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Inertia groups of high-dimensional complex projective spaces

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For a complex projective space the inertia group, the homotopy inertia group and the concordance inertia group are isomorphic. In complex dimension $4n+1$, these groups are related to computations in stable cohomotopy. Using stable homotopy theory, we make explicit computations to show that the inertia group is nontrivial in many cases. In complex dimension 9, we deduce some results on geometric structures on homotopy complex projective spaces and complex hyperbolic manifolds.

57R55, 57R60; 55P25, 55P42

1 Introduction

The study of manifolds in differential topology presents itself through four different classes of equivalence: homotopy equivalence, homeomorphism, PL homeomorphism and diffeomorphism. The classification of manifolds up to these equivalences is a fundamental question in geometry and topology.

One of the first results in this subject is that of Milnor [22] that there exist smooth manifolds which are homeomorphic to S^7 but not diffeomorphic. This raises the possibility of nondiffeomorphic smooth structures for a given a topological manifold. We refer to these as inequivalent smoothings. For example, the 7–sphere has 28 inequivalent smoothings.[1]

A smooth manifold homeomorphic to S^m is known as a smooth homotopy m–sphere.[2] The existence of smooth homotopy m–spheres was studied in the amazing work of Kervaire and Milnor [16]. The set of diffeomorphism classes of smooth homotopy spheres Θ_m for $m \geq 5$ forms a group under the operation of connected sum. It was shown that there exist exotic spheres in a vast majority of dimensions but also that in each dimension there are only finitely many. The proof of the result points to a curious

[1] This actually occurs as a consequence of the results of Kervaire and Milnor [16].
[2] We know that if $m \geq 5$ as an m–manifold homotopy equivalent to S^m is actually homeomorphic to S^m.

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connection to the stable homotopy groups of spheres (denoted by π_n^s for $n \geq 0$) whose values are mysterious but are computable using algebraic techniques, especially in low dimensions.

A possible way to change smooth structure on a smooth manifold M^m, without changing its homeomorphism type, is to take its connected sum $M^m \# \Sigma^m$ with a smooth homotopy sphere Σ^m. This induces a group action of Θ_m on the set of smooth structures on the topological manifold M. The collection of smooth homotopy spheres Σ^m which admit a diffeomorphism $M^m \# \Sigma^m \to M^m$ forms a subgroup $I(M)$ of Θ_m, called the inertia group of M^m.

The calculation of $I(M)$ for an arbitrary manifold M has proven to be a hard problem in general but there are results in certain cases. Tamura [27] constructed explicit nontrivial elements in the inertia group for certain 3–sphere bundles over S^4. Examples with nontrivial inertia group are also constructed by Brown and Steer [4]. In every dimension m, Winkelnkemper [30] proved that there exists a manifold M^m such that $I(M^m) = \Theta_m$. Levine [21] constructed certain nontrivial elements in the inertia group for many manifolds, most notably for simply connected nonspin manifolds in dimensions $8n+2$.

There are certain cases when the inertia group is 0. It was proved by Schultz [26] that $I(S^p \times S^q) = 0$ when $p+q \geq 5$. Kawakubo [14] proved that $I(\mathbb{C}P^m) = 0$ for $m \leq 8$. Limitations on the size of the inertia group have been given by Wall [29], Browder [3], Kosiński [19] and Novikov [24]. There is no systematic approach for computing the inertia groups in general, and many problems are open. In this paper we are interested in the problem: *What are the inertia groups of $\mathbb{C}P^m$ if $m \geq 9$?*

Analogous to the inertia group, for a manifold M one may define the homotopy inertia group $I_h(M)$ and the concordance inertia group $I_c(M)$. $I_h(M)$ (resp. $I_c(M)$) consists of those $\Sigma \in I(M)$ for which the diffeomorphism $M \# \Sigma \cong M$ is homotopic (resp. concordant) to the identity. Kasilingam [12] proved that for a complex projective space all these groups are the same.

The concordance inertia groups may be understood using homotopy theory. In dimensions $8n+2$, there is a $\mathbb{Z}/2$ summand of π_{8n+2}^s generated by μ_{8n+2} that corresponds to an exotic sphere $\Sigma^{8n+2} \in \Theta_{8n+2}$. Farrell and Jones [8] proved that $\mathbb{C}P^{4n+1} \# \Sigma^{8n+2}$ is not concordant to $\mathbb{C}P^{4n+1}$ using certain relations in KO^*, the cohomology theory induced by real K–theory. Hence this also implies that the element Σ^{8n+2} does not lie in the inertia group.

In this paper we try to compute the inertia groups of $\mathbb{C}P^{4n+1}$ using stable cohomotopy. We prove:

Theorem A (a) $I(\mathbb{C}P^9) \cong \mathbb{Z}/2$ *or* $\mathbb{Z}/4$ *as a subgroup of* $\Theta_{18} = \mathbb{Z}/2 \oplus \mathbb{Z}/8$.

(b) $I(\mathbb{C}P^{13}) \supseteq \mathbb{Z}/2$.

(See Theorem 3.11.) This is the first example of a nontrivial inertia group among the inertia groups of complex projective spaces.[3] These provide examples of exotic spheres whose α–invariant is 0 and where connected sum with projective spaces does not change the smooth structure. For the projective space $\mathbb{C}P^9$, we also get examples of exotic spheres with α–invariant 0 which change the smooth structure. The techniques involved are explicit calculations using Spanier–Whitehead duality and the knowledge of the stable homotopy groups of spheres in low dimensions at the prime 2.

We also make computations at odd primes p, making use of the nontriviality of certain elements in the stable homotopy groups proved by Lee [20] and deduce:

Theorem B *There are infinitely many values of* n *for which there exist nontrivial elements in the inertia group of* $\mathbb{C}P^{4n+1}$.

(For a more accurate statement see Theorem 3.11.)

