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A second cohomology class of the symplectomorphism group with the discrete topology

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A second cohomology class of the group cohomology of the symplectomorphism group is defined for a symplectic manifold with first Chern class proportional to the class of symplectic form and with trivial first real cohomology. Some properties of it are studied. In particular, it is characterized in terms of cohomology classes of the universal symplectic fiber bundle over the classifying space of the symplectomorphism group with the discrete topology.

57R17, 57R50; 55R40, 58H10

1 Introduction

A two cocycle on the symplectomorphism group of an exact symplectic manifold with trivial first real cohomology was defined and studied by Ismagilov, Losik and Michor [12]. Gal and Kędra [8] carried investigation of it further. They provided another definition of the two cocycle and obtained vanishing and nonvanishing results of its cohomology class. They also gave some applications of it to foliated symplectic bundles and Hamiltonian actions. This two cocycle is constructed by using a primitive of the symplectic form, the existence of which is the starting point of the construction.

In this paper we consider a symplectic manifold (M,ω) of dimension $2n$ with first Chern class $c_1(M)$ proportional to the class $[\omega]$ of the symplectic form ω, that is, we assume the equality $c_1(M)+\kappa[\omega]=0$ in $H^2(M;\mathbb{R})$ for some $\kappa\in\mathbb{R}$. This is our assumption instead of the exactness of the symplectic form. We define a second cohomology class $\sigma\in H^2(G,\mathbb{R})$ in the group cohomology of the symplectomorphism group $G=\mathrm{Symp}(M,\omega)$ of (M,ω) with values in $\mathbb{R}$ in a similar way to another definition by Gal and Kędra, mentioned above. In order to define σ, we need the symplectic twistor space of (M,ω), which is a fiber bundle $p\colon \mathcal{J}\to M$ over M with fiber the Siegel upper half space; see Albuquerque and Rawnsley [3]. In particular we use the fact that the standard action of G on M induces that as $U(n)$–bundle isomorphisms on the pullback bundle p^*TM over $\mathcal{J}$ of the tangent bundle TM of M by p, which has a canonical Hermitian structure. The class σ is defined in terms

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of differential forms on $\mathcal{J}$ using a unitary connection of p^*TM and other ingredients. We can show vanishing results for the restrictions of it to certain subgroups.

The group cohomology $H^*(G,\mathbb{R})$ is identified with $H^*(BG^\delta;\mathbb{R})$, the singular cohomology of the classifying space BG^δ of G^δ, which is G with the discrete topology. In order to give the class $\|\sigma\|$ in $H^2(BG^\delta;\mathbb{R})$ corresponding to σ, we need to handle differential forms on the fiber bundle $EG^\delta \times_{G^\delta} \mathcal{J}$ with fiber $\mathcal{J}$ associated with the universal principal bundle $EG^\delta \to BG^\delta$. Therefore we give the bundles as simplicial manifolds and we do calculus of differential forms on them; see Dupont [6; 7]. The class $\|\sigma\|$ is characterized so that its pullback by the projection of the associated (M,ω)–bundle $EG^\delta \times_{G^\delta} M \to BG^\delta$ is the sum of the first Chern class of the tangent bundle along the fibers of it and κ times the transverse symplectic class of it. This is a similar result to that for the transverse symplectic class of a foliated surface bundle with area-preserving holonomy in Kotschick and Morita [19], and is used to show the nontriviality of σ for the example in this paper. Let BG be the classifying space of G with the C^∞ topology. The identity map of G induces a continuous map $BG^\delta \to BG$, hence a homomorphism $\iota^*\colon H^*(BG;\mathbb{R}) \to H^*(BG^\delta;\mathbb{R})$. If M is closed, a constant multiple of the cohomology class $\|\sigma\|$ is shown to be the image ι^*c by ι^* of a certain characteristic class $c \in H^2(BG;\mathbb{R})$ of (M,ω)–bundles obtained by fiber integration; see Januszkiewicz and Kędra [13]. Thus $\|\sigma\|$, and hence σ, is essentially a known characteristic class in the closed case. Therefore similar results for ι^*c to those for σ hold.

Finally we give a nontrivial example of σ. We consider the moduli space $\mathcal{R}_g$ of SU(2)–representations of the fundamental group of a compact oriented surface of genus g with one boundary whose holonomy along the boundary is $-I$. It is a closed, simply connected symplectic manifold satisfying our assumptions. Using some facts on mapping class groups of surfaces, we can show the nontriviality of the class σ of the symplectomorphism group of the symplectic manifold $\mathcal{R}_g$ for $g \geqq 4$. This result implies that the class ι^*c is also nontrivial in this example, as stated above, which tells us that a characteristic class obtained by fiber integration is generally nontrivial on discrete symplectomorphism groups, in particular of closed simply connected symplectic manifolds. On the other hand, Reznikov defined cohomology classes of the symplectomorphism group of a general symplectic manifold and showed a similar result to this example in Reznikov [23]. One may be curious about the connection between Reznikov's second cohomology class and σ, but we leave it as a question.

This paper is organized as follows. In Section 2, we define the second cohomology class σ and state main results. In Section 3, we show the well-definedness of σ and vanishing results for restriction of σ to certain subgroups of G. In Section 4, we recall some properties of simplicial manifolds and introduce simplicial manifolds $N\bar{\mathcal{J}}$ and others related with G. In Sections 5 and 6, we define a simplicial 2–form P on $N\bar{\mathcal{J}}$

and show a relation between it and σ. In Section 7, we define a second cohomology class $\|P\|$ of the fat realization of $N\bar{\mathcal{J}}$ corresponding to P and prove other main results. In Section 8, we give a nontrivial example of σ.

2 Definitions and main results

Let (M,ω) be a connected, symplectic manifold of dimension $2n$ and let $G = \mathrm{Symp}(M,\omega)$ the symplectomorphism group of it. Let $\mathcal{P} \to M$ be the principal symplectic frame bundle for (M,ω). Let $\mathcal{J}$ be the quotient space $\mathcal{P}/U(n)$; then the projection $p\colon \mathcal{J} \to M = \mathcal{P}/\mathrm{Sp}(2n,\mathbb{R})$ is a fiber bundle over M with fiber $\mathrm{Sp}(2n,\mathbb{R})/U(n)$. It is a smooth manifold of dimension $2n+n(n+1)$. The fiber $\mathrm{Sp}(2n,\mathbb{R})/U(n)$ is identified with the Siegel upper half space and is contractible. It is also identified with the space of positive compatible complex structures on the standard symplectic vector space $\mathbb{R}^n$. Therefore we can consider $\mathcal{J}$ as the fiber bundle over M whose fiber $\mathcal{J}_x$ at $x \in M$ is the space of positive compatible complex structures on the symplectic vector space (T_xM,ω_x) [3; 21; 25]. Since the pullback bundle p^*TM over $\mathcal{J}$ of the tangent bundle TM by the projection p is also given as $p^*TM = \mathcal{P} \times_{U(n)} \mathbb{C}^n$, it is obvious it is a Hermitian vector bundle. It is easily checked that its Hermitian form on the fiber $p^*TM|_{j_x} = T_xM$ at $j_x \in \mathcal{J}$ with $p(j_x) = x$ is given by $h_{j_x}(\cdot,\cdot) = \omega_x(\cdot, j_x\cdot) - \sqrt{-1}\omega_x(\cdot,\cdot)$. The standard action of G on M induces those on $\mathcal{P}$, $\mathcal{J}$ and p^*TM. Since the construction of the bundles is natural, the projections $p^*TM \to \mathcal{J} \to M$ are G–equivariant. Moreover G acts on p^*TM by $U(n)$–bundle isomorphisms. Therefore G acts also on the space of $U(n)$–connections of p^*TM. Let A be a $U(n)$–connection of p^*TM; then we have the pushforward connection of A by $\varphi \in G$, which is denoted by φ_*A. We remark that the actions of $\varphi \in G$ on M and on $\mathcal{J}$ are denoted by $\varphi(x)$ for $x \in M$ and by $\varphi(j)$ for $j \in \mathcal{J}$, respectively, without distinction. In particular its induced action by pullback on the space $\Omega^*(\mathcal{J})$ of forms on $\mathcal{J}$ at $\alpha \in \Omega^*(\mathcal{J})$ is denoted by $\varphi^*\alpha$. But we mainly use the left action given by the pullback $\varphi_*\alpha = (\varphi^{-1})^*\alpha$ of the inverse.

Let $c_1(M)$ and $[\omega]$ be the first Chern class of (M,ω) and the cohomology class of the symplectic form ω respectively in the second cohomology group $H^2(M)$ of M. Here and hereafter singular cohomology groups and de Rham cohomology groups are identified and the coefficients of them are $\mathbb{R}$ if not mentioned explicitly.

We assume that the equality $c_1(M) + \kappa[\omega] = 0$ holds in $H^2(M)$ for some $\kappa \in \mathbb{R}$. Pulling back by the projection p, we have $c_1(p^*TM) + \kappa[p^*\omega] = 0$ in $H^2(\mathcal{J})$. Thus there exists a 1–form $\mu \in \Omega^1(\mathcal{J})$ on $\mathcal{J}$ satisfying

$$c_1(F_A) + \kappa p^*\omega + d\mu = 0. \tag{2-1}$$

Here F_A is the curvature form of A and c_1 is the $\mathrm{Ad}_{U(n)}$–invariant linear form on the Lie algebra $u(n)$ corresponding to the first Chern class. Put

$$f(\varphi) = c_1(\varphi_* A - A) + \varphi_* \mu - \mu$$

for each $\varphi \in G$. It is easy to see that $f(\varphi)$ belongs to the space $Z^1(\mathcal{J})$ of closed 1–forms on $\mathcal{J}$. Thus we have a map $f\colon G \to Z^1(\mathcal{J})$.

Lemma 2.1 *The map f is a crossed homomorphism and its cohomology class $[f] \in H^1(G, Z^1(\mathcal{J}))$ in the group cohomology of G with coefficients in $Z^1(\mathcal{J})$ is independent of the choice of A and μ.*

This lemma is explained in the beginning of Section 3. But its detailed proof is not given there because it was proved in a general setting in [15].

If the first cohomology group of M is nontrivial, the class $[f]$ descents to a class in $H^1(G, H^1(M))$ by considering the projection $Z^1(\mathcal{J}) \to H^1(\mathcal{J}) \cong H^1(M)$. There are some works about it in [15] and related ones in [4; 16; 18; 19].

From now on we assume in addition $H^1(M) = \{0\}$. Under the assumptions, considering the de Rham complex of $\mathcal{J}$, we have a short exact sequence $\{0\} \to \mathbb{R} \hookrightarrow \Omega^0(\mathcal{J}) \xrightarrow{d} B^1(\mathcal{J}) = Z^1(\mathcal{J}) \to \{0\}$ of G–modules, where $B^1(\mathcal{J})$ denotes the space of exact 1–forms on $\mathcal{J}$. It induces a long exact sequence of cohomology groups of G. In particular we have the connecting homomorphism $\delta\colon H^1(G, Z^1(\mathcal{J})) \to H^2(G, \mathbb{R})$ in it [5]. The following proposition is proved in Section 3.

Proposition 2.2 *Let (M, ω) be a connected symplectic manifold of dimension $2n$ with $H^1(M) = \{0\}$. Assume $c_1(M) + \kappa[\omega] = 0$ in $H^2(M)$ for some $\kappa \in \mathbb{R}$. Then $\sigma := \delta([f]) \in H^2(G, \mathbb{R})$ is uniquely defined, that is, it is independent of the construction.*

By the definition of the connecting homomorphism δ, a 2–cocycle representing the class σ can be expressed in terms of f. Using this expression, we can show vanishing results of restrictions of σ to certain subgroups of G. For any $j \in \mathcal{J}$ and any positive compatible almost complex structure J on (M, ω), let G_j and G_J be the subgroups of G consisting of all elements which preserve j and J, respectively. The following theorem is proved in Section 3.

Theorem 2.3 *Let K be any subgroup of G_j for some j or of G_J for some J. If $K \subset G_J$, we assume in addition that M is closed. Then the restriction of the class σ to K vanishes, that is, the equality $\iota^* \sigma = 0$ in $H^2(K, \mathbb{R})$ holds, where $\iota\colon K \hookrightarrow G$ is the inclusion.*

Here we remark that the cohomology group of K in this theorem is that in group theory. In this paper, cohomology groups of groups denote those in group theory like $H^*(G,\mathbb{R})$ for G, as mentioned in the introduction.

Let G^δ be G with the discrete topology and $EG^\delta \to BG^\delta$ the universal principal G^δ–bundle over the classifying space BG^δ of G^δ. Let $\pi\colon E^\delta M = EG^\delta \times_{G^\delta} M \to BG^\delta$ be the universal (M,ω)–bundle over BG^δ and $E^\delta TM \to E^\delta M$ its tangent bundle along the fibers. Since $E^\delta TM$ is a symplectic vector bundle of rank $2n$, we have the first Chern class $c_1(E^\delta TM) \in H^2(E^\delta M)$. The symplectic form ω on M defines a cohomology class $\|\omega\| \in H^2(E^\delta M)$, which is precisely given in Section 7. Since $H^2(BG^\delta)$ is naturally identified with the group cohomology $H^2(G,\mathbb{R})$, we have a homomorphism $H^2(G,\mathbb{R}) = H^2(BG^\delta) \xrightarrow{\pi^*} H^2(E^\delta M)$, which is also denoted by π^*. The following theorem is proved in Section 7.

Theorem 2.4 *Under the setting of Proposition 2.2, the homomorphism*

$$\pi^*\colon H^2(G,\mathbb{R}) \to H^2(E^\delta M)$$

is injective and the equality $\pi^*\sigma = c_1(E^\delta TM) + \kappa\|\omega\|$ holds in $H^2(E^\delta M)$.

Let G_b be the subgroup of G consisting of all elements which preserve a given basepoint $b \in M$ and $D\colon G_b \to \mathrm{Sp}(2n,\mathbb{R})$ a homomorphism obtained from the differentials $d\varphi_b\colon T_bM \to T_bM$ at b for all $\varphi \in G_b$. Here we take a symplectic basis for (T_bM,ω_b), so we can consider the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ as the target of D. Let $c_1 \in H^2(\mathrm{Sp}(2n,\mathbb{R}),\mathbb{R})$ be the class in the group cohomology of $\mathrm{Sp}(2n,\mathbb{R})$ corresponding to the first Chern class of flat symplectic vector bundles. Then we have $D^*c_1 \in H^2(G_b;\mathbb{R})$. It is independent of the choice of the symplectic basis for (T_bM,ω_b). Let $\iota_b\colon G_b \hookrightarrow G$ be the inclusion. The following corollary is proved in Section 7 and is used to show that σ in the example below is nontrivial.

