l which is locally minimal (maximal) under h is called a *minimal* (*maximal*) *vertex*. We say that a vertex is a *critical vertex* if it is either minimal or maximal. A vertex which is neither minimal nor maximal is called a *vertical vertex*.

Definition 3.5 Let r be a regular value of $h|_T$. We call each component of $h^{-1}(r) \cap V$ a *horizontal piece*.

Let v be a vertex in l, regard v as a component of M (see the discussion at the end of Section 2) and let P be a horizontal piece in v. We want to describe the intersection of l with P. If v is a maximal vertex, then the two adjacent vertices in l are both below v.

Thus $l \cap v$ is an arc with one maximal point with respect to the height function h, and P intersects l in either zero, one or two points. Note that the intersection at one point is not transversal because if we move the piece slightly above or below, then the intersection would be zero or two points. A similar result holds for minimal vertices. If v is vertical, then $l \cap v$ is a monotonic arc, so every horizontal piece intersects l exactly once.

Another observation is that the two vertices adjacent to a critical vertex must both be vertical, so in any simple loop, vertical vertices must exist.

Lemma 3.6 *Let l be a simple loop in $\Gamma(V)$. Then, as a simple closed curve in V, we have that l (with any orientation) represents a generator of $H_1(V)$.*

Proof Let v be a vertical vertex of l, and pick a horizontal piece of v. Then P is a properly embedded surface in V and intersects l transversally once, so the algebraic intersection number of l and P is just ± 1; hence l must be a generator of $H_1(V)$. □

Proposition 3.7 *There is a unique simple loop in $\Gamma(V)$.*

Proof The existence of a loop follows from Corollary 3.3. To show the uniqueness, suppose, on the contrary, there are two different simple loops l_1, l_2. If l_1 and l_2 do not have the same vertical vertices, then there is a vertical vertex v with respect to one loop but not the other, say, with respect to l_1 but not l_2. Then pick a generic horizontal piece P in v and calculate the intersection number of l_2 and P. Since the geometric intersection number is even, the algebraic intersection number is also even and cannot be ± 1. But the algebraic intersection number of l_1 and P is ± 1. This contradicts the fact that both l_1 and l_2 are generators of $H_1(V)$.

Finally, observe that if the two simple loops have same vertical vertices, then they must be the same loop. We conclude that the simple loop must be unique. □

4 The inequalities

To calculate width, we use a technique coming from Zupan; see Section 5 in [9]. Let Z_2 be the free monoid generated by $\{a, b\}$, and let $\varphi\colon Z_2 \to \mathbb{Z}$ be a homomorphism such that $\varphi(a) = 2$, $\varphi(b) = -2$ and $\varphi(\alpha\beta) = \varphi(\alpha) + \varphi(\beta)$. For a word $x = \alpha_1\alpha_2\cdots\alpha_m \in Z_2$, where each α_j is a or b, write $x_i = \alpha_1\alpha_2\cdots\alpha_i \in Z_2$, and define

$$w(x) = \sum_{i=1}^{m} \varphi(x_i).$$

For each knot $k \in \mathcal{K}$ (see Definition 2.2), associate a word $x = x(k) \in Z_2$ to it as follows: suppose all the critical points of k, from the lowest to highest, are $p_1, p_2, \dots, p_m$; then define

$$x = x(k) = \alpha_1\alpha_2\cdots\alpha_m,$$

where $\alpha_i = a$ if p_i is a local minimal critical point and $\alpha_i = b$ if p_i is a local maximal critical point. Let $w_i(k)$ be defined as in Definition 2.2. It is not hard to see that $w_i(k) = \varphi(x_i)$ and

$$w(k) = \sum_{i=1}^{m} w_i(k) = \sum_{i=1}^{m} \varphi(x_i) = w(x).$$

Lemma 4.1 *Suppose $x = \alpha_1\alpha_2\cdots\alpha_m \in Z_2$ is a word.*

(i) *Suppose $\varphi(x_i) \geqslant 0$ for $i = 1, 2, \dots, m$. Let x' be a word obtained by deleting two letters α_i, α_j in F, where $i < j$, $\alpha_i = a$ and $\alpha_j = b$. Then $w(x) \geqslant w(x')$.*

(ii) *Suppose x' is obtained from x by exchanging two letters α_i, α_{i+1}, where $\alpha_i = a$ or $\alpha_{i+1} = b$. Then $w(x) \geqslant w(x')$.*

The proof is straightforward. We call the operation on words in (i) (or in (ii)) of this lemma the type I (or II) operation. The next lemma is useful when estimating $w(k)$:

Lemma 4.2 *Suppose n is a fixed positive integer, and $x = x(\widetilde{k})$ is a word, associated with a knot $\widetilde{k}$, which has the form $\omega_1\alpha_1^{s_1}\omega_2\alpha_2^{s_2}\cdots\omega_m\alpha_m^{s_m}\omega_{m+1}$, where $\omega_i = \beta_{i1}\beta_{i2}\cdots\beta_{it_i}$ is a word for $i = 1, 2, \dots, m+1$. Assume that each $s_i \geqslant n$.*

Furthermore, suppose that $x = \alpha_1\cdots\alpha_m$ is the word associated to another knot $\widehat{l}$, and

(1) *$\varphi(\omega_1\alpha_1^{s_1}\cdots\omega_i\alpha_i^{s_i}) \geqslant n\varphi(\alpha_1\cdots\alpha_i)$ if $\alpha_i = a$,*

(2) *$\varphi(\omega_1\alpha_1^{s_1}\cdots\omega_{i-1}\alpha_{i-1}^{s_{i-1}}\omega_i) \geqslant n\varphi(\alpha_1\cdots\alpha_{i-1})$ if $\alpha_i = b$.*

Then we have

$$w(\widetilde{k}) \geqslant n^2 w(\alpha_1\alpha_2\cdots\alpha_m) = n^2 w(\widehat{l}).$$

Proof Suppose $1 \leqslant i \leqslant m$. If $\alpha_i = a$, then for $0 \leqslant j \leqslant n-1$, we have

$$\varphi(\omega_1\alpha_1^{s_1}\cdots\omega_i\alpha_i^{s_i-j}) \geqslant n\varphi(\alpha_1\cdots\alpha_i) - 2j;$$

if $\alpha_i = b$, then for $1 \leqslant j \leqslant n$, we have

$$\begin{aligned}\varphi(\omega_1\alpha_1^{s_1}\cdots\omega_{i-1}\alpha_{i-1}^{s_{i-1}}\omega_i\alpha_i^{j}) &\geqslant n\varphi(\alpha_1\cdots\alpha_{i-1}) - 2j\\ &= n(\varphi(\alpha_1\cdots\alpha_i) + 2) - 2j\\ &= n\varphi(\alpha_1\cdots\alpha_i) + 2n - 2j.\end{aligned}$$

Since the word comes from a knot, $\varphi(x_l) \geqslant 0$ for any l. Hence we have

$$
\begin{aligned}
w(\widetilde{k}) &\geqslant \sum_{\alpha_i=a} \sum_{j=0}^{n-1} \varphi(\omega_1 \alpha_1^{s_1} \cdots \omega_i \alpha_i^{s_i-j}) + \sum_{\alpha_i=b} \sum_{j=1}^{n} \varphi(\omega_1 \alpha_1^{s_1} \cdots \omega_{i-1} \alpha_{i-1}^{s_{i-1}} \omega_i \alpha_i^{j}) \\
&\geqslant \sum_{i=1}^{m} n^2 \varphi(\alpha_1, \dots, \alpha_i) + \sum_{\alpha_i=a} \sum_{j=0}^{n-1} (-2j) + \sum_{\alpha_i=b} \sum_{j=1}^{n} 2n - 2j \\
&= n^2 w(\alpha_1, \dots, \alpha_n) - \sum_{\alpha_i=a} n(n-1) + \sum_{\alpha_i=b} n(n-1) \\
&= n^2 w(\alpha_1, \dots, \alpha_n) \\
&= n^2 w(\widehat{l}).
\end{aligned}
$$

□

Now suppose k is a satellite knot with companion j, and let V be a closed regular neighborhood of j that contains k. Without loss of generality, we can assume that V is a nice solid torus. Let l be the unique loop in $\Gamma(V)$ as in Proposition 3.7; then l can also be viewed as a knot in $\mathring{V} \subset S^3$. Denote by J (or K, L) the knot type of j (or k, l).