Computations of the inertia groups carry with them a number of geometric applications. Using previously known results on the triviality of the inertia group $I(\mathbb{C}P^m)$ for $m \leq 8$, it is possible to classify, up to diffeomorphism, all closed smooth manifolds homeomorphic to the complex projective n–space for $n = 3$ and 4; see Kasilingam [13]. Kasilingam [12] showed that a way to generate exotic spheres which are not in the inertia group[4] in dimensions $8n+2$ is by considering exotic spheres with a nontrivial α–invariant.[5] The exotic spheres outside the inertia group for complex projective spaces have consequences for complex hyperbolic manifolds; see Farrell and Jones [8].

In this paper, from the inertia group of $\mathbb{C}P^9$, we deduce examples of three inequivalent smooth structures of $\mathbb{C}P^9$, different from the standard one, such that one admits a metric of nonnegative scalar curvature and the other two do not (see Theorem 4.3). The example relies on the construction of an element outside the inertia group whose

[3] The reader may observe that $\mathbb{C}P^{4n+1}$ is a spin manifold.
[4] These are defined to be Farrell–Jones spheres.
[5] These are defined as Hitchin spheres.

α–invariant is 0. These examples extend the results in [12] to dimension 18. Following this example, we construct examples of closed negatively curved Riemannian 18–manifolds which are homeomorphic but not diffeomorphic to complex hyperbolic manifolds (see Theorem 4.4).

Organization of the paper In Section 2, we introduce some preliminaries on the inertia group $I(\mathbb{C}P^m)$ and prove a result relating this to a computation in stable cohomotopy for dimension $m = 4n+1$. In Section 3, we make some computations in stable homotopy related to the above question and prove the nontriviality of $I(\mathbb{C}P^{4n+1})$ for the different values of n. Finally, Section 4 contains some geometric applications.

Notation Denote by $O = \operatorname{colim}_n O(n)$, $\mathrm{Top} = \operatorname{colim}_n \mathrm{Top}(n)$ and $G = \operatorname{colim}_n G(n)$ the direct limit of the groups of orthogonal transformations, homeomorphisms and homotopy equivalences, respectively. In this paper all manifolds will be closed, smooth, oriented and connected, and all homeomorphisms and diffeomorphisms are assumed to preserve orientation, unless otherwise stated.

2 Inertia groups of complex projective spaces

In this section we recall some basic facts about inertia groups, specializing to the case of complex projective spaces, and provide the background of the arguments in the rest of the paper. We start by recalling some terminology from [16]:

Definition 2.1 (a) A homotopy m–sphere Σ^m is an oriented, smooth, closed manifold homotopy equivalent to the standard unit sphere S^m in $\mathbb{R}^{m+1}$.

(b) A homotopy m–sphere Σ^m is said to be exotic if it is not diffeomorphic to S^m.

(c) Two homotopy m–spheres Σ_1^m and Σ_2^m are said to be equivalent if there exists an orientation-preserving diffeomorphism $f\colon \Sigma_1^m \to \Sigma_2^m$.

The set of equivalence classes of homotopy m–spheres is denoted by Θ_m. The equivalence class of Σ^m is denoted by $[\Sigma^m]$. When $m \geq 5$, Θ_m forms an abelian group with group operation given by connected sum # and the zero element represented by the equivalence class of S^m. M Kervaire and J Milnor [16] showed that each Θ_m is a finite group.

Definition 2.2 Let M be a topological manifold. Let (N, f) be a pair consisting of a smooth manifold N together with a homeomorphism $f\colon N \to M$. Two such pairs (N_1, f_1) and (N_2, f_2) are concordant provided there exists a diffeomorphism

$g\colon N_1 \to N_2$ such that the composition $f_2 \circ g$ is topologically concordant to f_1, ie there exists a homeomorphism $F\colon N_1 \times [0,1] \to M \times [0,1]$ such that $F|_{N_1 \times 0} = f_1$ and $F|_{N_1 \times 1} = f_2 \circ g$. The set of all such concordance classes is denoted by $\mathcal{C}(M)$.

Note that there is a homeomorphism $h\colon M^m \# \Sigma^m \to M^m$ for $m \geq 5$ which is the inclusion map outside of homotopy sphere Σ^m and well defined up to topological concordance. We will denote the class of $(M^m \# \Sigma^m, h)$ in $\mathcal{C}(M)$ by $[M^m \# \Sigma^m]$. (Note that $[M^m \# S^m]$ is the class of (M^m, id_{M^m}).)

Definition 2.3 Let M^m be a closed smooth m–dimensional manifold. The inertia group $I(M) \subset \Theta_m$ is defined as the set of $\Sigma \in \Theta_m$ for which there exists a diffeomorphism $\phi\colon M \to M \# \Sigma$.

The homotopy inertia group $I_h(M)$ is defined as the set of all $\Sigma \in I(M)$ such that there exists a diffeomorphism $M \to M \# \Sigma$ which is homotopic to id: $M \to M \# \Sigma$.

The concordance inertia group $I_c(M)$ is defined as the set of all $\Sigma \in I_h(M)$ such that $M \# \Sigma$ is concordant to M.

We recall the following theorem about complex projective spaces:

Theorem 2.4 [12, Theorem 4.2] *For $n \geq 1$, $I_c(\mathbb{C}P^n) = I_h(\mathbb{C}P^n) = I(\mathbb{C}P^n)$.*

Next we recall a reformulation of inertia groups via homotopy theory. Let $f_M\colon M^m \to S^m$ be a degree 1 map. Note that f_M is well defined up to homotopy. Composition with f_M defines a homomorphism

$$f_M^*\colon [S^m, \mathrm{Top}/O] \to [M^m, \mathrm{Top}/O],$$

and in terms of the identifications

$$\Theta_m = [S^m, \mathrm{Top}/O] \quad \text{and} \quad \mathcal{C}(M^m) = [M^m, \mathrm{Top}/O]$$

given by [17, pages 25 and 194], f_M^* becomes $[\Sigma^m] \mapsto [M^m \# \Sigma^m]$. Therefore the concordance inertia group $I_c(M)$ can be identified with $\mathrm{Ker}(f_M^*)$.