Corollary 2.5 *The equality $\iota_b^*\sigma = D^*c_1$ holds in $H^2(G_b,\mathbb{R})$.*

Next we consider the case that M is closed. Let $EM \to BG$ be the universal (M,ω)–bundle over the classifying space BG of G with the C^∞ topology and $ETM \to EM$ the tangent bundle along the fibers of EM. Since ETM is a symplectic vector bundle, we have the first Chern class $c_1(ETM) \in H^2(EM)$ and its $(n+1)^{\text{st}}$ power $c_1(ETM)^{n+1} \in H^{2(n+1)}(EM)$. Since M is closed, the integration along the fibers induces a homomorphism $\pi_!\colon H^{2(n+1)}(EM) \to H^2(BG)$, which is also called the Gysin homomorphism. Put $c = \pi_!(c_1(ETM)^{n+1}) \in H^2(BG)$. It is one of the characteristic classes of (M,ω)–bundles with structure group G. Since the identity map of G induces a continuous homomorphism from G^δ to G with the C^∞ topology, and hence a continuous map $\iota\colon BG^\delta \to BG$, we have its induced homomorphism $\iota^*\colon H^2(BG) \to H^2(BG^\delta) = H^2(G,\mathbb{R})$. The following theorem is also proved in Section 7.

Theorem 2.6 *If, in the setting of Proposition 2.2, M is in addition supposed to be closed, then the equality $\iota^* c = (n+1)c_1(M)^n([M])\sigma$ holds in $H^2(G,\mathbb{R})$, where $c_1(M)^n([M])$ is the evaluation of $c_1(M)^n \in H^{2n}(M)$ on the fundamental class $[M]$ of M.*

For a subgroup $K \subset G$, let $\iota_K^*\colon H^2(BG) \xrightarrow{\iota^*} H^2(G,\mathbb{R}) \to H^2(K,\mathbb{R})$ be the composite map of ι^* with the induced homomorphism by the inclusion $K \hookrightarrow G$. For a closed manifold M, the following corollary is just Theorem 2.3 in terms of $c \in H^2(BG)$ using Theorem 2.6.

Corollary 2.7 *Under the assumptions of Theorem 2.6, the equality $\iota_K^* c = 0$ holds in $H^2(K,\mathbb{R})$ for any subgroup K of G_j or G_J in Theorem 2.3.*

Finally we give a nontrivial example of σ. Let Σ_g be a closed oriented surface of genus $g \geqq 2$ and $\Sigma_{g,1} \subset \Sigma_g$ the complement of an open disk in Σ_g. The moduli space $\mathcal{R}_g$ of SU(2)–representations of $\pi_1(\Sigma_{g,1})$ with holonomy $-I$ along the boundary $\partial\Sigma_{g,1}$ is given by

$$\mathcal{R}_g = \bigl\{\phi \in \mathrm{Hom}(\pi_1(\Sigma_{g,1}),\mathrm{SU}(2)) \mid \phi(\partial\Sigma_{g,1}) = -I\bigr\}/\mathrm{Ad}\,\mathrm{SU}(2),$$

where $\partial\Sigma_{g,1}$ also presents the homotopy class of itself. It is known to be a smooth, closed, 1–connected symplectic manifold of dimension $6g-6$ with symplectic form ω, whose cohomology class $[\omega]$ is twice the generator of $H^2(\mathcal{R}_g;\mathbb{Z}) \cong \mathbb{Z}$ (see [24] for properties of $\mathcal{R}_g$). Thus the assumptions of Proposition 2.2 are satisfied for $\mathcal{R}_g$. In particular we have a relation $c_1(\mathcal{R}_g) + \kappa[\omega] = 0$ for some $0 \neq \kappa \in \mathbb{R}$. Therefore we have our cohomology class $\sigma \in H^2(\mathrm{Symp}(\mathcal{R}_g,\omega),\mathbb{R})$.

Let $\Gamma_{g,1}$ be the mapping class group of $\Sigma_{g,1}$, that is, the group of the isotopy classes of diffeomorphisms of $\Sigma_{g,1}$ whose restrictions to $\partial\Sigma_{g,1}$ are the identity. The action of $\Gamma_{g,1}$ on $\pi_1(\Sigma_{g,1})$ induces a homomorphism $\rho\colon \Gamma_{g,1} \to \mathrm{Symp}(\mathcal{R}_g,\omega)$. Considering extension of the diffeomorphisms by the identity on $\Sigma_g \setminus \Sigma_{g,1}$, the group $\Gamma_{g,1}$ acts on $H^1(\Sigma_g;\mathbb{Z})$, hence we have a homomorphism $\tau\colon \Gamma_{g,1} \to \mathrm{Aut}(H^1(\Sigma_g;\mathbb{Z}), \cdot \cup \cdot) \cong \mathrm{Sp}(2g,\mathbb{Z}) \subset \mathrm{Sp}(2g,\mathbb{R})$ and the cohomology class $\tau^* c_1 \in H^2(\Gamma_{g,1},\mathbb{R})$, which is known to be nontrivial for $g \geqq 3$ (see [17; 22]). Here $c_1 \in H^2(\mathrm{Sp}(2g,\mathbb{R}),\mathbb{R})$ is the first Chern class appearing before. The following theorem is proved in Section 8.

Theorem 2.8 *Let $g \geqq 4$, then the equality $\rho^*\sigma = -3\tau^* c_1$ holds in $H^2(\Gamma_{g,1},\mathbb{R})$. Hence σ and $\iota^* c \in H^2(\mathrm{Symp}(\mathcal{R}_g,\omega),\mathbb{R})$ are nontrivial.*

This theorem implies that there is a foliated symplectic bundle over a surface with closed simply connected fiber $\mathcal{R}_g$ and with nontrivial smooth characteristic class given by fiber integration.

3 Proofs of Proposition 2.2 and Theorem 2.3

In this section we prove Proposition 2.2 and Theorem 2.3. We return to the situation above Lemma 2.1.

Lemma 3.1 (1) *The* 1*–form* $f(\varphi)$ *on* $\mathcal{J}$ *is closed for each* $\varphi \in G$*.*

(2) *The map* $f\colon G \to Z^1(\mathcal{J})$, $\varphi \mapsto f(\varphi)$*, is a crossed homomorphism.*

By definition, f is a crossed homomorphism if and only if it satisfies $\delta f(\varphi,\psi) := \varphi_*\{f(\psi)\} - f(\varphi\psi) + f(\varphi) = 0$ for any $\varphi,\psi \in G$, that is, it is a 1–cocycle in group cohomology theory. Another choice of connection B and $\nu \in \Omega^1(\mathcal{J})$ instead of A and μ gives a crossed homomorphism

$$g\colon G \to \Omega^1(\mathcal{J}), \quad g(\varphi) = c_1(\varphi_* B - B) + \varphi_*\nu - \nu.$$

Put $\alpha = \nu - \mu + c_1(B - A) \in \Omega^1(\mathcal{J})$.

Lemma 3.2 (1) α *is a closed* 1*–form on* $\mathcal{J}$

(2) $g(\varphi) - f(\varphi) = \varphi_*\alpha - \alpha =: \delta\alpha(\varphi)$ *for each* $\varphi \in G$*.*

Lemmas 3.1 and 3.2 are proved by direct computation using the following formulas. Their detailed proofs are omitted because they were proved in a general setting in [15]. Let A, B and C be $U(n)$–connections on p^*TM; then we have

(1) $c_1(A - B) + c_1(B - C) = c_1(A - C)$,

(2) $dc_1(B - A) = c_1(F_B) - c_1(F_A)$, and

(3) $c_1(F_{\varphi_* A}) = \varphi_* c_1(F_A)$ for each $\varphi \in G$.

Here we note that we consider $c_1(A - B)$, $c_1(F_A)$ and others as forms on $\mathcal{J}$, not on the total space $\mathcal{P}$ of the $U(n)$–principal bundle $\mathcal{P} \to \mathcal{P}/U(n) = \mathcal{J}$ associated with p^*TM.

Proof of Lemma 2.1 and Proposition 2.2 Since a crossed homomorphism is nothing but a 1–cocycle, we have a cohomology class $[f] \in H^1(G, Z^1(\mathcal{J}))$ by Lemma 3.1. Lemma 3.2 shows that the difference between 1–cocycles f and g is a coboundary. This implies that the class $[f]$ is independent of the construction, giving Lemma 2.1. Proposition 2.2 is a direct consequence of the uniqueness of the class $[f]$. □

The second cohomology class σ defined in Proposition 2.2 is explicitly given as follows. Since the target of f is $B^1(\mathcal{J})$, there exists a map

$$h\colon G \to \Omega^0(\mathcal{J}) \tag{3-1}$$

satisfying $dh(\varphi) = f(\varphi)$ for each $\varphi \in G$. Let

$$s(\varphi, \psi) := \delta h(\varphi, \psi) := \varphi_* h(\psi) - h(\varphi\psi) + h(\varphi)$$

for each $\varphi, \psi \in G$; then we have a 2–cocycle $s\colon G^2 \to \mathbb{R}$ and its cohomology class is $\sigma = [s] \in H^2(G, \mathbb{R})$. Here we note that $\mathbb{R}$ is identified with the constant functions in $\Omega^0(\mathcal{J})$.

Proof of Theorem 2.3 We have only to prove the theorem for $K = G_j$ and $K = G_J$.

In the case of $K = G_j$, we can take a map h in (3-1) satisfying the additional condition $h(\varphi)(j) = 0$ for all $\varphi \in G$. Such an h is uniquely determined. For the 2–cocycle $s = \delta h\colon G^2 \to \mathbb{R}$ and any $\varphi, \psi \in G_j$, we have

$$s(\varphi, \psi) = s(\varphi, \psi)(j) = \varphi_*\{h(\psi)\}(j) = h(\psi)(\varphi^{-1}(j)) = h(\psi)(j) = 0,$$

where the first equality is the identification of a constant function with its value at $j \in \mathcal{J}$. Thus we have $\iota^*\sigma = 0$ in $H^2(G_j, \mathbb{R})$.

In the case of $K = G_J$, we can take a map h in (3-1) satisfying the condition $\int_M J^*\{h(\varphi)\}\omega^n = 0$ for all $\varphi \in G$, where J is considered as a map $J\colon M \to \mathcal{J}$. Such an h is also unique. For the 2–cocycle $s = \delta h$, we have

$$\begin{aligned} s(\varphi, \psi) \int_M \omega^n &= \int_M J^*\{s(\varphi, \psi)\}\omega^n = \int_M J^*\{\varphi_* h(\psi)\}\omega^n \\ &= \int_M \varphi_*\{J^* h(\psi)\omega^n\} = \int_M J^*\{h(\psi)\}\omega^n = 0 \end{aligned}$$

for any $\varphi, \psi \in G_J$, which implies $\iota^*\sigma = 0$ in $H^2(G_J, \mathbb{R})$. □

4 Simplicial manifolds

In this section we recall simplicial manifolds, simplicial forms and so on, for which we refer mainly to [6; 7]. We also introduce some simplicial manifolds needed in this paper.

A simplicial set X is a sequence $X = \{X_q\}_{q=0,1,\dots}$ of sets together with face operators $\varepsilon_i\colon X_q \to X_{q-1}$ for $i = 0, \dots, q$ and degeneracy ones $\eta_i\colon X_q \to X_{q+1}$ for $i = 0, \dots, q$

which satisfy the identities

$$\begin{aligned}
\varepsilon_i\varepsilon_j &= \varepsilon_{j-1}\varepsilon_i && \text{if } i < j,\\
\eta_i\eta_j &= \eta_{j+1}\eta_i && \text{if } i \leqq j,\\
\varepsilon_i\eta_j &= \begin{cases} \eta_{j-1}\varepsilon_i & \text{if } i < j,\\ \mathrm{id} & \text{if } i = j,\ j+1,\\ \eta_j\varepsilon_{i-1} & \text{if } i > j+1. \end{cases}
\end{aligned}$$

Moreover, if all X_q are topological spaces and all face and degeneracy operators are continuous maps, then X is called a simplicial space. Furthermore, if all X_q are smooth manifolds and all face and degeneracy operators are smooth maps, then X is called a simplicial manifold.

A simplicial n–form P on a simplicial manifold $X = \{X_q\}$ is a sequence $P = \{P_q\}_{q=0,1,\ldots}$ of n–forms P_q on $\Delta^q \times X_q$ such that

$$(\varepsilon^i \times \mathrm{id})^* P_q = (\mathrm{id} \times \varepsilon_i)^* P_{q-1} \quad \text{on } \Delta^{q-1} \times X_q \text{ for } i = 0, \ldots, q \text{ and } q = 1, 2, \ldots,$$

where $\Delta^q \subset \mathbb{R}^{q+1}$ is the standard q–simplex given by

$$\Delta^q = \Big\{ t = (t_0, \ldots, t_q) \in \mathbb{R}^{q+1} \ \Big|\ t_i \geqq 0 \text{ for } i = 0, \ldots, q,\ \sum_{i=0}^{q} t_i = 1 \Big\} \tag{4-1}$$

and $\varepsilon^i \colon \Delta^{q-1} \to \Delta^q$ is the i^{th} face map given by

$$\varepsilon^i(t_0, \ldots, t_{q-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{q-1})$$

for $q = 0, 1, \ldots$.

Let $\mathbb{A}^n(X)$ be the set of simplicial n–forms on X. The exterior differentials on the $\Delta^q \times X_q$ define a differential $d \colon \mathbb{A}^n(X) \to \mathbb{A}^{n+1}(X)$ and the exterior multiplication of usual forms on them defines a multiplication $\wedge \colon \mathbb{A}^{n_1}(X) \otimes \mathbb{A}^{n_2}(X) \to \mathbb{A}^{n_1+n_2}(X)$ satisfying obvious identities. Let $\mathbb{A}^{k,l}(X)$ be the set of simplicial $k+l$–forms P whose restriction $P|_{\Delta^q \times X_q} := P_q$ is locally of the form

$$\sum a_{i_1\ldots i_k, j_1\ldots j_l}\, dt_{i_1} \wedge \cdots \wedge dt_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l},$$

where $t = (t_0, \ldots, t_q)$ are the coordinates of Δ^q in (4-1), (x_j) are local ones of X_q and the $a_{i_1\ldots i_k, j_1\ldots j_l}$ are smooth functions on $\Delta^q \times X_q$. Let $d' \colon \mathbb{A}^{k,l}(X) \to \mathbb{A}^{k+1,l}(X)$ be the exterior differential with respect to t and let $d'' \colon \mathbb{A}^{k,l}(X) \to \mathbb{A}^{k,l+1}(X)$ be the exterior differential with respect to (x_j) times $(-1)^k$. Then we have a double complex $(\mathbb{A}^{*,*}(X), d', d'')$ and its total complex $(\mathbb{A}^*(X), d)$ with $\mathbb{A}^n(X) = \bigoplus_{k+l=n} \mathbb{A}^{k,l}(X)$ and $d = d' + d''$.

We have another double complex $(\mathcal{A}^{*,*}(X), \delta', d'')$. Let $\mathcal{A}^{k,l}(X) = \Omega^l(X_k)$ be the set of l–forms on X_k. The differential $\delta'\colon \mathcal{A}^{k,l}(X) = \Omega^l(X_k) \to \mathcal{A}^{k+1,l}(X) = \Omega^l(X_{k+1})$ is given by $\delta' = \sum_{i=0}^{k+1}(-1)^i \varepsilon_i^*$ and the differential $d''\colon \mathcal{A}^{k,l}(X) = \Omega^l(X_k) \to \mathcal{A}^{k,l+1}(X) = \Omega^{l+1}(X_k)$ is given by $(-1)^k$ times the exterior differential on $\Omega^l(X_k)$. Put $\mathcal{A}^n(X) = \bigoplus_{k+l=n} \mathcal{A}^{k,l}(X)$ and $\delta = \delta' + d''$, then we have the total complex $(\mathcal{A}^*(X), \delta)$ of $(\mathcal{A}^{*,*}(X), \delta', d'')$.