Lemma 4.3 $$w(L) \geqslant w(J).$$

Proof Picking a horizontal piece P as in Lemma 3.1, and capping off all inessential boundaries of P near the boundary T, we get a meridian disk D of V such that $D \cap l = P \cap l$. By Section 3, l represents a generator of $H_1(V)$, so the algebraic intersection number of l and P is ± 1. Note also that l intersects P (and hence D) at most two points; see the discussion below Definition 3.5. So l must intersect the meridian disk D (transversally) only once and hence can be viewed as a composition of j and possibly another knot l'. Then by Lemma 2.7, $w(L) \geqslant w(J)$. □

Lemma 4.4 *Let P be a horizontal piece in a vertical vertex of l. Then the geometric intersection number of k and P is no less than the winding number of the pattern k.*

Proof Since the winding number for the pattern of k is n, and l represents a generator of $H_1(V) \cong \mathbb{Z}$, we have $[k] = \pm n[l] \in H_1(V)$. Since P is a vertical vertex, the algebraic intersection number (up to sign) of ± 1 and l is 1. Consequently, P and k must have algebraic intersection number (up to sign) $\pm n$, hence the geometric intersection number is at least n. □

Now we isotope l into an equivalent knot $\widehat{l}$ and change k into another knot $\widetilde{k}$ as follows.

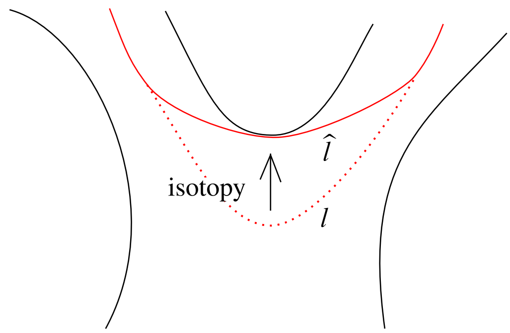

Figure 3: Isotopy between l and $\widehat{l}$

Suppose all the critical points of l are $q_1, q_2, \dots, q_m$, from the lowest to the highest. Each q_j corresponds to a critical vertex of $\Gamma(V)$ and hence corresponds to a component of $M = V - \left(\bigcup_{i=1}^n h^{-1}(c_i)\right)$ (see Definition 2.4), denoted by C_j. When q_j is a local minimal point of l, suppose C_j is bounded from above by $h^{-1}(c_{i_j})$. Since v_j has a product structure, we can move q_j up to a point $\widehat{q}_j$ so that $\widehat{q}_j$ is a critical point of $h|_T$ and $h(\widehat{q}_j) = c_{i_j}$. Furthermore, we can assume that no more critical points of l are created. Do similar operations on local maximal points of l and after all such operations, l becomes a new knot $\widehat{l}$. Obviously, $\widehat{l}$ and l are equivalent knots, and all the critical points of $\widehat{l}$ are $\widehat{q}_1, \widehat{q}_2, \dots, \widehat{q}_m$. See Figure 3 for the isotopy near a local minimal point of l.

Suppose $\widehat{q}_j$ is a local minimal point, then pick a regular value r_j slightly larger than c_{i_j} so that no other critical point of T, $\widehat{l}$, k lies between two level spheres $h^{-1}(r_j)$ and $h^{-1}(c_{i_j})$. There are two horizontal pieces P_j and Q_j on $h^{-1}(r_j)$ which respectively belong to the two vertical vertices adjacent to the vertex C_j. The pieces P_j and Q_j cut k into arcs, and each arc of $k - P_j - Q_j$ that lies in C_j is disjoint from any other C_t for $t \neq j$ since l is the unique simple loop. If β is such an arc intersecting C_j, we can create a new arc γ so that β and γ have the same end points, γ has exactly one inner critical point which is local minimal and γ is contained in $h^{-1}(c_{i_j}, r_j)$. Then replace β by γ and do this repeatedly until no arc of $k - P_j - Q_j$ intersects C_j. Then we finish the operation for a particular local minimal point of $\widehat{l}$. See Figure 4. Do similar replacements for local maximal points of $\widehat{l}$. After such replacement for all m critical points of $\widehat{l}$, we have that k becomes a new knot $\widetilde{k}$. The knots k and $\widetilde{k}$ may have different knot type, but it does not matter. We only need the following inequality.