Recall that the based homotopy classes $[X, G]$ can be identified with the 0^{th} stable cohomotopy group $\widetilde{\pi}^0(X)$. We also write f_M^* for the induced homomorphism $[S^m, G] = \widetilde{\pi}^0(S^m) = \pi_m^s \to [M^m, G] = \widetilde{\pi}^0(M^m)$ by a degree 1 map $f_M\colon M^m \to S^m$.

Now recollect some facts from smoothing theory [5]. The natural inclusions of H–spaces $O \subset \mathrm{Top} \subset G$ induce H–space maps $\phi\colon G \to G/O$ and $\psi\colon \mathrm{Top}/O \to G/O$

such that

$$\psi_*\colon \Theta_{8n+2} = \pi_{8n+2}(\mathrm{Top}/O) \to \pi_{8n+2}(G/O)$$

is an isomorphism for $n \geq 1$. The homotopy groups of G are the stable homotopy groups of spheres π_m^s; ie $\pi_m(G) = \pi_m^s$ for $m \geq 1$. For $n \geq 1$,

$$\phi_*\colon \pi_{8n+2}^s \to \pi_{8n+2}(G/O)$$

is an isomorphism.

Theorem 2.5 *Let M^{8n+2} be a closed, smooth $8n+2$–manifold homotopy equivalent to $\mathbb{C}P^{4n+1}$ with $n \geq 1$. Then $I_c(M) = \mathrm{Ker}(\Phi)$, where*

$$\Phi\colon \Theta_{8n+2} \xrightarrow{\psi_*} \pi_{8n+2}(G/O) \xrightarrow{\phi_*^{-1}} \widetilde{\pi}^0(S^{8n+2}) \xrightarrow{f_M^*} \widetilde{\pi}^0(M^{8n+2}).$$

The proof of Theorem 2.5 requires two facts we prove below:

Lemma 2.6 *Let M^{2m} be a closed smooth $2m$–manifold homotopy equivalent to $\mathbb{C}P^m$ with $m \geq 1$. Then the homomorphism $\psi_*\colon [M^{2m}, \mathrm{Top}/O] \mapsto [M^{2m}, G/O]$ is monic.*

Proof Consider the Barratt–Puppe sequence for the inclusion $i\colon \mathbb{C}P^{m-1} \hookrightarrow \mathbb{C}P^m$, which induces the exact sequence on taking homotopy classes $[-, \Omega(G/\mathrm{Top})]$

$$\cdots \to [S\mathbb{C}P^{m-1}, \Omega(G/\mathrm{Top})] \to [S^{2m}, \Omega(G/\mathrm{Top})] \xrightarrow{f^*_{\mathbb{C}P^m}} [\mathbb{C}P^m, \Omega(G/\mathrm{Top})] \\ \xrightarrow{i^*} [\mathbb{C}P^{m-1}, \Omega(G/\mathrm{Top})] \to \cdots,$$

and, by identifying

$$[S^{2m}, \Omega(G/\mathrm{Top})] = [S^{2m+1}, G/\mathrm{Top}] = L_{2m+1}(e) = 0,$$

where $L_k(e)$ is the simply connected surgery obstruction group, and

$$[\mathbb{C}P^1, \Omega(G/\mathrm{Top})] = 0,$$

we can prove that $[\mathbb{C}P^m, \Omega(G/\mathrm{Top})] = 0$ for all m. Now consider the long exact sequence associated to the fibration $\mathrm{Top}/O \to G/O \to G/\mathrm{Top}$,

$$\cdots \to [M^{2m}, \Omega(G/\mathrm{Top})] \to [M^{2m}, \mathrm{Top}/O] \xrightarrow{\psi_*} [M^{2m}, G/O] \to [M^{2m}, G/\mathrm{Top}],$$

and using the fact that $[M^{2m}, \Omega(G/\mathrm{Top})] = [\mathbb{C}P^m, \Omega(G/\mathrm{Top})] = 0$, we have that the homomorphism ψ_* is monic. □

Lemma 2.7 *Let M^{2m} be a closed, smooth $2m$–manifold homotopy equivalent to $\mathbb{C}P^m$ with $m \geq 1$. Then the homomorphism $\phi_*\colon [M^{2m}, G] \to [M^{2m}, G/O]$ is monic.*

Proof Since M^{2m} is homotopy equivalent to $\mathbb{C}P^m$, let $g\colon M^{2m} \to \mathbb{C}P^m$ be a homotopy equivalence. The induced map $g^*\colon [\mathbb{C}P^m, -] \to [M^{2m}, -]$ fits into the following commutative diagram:

$$\begin{array}{ccc} [\mathbb{C}P^m, G] & \xrightarrow{\phi_*} & [\mathbb{C}P^m, G/O] \\ \cong \big\downarrow g^* & & \cong \big\downarrow g^* \\ [M^{2m}, G] & \xrightarrow{\phi_*} & [M^{2m}, G/O] \end{array}$$

Brumfiel [6, page 77] has shown that

$$\phi_*\colon [\mathbb{C}P^m, G] \to [\mathbb{C}P^m, G/O]$$

is monic for all $m \geq 1$. This implies that the homomorphism $\phi_*\colon [M^{2m}, G] \to [M^{2m}, G/O]$ is monic. □

Proof of Theorem 2.5 Consider the following commutative diagram:

$$\begin{array}{ccc} [S^{2m}, \mathrm{Top}/O] = \Theta_{2m} & \xrightarrow{f^*_{M^{2m}}} & [M^{2m}, \mathrm{Top}/O] = \mathcal{C}(M^{2m}) \\ \big\downarrow \psi_* & & \big\downarrow \psi_* \\ [S^{2m}, G/O] & \xrightarrow{f^*_{M^{2m}}} & [M^{2m}, G/O] \\ \phi_* \big\uparrow & & \phi_* \big\uparrow \\ [S^{2m}, G] = \pi^s_{2m} & \xrightarrow{f^*_{M^{2m}}} & [M^{2m}, G] \end{array}$$

Recall that the concordance class $[M^{2m} \# \Sigma] \in [M^{2m}, \mathrm{Top}/O]$ of $M^{2m} \# \Sigma$ is $f^*_{M^{2m}}([\Sigma])$ when $m > 2$, and that $[M^{2m}] = [M^{2m} \# S^{2m}]$ is the zero element of this group. Now Lemmas 2.6 and 2.7, used in conjunction with a simple diagram chase for $m = 4n+1$, show that $I_c(M^{8n+2}) = \mathrm{Ker}(\Phi = f^*_M \circ \phi_*^{-1} \circ \psi_*)$, thus proving Theorem 2.5. □

Identifying the group Θ_{8n+2} with $\tilde{\pi}^0(S^{8n+2})$ in view of Theorem 2.5 we consider the following question, related to inertia groups:

Problem 2.8 *What is the kernel of $f^*_{\mathbb{C}P^{4n+1}}\colon \tilde{\pi}^0(S^{8n+2}) \to \tilde{\pi}^0(\mathbb{C}P^{4n+1})$?*

We explore some cases of this question in the next section.