Now we have two double complexes, $(\mathbb{A}^{*,*}(X), d', d'')$ and $(\mathcal{A}^{*,*}(X), \delta', d'')$. Restricting elements of $\mathbb{A}^{k,l}(X)$ to $\Delta^k \times X_k$ and integrating them over Δ^k, we obtain a map $I_{\Delta^k}\colon \mathbb{A}^{k,l}(X) \to \mathcal{A}^{k,l}(X)$ for each k. The collection $I_\Delta = \{I_{\Delta^k}\}$ clearly defines a map of double complexes. Moreover, it induces a natural isomorphism $I_\Delta = I_{\Delta,X}\colon H^*(\mathbb{A}^*(X)) \cong H^*(\mathcal{A}^*(X))$ on homology groups of their total complexes, where we use the same symbol I_Δ for the induced map.

We need the fat realization $\|X\|$ and the geometric one $|X|$, of a simplicial space $X = \{X_q\}$. They are the topological spaces given by

$$\|X\| = \coprod_{q \geqq 0} \Delta^q \times X_q/{\sim}$$

with the identifications

(4-2) $\qquad (\varepsilon^i t, x) \sim (t, \varepsilon_i x) \quad \text{for } t \in \Delta^{q-1},\ x \in X_q, i = 0, \dots, q,\ q = 1, 2, \dots,$

and by

$$|X| = \coprod_{q \geqq 0} \Delta^q \times X_q/{\sim}, {\sim'}$$

with the identifications (4-2) and

$$(\eta^i t, x) \sim' (t, \eta_i x) \quad \text{for } t \in \Delta^{q+1},\ x \in X_q,\ i = 0, \dots, q,\ q = 0, 1, \dots.$$

In particular we have a natural map $\|X\| \to |X|$.

A simplicial map $f\colon X \to Y$ of simplicial spaces is a collection $f = \{f_q\colon X_q \to Y_q\}$ of continuous maps that commutes with the face and degeneracy operators. It induces a continuous map $\|f\|\colon \|X\| \to \|Y\|$ of fat realizations. Moreover, if X and Y are simplicial manifolds and all the f_q are differential maps, f is called a differential simplicial map. In this case, it induces homomorphisms $f^*\colon H^*(\mathcal{A}^*(Y)) \to H^*(\mathcal{A}^*(X))$ and $f^*\colon H^*(\mathbb{A}^*(Y)) \to H^*(\mathbb{A}^*(X))$ of cohomologies.

If X is a simplicial manifold, we have an isomorphism

(4-3) $\qquad \mathcal{I} = \mathcal{I}_X\colon H^*(\mathbb{A}^*(X)) \to H^*(\|X\|),$

which is the composition of isomorphisms

$$H^*(\mathbb{A}^*(X)) \xrightarrow[\cong]{I_\Delta} H^*(\mathcal{A}^*(X)) \xrightarrow[\cong]{I_{\mathrm{dR}}} H^*(\|X\|),$$

where $I_{\mathrm{dR}} = I_{\mathrm{dR},X}$ is the isomorphism in the de Rham theorem in this version. These isomorphisms are natural, that is, the diagram

$$\begin{tikzcd} \mathcal{I}\colon H^*(\mathbb{A}^*(X)) \arrow[r, "I_\Delta"] & H^*(\mathcal{A}^*(X)) \arrow[r, "I_{\mathrm{dR}}"] & H^*(\|X\|) \\ \mathcal{I}\colon H^*(\mathbb{A}^*(Y)) \arrow[r, "I_\Delta"] \arrow[u, "f^*"] & H^*(\mathcal{A}^*(Y)) \arrow[r, "I_{\mathrm{dR}}"] \arrow[u, "f^*"] & H^*(\|Y\|) \arrow[u, "\|f\|^*"] \end{tikzcd} \tag{4-4}$$

commutes for any differential simplicial map $f\colon X \to Y$ of simplicial manifolds. The isomorphism $\mathcal{I}$ is multiplicative with respect to the wedge product on the source and the cup product on the target.

We need a spectral sequence of the double complex $(\mathcal{A}^{k,l}(X), \delta', d'')$. Let $F^n\mathcal{A}^*(X) = \bigoplus_{p \geqq n} \mathcal{A}^{p,*}(X)$ be a subcomplex of $(\mathcal{A}^*(X), \delta)$ for each $n \geqq 0$. Then we have a filtration $\mathcal{A}^*(X) = F^0\mathcal{A}^*(X) \supseteqq \cdots \supseteqq F^n\mathcal{A}^*(X) \supseteqq \cdots$. The spectral sequence obtained from it has $E_1^{p,q} = H^{p+q}(F^p\mathcal{A}^*(X)/F^{p+1}\mathcal{A}^*(X), \delta)$ and the differential d_1 is identified as

$$[d_1\colon E_1^{p,q} \to E_1^{p+1,q}] = [\delta'\colon H^q(\mathcal{A}^{p,*}(X), d'') \to H^q(\mathcal{A}^{p+1,*}(X), d'')]. \tag{4-5}$$

Put $F^nH^*(\mathcal{A}^*(X)) = \mathrm{Im}[H^*(F^n\mathcal{A}^*(X)) \to H^*(\mathcal{A}^*(X))]$; then the E_∞–term is given by $E_\infty^{p,q} \cong F^pH^{p+q}(\mathcal{A}^*(X))/F^{p+1}H^{p+q}(\mathcal{A}^*(X))$. Therefore we have short exact sequences $\{0\} \to F^{p+1}H^{p+q}(\mathcal{A}^*(X)) \to F^pH^{p+q}(\mathcal{A}^*(X)) \to E_\infty^{p,q} \to \{0\}$.

For any set Y, we have the simplicial set

$$NY = \{NY_q = Y\}_{q=0,1,\ldots} \tag{4-6}$$

with all face and degeneracy operators equal to the identity. If Y is a smooth manifold, we have a simplicial manifold NY.

Lemma 4.1 *For the spectral sequence for the double complex $(\mathcal{A}^{*,*}(NY), \delta', d'')$, the following hold:*

(1) $E_r^{p,q} = 0$ *for* $p \neq 0$ *and* $r = 2, 3, \ldots, \infty$.

(2) $H^q(\mathcal{A}^*(NY)) \cong E_\infty^{p,q} = E_2^{0,q} \cong H^q(Y)$.

Proof Since $\mathcal{A}^{p,q}(NY) = \Omega^q(Y)$, we have $E_1^{p,q} = H^q(\Omega^*(Y), (-1)^p d) = H^q(Y)$ and $d_1 = \delta' = \sum_{i=0}^{p+1}(-1)^i\varepsilon_i^*$ on $E_1^{p,q}$ under the identification (4-5). Since $\varepsilon_i = \mathrm{id}$ for all i, we get $d_1 = \mathrm{id}$ if p is odd and $d_1 = 0$ if p is even. Thus we have

$$(E_1^{*,q}, d_1)\colon 0 \to E_1^{0,q} \xrightarrow{0} E_1^{1,q} \xrightarrow{\mathrm{id}} E_1^{2,q} \xrightarrow{0} \cdots$$

for all q, hence $E_2^{0,q} = E_1^{0,q} = H^q(Y)$ and $E_2^{p,q} = \{0\}$ for $p \neq 0$. These imply the lemma. □

The identity map $NY_q = Y \xrightarrow{\mathrm{id}} Y$ induces a natural map $\|NY\| \to |NY| = Y$ and vertical maps in the commutative diagram

$$
\begin{tikzcd}
H^*(\Omega^*(Y)) \arrow[r, equal] \arrow[d] & H^*(\mathcal{A}^{0,*}(NY)) \arrow[r, "I_{\mathrm{dR}}"] \arrow[d, "\cong"] & H^*(Y) \arrow[d] \\
\mathcal{I}\colon H^*(\mathbb{A}^*(NY)) \arrow[r, "I_\Delta"] & H^*(\mathcal{A}^*(NY)) \arrow[r, "I_{\mathrm{dR}}"] & H^*(\|NY\|)
\end{tikzcd}
\tag{4-7}
$$

Since all the horizontal maps are isomorphisms and the middle vertical map gives the isomorphism (2) in Lemma 4.1, the other vertical ones are also isomorphisms.

Next we define the simplicial manifolds needed in this paper. Recall that we have the sequence $p^*TM \to \mathcal{J} \to M$ of fiber bundles defined in Section 2. In particular, the first is a $U(n)$–vector bundle. In the rest of this section, we write simply $G = G^\delta$, where G^δ is the symplectomorphism group $G = \mathrm{Symp}(M,\omega)$ of (M,ω) with the discrete topology.

Put $N\bar{\mathcal{J}}_q = \mathcal{J} \times G^q$. Then we have the simplicial manifold $N\bar{\mathcal{J}} = \{N\bar{\mathcal{J}}_q\}_{q=0,1,\dots}$ equipped with face operators

$$
\begin{aligned}
\varepsilon_i\colon N\bar{\mathcal{J}}_q &\to N\bar{\mathcal{J}}_{q-1}, \\
(J,\varphi_1,\dots,\varphi_q) &\mapsto \begin{cases} (\varphi_1^{-1}J,\varphi_2,\dots,\varphi_q) & \text{if } i=0, \\ (J,\varphi_1,\dots,\varphi_i\varphi_{i+1},\dots,\varphi_q) & \text{if } 0<i<q, \\ (J,\varphi_1,\dots,\varphi_{q-1}) & \text{if } i=q, \end{cases}
\end{aligned}
\tag{4-8}
$$

for $q = 1,2,\dots$ and degeneracy ones

$$
\begin{aligned}
\eta_i\colon N\bar{\mathcal{J}}_q &\to N\bar{\mathcal{J}}_{q+1}, \\
(J,\varphi_1,\dots,\varphi_q) &\mapsto (J,\varphi_1,\dots,\varphi_i,\mathrm{id},\varphi_{i+1},\dots,\varphi_q),
\end{aligned}
\qquad \text{for } i=0,1,\dots,q
\tag{4-9}
$$

for $q = 0,1,\dots$.

For simplicity, we put $T = p^*TM$. Let $N\bar{T}_q = T \times G^q$; then the collection $N\bar{T} = \{N\bar{T}_q\}_{q=0,1,\dots}$ with ε_i and η_i given by the same expression (4-8) and (4-9) with $v \in T$ instead of $J \in \mathcal{J}$ is a simplicial manifold.

Similarly we have the simplicial manifold $N\bar{M} = \{N\bar{M}_q\}_{q=0,1,\dots}$ with $N\bar{M}_q = M \times G^q$.

Let $NG_q = G^q$; then we have the simplicial space $NG = \{NG_q\}_{q=0,1,\dots}$ equipped with face operators

$$\varepsilon_i\colon NG_q \to NG_{q-1}, \quad (\varphi_1,\dots,\varphi_q) \mapsto \begin{cases} (\varphi_2,\dots,\varphi_q) & \text{if } i=0, \\ (\varphi_1,\dots,\varphi_i\varphi_{i+1},\dots,\varphi_q) & \text{if } 0<i<q, \\ (\varphi_1,\dots,\varphi_{q-1}) & \text{if } i=q, \end{cases}$$

for $q=1,2,\dots$ and degeneracy ones

$$\begin{aligned} &\eta_i\colon NG_q \to NG_{q+1}, \\ &(\varphi_1,\dots,\varphi_q) \mapsto (\varphi_1,\dots,\varphi_i,\mathrm{id},\varphi_{i+1},\dots,\varphi_q), \end{aligned} \qquad \text{for } i=0,1,\dots,q$$

for $q=0,1,\dots$.

We note that if we consider all NG_q as 0–dimensional manifolds, namely, disjoint unions of points, we can also consider NG as a simplicial manifold. In this case the face and degeneracy operators are also given by the same formulas (4-8) and (4-9) with one point $\{*\}$ with trivial G action instead of $\mathcal{J}$.

Since the actions of G on the fiber bundles $T \to \mathcal{J} \to M \to \{*\}$ commute with the projections of them, we have a sequence of simplicial maps

$$N\overline{T} \to N\overline{\mathcal{J}} \to N\overline{M} \to NG.$$

The first map is a simplicial $U(n)$–vector bundle, and the second and the last are simplicial fiber bundles with fibers $\mathcal{J}_x$ for $x \in M$ and M, respectively.

These simplicial manifolds can be also obtained as follows. Let $N\overline{G}_q = G^{q+1}$; then we have the simplicial space $N\overline{G} = \{N\overline{G}_q\}$ with face and degeneracy operators

$$\begin{aligned} \varepsilon_i(\varphi_0,\dots,\varphi_q) &= (\varphi_0,\dots,\varphi_{i-1},\varphi_{i+1},\dots,\varphi_q), \\ \eta_i(\varphi_0,\dots,\varphi_q) &= (\varphi_0,\dots,\varphi_i,\varphi_i,\dots,\varphi_q). \end{aligned}$$

It is a simplicial principal G–bundle over NG with G–action $(\varphi_0,\dots,\varphi_q)\cdot\varphi = (\varphi_0\varphi,\dots,\varphi_q\varphi)$ for $((\varphi_0,\dots,\varphi_q),\varphi) \in N\overline{G}_q \times G$. The simplicial manifolds $N\overline{F}$ for $F = T, \mathcal{J}, M$ can be obtained as $N\overline{F} = \{N\overline{G}_q \times_G F\}$, that is, these simplicial manifolds are associated bundles with the simplicial principal G–bundle $N\overline{G}$ over NG. Here we remark that if we take appropriate identifications of $N\overline{G}_q \times_G F$ with $N\overline{F}_q = F \times G^q$, we have the expressions (4-8) and (4-9) of face and degeneracy operators of $N\overline{F}$. If we take other identifications, we of course obtain other expressions of face and degeneracy operators.

The fat realization $\|NG\|$ is known to be an explicit construction of the classifying space BG of G. The singular cohomology $H^*(\|NG\|)$ with coefficients in $\mathbb{R}$ is identified with the group cohomology $H^*(G,\mathbb{R})$. It is easy to see it if we use the de Rham isomorphism $I_{\mathrm{dR}}\colon H^*(\mathcal{A}^*(NG)) \cong H^*(\|NG\|)$. In fact, since $\mathcal{A}^{p,q}(NG) = \{0\}$ for $q > 0$ and $d'' = 0$, we have $\mathcal{A}^p(NG) = \mathcal{A}^{p,0}(NG) = \Omega^0(NG_p) = \mathrm{Map}(G^p, \mathbb{R})$ and $\delta = \delta'$. This implies

$$\begin{aligned}
&(\delta a)(\varphi_1, \dots, \varphi_{p+1}) \\
&= \Big(\sum_{i=0}^{p+1} (-1)^i \varepsilon_i^* a\Big)(\varphi_1, \dots, \varphi_{p+1}) \\
&= a(\varphi_2, \dots, \varphi_{p+1}) + \sum_{i=1}^{p} (-1)^i a(\varphi_1, \dots, \varphi_i\varphi_{i+1}, \dots, \varphi_{p+1}) + (-1)^{p+1} a(\varphi_1, \dots, \varphi_p)
\end{aligned}$$

for $a \in \mathrm{Map}(G^p, \mathbb{R})$ and $(\varphi_1, \dots, \varphi_{p+1}) \in G^{p+1}$. Thus the complex $(\mathcal{A}^*(NG), \delta)$ is nothing but the cochain complex $C^*(G, \mathbb{R})$ of G with coefficients in $\mathbb{R}$ in group cohomology theory [5]. Usually the cochain complex obtained from the CW complex decomposition of $\|NG\|$ in the obvious way is identified with the cochain complex of G, but we identify $C^*(G, \mathbb{R})$ and its cohomology group $H^*(G, \mathbb{R})$ with $(\mathcal{A}^*(NG), \delta)$ and $H^*(\mathcal{A}^*(NG))$, respectively, in this paper.