Lemma 4.5 $$w(k) \geqslant w(\widetilde{k}).$$

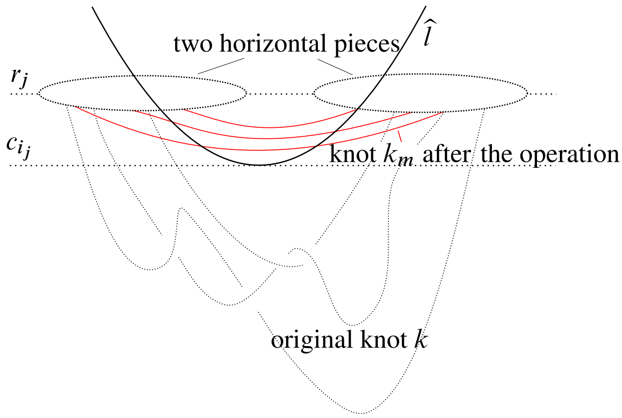

Figure 4: Operation on k

Proof We study how k becomes $\widetilde{k}$. Let $\widehat{q}_j$ be a local minimal point of $\widehat{l}$, and let P_j, Q_j be as above. Also let β be an arc in $k - P_j - Q_j$ which has end points in $P_i \cup Q_i$ and has interior below them. Then the operation of creating γ and replacing β can be done by two steps. the first step is to cancel pairs of maximal and minimal points of β to make β to have a unique critical point which is minimal. Since the interior of β is below its two end points, we can always pair a maximal point with another minimal point which is lower. This corresponds to the type I operation on words and, by Lemma 4.1, will not increase width. The condition that $\varphi(y_i) > 0$ in Lemma 4.1 holds because, after cancelling each pair of points, k still remains a knot. The second step is to lift the unique minimal point of β above the level c_{i_j}. This corresponds to type II operation on words and will not increase width. Similar arguments apply to local maximal points of $\widehat{l}$. □

Lemma 4.6 *$w(\widetilde{k}) \geqslant n^2 w(\widehat{l})$, where n is the winding number.*

Proof Now we need to estimate $w(\widetilde{k})$. The difficulty is that we do not know every critical point of $\widetilde{k}$ but only the ones near a critical point of $\widehat{l}$. For a local minimal point $\widehat{q}_j$ of $\widehat{l}$, let r_i be a regular value slightly larger than c_{i_j} as in the discussion above. Then $|h^{-1}(r_j) \cap \widehat{l}| = \omega_j(\widehat{l})$, so there are exactly $\omega_j(\widehat{l})$ horizontal pieces of V on $h^{-1}(r_j)$ which intersect $\widehat{l}$. By Lemma 4.4, we have

$$|h^{-1}(r_j) \cap \widetilde{k}| \geqslant n\omega_j(\widehat{l}).$$

A similar argument applies when $\widehat{q}_i$ is local maximal point. The difference is that we should pick a regular level r_i slightly lower than c_{i_j}, and hence

$$|h^{-1}(r_j) \cap \widetilde{k}| \geqslant n(\omega_{j-1}(\widehat{l})).$$

Note that ω_{j-1} appears on the right because we pick a regular level slightly lower than the critical level c_{i_j}. Compare Definition 2.2.

Suppose the word for $\widehat{l}$ is $\alpha_1\alpha_2\cdots\alpha_m$. Then the word for $\widetilde{k}$ can be written as $\omega_1\alpha_1^{s_1}\omega_2\alpha_2^{s_2}\cdots\omega_m\alpha_m^{s_m}\omega_{m+1}$, where ω_i is an arbitrary word for $i=1,2,\dots,m+1$ and $s_i \geqslant n$ for all $i=1,2,\dots,m$. The argument above shows that the words for $\widetilde{k}$ and $\widehat{l}$ satisfy the conditions of Lemma 4.2, and hence we have

$$w(\widetilde{k}) \geqslant n^2 w(\alpha_1\alpha_2\cdots\alpha_m) = n^2 w(\widehat{l}). \qquad \square$$

Theorem 4.7 *Let K be a satellite knot with companion J and winding number n. Then*

$$w(K) \geqslant n^2 w(J).$$

Proof Choose $k \in \mathcal{K}$ which realizes the width of its knot type. Without loss of generality, we can assume that V is a closed regular neighborhood of j which is nice and contains k. Furthermore, we could assume that all critical points of ∂V and k are in distinct levels. Construct $\Gamma(V)$ and pick the unique loop l by Proposition 3.7. Combining Lemmas 4.3, 4.5 and 4.6, we have

$$w(K) = w(k) \geqslant n^2 w(\widetilde{k}) \geqslant n^2 w(\widehat{l}) \geqslant n^2 w(L) \geqslant n^2 w(J). \qquad \square$$

Corollary 4.8 *Let K be a satellite knot with knotted companion J, and suppose the winding number of the pattern is n. If K has a braid pattern, then*

$$w(K) = n^2 w(J).$$

Proof Suppose j realizes the width of its knot type and has the associated word $x = \alpha_1\alpha_2\cdots\alpha_t$. Since k has a braid pattern, we can embed k so that the word associated to it is $y = \alpha_1^n\alpha_2^n\cdots\alpha_m^n$. By direct calculation, we have

$$w(K) \leqslant w(k) = w(y) = n^2 w(x) = n^2 w(j) = n^2 w(J).$$

Together with Theorem 4.7, we conclude that

$$w(K) = n^2 w(J). \qquad \square$$

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