3 Computations in stable homotopy

In this section we make computations relating to Problem 2.8. We work in the category of spectra. Throughout this section we use the notation X for both the space and the spectrum $\Sigma^\infty X$ and the notation $\{X, Y\}$ for the stable homotopy classes of maps from X to Y.

Recall [7; 11] that there are models of the category of spectra which are a closed symmetric monoidal category with monoidal structure given by the smash product $\wedge$ and the mapping spectrum denoted by $F(-,-)$. The sphere spectrum S^0 is the unit of the monoidal structure. In this category, for a spectrum X one may form the dual spectrum $DX = F(X, S^0)$. This notion appeared earlier for finite spectra as the Spanier–Whitehead dual of X [2].

One notes that if X is a finite cellular spectrum then so is the dual DX. We briefly recall this construction. Let the cellular structure on X be given by $X = \operatorname{colim}_n X^{(n)}$ such that $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching a cell of dimension a_n. That is, there are cofibre sequences

$$S^{a_n-1} \to X^{(n-1)} \to X^{(n)}.$$

The dual structure on DX is given by $DX = \operatorname{colim}_n D(X/X^{(n)})$, with cofibre sequences

$$S^{-a_n-1} \to D(X/X^{(n)}) \to D(X/X^{(n-1)})$$

obtained by dualizing the cofibre

$$S^{a_n} \to X/X^{(n-1)} \to X/X^{(n)} \to S^{a_n+1}.$$

Thus, in the colimit $DX = \operatorname{colim}_n D(X/X^{(n)})$, $D(X/X^{(n-1)})$ is obtained from $D(X/X^{(n)})$ by attaching a cell of dimension $-a_n$. Therefore, DX is also a finite cellular spectrum with a $-n$–cell for every n–cell of X.

Note that $\tilde{\pi}^0(X) \cong \{X, S^0\}$. Note also that $\{X, S^0\} = \pi_0 F(X, S^0) = \pi_0 D(X)$. Therefore, for a map $f\colon X \to Y$ of spectra, the map $\tilde{\pi}^0(f)\colon \tilde{\pi}^0(Y) \to \tilde{\pi}^0(X)$ is equivalent to the map $\pi_0(D(f))\colon \pi_0 D(Y) \to \pi_0 D(X)$ induced by the dual map $D(f)\colon D(Y) \to D(X)$.

For Problem 2.8 we wish to compute $\tilde{\pi}^0(f)$, where $f\colon \mathbb{C}P^{4n+1} \to S^{8n+2}$ is the usual degree 1 map. Our approach is to compute $\pi_0(D(f))$.

3.1 Computations in dimension 18

We begin by noting that $D\mathbb{C}P^9$ has the filtration

$$S^{-18} = D(\mathbb{C}P^9/\mathbb{C}P^8) \to D(\mathbb{C}P^9/\mathbb{C}P^7) \to \cdots \to D(\mathbb{C}P^9/\mathbb{C}P^0) = D(\mathbb{C}P^9) \tag{3-1}$$

and there are cofibre sequences, for $1 \le k \le 8$,

$$S^{-2k-1} \to D(\mathbb{C}P^9/\mathbb{C}P^k) \to D(\mathbb{C}P^9/\mathbb{C}P^{k-1}) \to S^{-2k}.$$

The degree 1 map $\mathbb{C}P^9 \to S^{18}$ dualises to the inclusion of the bottom cell $S^{-18} \to D(\mathbb{C}P^9)$. Thus we are interested in the question: which elements of $\pi^s_{18} = \pi_0 S^{-18}$ map to 0 under the map above? Recall that the homotopy group $\pi_0 S^{-18} = \pi^s_{18} = \mathbb{Z}/2 \oplus \mathbb{Z}/8$. The $\mathbb{Z}/2$ summand is generated by the element μ_{18} [1]. From [8] it follows that this element maps nontrivially into $\pi_0 D\mathbb{C}P^9$. Therefore the question remains about the other summand, $\mathbb{Z}/8$. In this computation we need formulas for the action of the Steenrod operations on the cohomology of $\mathbb{C}P^n$. We recall them below:

Proposition 3.1 *In $H^*(\mathbb{C}P^n;\mathbb{Z}/2)$ we have the formulas*

$$\mathrm{Sq}^2(x^k) = x^{k+1} \iff k \text{ is odd},$$
$$\mathrm{Sq}^4(x^k) = x^{k+2} \iff k \equiv 2 \text{ or } 3 \pmod 4.$$

Recall that the Steenrod operation Sq^2 detects the Hopf map η, which in our notation is h_1, and Sq^4 detects the map ν, which in our notation is h_2 (modulo $2h_2$). We prove:

Proposition 3.2 *The map $\pi^s_{18} = \pi_0 S^{-18} \to \pi_0(D(\mathbb{C}P^9/\mathbb{C}P^3))$ is injective.*

Proof We compute this map using the filtration (3-1). Note that the group π^s_{18} is 2–torsion so it suffices to work in the 2–local category. We use the notation in [25, Appendix 3.3]. In terms of this notation, π^s_{18} is $\mathbb{Z}/2\{h_1P^2h_1\} \oplus \mathbb{Z}/8\{h_2h_4\}$. It helps to note that the element h_2h_4 is indecomposable in the algebra π^s_* of stable homotopy groups (since the element h_4 supports the differential $d_2(h_4) = h_0h_3^2$ and all the hidden extensions in the range are written in Corollary 4.4.50).