5 Simplicial 2–forms

In this section we construct a simplicial 2–form P on the simplicial manifold $N\bar{\mathcal{J}}$, which corresponds to the cohomology class σ defined in Proposition 2.2. Let (M, ω) be a connected, symplectic manifold of dimension $2n$, and $N\bar{T}$ and $N\bar{\mathcal{J}}$ the simplicial manifolds defined in Section 4. We assume that $c_1(M) + \kappa[\omega] = 0$ for some $\kappa \in \mathbb{R}$ and $H^1(M) = \{0\}$.

Since $N\bar{T} \to N\bar{\mathcal{J}}$ is a simplicial $U(n)$–vector bundle, we can consider the first Chern class $c_1(N\bar{T}) \in H^2(\mathbb{A}^*(N\bar{\mathcal{J}}))$ of it (see [6; 7] for characteristic classes of simplicial vector bundles). We define some simplicial 2–forms on $N\bar{\mathcal{J}}$. Let A be the $U(n)$–connection on $T = p^*TM$ and μ the 1–form on $\mathcal{J}$ chosen in Section 2, which satisfies the equality (2-1). Then we have the crossed homomorphism $f\colon G \to Z^1(\mathcal{J})$ in Lemma 2.1 or 3.1, the map $h\colon G \to \Omega^0(\mathcal{J})$ (3-1) and the 2–cocycle $s\colon G^2 \to \mathbb{R} \subset \Omega^0(\mathcal{J})$. Let $F \in Z^1(N\bar{\mathcal{J}}_1)$ be the closed 1–form on $N\bar{\mathcal{J}}_1$ corresponding to f, namely it is defined by $F(J, \varphi) = f(\varphi)(J)$ for $(J, \varphi) \in N\bar{\mathcal{J}}_1 = \mathcal{J} \times G$. Here we remark that the letter F is used also as F_B to express the curvature of a connection B, but we don't mind it since we can easily distinguish them. Similarly let $H \in \Omega^0(N\bar{\mathcal{J}}_1)$ and $S \in \Omega^0(N\bar{\mathcal{J}}_2)$ be the functions on $N\bar{\mathcal{J}}_1$ and $N\bar{\mathcal{J}}_2$ corresponding to h and s, respectively. Then we have $dH = F$, where d is the exterior derivative on $N\bar{\mathcal{J}}_1$ as an ordinary manifold, and $dS = 0$ since the restriction of S to each component $\mathcal{J} \times \{(\varphi_1, \varphi_2)\} \subset N\bar{\mathcal{J}}_2$ is constant.

Next we consider induced homomorphisms $\Omega^0(N\bar{\mathcal{J}}_1) \xrightarrow{\varepsilon_i^*} \Omega^0(N\bar{\mathcal{J}}_2) \xrightarrow{\varepsilon_j^*} \Omega^0(N\bar{\mathcal{J}}_3)$ by the face operators ε_i of $N\bar{\mathcal{J}}$.

Lemma 5.1 *The following equalities hold:*

(1) $$(\varepsilon_i^* H)(J,\varphi_1,\varphi_2) = \begin{cases} \varphi_{1*}\{h(\varphi_2)\}(J) & \text{if } i=0,\\ h(\varphi_1\varphi_2)(J) & \text{if } i=1,\\ h(\varphi_1)(J) & \text{if } i=2,\end{cases}$$

for $\varepsilon_i\colon N\bar{\mathcal{J}}_2 \to N\bar{\mathcal{J}}_1$ *and* $(J,\varphi_1,\varphi_2) \in N\bar{\mathcal{J}}_2$,

(2) $\varepsilon_0^* H - \varepsilon_1^* H + \varepsilon_2^* H = S$ *in* $\Omega^0(N\bar{\mathcal{J}}_2)$, *and*

(3) $\varepsilon_0^* S - \varepsilon_1^* S + \varepsilon_2^* S - \varepsilon_3^* S = 0$ *in* $\Omega^0(N\bar{\mathcal{J}}_3)$.

Proof (1) For $i=0$, we have

$$\begin{aligned}(\varepsilon_0^* H)(J,\varphi_1,\varphi_2) &= H(\varepsilon_0(J,\varphi_1,\varphi_2)) = H(\varphi_1^{-1}J,\varphi_2)\\ &= h(\varphi_2)(\varphi_1^{-1}J) = \varphi_{1*}\{h(\varphi_2)\}(J).\end{aligned}$$

The rest are similarly obtained: (2) is the definition of s in terms of S and H; (3) means $(\delta')^2 = 0$, so it is clear or is easily checked by a direct computation using properties of the ε_i. □

For each $0 \leqq m \leqq q$, let $\tau_{i_0,i_1,\ldots,i_m} = \tau^q_{i_0,i_1,\ldots,i_m}\colon N\bar{\mathcal{J}}_q \to N\bar{\mathcal{J}}_m$ be the composite map defined by $\tau_{i_0,i_1,\ldots,i_m} = \varepsilon_0 \ldots \widehat{\varepsilon}_{i_0} \cdots \widehat{\varepsilon}_{i_m} \cdots \varepsilon_q$ for $0 \leqq i_0 < i_1 < \cdots < i_m \leqq q$, where $\widehat{\varepsilon}_{i_j}$ means that ε_{i_j} is removed and $\tau^q_{0,1,\ldots,q} = \mathrm{id}$. We need the following lemma.

Lemma 5.2 *The following equalities hold for* H, S *and* $\tau_{i,\ldots} = \tau^q_{i,\ldots}$*:*

(1) $\tau_{i_0,i_1,\ldots,i_m}\varepsilon_j = \tau_{i_0,\ldots,i_{k-1},i_k+1,\ldots,i_m+1}$ *for* $i_{k-1} < j \leqq i_k$ *and* $0 \leqq m \leqq q$,

(2) $\tau^*_{1,j+1}H - \tau^*_{0,j+1}H + \tau^*_{0,1}H - \tau^*_{0,1,j+1}S = 0$ *for* $1 \leqq j \leqq q-1$ *and* $q \geqq 2$, *and*

(3) $\tau^*_{0,1,l+1}S - \tau^*_{0,1,k+1}S + \tau^*_{1,k+1,l+1}S - \tau^*_{0,k+1,l+1}S = 0$ *for* $1 \leqq k < l \leqq q-1$ *and* $q \geqq 3$.

Proof Since $\varepsilon_i\varepsilon_j = \varepsilon_{j-1}\varepsilon_i$ for $i<j$, we have $\varepsilon_k\varepsilon_l = \varepsilon_l\varepsilon_{k+1}$ for $k \geqq l$. Using them, we obtain the equality (1). Similarly we can show that the left-hand sides of the equalities (2) and (3) are equal to $\tau^*_{0,1,j+1}(\varepsilon_0^*H - \varepsilon_1^*H + \varepsilon_2^*H - S)$ and $\tau^*_{0,1,k+1,l+1}(\varepsilon_2^*S - \varepsilon_3^*S + \varepsilon_0^*S - \varepsilon_1^*S)$, respectively. We get (2) and (3) by Lemma 5.1. □

Let $\bar{\omega}_q$ be the 2–form on $\Delta^q \times N\bar{\mathcal{J}}_q$ which is the pullback of the symplectic form ω on M by the composition $\Delta^q \times N\bar{\mathcal{J}}_q = \Delta^q \times \mathcal{J} \times G^q \to \mathcal{J} \xrightarrow{p} M$ of projections for each q. Moreover, we define 1–forms Q_q and R_q on $\Delta^q \times N\bar{\mathcal{J}}_q$ for each q by

$$Q_q = \sum_{j=1}^{q} dt_j \cdot \tau_{0,j}^* H + \sum_{1 \leqq k < l \leqq q} (t_k dt_l - t_l dt_k) \cdot \tau_{0,k,l}^* S,$$

$$R_q = \sum_{l=0}^{q} t_l \tau_l^* \mu,$$

where $\tau_{j,\dots} = \tau^q_{j,\dots}$, $\mu \in \Omega^1(N\bar{\mathcal{J}}_0) = \Omega^1(\mathcal{J})$ and empty sums mean zero. We remark that here and hereafter functions and forms on Δ^q and on $N\bar{\mathcal{J}}_q$ can be considered as those on $\Delta^q \times N\bar{\mathcal{J}}_q$ by pulling them back by the projections $\Delta^q \times N\bar{\mathcal{J}}_q \to \Delta^q, N\bar{\mathcal{J}}_q$.

Recall that $\tau_l \colon N\bar{\mathcal{J}}_q \to N\bar{\mathcal{J}}_0, (t_0, \dots, t_q) \in \Delta^q$ and that A is a $U(n)$–connection of the vector bundle $T = N\bar{T}_0$ over $\mathcal{J} = N\bar{\mathcal{J}}_0$. Thus the sum $B_q = \sum_{l=0}^{q} t_l \widetilde{\tau_l^* A}$ is a $U(n)$–connection of $\Delta^q \times N\bar{T}_q$, where $\widetilde{\tau_l^* A}$ is the product connection of $\tau_l^* A$ with the trivial one in the direction of Δ^q for each l. We have its first Chern form $c_1(F_{B_q}) \in \Omega^2(\Delta^q \times N\bar{\mathcal{J}}_q)$. Put

$$P_q = c_1(F_{B_q}) + \kappa \bar{\omega}_q + d(Q_q + R_q); \tag{5-1}$$

then we have $P_q \in \Omega^2(\Delta^q \times N\bar{\mathcal{J}}_q)$.

Lemma 5.3 (1) *$\bar{\omega} = \{\bar{\omega}_q\}$ and $c_1 = \{c_{1,q} = c_1(F_{B_q})\}$ are closed simplicial 2–forms on $N\bar{\mathcal{J}} = \{N\bar{\mathcal{J}}_q\}$.*

(2) *$Q = \{Q_q\}$ and $R = \{R_q\}$ are simplicial 1–forms on $N\bar{\mathcal{J}}$.*

(3) *$P = \{P_q\}$ is a closed simplicial 2–form on $N\bar{\mathcal{J}}$. Its cohomology class satisfies $[P] = [c_1 + \kappa\bar{\omega}] \in H^2(\mathbb{A}^*(N\bar{\mathcal{J}}))$ and is independent of the choice of the ingredients, in particular of A.*

Proof (1) It is clear for $\bar{\omega}$. For c_1, since $\tau_l \varepsilon_i = \tau_l$ for $l \leqq i-1$ and $\tau_l \varepsilon_i = \tau_{l+1}$ for $l \geqq i$ by Lemma 5.2, we have

$$\begin{aligned}(\mathrm{id} \times \varepsilon_i)^* B_{q-1} &= \sum_{l=0}^{q-1} t_l \widetilde{(\tau_l \varepsilon_i)^* A} = \sum_{l=0}^{i-1} t_l \widetilde{\tau_l^* A} + \sum_{l=i}^{q-1} t_l \widetilde{\tau_{l+1}^* A} \\ &= \sum_{l=0}^{i-1} t_l \widetilde{\tau_l^* A} + \sum_{l=i+1}^{q} t_{l-1} \widetilde{\tau_l^* A} = (\varepsilon^i \times \mathrm{id})^* \left(\sum_{l=0}^{q} t_l \widetilde{\tau_l^* A} \right) = (\varepsilon^i \times \mathrm{id})^* B_q\end{aligned}$$

as connections on $\Delta^{q-1}\times N\overline{T}_q$, hence we obtain $(\varepsilon^i\times\mathrm{id})^*c_{1,q}=(\mathrm{id}\times\varepsilon_i)^*c_{1,q-1}$ for each q. Thus c_1 is a simplicial 2–form on $N\overline{\mathcal{J}}$. Its closedness is obvious.

(2) On $\Delta^{q-1}\times N\overline{\mathcal{J}}_q$, for $i=1,\dots,q$ we have

$$\begin{aligned}
&(\mathrm{id}\times\varepsilon_i)^*Q_{q-1}\\
&=\sum_{j=1}^{q-1}dt_j\cdot(\tau_{0,j}\varepsilon_i)^*H+\sum_{1\leqq k<l\leqq q-1}(t_k\,dt_l-t_l\,dt_k)\cdot(\tau_{0,k,l}\varepsilon_i)^*S\\
&=\sum_{j=1}^{i-1}dt_j\cdot\tau_{0,j}^*H+\sum_{j=i}^{q-1}dt_j\cdot\tau_{0,j+1}^*H+\sum_{1\leqq i\leqq k<l\leqq q-1}(t_k\,dt_l-t_l\,dt_k)\cdot\tau_{0,k+1,l+1}^*S\\
&\qquad+\sum_{1\leqq k<i\leqq l\leqq q-1}(t_k\,dt_l-t_l\,dt_k)\cdot\tau_{0,k,l+1}^*S+\sum_{1\leqq k<l<i\leqq q}(t_k\,dt_l-t_l\,dt_k)\cdot\tau_{0,k,l}^*S.
\end{aligned}$$

On the other hand, for $i=1,\dots,q$ we have

$$\begin{aligned}
&(\varepsilon^i\times\mathrm{id})^*Q_q\\
&=\sum_{j=1}^{i-1}dt_j\cdot\tau_{0,j}^*H+\sum_{j=i+1}^{q}dt_{j-1}\cdot\tau_{0,j}^*H\\
&\qquad+\Bigg(\sum_{1\leqq i<k<l\leqq q}+\sum_{1\leqq k<i<l\leqq q}+\sum_{1\leqq k<l<i\leqq q}\Bigg)(\varepsilon^i)^*(t_k\,dt_l-t_l\,dt_k)\cdot\tau_{0,k,l}^*S,
\end{aligned}$$

where we used $(\varepsilon^i)^*(t_k\,dt_l-t_l\,dt_k)=0$ for $i=k,l,$, so

$$\begin{aligned}
&(\varepsilon^i\times\mathrm{id})^*Q_q\\
&=\sum_{j=1}^{i-1}dt_j\cdot\tau_{0,j}^*H+\sum_{j=i}^{q-1}dt_j\cdot\tau_{0,j+1}^*H+\sum_{1\leqq i<k<l\leqq q}(t_{k-1}\,dt_{l-1}-t_{l-1}\,dt_{k-1})\cdot\tau_{0,k,l}^*S\\
&\qquad+\sum_{1\leqq k<i<l\leqq q}(t_k\,dt_{l-1}-t_{l-1}\,dt_k)\cdot\tau_{0,k,l}^*S+\sum_{1\leqq k<l<i\leqq q}(t_k\,dt_l-t_l\,dt_k)\cdot\tau_{0,k,l}^*S,
\end{aligned}$$

which is equal to the result of the computation above if we change the variables k and l appropriately. Thus we have $(\varepsilon^i\times\mathrm{id})^*Q_q=(\mathrm{id}\times\varepsilon_i)^*Q_{q-1}$ for $i=1,\dots,q$.