We start proceeding along the sequence (3-1) with the spectrum $D(\mathbb{C}P^9/\mathbb{C}P^7)$. This fits into a cofibre

$$S^{-17} \to S^{-18} \to D(\mathbb{C}P^9/\mathbb{C}P^7).$$

The map $S^{-17} \to S^{-18} \in \pi^s_1 = \mathbb{Z}/2\{h_1\}$. Whether it is nontrivial or not is determined by the action of the Steenrod operation Sq^2 on the cone and hence determined by the action of Sq^2 on $\mathbb{C}P^9/\mathbb{C}P^7$. Note from Proposition 3.1 that $\mathrm{Sq}^2(x^8)=0$. Therefore the map is trivial and $D(\mathbb{C}P^9/\mathbb{C}P^7)\simeq S^{-18}\vee S^{-16}$. It follows that the map from $\pi_{18}^s\to\pi_0D(\mathbb{C}P^9/\mathbb{C}P^7)$ is injective.

The next term is $D(\mathbb{C}P^9/\mathbb{C}P^6)$. We have the cofibre sequence

$$S^{-15}\to D(\mathbb{C}P^9/\mathbb{C}P^7)\to D(\mathbb{C}P^9/\mathbb{C}P^6). \tag{3-2}$$

The map $S^{-15}\to D(\mathbb{C}P^9/\mathbb{C}P^7)\simeq S^{-18}\vee S^{-16}$ is an element of $\pi_1^s\oplus\pi_3^s=\mathbb{Z}/2\{h_1\}\oplus\mathbb{Z}/8\{h_2\}$. Note that on $\mathbb{C}P^9/\mathbb{C}P^6$ the Steenrod operations satisfy the formulas $\mathrm{Sq}^2(x^7)=x^8$ and $\mathrm{Sq}^4(x^7)=x^9$. Thus the 16–cell in $\mathbb{C}P^9/\mathbb{C}P^6$ attaches onto the 14–cell by h_1 and the 18–cell attaches via h_2 onto the 14–cell (or some other odd multiple which does not change the computations below). Therefore the map $S^{-15}\to S^{-16}\vee S^{-18}$ is given by (h_1,h_2). On π_0, we have the sequence

$$\pi_{15}^s\xrightarrow{(h_1,h_2)}\pi_{16}^s\oplus\pi_{18}^s\to\pi_0D(\mathbb{C}P^9/\mathbb{C}P^6).$$

Observe that multiplication by h_2 from π_{15}^s to π_{18}^s is 0. This may be read off from [25, Table A.3.1a] and the fact that h_4, h_0h_4 and $h_0^2h_4$ support nontrivial differentials (see also [18, Figure 5.11]). It follows that the map from $\pi_{18}^s\to\pi_0D(\mathbb{C}P^9/\mathbb{C}P^6)$ is injective.

The next term in the sequence (3-1) is $D(\mathbb{C}P^9/\mathbb{C}P^5)$ and is formed from $D(\mathbb{C}P^9/\mathbb{C}P^6)$ by the cofibre

$$S^{-13}\to D(\mathbb{C}P^9/\mathbb{C}P^6)\to D(\mathbb{C}P^9/\mathbb{C}P^5).$$

The group $\pi_{13}^s=0$ and so the map from $\pi_{18}^s\to\pi_0D(\mathbb{C}P^9/\mathbb{C}P^5)$ is injective.

Next we analyse the cofibre

$$S^{-11}\to D(\mathbb{C}P^9/\mathbb{C}P^5)\to D(\mathbb{C}P^9/\mathbb{C}P^4)$$

and compute the image $\pi_{11}^s=\mathbb{Z}/8\{Ph_2\}\to\pi_0D(\mathbb{C}P^9/\mathbb{C}P^5)$. Note that the methods above imply that $D(\mathbb{C}P^9/\mathbb{C}P^5)$ is the cofibre

$$S^{-13}\vee S^{-15}\xrightarrow{A}S^{-16}\vee S^{-18}\to D(\mathbb{C}P^9/\mathbb{C}P^5), \tag{3-3}$$

where the matrix A is given by

$$\begin{bmatrix} h_2 & h_1\\ 0 & h_2\end{bmatrix}.$$

The map $S^{-11} \to D(\mathbb{C}P^9/\mathbb{C}P^5)$ may be described by a map $S^{-11} \to S^{-12} \vee S^{-14}$ (hence in $\pi_1^s \oplus \pi_3^s = \mathbb{Z}/2\{h_1\} \oplus \mathbb{Z}/8\{h_2\}$) together with a choice of null homotopy after composition by A to $S^{-15} \vee S^{-17}$. The choice of null homotopy lies in $\{S^{-11}, S^{-16} \vee S^{-18}\} = \pi_5^s \oplus \pi_7^s = \mathbb{Z}/16\{h_3\}$.

The map $S^{-11} \to S^{-12} \vee S^{-14}$ is also the attaching map of the -10–cell for the spectrum $D(\mathbb{C}P^7/\mathbb{C}P^4)$. Hence one may try to compute its homotopy class via Steenrod operations. From the formulas in Proposition 3.1 we have $\mathrm{Sq}^2(x^5) = x^6$ and $\mathrm{Sq}^4(x^5) = 0$. So the map onto the -12–cell is h_1 and the map onto the -14–cell is some even multiple of h_2. It follows from the formulas in [23, Proposition 5.2] that the second map is 0.

Now take a class $a \in \pi_{11}^s$, so that a is some multiple of Ph_2. To compute its image onto $\pi_{12}^s \oplus \pi_{14}^s$, one has to multiply by the class $(h_1, 0)$. Notice that $h_1 Ph_2 = 0$. Therefore the image must be zero.

Our case of interest is the image in π_{18}^s. This can be computed using Toda brackets. More precisely, the image $\pi_{11}^s \to \pi_0 D(\mathbb{C}P^9/\mathbb{C}P^5)$ lies in the image of

$$\pi_0(S^{-16} \vee S^{-18}) \to \pi_0 D(\mathbb{C}P^9/\mathbb{C}P^5),$$

which is computed as the image of the set $\langle \pi_{11}^s, (h_1, 0), A\rangle$. We obtain the image onto $\pi_{18}^s \subset \pi_0 D(\mathbb{C}P^9/\mathbb{C}P^5)$ as the projection of $\langle \pi_{11}^s, (h_1, 0), A\rangle \in \pi_{16}^s \oplus \pi_{18}^s$ onto π_{18}^s. As $h_2 \cdot \pi_{15}^s$ and $\pi_{11}^s \cdot h_3$ are 0 (from [25, Table A.3.1a]), the indeterminacy of the bracket maps to 0 in π_{18}^s. Now, the -14–cell is the only cell in $D(\mathbb{C}P^9/\mathbb{C}P^5)$ which attaches nontrivially down to the -18–cell and the map from S^{-11} is null-homotopic on the -14–cell, so the above bracket projected down to π_{18}^s must contain 0. Hence the map from $\pi_{18}^s \to \pi_0 D(\mathbb{C}P^9/\mathbb{C}P^4)$ is injective.

The next term in the sequence (3-1) is $D(\mathbb{C}P^9/\mathbb{C}P^3)$. This fits into a cofibre

$$S^{-9} \to D(\mathbb{C}P^9/\mathbb{C}P^4) \to D(\mathbb{C}P^9/\mathbb{C}P^3)$$

and we compute the image $\pi_9^s = \mathbb{Z}/2\{h_2^3 = h_1^2 h_3, h_1 c_0, Ph_1\} \to \pi_0 D(\mathbb{C}P^9/\mathbb{C}P^4)$. Compose the map $S^{-9} \to D(\mathbb{C}P^9/\mathbb{C}P^4) \to S^{-10}$ to the top cell. This can be detected by computing Sq^2. As $\mathrm{Sq}^2(x^4) = 0$, this map is null-homotopic. Therefore the attaching map goes down to $D(\mathbb{C}P^9/\mathbb{C}P^5)$, which we compute using the cofibre (3-3).

The map $S^{-9} \to D(\mathbb{C}P^9/\mathbb{C}P^5)$ is given by a map $S^{-9} \to S^{-12} \vee S^{-14}$ (hence in $\pi_3^s \oplus \pi_5^s = \mathbb{Z}/8\{h_2\}$) and a choice of null homotopy after composition by A to $S^{-15} \vee S^{-17}$. The choice of null homotopy lies in $\{S^{-9}, S^{-16} \vee S^{-18}\} = \pi_7^s \oplus \pi_9^s =$

$\mathbb{Z}/16\{h_3\}\oplus\mathbb{Z}/2\{h_2^3, h_1c_0, Ph_1\}$. We have $\mathrm{Sq}^2(x^4)=0$ and hence the map is of the form $2kh_2$. From the formulas in [23, Proposition 5.2] it follows that $k=1$. Now take a class $a\in\pi_9^s$. Note $h_2\cdot\pi_9^s=0$. Thus the image in π_{12}^s is 0.

Our interest is the image in π_{18}^s. This can again be computed using Toda brackets as in the previous case. The image onto π_{18}^s must be obtained by map onto the factor S^{-14} which is the only factor which attaches down to the -18–cell. But the attachment of S^{-9} onto this cell is 0 and hence the entire Toda bracket is forced to contain zero. Also the indeterminacy may be computed to be 0. Hence the map from $\pi_{18}^s\to\pi_0D(\mathbb{C}P^9/\mathbb{C}P^3)$ is injective. □

The above computation implies that the classes in π_{18}^s survive in the sequence all the way up to $\pi_0D(\mathbb{C}P^9/\mathbb{C}P^3)$. However, in the next step we obtain a nontrivial kernel. We use some formulas for Toda brackets from [18].

Theorem 3.3 *The class $4h_2h_4\in\pi_{18}^s$ maps to 0 in $\pi_0(D(\mathbb{C}P^9/\mathbb{C}P^2))$. It follows that the kernel of the map $\pi_{18}^s\to\pi_0(D\mathbb{C}P^9)$ is at least $\mathbb{Z}/2$.*

Proof The second statement follows from the first and the fact that h_2h_4 represents an element of order 8 in π_{18}^s. Thus we need prove only the first. We have the cofibre

$$S^{-7}\to D(\mathbb{C}P^9/\mathbb{C}P^3)\to D(\mathbb{C}P^9/\mathbb{C}P^2).$$

We compute the image $\pi_7^s=\mathbb{Z}/16\{h_3\}\to\pi_0D(\mathbb{C}P^9/\mathbb{C}P^3)$. Compose the map $S^{-7}\to D(\mathbb{C}P^9/\mathbb{C}P^3)\to S^{-8}$ to the top cell. This composite is h_1 by the formula $\mathrm{Sq}^2(x^3)=x^4$ in Proposition 3.1. The map $h_1\colon\pi_7^s\to\pi_8^s$ has kernel $\mathbb{Z}/8\{2h_3\}$.

First we prove that the map $\pi_7^s\to\pi_0D(\mathbb{C}P^7/\mathbb{C}P^3)$ has kernel $\mathbb{Z}/8\{2h_3\}$. That is, we show that $2h_3$ maps to 0 in the latter group. Note from the proof of Proposition 3.2 that none of the cells in dimension -8, -10 or -12 attach to the -14–cell. Thus we have $D(\mathbb{C}P^7/\mathbb{C}P^3)\simeq D(\mathbb{C}P^6/\mathbb{C}P^3)\vee S^{-14}$.