For $i=0$, eliminating t_0 by use of the equality $\sum_{j=0}^{q-1}t_j=1$ on Δ^{q-1} and a similar computation as above implies

$$
\begin{aligned}
&(\varepsilon^0 \times \mathrm{id})^* Q_q - (\mathrm{id} \times \varepsilon_0)^* Q_{q-1} \\
&\quad = \sum_{j=1}^{q-1} dt_j \cdot (-\tau_{1,j+1}^* H + \tau_{0,j+1}^* H - \tau_{0,1}^* H + \tau_{0,1,j+1}^* S) \\
&\qquad\quad + \sum_{1 \leqq k < l \leqq q-1} (t_k dt_l - t_l dk_k) \\
&\qquad\qquad\qquad \cdot (-\tau_{0,1,l+1}^* S + \tau_{0,1,k+1}^* S - \tau_{1,k+1,l+1}^* S + \tau_{0,k+1,l+1}^* S) \\
&\quad = 0.
\end{aligned}
$$

Here the last equality is obtained by Lemma 5.2. Thus we have $(\varepsilon^i \times \mathrm{id})^* Q_q = (\mathrm{id} \times \varepsilon_i)^* Q_{q-1}$ on $\Delta^{q-1} \times N\bar{\mathcal{J}}_q$ for $0 \leqq i \leqq q$ and $q = 1, 2, \ldots$, hence $Q = \{Q_q\}$ is a simplicial 1–form on $N\bar{\mathcal{J}}$. R is shown to be a simplicial 1–form in a similar way as c_1 is, so its proof is omitted.

(3) This is clear from (1) and (2) of this lemma, the definition (5-1) of P_q, and properties of the first Chern form. □

Let $a_l = \tau_l^* A - \tau_0^* A$ for $l = 0, 1, \ldots, q$. Since the $\tau_l^* A$ are connections of $N\bar{T}_q$, the a_l are $\mathrm{End}_{\mathbb{C}}\, N\bar{T}_q$–valued 1–forms on $N\bar{\mathcal{J}}_q$, hence the $c_1(a_l)$ are 1–forms on $N\bar{\mathcal{J}}_q$. They are considered as 1–forms on $\Delta^q \times N\bar{\mathcal{J}}_q$, as mentioned before.

Lemma 5.4 (1) $c_1(F_{B_q}) = \sum_{l=0}^{q} t_l c_1(F_{\tau_l^* A}) + \sum_{l=1}^{q} dt_l \wedge c_1(a_l)$.

(2) $\tau_{0,l}^* F = c_1(a_l) + \tau_l^* \mu - \tau_0^* \mu$ *for* $0 \leqq l \leqq q$, *where* F *is the closed* 1*–form on* $N\bar{\mathcal{J}}_1$ *defined at the beginning of this section.*

Proof (1) On $\Delta^q \times N\bar{\mathcal{J}}_q$, since $B_q = \widetilde{\tau_0^* A} + \sum_{l=1}^{q} t_l a_l$, we have

$$F_{B_q} = F_{\widetilde{\tau_0^* A}} + d_{\widetilde{\tau_0^* A}}\Big(\sum_{l=1}^{q} t_l a_l\Big) + \Big(\sum_{l=1}^{q} t_l a_l\Big) \wedge \Big(\sum_{l=1}^{q} t_l a_l\Big),$$

hence

$$c_1(F_{B_q}) = c_1\Big(F_{\widetilde{\tau_0^* A}} + d_{\widetilde{\tau_0^* A}}\Big(\sum_{l=1}^{q} t_l a_l\Big)\Big),$$

where $d_{\widetilde{\tau_0^* A}}$ is the covariant exterior derivative with respect to $\widetilde{\tau_0^* A}$. Since

$$d_{\widetilde{\tau_0^* A}}\Big(\sum_{l=1}^{q} t_l a_l\Big) = \sum_{l=1}^{q} (dt_l \wedge a_l + t_l d_{\tau_0^* A} a_l),$$

we obtain

$$c_1(F_{B_q}) = c_1\Big(F_{\tau_0^* A} + \sum_{l=1}^{q} t_l d_{\tau_0^* A} a_l\Big) + \sum_{l=1}^{q} dt_l \wedge c_1(a_l)$$

$$= \sum_{l=0}^{q} t_l c_1(F_{\tau_0^* A} + d_{\tau_0^* A} a_l) + \sum_{l=1}^{q} dt_l \wedge c_1(a_l)$$
$$= \sum_{l=0}^{q} t_l c_1(F_{\tau_l^* A}) + \sum_{l=1}^{q} dt_l \wedge c_1(a_l),$$

where we used $\sum_{l=0}^{q} t_l = 1$, $a_0 = 0$, and $F_{\widetilde{\tau_0^* A}} = F_{\tau_0^* A}$ as forms on $\Delta^q \times N\bar{\mathcal{J}}_q$.

(2) Since $F \in \Omega^1(N\bar{\mathcal{J}}_1)$ and $\tau_{0,l}\colon N\bar{\mathcal{J}}_q \to N\bar{\mathcal{J}}_1$, we have $\tau_{0,l}^* F \in \Omega^1(N\bar{\mathcal{J}}_q)$. Put $\Phi_l := \Phi_l(\varphi_1, \dots, \varphi_q) := \varphi_1 \varphi_2 \cdots \varphi_l$ for $(J, \varphi_1, \dots, \varphi_q) \in N\bar{\mathcal{J}}_q = \mathcal{J} \times G^q$. Then we have $\tau_{0,l}(J, \varphi_1, \dots, \varphi_q) = \varepsilon_1 \cdots \widehat{\varepsilon}_l \cdots \varepsilon_q(J, \varphi_1, \dots, \varphi_q) = (J, \Phi_l)$ and

$$\tau_{0,l}^* F = f(\Phi_l) = c_1((\Phi_l)_* A - A) + (\Phi_l)_* \mu - \mu = c_1(a_l) + \tau_l^* \mu - \tau_0^* \mu$$

on $\mathcal{J} = \mathcal{J} \times \{(\varphi_1, \dots, \varphi_q)\} \subset N\bar{\mathcal{J}}_q$, where $\tau_l = \tau_l^q\colon N\bar{\mathcal{J}}_q \to N\bar{\mathcal{J}}_0 = \mathcal{J}$ is given by $\tau_l(J, \varphi_1, \dots, \varphi_q) = \Phi_l^{-1} J$ for $1 \leqq l \leqq q$ and $\tau_0(J, \varphi_1, \dots, \varphi_q) = J$. □

Lemma 5.5 $P_q = 2 \sum_{1 \leqq k < l \leqq q} dt_k \wedge dt_l \cdot \tau_{0,k,l}^* S$.

Proof On $\Delta^q \times N\bar{\mathcal{J}}_q$, we have $dR_q = \sum_{l=0}^{q} dt_l \wedge \tau_l^* \mu - \sum_{l=0}^{q} t_l c_1(F_{\tau_l^* A}) - \kappa \bar{\omega}_q$, where we used the equality (2-1) and $\sum_{l=0}^{q} t_q = 1$ on Δ^q. Using this, Lemma 5.4 and $\sum_{l=0}^{q} dt_l = 0$, we have

$$c_1(F_{B_q}) + \kappa \bar{\omega}_q + dR_q = \sum_{l=1}^{q} dt_l \wedge c_1(a_l) + \sum_{l=0}^{q} dt_l \wedge \tau_l^* \mu$$
$$= \sum_{l=1}^{q} dt_l \wedge \{c_1(a_l) + \tau_l^* \mu - \tau_0^* \mu\} = \sum_{l=1}^{q} dt_l \wedge \tau_{0,l}^* F.$$

Since $dH = F$ and $dS = 0$, we have

$$dQ_q = -\sum_{j=1}^{q} dt_j \wedge \tau_{0,j}^* F + 2 \sum_{1 \leqq k < l \leqq q} dt_k \wedge dt_l \cdot \tau_{0,k,l}^* S.$$

The definition (5-1) of P_q and the computations above imply this lemma. □

Corollary 5.6 *$P \in \mathbb{A}^{2,0}(N\bar{\mathcal{J}})$ and $I_\Delta(P) = S \in \mathcal{A}^{2,0}(N\bar{\mathcal{J}})$.*

Proof By Lemma 5.5, the former is clear. We then have the latter:

$$I_\Delta(P) = \int_{\Delta^2} P_2 = \tau_{0,1,2}^* S = S \in \Omega^0(N\bar{\mathcal{J}}_2) = \mathcal{A}^{2,0}(N\bar{\mathcal{J}})$$

by the definition of I_Δ and since $\int_{\Delta^2} dt_1 \wedge dt_2 = \frac{1}{2}$ and $\tau_{0,1,2} = \mathrm{id}$ on $N\bar{\mathcal{J}}_2$. □

6 Spectral sequence of $\mathcal{A}^{*,*}(N\bar{\mathcal{J}})$

In this section we consider the double complex $(\mathcal{A}^{*,*}(X), \delta', d'')$ in Section 4 with $X = N\bar{\mathcal{J}}$. Its spectral sequence has $E_1^{p,q} = H^q(\mathcal{A}^{p,*}(N\bar{\mathcal{J}}), d'')$ and

$$d_1 = \delta' \colon H^q(\mathcal{A}^{p,*}(N\bar{\mathcal{J}}), d'') \to H^q(\mathcal{A}^{p+1,*}(N\bar{\mathcal{J}}), d'').$$

Since $N\bar{\mathcal{J}}_p = \mathcal{J} \times G^p$ and the topology of G is the discrete one, we have the identification

$$\mathcal{A}^{p,*}(N\bar{\mathcal{J}}) = \Omega^*(N\bar{\mathcal{J}}_p) \cong \mathrm{Map}(G^p, \Omega^*(\mathcal{J})) = \mathrm{Map}(NG_p, \Omega^*(\mathcal{J})),$$

where Map denotes the set of all maps. Under this identification, the differential d'' on $\mathcal{A}^{p,*}(N\bar{\mathcal{J}})$ corresponds to $(-1)^p$ times the exterior differential on the target $\Omega^*(\mathcal{J})$ of the maps $NG_p \to \Omega^*(\mathcal{J})$. Thus we get $E_1^{p,q} = H^q(\mathcal{A}^{p,*}(N\bar{\mathcal{J}}), d'') \cong \mathrm{Map}(NG_p, H^q(\mathcal{J}))$ and the differential

$$d_1 = \delta' \colon \mathrm{Map}(NG_p, H^q(\mathcal{J})) \to \mathrm{Map}(NG_{p+1}, H^q(\mathcal{J}))$$

given by

$$\begin{aligned}
\text{(6-1)} \quad (\delta' a)(\varphi_1, \dots, \varphi_{p+1})
&= \Big(\sum_{i=0}^{p+1} (-1)^i \varepsilon_i^* a\Big)(\varphi_1, \dots, \varphi_{p+1}) \\
&= \varphi_{1*} a(\varphi_2, \dots, \varphi_{p+1}) + \sum_{i=1}^{p} (-1)^i a(\varphi_1, \dots, \varphi_i\varphi_{i+1}, \dots, \varphi_{p+1}) \\
&\qquad\qquad + (-1)^{p+1} a(\varphi_1, \dots, \varphi_p)
\end{aligned}$$

for $a \in \mathrm{Map}(NG_p, H^q(\mathcal{J}))$ and $(\varphi_1, \dots, \varphi_{p+1}) \in NG_{p+1} = G^{p+1}$, where the ε_i are given by (4-8).

On the other hand, we have cohomology groups of G in group theory [5]. In particular, we consider the group cohomology $H^*(G, H^q(\mathcal{J}))$ with values in the G–module $H^q(\mathcal{J})$ whose G–action is given by $(\varphi, a) \mapsto \varphi_* a := (\varphi^{-1})^* a$ for any $(\varphi, a) \in G \times H^q(\mathcal{J})$. Its cochain group is given by $C^p(G, H^q(\mathcal{J})) = \mathrm{Map}(G^p, H^q(\mathcal{J}))$ and its coboundary operator $\delta \colon C^p(G, H^q(\mathcal{J})) \to C^{p+1}(G, H^q(\mathcal{J}))$ by the same expression (6-1). Thus the differential $d_1 = \delta'$ of the spectral sequence is identified with the coboundary operator δ of the cochain complex $C^*(G, H^q(\mathcal{J}))$ and the E_2–term is given by $E_2^{p,q} = H^p(G, H^q(\mathcal{J}))$.

Let $\pi \colon N\bar{\mathcal{J}} \to N\bar{M} \to NG$ be the composition of simplicial bundle projections.

Lemma 6.1 *Assume $H^1(M)=\{0\}$, then there exists an exact sequence*

$$\{0\} \to H^2(G,\mathbb{R}) \xrightarrow{\pi^*} H^2(\mathcal{A}^*(N\bar{\mathcal{J}})) \to H^2(\mathcal{J})^G, \tag{6-2}$$

where the middle map π^* is the induced homomorphism by π under the identification $H^2(G,\mathbb{R})=H^2(\mathcal{A}^*(NG))$, and $H^2(\mathcal{J})^G$ is the invariant subspace of $H^2(\mathcal{J})$ under the action of G.

Proof We consider the spectral sequence above. The assumption $H^1(\mathcal{J})\cong H^1(M)=\{0\}$ implies $E_2^{p,1}=H^p(G,H^1(\mathcal{J}))=\{0\}$ for all p. Hence we have $H^2(\mathcal{A}^*(N\bar{\mathcal{J}}))\cong E_\infty^{0,2}\oplus\{0\}\oplus E_\infty^{2,0}$, more precisely a short exact sequence

$$\{0\} \to E_\infty^{2,0} \to H^2(\mathcal{A}^*(N\bar{\mathcal{J}})) \to E_\infty^{0,2} \to \{0\}. \tag{6-3}$$

Moreover we can easily check that $E_\infty^{0,2}=E_4^{0,2}\subset E_2^{0,2}=H^0(G,H^2(\mathcal{J}))=H^2(\mathcal{J})^G$ and $E_\infty^{2,0}=E_2^{2,0}\cong H^2(G,\mathbb{R})$. Thus we obtain the exact sequence (6-2). The middle map is given as follows. By considering NG as a simplicial manifold as mentioned before, we have $H^2(\mathcal{A}^*(NG))\cong\{0\}\oplus\{0\}\oplus E_\infty^{2,0}(NG)$ from the spectral sequence $\{E_r^{p,q}(NG)\}$ for the double complex $(\mathcal{A}^{*,*}(NG),\delta',d'')$. The induced map $\pi^*\colon H^2(\mathcal{A}^*(NG))\to H^2(\mathcal{A}^*(N\bar{\mathcal{J}}))$ preserves the decompositions by the E_∞–terms of them because π is a simplicial map. This and our convention $H^2(G,\mathbb{R})=H^2(\mathcal{A}^*(NG))$ imply the middle map is given as in the statement of this lemma. □

We return to the situation in Section 5, namely, we assume $c_1(M)+\kappa[\omega]=0$ and $H^1(M)=\{0\}$. Thus we have the cohomology class $[P]$ appearing in Lemma 5.3. By Corollary 5.6 and the proof of Lemma 6.1, $I_\Delta([P])$ belongs to the image of π^*.

Lemma 6.2 *The equality $\pi^*\sigma=I_\Delta([P])$ holds in $H^2(\mathcal{A}^*(N\bar{\mathcal{J}}))$. Moreover, if $[P]\neq 0\in H^2(\mathbb{A}^*(N\bar{\mathcal{J}}))$, then $\sigma\neq 0\in H^2(G,\mathbb{R})$.*

Proof The image of $s\in\mathrm{Map}(G^2,\mathbb{R})$ under the map $\mathrm{Map}(G^2,\mathbb{R})=\mathcal{A}^2(NG)\xrightarrow{\pi^*}\mathcal{A}^2(N\bar{\mathcal{J}})$ is nothing but $S\in\mathcal{A}^2(N\bar{\mathcal{J}})$. Thus we get $\pi^*\sigma=[S]$ in $H^2(\mathcal{A}^*(N\bar{\mathcal{J}}))$. Corollary 5.6 implies the required equality.

Since $I_\Delta\colon H^*(\mathbb{A}^*(N\bar{\mathcal{J}}))\to H^*(\mathcal{A}^*(N\bar{\mathcal{J}}))$ is an isomorphism and π^* is injective on the second cohomology by Lemma 6.1, we have the rest. □

7 Cohomology classes

In this section we define the cohomology class in $H^2(\|N\bar{\mathcal{J}}\|)$ corresponding to $[P]$ and prove Theorem 2.4, Corollary 2.5 and Theorem 2.6.

The fat realization $\|N\bar{T}\| \to \|N\bar{\mathcal{J}}\|$ of the simplicial $U(n)$–vector bundle $N\bar{T} \to N\bar{\mathcal{J}}$ is an ordinary $U(n)$–vector bundle. Thus we have the first Chern class $c_1(\|N\bar{T}\|) \in H^2(\|N\bar{\mathcal{J}}\|)$. On the other hand we defined closed simplicial 2–forms $\bar{\omega}$ and P on $N\bar{\mathcal{J}}$ in Lemma 5.3, hence we have cohomology classes $[\bar{\omega}]$ and $[P]$ in $H^2(\mathbb{A}^*(N\bar{\mathcal{J}}))$. Put $\|\bar{\omega}\| = \mathcal{I}([\bar{\omega}])$ and $\|P\| = \mathcal{I}([P])$ in $H^2(\|N\bar{\mathcal{J}}\|)$, where $\mathcal{I} = \mathcal{I}_{N\bar{\mathcal{J}}}$ is the isomorphism (4-3) with $X = N\bar{\mathcal{J}}$. Since the map $\mathcal{I}$ is natural and Chern–Weil theory holds for simplicial vector bundles (see [6; 7]), Lemma 5.3(3) implies

$$\|P\| = c_1(\|N\bar{T}\|) + \kappa\|\bar{\omega}\| \in H^2(\|N\bar{\mathcal{J}}\|). \tag{7-1}$$

Consider the Leray–Serre spectral sequence $\{E_r^{p,q}\}$ for the bundle $\|\pi\|\colon \|N\bar{\mathcal{J}}\| \to \|NG\|$ with fiber $\mathcal{J}$ which is the fat realization of $\pi\colon N\bar{\mathcal{J}} \to NG$. The E_2–term is given by $E_2^{p,q} = H^p(\|NG\|, \mathcal{H}^q)$, where $\mathcal{H}^q$ is the local system associated with this bundle, the coefficients of which are in $H^q(\mathcal{J})$. In the same way as the proof of Lemma 6.1, we have a short exact sequence

$$\{0\} \to E_\infty^{2,0} \to H^2(\|N\bar{\mathcal{J}}\|) \to E_\infty^{0,2} \to \{0\}, \tag{7-2}$$

$E_\infty^{2,0} = E_2^{2,0} = H^2(\|NG\|)$ and $E_\infty^{0,2} = E_4^{0,2} \subset E_2^{0,2} = H^0(\|NG\|, \mathcal{H}^2) = H^2(\mathcal{J})^G$. We then obtain the exact sequence

$$\{0\} \to H^2(\|NG\|) \xrightarrow{\|\pi\|^*} H^2(\|N\bar{\mathcal{J}}\|) \to H^2(\mathcal{J})^G, \tag{7-3}$$

where the last map is the induced homomorphism by the inclusion $\mathcal{J} \hookrightarrow \|N\bar{\mathcal{J}}\|$ of the fiber at the 0–cell $\Delta^0 \times NG_0$.

Let $\|\sigma\| \in H^2(\|NG\|)$ be the image of $\sigma \in H^2(G, \mathbb{R})$ under the map

$$H^2(G, \mathbb{R}) = H^2(\mathcal{A}^*(NG)) \xrightarrow{I_{\mathrm{dR}}} H^2(\|NG\|).$$

Lemma 7.1 *Let $\{E_r^{p,q}\}$ be the Leray–Serre spectral sequence for the bundle*

$$\|\pi\|\colon \|N\bar{\mathcal{J}}\| \to \|NG\|,$$

then the following hold in the exact sequence (7-2) or (7-3):

(1) $\|\bar{\omega}\| \in E_\infty^{0,2}$*; precisely, the image of* $\|\bar{\omega}\|$ *in* $E_\infty^{0,2}$ *is nonzero if* ω *is not exact;*

(2) $\|P\| \in E_\infty^{2,0}$*; precisely,* $\|P\|$ *belongs to the image of* $E_\infty^{2,0}$ *in* $H^2(\|N\bar{\mathcal{J}}\|)$*; and*

(3) $\|\pi\|^*(\|\sigma\|) = \|P\|$.

Proof Let $N\mathcal{J}$ be the simplicial manifold (4-6) with $Y = \mathcal{J}$. The injections $N\mathcal{J}_q = \mathcal{J} \to \mathcal{J} \times G^q = N\bar{\mathcal{J}}_q$, $J \mapsto (J, \mathrm{id}, \dots, \mathrm{id})$, induce a simplicial map $\iota\colon N\mathcal{J} \to N\bar{\mathcal{J}}$.

Thus we have simplicial maps $N\mathcal{J} \xrightarrow{\iota} N\bar{\mathcal{J}} \xrightarrow{\pi} NG$. By the naturality of the de Rham isomorphism I_{dR}, the diagram

$$\begin{tikzcd} H^2(\mathcal{A}^*(NG)) \arrow[r, "\pi^*"] \arrow[d, "I_{\mathrm{dR}}"] & H^2(\mathcal{A}^*(N\bar{\mathcal{J}})) \arrow[r, "\iota^*"] \arrow[d, "I_{\mathrm{dR}}"] & H^2(\mathcal{A}^*(N\mathcal{J})) \arrow[d, "I_{\mathrm{dR}}"] \\ H^2(\|NG\|) \arrow[r, "\|\pi\|^*"] & H^2(\|N\bar{\mathcal{J}}\|) \arrow[r, "\|\iota\|^*"] & H^2(\|N\mathcal{J}\|) \end{tikzcd} \tag{7-4}$$

commutes. Moreover we have $H^*(\mathcal{A}^*(N\mathcal{J})) \cong H^*(\mathcal{J}) \cong H^*(\|N\mathcal{J}\|)$ by Lemma 4.1 and the diagram (4-7). The commutative diagram (7-4) implies that the exact sequences (6-2) and (7-3), and hence (6-3) and (7-2), are isomorphic through I_{dR}.

Since $\bar{\omega} \in \mathbb{A}^{0,2}(N\bar{\mathcal{J}})$, we have $I_\Delta(\bar{\omega}) = p^*\omega$ in $\mathcal{A}^{0,2}(N\bar{\mathcal{J}}) = \Omega^2(\mathcal{J})$, hence $[I_\Delta(\bar{\omega})] = [p^*\omega] \neq 0$ in $E_\infty^{0,2} \subset H^2(\mathcal{J})^G$ of (6-3). Since $P \in \mathbb{A}^{2,0}(N\bar{\mathcal{J}})$ by Corollary 5.6, we have $I_\Delta(P) \in \mathcal{A}^{2,0}(N\bar{\mathcal{J}})$, hence $[I_\Delta(P)] \in E_\infty^{2,0}$ in (6-3). Thus we have $\|\bar{\omega}\| = I_{\mathrm{dR}}([I_\Delta(\bar{\omega})]) \in E_\infty^{0,2}$ and $\|P\| = I_{\mathrm{dR}}\big([I_\Delta(P)]\big) \in E_\infty^{2,0}$ in (7-2). This completes the proof of (1) and (2). (3) is clear from Lemma 6.2 and the commutativity of the first square of the diagram (7-4). □

Recall that we defined the sequence of fiber bundles $N\bar{T} \to N\bar{\mathcal{J}} \xrightarrow{\bar{\beta}} N\bar{M} \to NG$ of simplicial manifolds in Section 4, where the second simplicial map is named $\bar{\beta}$. Taking the fat realization of it, we have a sequence of fiber bundles

$$\|N\bar{T}\| \xrightarrow{\alpha} \|N\bar{\mathcal{J}}\| \xrightarrow{\beta = \|\bar{\beta}\|} \|N\bar{M}\| \xrightarrow{\gamma} \|NG\|,$$

where the projections are (re)named as written. We note that $\gamma \circ \beta = \|\pi\|$. Since the fibers of β are contractible, there exists a section $j\colon \|N\bar{M}\| \to \|N\bar{\mathcal{J}}\|$ of it. In the same way as the definition of $\bar{\omega}$, the symplectic form ω on M defines a simplicial closed 2–form $\omega' \in \mathbb{A}^2(N\bar{M})$. It satisfies $\bar{\omega} = \bar{\beta}^*\omega'$ in $\mathbb{A}^2(N\bar{\mathcal{J}})$. Put $\|\omega\| = \mathcal{I}([\omega']) \in H^2(\|N\bar{M}\|)$; then we have $j^*\|\bar{\omega}\| = \|\omega\|$.

Let $\zeta \to \|N\bar{M}\|$ be the tangent bundle along the fibers of the M–bundle $\gamma\colon \|N\bar{M}\| \to \|NG\|$. Since ζ is a symplectic vector bundle, we have the first Chern class $c_1(\zeta) \in H^2(\|N\bar{M}\|)$, which agrees with $j^*c_1(\|N\bar{T}\|)$.

Corollary 7.2 *Let $\{E_r^{p,q}\}$ be the Leray–Serre spectral sequence for the bundle*

$$\gamma\colon \|N\bar{M}\| \to \|NG\|;$$

then the following hold:

1. $\|\omega\| \in E_\infty^{0,2}$,
2. $\gamma^*\|\sigma\| \in E_\infty^{2,0}$, *and*
3. $\gamma^*\|\sigma\| = c_1(\zeta) + \kappa\|\omega\|$ *in* $H^2(\|N\bar{M}\|)$.

Proof The spectral sequences $\{E_r^{p,q}\}$ in Lemma 7.1 and in this corollary are isomorphic to each other for $r \geqq 1$ under the induced homomorphisms by fiber-preserving maps $\beta\colon \|N\bar{\mathcal{J}}\| \rightleftarrows \|N\overline{M}\| : j$ over the identity of $\|NG\|$ since the fibers of β are contractible and j is a section of β. Considering the pullback by j, we obtain the results from Lemma 7.1 and the equality (7-1) because of $\|\pi\| \circ j = \gamma$. $\square$

Proof of Theorem 2.4 By the definition of $\|\sigma\|$, Corollary 7.2(3) is nothing but the equality of Theorem 2.4, because $\gamma\colon \|N\overline{M}\| \to \|NG\|$ is a realization of $\pi\colon E^\delta M \to BG^\delta$ in Theorem 2.4. The injectivity of π^*, namely that of γ^*, is obtained from that of $\|\pi\|^*$ in the exact sequence (7-3) and the equality $\|\pi\| \circ j = \gamma$ since the section j induces an isomorphism on cohomology. $\square$

Take a basepoint $b \in M$ and fix it. Let $G_b = \{\varphi \in G \mid \varphi(b) = b\}$ be a subgroup of G. Replacing G with G_b in the definitions of NG, $N\overline{M}$, $N\bar{\mathcal{J}}$ and $N\overline{T}$, and considering the inclusion $G_b \hookrightarrow G$, we have a sequence of simplicial fiber bundles $N\overline{T}_b \to N\bar{\mathcal{J}}_b \to N\overline{M}_b \to NG_b$ and a commutative diagram of simplicial maps

$$\begin{tikzcd}
N\overline{T}_b \arrow[r] \arrow[d] & N\bar{\mathcal{J}}_b \arrow[r] \arrow[d] & N\overline{M}_b \arrow[r, shift left] \arrow[d, "\bar{\mu}"] & NG_b \arrow[l, shift left, "s"] \arrow[d] \\
N\overline{T} \arrow[r] & N\bar{\mathcal{J}} \arrow[r, "\bar{\beta}"] & N\overline{M} \arrow[r] & NG
\end{tikzcd}$$

Here maps needed later are named as in the diagram and $s = \{s_q\}\colon NG_b \to N\overline{M}_b$ is the simplicial map given by

$$s_q\colon NG_{b,q} = G_b^q \to M \times G_b^q = N\overline{M}_{b,q}, \quad s_q(\varphi_1, \dots, \varphi_q) = (b, \varphi_1, \dots, \varphi_q),$$

which is a section of the simplicial fiber bundle $N\overline{M}_b \to NG_b$.

Taking the fat realization of the right square of the diagram above, we have the following commutative one:

$$\begin{tikzcd}
\|N\overline{M}_b\| \arrow[r, shift left, "\gamma_b"] \arrow[d, "\mu = \|\bar{\mu}\|"] & \|NG_b\| \arrow[l, shift left, "\|s\|"] \arrow[d, "\iota"] \\
\|N\overline{M}\| \arrow[r, "\gamma"] & \|NG\|
\end{tikzcd} \tag{7-5}$$

Here the horizontal arrows are projections of fiber bundles and the vertical ones are inclusions, some of which are (re)named as in the diagram. The simplicial map $s\colon NG_b \to N\overline{M}_b$ induces a continuous section $\|s\|\colon \|NG_b\| \to \|N\overline{M}_b\|$ of γ_b and it satisfies $\gamma \circ \mu \circ \|s\| = \iota$.

Proof of Corollary 2.5 Corollary 2.5 is equivalent to that the equality $\iota^*(\|\sigma\|) = c_1((\mu \circ \|s\|)^*\zeta)$ holds in $H^2(\|NG_b\|)$. Thus we show this.