Observe that the -8–cell of $D(\mathbb{C}P^7/\mathbb{C}P^3)$ does not attach to the -10–cell. Together with the fact that π_{12}^s and π_{13}^s are 0, we get that $\pi_0D(\mathbb{C}P^6/\mathbb{C}P^3)\to\pi_0(S^{-8}\vee S^{-10})$ is an isomorphism. Now observe that the composite map $S^{-7}\to S^{-8}\vee S^{-10}$ is h_1 on the first factor and h_2 on the second factor. Hence the kernel of the map $\pi_7^s\to\pi_0D(\mathbb{C}P^6/\mathbb{C}P^3)$ is $\mathbb{Z}/8\{2h_3\}$.

Observe that Proposition 5.6 of [23] implies that the 14–cell in $\mathbb{C}P^7/\mathbb{C}P^2$ does not attach to the cells in dimension 8, 10 or 12, and attaches onto the 6–cell by the map $2h_3$. Hence the map $S^{-7} \to D(\mathbb{C}P^6/\mathbb{C}P^3) \vee S^{-14} \to S^{-14}$ is given by $2h_3$. Thus the map $\pi_7^s \to \pi_{14}^s$ is multiplication by $2h_3$, which is 0. Thus we have that the kernel of the map $\pi_7^s \to \pi_0 D(\mathbb{C}P^7/\mathbb{C}P^3)$ is $\mathbb{Z}/8\{2h_3\}$.

Next we extend the above result to the map $\pi_7^s \to \pi_0 D(\mathbb{C}P^8/\mathbb{C}P^3)$. The map $2h_3\colon S^0 \to S^{-7}$ factors in the diagram as below:

$$\begin{tikzcd}
S^0 \arrow[r, "2h_3"] \arrow[dr, "h_3"'] & S^{-7} \arrow[r] & D(\mathbb{C}P^8/\mathbb{C}P^3) \arrow[r] & S^{-8} \\
& S^{-7} \arrow[u, "2"'] \arrow[r, "\alpha"] & D(\mathbb{C}P^8/\mathbb{C}P^4) \arrow[u] &
\end{tikzcd}$$

We prove that the composite

$$S^0 \xrightarrow{h_3} S^{-7} \xrightarrow{\alpha} D(\mathbb{C}P^8/\mathbb{C}P^4)$$

is 0. The above argument shows that

$$S^0 \xrightarrow{h_3} S^{-7} \xrightarrow{\alpha} D(\mathbb{C}P^8/\mathbb{C}P^4) \to D(\mathbb{C}P^7/\mathbb{C}P^4) \simeq D(\mathbb{C}P^6/\mathbb{C}P^4) \vee S^{-14}$$

is 0. It remains to compute the map onto

$$\pi_{16}^s = \pi_0 S^{-16},$$

the bottom cell of $D(\mathbb{C}P^8/\mathbb{C}P^4)$. This may be computed via the cofibre

$$S^{-16} \to D(\mathbb{C}P^8/\mathbb{C}P^4) \xrightarrow{q} D(\mathbb{C}P^7/\mathbb{C}P^4) \to S^{-15}$$

as an element of the sum of Toda brackets $\langle h_3, q \circ \alpha, D(\mathbb{C}P^6/\mathbb{C}P^4) \to S^{-15}\rangle$ and $\langle h_3, q \circ \alpha, S^{-14} \to S^{-15}\rangle$.

We use that the attachment of the -6–cell to the -14–cell is $2h_3$, as noted above. Thus the attachment of α onto the -14–cell is given by $4h_3$. Hence the latter bracket equals $\langle h_3, 2h_3, h_1\rangle = 2h_1h_4 = 0$ modulo trivial indeterminacy.

The first bracket above is the 4–fold bracket $\langle h_3, 2h_2, h_1, h_2\rangle$. The indeterminacy of this bracket lies in the three-fold Toda bracket $\langle Ph_2, h_1, h_2\rangle$. Note that this bracket is in the kernel of the map $\pi_{16}^s \to \pi_0 D(\mathbb{C}P^8/\mathbb{C}P^4)$ killed by the attachment of the -10–cell. Modulo the above indeterminacy, the bracket $\langle h_3, 2h_2, h_1, h_2\rangle$ is a multiple of 2 and hence 0. Thus, the kernel of $\pi_7^s \to \pi_0 D(\mathbb{C}P^8/\mathbb{C}P^3)$ is $2h_3$.

The class $2h_3$ maps to 0 under $S^{-7} \to D(\mathbb{C}P^9/\mathbb{C}P^3) \to D(\mathbb{C}P^8/\mathbb{C}P^3)$. Thus it maps to $\pi_0 S^{-18}$. We prove that it maps to the class $4h_2h_4$ under this map. We have the factorization, as above:

$$\begin{tikzcd} S^0 \arrow[r, "2h_3"] \arrow[dr, "h_3"'] & S^{-7} \arrow[r] & D(\mathbb{C}P^9/\mathbb{C}P^3) \arrow[r] & S^{-8} \\ & S^{-7} \arrow[u, "2"] \arrow[r, "\alpha"] & D(\mathbb{C}P^9/\mathbb{C}P^4) \arrow[u] & \end{tikzcd}$$

We write $D(\mathbb{C}P^9/\mathbb{C}P^4)$ as the cofibre

$$\Sigma^{-1}D(\mathbb{C}P^6/\mathbb{C}P^4)\vee S^{-15}\to S^{-16}\vee S^{-18}\to D(\mathbb{C}P^9/\mathbb{C}P^4)$$
$$\to D(\mathbb{C}P^6/\mathbb{C}P^4)\vee S^{-14}\to S^{-15}\vee S^{-17}.$$

Hence the map onto π_{18}^s is a sum of Toda brackets $\langle h_3, \alpha, D(\mathbb{C}P^6/\mathbb{C}P^4)\to S^{-17}\rangle$ and $\langle h_3, \alpha, S^{-14}\to S^{-17}\rangle$. The latter map is the Toda bracket $\langle h_3, 4h_3, h_2\rangle = 2h_2h_4$ modulo trivial indeterminacy.

For the first map, note that the attaching map $D(\mathbb{C}P^6/\mathbb{C}P^4)\to S^{-17}$ restricting to the bottom cell S^{-12} is trivial and on the top cell is h_3 as computed by the Steenrod operation $\mathrm{Sq}^8(x^5) = x^9$. Therefore the first map is $\langle h_3, 2h_2, h_3\rangle$. This bracket is computed in [18, page 251] as $2h_2h_4$.