The equality $(\bar{\mu} \circ s)^*\omega' = 0$ in $\mathbb{A}^2(NG_b)$ is obvious. Considering the commutative diagram

$$\begin{tikzcd} H^*(\mathbb{A}^*(N\overline{M})) \arrow[r, "\bar{\mu}^*"] \arrow[d, "\mathcal{I}"'] & H^*(\mathbb{A}^*(N\overline{M}_b)) \arrow[r, "s^*"] \arrow[d, "\mathcal{I}"'] & H^*(\mathbb{A}^*(NG_b)) \arrow[d, "\mathcal{I}"'] \\ H^*(\|N\overline{M}\|) \arrow[r, "\mu^*=\|\bar{\mu}\|^*"] & H^*(\|N\overline{M}_b\|) \arrow[r, "\|s\|^*"] & H^*(\|NG_b\|) \end{tikzcd}$$

obtained from the naturality (4-4) of $\mathcal{I}$ for simplicial maps, we have

$$\|s\|^*\mu^*(\|\omega\|) = \|s\|^*\mu^*\mathcal{I}([\omega']) = \mathcal{I}((\bar{\mu} \circ s)^*[\omega']) = 0.$$

This implies

$$\iota^*(\|\sigma\|) = \|s\|^*\mu^*\gamma^*(\|\sigma\|) = \|s\|^*\mu^*(c_1(\zeta) + \kappa\|\omega\|) = c_1\big((\mu \circ \|s\|)^*\zeta\big),$$

where we used the commutative diagram (7-5) and (3) of Corollary 7.2. □

Remark 7.3 Let $[\mathfrak{G}]$ be the second cohomology class of G for an exact symplectic manifold with trivial first real cohomology group defined by Ismagilov, Losik and Michor [12] mentioned in the beginning of the introduction. Here the symbol $[\mathfrak{G}]$ is that of Gal and Kędra in [8]. We remark that $[\mathfrak{G}]$ is generally different from our cohomology class σ even if the first Chern class of the exact symplectic manifold is trivial. In this case, we can take any $\kappa \in \mathbb{R}$. We apply Corollary 2.5 to the standard symplectic vector space $\mathbb{R}^{2n}$ with origin b. The inclusion $\iota\colon \mathrm{Sp}(2n, \mathbb{R}) \hookrightarrow G = \mathrm{Symp}(\mathbb{R}^{2n})$ factors through $\iota_b\colon G_b \hookrightarrow G$. Since the restriction of D to $\mathrm{Sp}(2n, \mathbb{R})$ is the identity map under the obvious identification, we have $\iota^*\sigma = c_1$ in $H^2(\mathrm{Sp}(2n, \mathbb{R}), \mathbb{R})$ by Corollary 2.5. On the other hand we have $\iota^*([\mathfrak{G}]) = 0$ since the restriction of $[\mathfrak{G}]$ to G_b vanishes by Example 3.9(1) in [8]. Thus σ and $[\mathfrak{G}]$ are different in this example because of the term of the first Chern class.

Proof of Theorem 2.6 Let $\{E_r^{p,q}\}$ be the Leray–Serre spectral sequence for the bundle $\gamma\colon \|N\overline{M}\| \to \|NG\|$. We have $E_2^{p,q} = H^p(\|NG\|, \mathcal{H}^q)$, where $\mathcal{H}^q$ is the local system associated with this bundle, the coefficients of which are in $H^q(M)$. Since M is a $2n$–dimensional manifold, we have $E_2^{p,q} = \{0\}$ for $q > 2n$, hence

$$H^{2+2n}(\|N\overline{M}\|) = \{0\} \oplus \cdots \oplus \{0\} \oplus E_\infty^{2,2n} \oplus E_\infty^{3,2n-1} \oplus \cdots \oplus E_\infty^{2+2n,0}.$$

Thus we get the Gysin homomorphism

$$\pi_!\colon H^{2+2n}(\|N\overline{M}\|) \to E_\infty^{2,2n} \subset E_2^{2,2n} = H^2(\|NG\|, \mathcal{H}^{2n}) \cong H^2(\|NG\|),$$

where we used the fact that the local system $\mathcal{H}^{2n} = H^{2n}(M) \cong \mathbb{R}$ is trivial.

On the other hand, we have $H^2(\|N\overline{M}\|) = E_\infty^{0,2} \oplus E_\infty^{2,0} = E_4^{0,2} \oplus E_2^{2,0}$ and $c_1(\zeta) = \gamma^*\|\sigma\| - \kappa\|\omega\|$ with $\gamma^*\|\sigma\| \in E_\infty^{2,0}$ and $\|\omega\| \in E_\infty^{0,2}$ by Corollary 7.2. Then we have

$$\begin{aligned} c_1(\zeta)^{n+1} &= \sum_{k=0}^{n+1} \binom{n+1}{k} (\gamma^*\|\sigma\|)^k (-\kappa\|\omega\|)^{n+1-k} \\ &= (-\kappa)^{n+1}\|\omega\|^{n+1} + (n+1)(-\kappa)^n(\gamma^*\|\sigma\|)\|\omega\|^n + \cdots, \end{aligned}$$

where the multiplication is the cup product of $H^*(\|N\overline{M}\|)$.

Here we recall the cup product in the Leray–Serre spectral sequence [20]. The cup products $H^p(\|NG\|, \mathcal{H}^q) \otimes H^{p'}(\|NG\|, \mathcal{H}^{q'}) \to H^{p+p'}(\|NG\|, \mathcal{H}^{q+q'})$ in cohomologies with local coefficients induce products

$$\vee\colon E_r^{p,q} \otimes E_r^{p',q'} \to E_r^{p+p',q+q'}$$

for $r = 2, 3, \ldots, \infty$. Let ab be the cup product of $a \in E_\infty^{p,q}$ and $b \in E_\infty^{p',q'}$ in $H^*(\|N\overline{M}\|)$, then we have $ab = (-1)^{qp'} a \vee b$.

Taking this fact into consideration, we have $(\gamma^*\|\sigma\|)^k\|\omega\|^{n+1-k} \in E_\infty^{2k,2(n-k+1)}$ and $\|\omega\|^{n+1} = 0$. Therefore we obtain

$$\pi_!(c_1(\zeta)^{n+1}) = (n+1)(-\kappa)^n[\omega]^n([M])\|\sigma\| = (n+1)c_1(M)^n([M])\|\sigma\|$$

in $H^2(\|NG\|)$, where we remark that $\gamma^*\|\sigma\| \in E_\infty^{2,0}$ is $\|\sigma\| \in H^2(\|NG\|)$ under the identification $E_\infty^{2,0} = E_2^{2,0} = H^2(\|NG\|)$. The sequence of fiber bundles $E^\delta TM \to E^\delta M \to BG^\delta$ is isomorphic to the pullback of that of $ETM \to EM \to BG$ by the map $\iota\colon BG^\delta \to BG$ induced from the identity map of G appearing above Theorem 2.6. Thus we have $\iota^* c = \pi_!(c_1(E^\delta TM)^{n+1})$ by the naturality of the Gysin homomorphism. Since $\zeta \to \|N\overline{M}\|$ is a realization of $E^\delta TM \to E^\delta M$, we obtain Theorem 2.6 under appropriate identification. □

Example 7.4 We give an application of Corollary 2.7. We refer to [14; 1; 2] for ingredients in this example. Let $\omega_{\mathbb{C}P^i}$ be the Fubini–Study form on $\mathbb{C}P^i$ normalized so that $\frac{1}{2\pi}\int_{\mathbb{C}P^1} \omega_{\mathbb{C}P^i} = 1$ for a line $\mathbb{C}P^1 \subset \mathbb{C}P^i$ for $i = 1, 2$. Let

$$M = \{([z_0, z_1, z_2], [w_1, w_2]) \in \mathbb{C}P^2 \times \mathbb{C}P^1 \mid z_1 w_2^k = z_2 w_1^k\}$$

be the k^{th} Hirzebruch surface for a nonnegative integer k. For any positive numbers r and s, let ω be the Kähler form on M which is the restriction to M of the Kähler form $s\pi_1^*\omega_{\mathbb{C}P^2} + r\pi_2^*\omega_{\mathbb{C}P^1}$ on $\mathbb{C}P^2 \times \mathbb{C}P^1$, where π_1 and π_2 are the projections from $\mathbb{C}P^2 \times \mathbb{C}P^1$ to the first factor $\mathbb{C}P^2$ and the second $\mathbb{C}P^1$, respectively. The effective action of $\mathbb{T} = \{(\alpha, \beta) \in \mathbb{C}^2 : |\alpha| = |\beta| = 1\}$ on M given by

$$(\alpha, \beta) \cdot ([z_0, z_1, z_2], [w_1, w_2]) = ([\alpha z_0, z_1, \beta^k z_2], [w_1, \beta w_2])$$

is a Kähler one. It is also a Hamiltonian action and its moment map is given by

$$\Phi([z_0,z_1,z_2],[w_1,w_2]) = \left(\frac{-s|z_0|^2}{|z_0|^2+|z_1|^2+|z_2|^2}, \frac{-sk|z_2|^2}{|z_0|^2+|z_1|^2+|z_2|^2} + \frac{r|w_1|^2}{|w_1|^2+|w_2|^2}\right).$$

Thus the Kähler manifold M with Kähler form ω endowed with this toric action is a Kähler toric variety. Its Delzant polytope $\mathrm{Im}(\Phi)$ is checked to be the quadrilateral with vertices $(s,0)$, (s,r), $(0,-sk)$ and $(0,r)$. After changing the basis of $\mathbb{T}$ and adding a constant vector to the moment map Φ, it becomes the quadrilateral with vertices $(0,0)$, $(r+ks,0)$, (r,s) and $(0,s)$ [14].

Let $E\mathbb{T} \to B\mathbb{T}$ be the universal principal $\mathbb{T}$–bundle over the classifying space $B\mathbb{T}$ of $\mathbb{T}$ with the usual topology and $E_{\mathbb{T}}M = E\mathbb{T} \times_{\mathbb{T}} M \to B\mathbb{T}$ its associate M–bundle. Then we have cohomology classes in $H^*(B\mathbb{T})$ obtained from Chern classes of the tangent bundle $E_{\mathbb{T}}TM$ along the fibers of $E_{\mathbb{T}}M$ by the fiber integration $\pi_!\colon H^*(E_{\mathbb{T}}M) \to H^{*-4}(B\mathbb{T})$. In particular we consider the fiber integral $\pi_!(c_1(E_{\mathbb{T}}TM)^3) \in H^2(B\mathbb{T})$ of the third power of the first Chern class of $E_{\mathbb{T}}TM$.

Let $r = 1+\lambda-\frac{1}{2}(k-1) > 0$ and $\lambda > -1$ if k is odd, and $r = 1+\lambda-\frac{1}{2}k > 0$ and $\lambda \geqq 0$ if k is even, and $s=1$. Then the Delzant polytope of M is the quadrilateral with vertices $(0,0)$, $\big(2+\lambda+\frac{1}{2}(k-1),0\big)$, $\big(1+\lambda-\frac{1}{2}(k-1),1\big)$, $(0,1)$ if k is odd, and with ones $(0,0)$, $\big(1+\lambda+\frac{1}{2}k,0\big)$, $\big(1+\lambda-\frac{1}{2}k,1\big)$, $(0,1)$ if k is even. If $1+\lambda > \big[\frac{1}{2}k\big]$, then Januszkiewicz and Kędra computed the fiber integrals of characteristic classes of $E_{\mathbb{T}}TM$ in terms of k and λ in Proposition 4.11 in [13], which holds more generally for symplectic toric varieties whose Delzant polytopes are given as above. Using a formula in the proposition, we obtain $\pi_!(c_1(E_{\mathbb{T}}TM)^3) = -2k(kx-2y) \in H^2(B\mathbb{T}) = \mathbb{R}[x,y]$, where x and y are the cohomology classes used in the proposition.

For $k \geqq 2$, ω is not proportional to the first Chern class $c_1(M)$ of M because $c_1(M)$ takes both positive and nonpositive values on the classes of holomorphic spheres in M. Thus we take $k=1$. In this case M is also known as the one point blow-up of $\mathbb{C}P^2$. Let E be the exceptional divisor and F the inverse image of π_2 at any point of $\mathbb{C}P^1$. Since E and F are $\mathbb{C}P^1$ and their self-intersection numbers are -1 and 0, respectively, the first Chern class of M takes values 1 on E and 2 on F. On the other hand, we have $\int_E \omega = 2\pi r = 2\pi(1+\lambda)$ and $\int_F \omega = 2\pi s = 2\pi$. Therefore it must be $\lambda = -\frac{1}{2}$, which satisfies the condition of Januszkiewicz and Kędra's proposition, for $[\omega]$ to be proportional to $c_1(M)$.

We take $k=1$, $\lambda = -\frac{1}{2}$ and $s=1$, hence $r=\frac{1}{2}$, as explained above. The cohomology class $c = \pi_!(c_1(ETM)^3) \in H^2(BG)$ is mapped to nonzero the element $j^*c = -2(x-2y) \in H^2(B\mathbb{T})$ by the homomorphism j^* induced from the injection $j\colon \mathbb{T} \hookrightarrow G$. Let J be the complex structure of M. Since the action of $\mathbb{T}$ is Kähler, the image of j is a subgroup of $G_J \subset G$. Let $\mathbb{T}^\delta$ be $\mathbb{T}$ with the discrete topology. By Corollary 2.7, the image of j^*c by the homomorphism $H^2(B\mathbb{T}) \to H^2(B\mathbb{T}^\delta)$ induced from the continuous map $\mathbb{T}^\delta \to \mathbb{T}$ is zero.

8 An example

In this section we give a nontrivial example of the class σ. Let Σ_g be a closed oriented surface of genus $g \geqq 2$. Let $\Sigma_{g,1}$ be a subsurface of Σ_g which is the complement of a small open disk in Σ_g. Take a basepoint $* \in \partial\Sigma_{g,1} \subset \Sigma_g$. Put $\widetilde{\mathcal{R}}_g = \{\phi \in \operatorname{Hom}(\pi_1(\Sigma_{g,1}), \mathrm{SU}(2)) \mid \phi(\partial\Sigma_{g,1}) = -I\}$, where $\partial\Sigma_{g,1} \cong S^1$ also presents the homotopy class of itself in $\pi_1(\Sigma_{g,1}) := \pi_1(\Sigma_{g,1}, *)$ and $I \in \mathrm{SU}(2)$ is the identity matrix. We consider the moduli space $\mathcal{R}_g = \widetilde{\mathcal{R}}_g / \operatorname{Ad}\mathrm{SU}(2)$ of $\mathrm{SU}(2)$–representations of $\pi_1(\Sigma_{g,1})$ with holonomy $-I$ along the boundary $\partial\Sigma_{g,1}$. It is known as a closed, 1–connected, symplectic manifold of real dimension $6g-6$. Its first Chern class $c_1(\mathcal{R}_g)$ is given by twice the generator of $H^2(\mathcal{R}_g;\mathbb{Z}) \cong \mathbb{Z}$ [24]. Let ω be the symplectic form of $\mathcal{R}_g$; then we have $c_1(\mathcal{R}_g) + \kappa[\omega] = 0$ in $H^2(\mathcal{R}_g)$ for some $0 \neq \kappa \in \mathbb{R}$. Therefore we have the second cohomology class $\sigma \in H^2(\mathrm{Symp}(\mathcal{R}_g,\omega),\mathbb{R})$ in Proposition 2.2.

We show the nontriviality of the class σ. Let $\Sigma_{g-1,1} \subset \Sigma_{g,1}$ be the subsurface of $\Sigma_{g,1}$ the complement of which is diffeomorphic to a torus with two open disks deleted as depicted in Figure 1. Let $\Gamma_{g,1}$ be the mapping class group of $\Sigma_{g,1}$, that is, it is the group of the isotopy classes of diffeomorphisms of $\Sigma_{g,1}$ which preserve the boundary pointwise, and $\Gamma_{g-1,1}$ that of $\Sigma_{g-1,1}$. $\Gamma_{g-1,1}$ is considered as a subgroup of $\Gamma_{g,1}$ if we extend diffeomorphisms on $\Sigma_{g-1,1}$ by the identity on $\Sigma_{g,1} \setminus \Sigma_{g-1,1}$. $\Gamma_{g,1}$ acts symplectically on $\mathcal{R}_g$ by $\Gamma_{g,1} \times \mathcal{R}_g \to \mathcal{R}_g$, $(\varphi,[\phi]) \mapsto [\varphi_*\phi]$, where $\varphi_*\phi = (\varphi^{-1})^*\phi$ is the pullback of ϕ by the inverse of φ. This action induces a homomorphism $\rho\colon \Gamma_{g,1} \to \mathrm{Symp}(\mathcal{R}_g,\omega)$.

Let a_i and b_i for $1 \leqq i \leqq g$ be the elements of $\pi_1(\Sigma_{g,1})$ given as depicted in Figure 1. Since $\pi_1(\Sigma_{g,1})$ is the free group generated by them, the homomorphism $\phi_0\colon \pi_1(\Sigma_{g,1}) \to \mathrm{SU}(2)$ given by

$$\begin{aligned}
\phi_0(a_i) &= \phi_0(b_i) = I \quad \text{for } 1 \leqq i \leqq g-1,\\
\phi_0(a_g) &= \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix},\\
\phi_0(b_g) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

is well-defined.

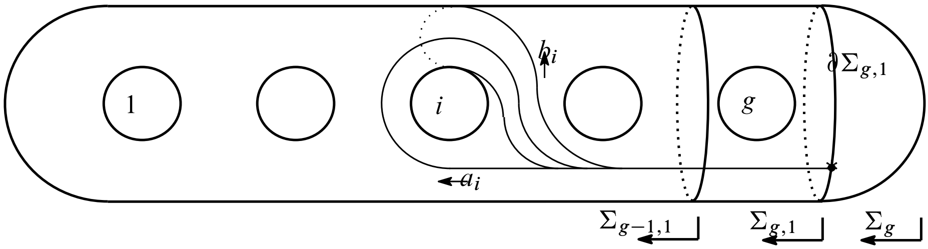

Figure 1: Surfaces and generators

Lemma 8.1 (1) *The representation ϕ_0 belongs to $\widetilde{\mathcal{R}}_g$, hence it defines a point $[\phi_0] \in \mathcal{R}_g$.*

(2) *The restriction of the action of $\Gamma_{g,1}$ on $\mathcal{R}_g$ to $\Gamma_{g-1,1}$ fixes $[\phi_0]$.*

Proof (1) We have only to show the equality $\phi_0\bigl(\prod_{i=1}^g [a_i, b_i]\bigr) = -I$. It is easily checked.

(2) For any $\varphi \in \Gamma_{g-1,1}$, the elements $\varphi_*(a_i)$ and $\varphi_*(b_i)$ for $1 \leqq i \leqq g-1$ belong to the subgroup S of $\pi_1(\Sigma_{g,1})$ generated by $\{a_i, b_i \mid 1 \leqq i \leqq g-1\}$, and $\varphi_*(a_g) = a_g$ and $\varphi_*(b_g) = b_g$. Since $\phi_0(S) = \{I\}$, we have $\varphi_*\phi_0 = \phi_0$. □

By Lemma 8.1, the differential of the action of $\Gamma_{g-1,1}$ at $[\phi_0]$ induces a homomorphism $D\colon \Gamma_{g-1,1} \to \mathrm{Aut}(T_{[\phi_0]}\mathcal{R}_g, \omega_{[\phi_0]})$. For any $\phi \in \widetilde{\mathcal{R}}_g$, the tangent space $T_{[\phi]}\mathcal{R}_g$ and the symplectic form $\omega_{[\phi]}$ on it are given as follows. Let Z^1_ϕ be the vector space of Ad_ϕ–crossed homomorphisms $u\colon \pi_1(\Sigma_{g,1}) \to \mathrm{su}(2)$ satisfying $u\bigl(\prod_{i=1}^g [a_i, b_i]\bigr) = 0$ and B^1_ϕ that of principal ones, namely, each $u \in B^1_\phi$ is given by $u(c) = \mathrm{Ad}_{\phi(c)}\, w - w$ for $c \in \pi_1(\Sigma_{g,1})$ for some $w \in \mathrm{su}(2)$. Then we have $Z^1_\phi \supset B^1_\phi$. On the other hand, ϕ descends to an SO(3)–representation $\bar{\phi}\colon \pi_1(\Sigma_g) := \pi_1(\Sigma_g, *) \to \mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm I\}$. Let $m_{\bar{\phi}}$ be the $\pi_1(\Sigma_g)$–module so(3) with respect to the adjoint representation $\mathrm{Ad}_{\bar{\phi}}$ of $\bar{\phi}$. Under a natural identification of so(3) with su(2), the spaces $Z^1_{\bar{\phi}}$ of $\mathrm{Ad}_{\bar{\phi}}$–crossed homomorphisms from $\pi_1(\Sigma_g)$ to so(3) and $B^1_{\bar{\phi}}$ of principal ones are identified with Z^1_ϕ and B^1_ϕ, respectively, in an obvious way. Thus $T_{[\phi]}\mathcal{R}_g = Z^1_\phi/B^1_\phi \cong Z^1_{\bar{\phi}}/B^1_{\bar{\phi}} = H^1(\pi_1(\Sigma_g), m_{\bar{\phi}})$. Under this isomorphism, the symplectic form $\omega_{[\phi]}$ is given by $\omega_{[\phi]}(x, y) = \langle B_*(x \cup y), [z]\rangle$ for $x, y \in H^1(\pi_1(\Sigma_g), m_{\bar{\phi}})$. Here $z = \sum_{i=1}^g \{(R_{i-1}, a_i) - (R_i b_i, a_i) + (R_{i-1} a_i, b_i) - (R_i, b_i)\}$ with $R_i = \prod_{j=1}^i [a_j, b_j]$ for $0 \leqq i \leqq g$ is a 2–cycle representing the generator $[z] \in H_2(\pi(\Sigma_g), \mathbb{Z})$,

$$B_*(\cdot \cup \cdot)\colon H^1(\pi_1(\Sigma_g), m_{\bar{\phi}}) \otimes H^1(\pi_1(\Sigma_g), m_{\bar{\phi}}) \to H^2(\pi_1(\Sigma_g), \mathbb{R})$$

is the cup product together with coefficient pairing B, which is the Killing form, and $\langle\cdot,\cdot\rangle$ denotes the natural pairing of cohomology and homology classes [9].

Next we consider the case of $\phi = \phi_0$. Let $\mathbb{T} = \{u \in Z^1_{\phi_0} \mid u(a_g) = u(b_g) = 0\}$ be a subspace of $Z^1_{\phi_0}$. It is isomorphic to $\mathrm{su}(2)^{2(g-1)}$. We can easily check that

$$B^1_{\phi_0} = \left\{ \phi \in Z^1_{\phi_0} \;\middle|\; \phi(a_i) = \phi(b_i) = 0 \text{ for } 1 \leqq i \leqq g-1, \right.$$
$$\left. \phi(a_g) = \begin{pmatrix} 0 & -\bar{\alpha} \\ \alpha & 0 \end{pmatrix}, \ \phi(b_g) = \begin{pmatrix} \sqrt{-1}b & \frac{1}{2}(\alpha - \bar{\alpha}) \\ \frac{1}{2}(\alpha - \bar{\alpha}) & -\sqrt{-1}b \end{pmatrix}, \ \alpha \in \mathbb{C}, \ b \in \mathbb{R} \right\}$$

and $Z^1_{\phi_0} = \mathbb{T} \oplus B^1_{\phi_0}$. Thus the projection $Z^1_{\phi_0} \to Z^1_{\phi_0}/B^1_{\phi_0}$ induces an isomorphism $\mathbb{T} \to T_{[\phi_0]}\mathcal{R}_g$.

Let H be the vector space

$$H^1(\Sigma_{g-1,1}, \partial\Sigma_{g-1,1}; \mathbb{R})$$

and Ω_H the symplectic form on H given by the cup product and the evaluation on the fundamental class $[\Sigma_{g-1,1}, \partial\Sigma_{g-1,1}]$. Let (H^3, Ω) be the direct sum of three copies of the symplectic vector space (H, Ω_H). Since $-\Omega$ is also a symplectic form on H^3, we get a symplectic vector space $(H^3, -\Omega)$. Take an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathrm{su}(2)$ with respect to minus the Killing form $-B$. We can consider H as a subspace of $\mathrm{Hom}(\pi_1(\Sigma_{g,1}, \mathbb{R}), \mathbb{R})$ by the injection

$$H \cong H^1(\Sigma_{g,1}, \Sigma_{g,1} \setminus \Sigma_{g-1,1}; \mathbb{R}) \hookrightarrow H^1(\Sigma_{g,1}; \mathbb{R}) = \mathrm{Hom}(\pi_1(\Sigma_{g,1}), \mathbb{R}).$$

Then the map $H^3 \to \mathbb{T}$ given by $w \mapsto \sum_{i=1}^3 w_i e_i$ for $w = (w_1, w_2, w_3) \in H^3$ is well-defined homomorphism by the definition of ϕ_0. Let $\xi\colon H^3 \to T_{[\phi_0]}\mathcal{R}_g$ be the composition of it with the isomorphism $\mathbb{T} \cong T_{[\phi_0]}\mathcal{R}_g$ above. Since the standard action of $\Gamma_{g-1,1}$ on H is symplectic, its diagonal action on $(H^3, -\Omega)$ is also symplectic.

Lemma 8.2 *The map $\xi\colon (H^3, -\Omega) \to (T_{[\phi_0]}\mathcal{R}_g, \omega_{[\phi_0]})$ is an isomorphism of symplectic vector spaces. Moreover, the restricted action of $\Gamma_{g-1,1}$ on $T_{[\phi_0]}\mathcal{R}_g$ is isomorphic to the diagonal action of it on H^3 under the isomorphism ξ, that is, the following diagram commutes:*

$$\begin{tikzcd} \Gamma_{g-1,1} \times H^3 \arrow[r] \arrow[d, "\mathrm{id}\times\xi"] & H^3 \arrow[d, "\xi"] \\ \Gamma_{g-1,1} \times T_{[\phi_0]}\mathcal{R}_g \arrow[r] & T_{[\phi_0]}\mathcal{R}_g \end{tikzcd}$$

where the horizontal arrows are given by the actions.

Proof Since the map $H^3 \to \mathbb{T}$ above is clearly an isomorphism, so is ξ. For any $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in H^3$, we have

$$\begin{aligned}\xi^*\omega_{\phi_0}(x,y) = \omega_{\phi_0}(\xi(x),\xi(y)) &= \left\langle B_*\left(\left[\sum_i x_i e_i\right]\cup\left[\sum_j y_j e_j\right]\right),[z]\right\rangle \\ &= \sum_{i,j}\langle x_i\cup y_j,[z]\rangle B(e_i,e_j) = -\sum_i\langle x_i\cup y_i,[z]\rangle = -\Omega(x,y),\end{aligned}$$

where we note that $B(e_i,e_j)=-\delta_{ij}$ and that we can compute these equalities like those in trivial module since $\phi_0(S)=\{I\}$ and $x(c)=y(c)=0$ for $c=a_g,b_g$. Thus we have a symplectic isomorphism ξ.

For the latter part, we have only to check $\xi(\varphi_* w)=\varphi_*\xi(w)$ for all $\varphi\in\Gamma_{g-1,1}$ and $w\in H^3$, where $\varphi_* w=(\varphi_* w_1,\varphi_* w_2,\varphi_* w_3)$. Since we can compute them like with trivial coefficients for the same reason as above, we can easily check the equality. □

Proof of Theorem 2.8 Let $\tau\colon\Gamma_{g,1}\to\mathrm{Sp}(2g,\mathbb{R})$ be the homomorphism in Theorem 2.8 for each $g\geqq 1$. Since $\mathrm{Aut}(H^1(\Sigma_g;\mathbb{Z}),\cdot\cup\cdot)=\mathrm{Aut}(H^1(\Sigma_g;\mathbb{Z}),-(\cdot\cup\cdot))$ as subsets of the automorphisms of $H^1(\Sigma_g;\mathbb{Z})$, we also have a homomorphism $\tau^-\colon\Gamma_{g,1}\to\mathrm{Aut}(H^1(\Sigma_g;\mathbb{Z}),-(\cdot\cup\cdot))\cong\mathrm{Sp}(2g,\mathbb{Z})\subset\mathrm{Sp}(2g,\mathbb{R})$ for each $g\geqq 1$. We simply use the symbols τ and τ^- for different g. On the other hand we denote $c_1\in H^2(\mathrm{Sp}(2g,\mathbb{R}),\mathbb{R})$ by $c_{1,g}$ for each g. Then we have $(\tau^-)^*c_{1,g}=-\tau^*c_{1,g}$.

Applying Corollary 2.5 to the situation here, we have $\iota_b^*\sigma=D^*c_{1,3(g-1)}$ in $H^2(G_b,\mathbb{R})$, where $G=\mathrm{Symp}(\mathcal{R}_g,\omega)$ and $b=[\phi_0]$. By Lemma 8.2, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Sp}(2(g-1),\mathbb{R})^3 \arrow[d, "\iota_2"'] & \Gamma_{g-1,1} \arrow[l, "\Delta"'] \arrow[r, "\iota_1"] \arrow[d, "\rho|_{\Gamma_{g-1,1}}"] & \Gamma_{g,1} \arrow[r, "\tau"] \arrow[d, "\rho"] & \mathrm{Sp}(2g,\mathbb{R}) \\
\mathrm{Sp}(6(g-1),\mathbb{R}) & G_b \arrow[l, "D"'] \arrow[r, "\iota_b"] & G &
\end{tikzcd}$$

where $\Delta=\tau^-\oplus\tau^-\oplus\tau^-$, and ι_1 and ι_2 are inclusions. Thus we have

$$\iota_1^*\rho^*\sigma=(\rho|_{\Gamma_{g-1,1}})^*\iota_b^*\sigma=(\rho|_{\Gamma_{g-1,1}})^*D^*c_{1,3(g-1)}=\Delta^*\iota_2^*c_{1,3(g-1)}$$

in $H^2(\Gamma_{g-1,1},\mathbb{R})$, and

$$\Delta^*\iota_2^*c_{1,3(g-1)}=3(\tau^-)^*c_{1,g-1}=-3\tau^*c_{1,g-1}=-3\iota_1^*\tau^*c_{1,g}$$

by properties of the first Chern class. Thus we obtain $\iota_1^*\rho^*\sigma=-3\iota_1^*\tau^*c_{1,g}$. Since $\iota_1^*\colon H^2(\Gamma_{g,1},\mathbb{R})\to H^2(\Gamma_{g-1,1},\mathbb{R})$ is injective for $g\geqq 4$ [10; 11; 17], we have $\rho^*\sigma=-3\tau^*c_{1,g}$. The nontriviality of the class $\tau^*c_{1,g}\in H^2(\Gamma_{g,1},\mathbb{R})$ for $g\geqq 3$ is a well-known fact [22]. This completes the proof of Theorem 2.8. □

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