Therefore the sum of the two maps is $4h_2h_4$. It follows that the element $4h_2h_4$ maps to 0 in $\pi_0 D(\mathbb{C}P^9/\mathbb{C}P^2)$. This completes the proof. □

It follows from the above result that the kernel of $\pi_{18}^s\to\{\mathbb{C}P^9, S^0\}$ is at least $\mathbb{Z}/2$ and is a subgroup of $\mathbb{Z}/8$. We prove that it cannot be $\mathbb{Z}/8$, so that the kernel can be $\mathbb{Z}/2$ or $\mathbb{Z}/4$.

Theorem 3.4 *The class $h_2h_4\in\pi_{18}^s$ maps to a nontrivial class in $\{\mathbb{C}P^9, S^0\}$.*

Proof We use some computations from [28]. Recall that the element h_2h_4 is denoted by ν^*. For such a stable class, the sphere of origin is the first sphere where this class desuspends to. For the class ν^* the sphere of origin is S^{12} and it desuspends to the class $\xi_{12}\colon S^{30}\to S^{12}$. The class ν^* also desuspends to $\nu_{16}^*\colon S^{34}\to S^{16}$.

From Lemma 12.14 of [28] we know that $H(\nu_{16}^*) = \nu_{31} \pmod{2\nu_{31}}$. The latter maps isomorphically to the stable range and is equivalent to (an odd multiple of) the map $h_2\in\pi_3^s$. We have the commutative diagram:

$$\begin{tikzcd} {[S^{34}, S^{16}]} \arrow[r, "H_\#"] \arrow[d] & {[S^{34}, S^{31}]} \arrow[r, "\cong"] \arrow[d] & \{S^{18}, S^{15}\} \arrow[d] \\ {[\Sigma^{16}\mathbb{C}P^9, S^{16}]} \arrow[r, "H_\#"] & {[\Sigma^{16}\mathbb{C}P^9, S^{31}]} \arrow[r, "\cong"] & \{\mathbb{C}P^9, S^{15}\} \end{tikzcd} \tag{3-4}$$

In terms of the Spanier–Whitehead duality the last map is induced by $\pi_{-15}S^{-18} \to \pi_{-15}D\mathbb{C}P^9$ from the inclusion of the bottom cell. Note that the map $D(\mathbb{C}P^9/\mathbb{C}P^6) \to D\mathbb{C}P^9$ is an isomorphism on π_{-15}. From the cofibre sequence (3-2), we have the exact sequence

$$\cdots \to \pi_{-15}S^{-15} \to \pi_{-15}S^{-18} \oplus \pi_{-15}S^{-16} \to \pi_{-15}D(\mathbb{C}P^9/\mathbb{C}P^6) \to \cdots .$$

The left map sends $1 \in \mathbb{Z}$ to $(h_2, h_1) \in \pi_{-15}S^{-18} \oplus \pi_{-15}S^{-16}$. It follows that $2h_2$ maps to 0 in $\pi_{-15}D(\mathbb{C}P^9/\mathbb{C}P^6)$ and the class h_2 maps nontrivially. Hence, in the diagram (3-4) the element ν_{16}^* in the top-left corner maps to a nontrivial element in the bottom-right group $\{\mathbb{C}P^9, S^{15}\}$. Therefore it maps nontrivially in $[\Sigma^{16}\mathbb{C}P^9, S^{16}]$.

Next we show that this implies that the image in $\{\mathbb{C}P^9, S^0\}$ is nontrivial. We have the diagram:

$$\begin{tikzcd}
{[\Sigma^{16}\mathbb{C}P^9/\mathbb{C}P^7, S^{16}]} \arrow[r] \arrow[d] & \{\mathbb{C}P^9/\mathbb{C}P^7, S^0\} \arrow[d] \\
{[\Sigma^{16}\mathbb{C}P^9, S^{16}]} \arrow[r] \arrow[d] & \{\mathbb{C}P^9, S^0\} \arrow[d] \\
{[\Sigma^{16}\mathbb{C}P^7, S^{16}]} \arrow[r, "\cong"] & \{\mathbb{C}P^7, S^0\}
\end{tikzcd}$$

Note that the vertical arrows are exact. It follows that the kernel of $[\Sigma^{16}\mathbb{C}P^9, S^{16}] \to \{\mathbb{C}P^9, S^0\}$ is isomorphic to the kernel of $[\Sigma^{16}\mathbb{C}P^9/\mathbb{C}P^7, S^{16}] \to \{\mathbb{C}P^9/\mathbb{C}P^7, S^0\}$. We have observed in the proof of Proposition 3.2 that $\mathbb{C}P^9/\mathbb{C}P^7 \simeq S^{16} \vee S^{18}$. Hence, $[\Sigma^{16}\mathbb{C}P^9/\mathbb{C}P^7, S^{16}] \cong [S^{32}, S^{16}] \oplus [S^{34}, S^{16}]$, so that the above kernel is the direct sum

$$\mathrm{Ker}\big([S^{32}, S^{16}] \to \{S^{16}, S^0\}\big) \oplus \mathrm{Ker}\big([S^{34}, S^{16}] \to \{S^{18}, S^0\}\big).$$

The element $\nu_{16}^* \in \pi_{34}S^{16}$ maps to $\nu^* \in \pi_{18}^s$ and so does not lie in the kernel above. It follows that the image of ν_{16}^* maps nontrivially to $\{\mathbb{C}P^9, S^0\}$. Thus the element h_2h_4 is mapped nontrivially in $\{\mathbb{C}P^9, S^0\}$. □

Summarizing the computations in Proposition 3.2 and Theorems 3.3 and 3.4, we have the following result for Problem 2.8:

Corollary 3.5 *The kernel* $\mathrm{Ker}(f_{\mathbb{C}P^9}^*) \subset \pi_{18}^s = \mathbb{Z}/2 \oplus \mathbb{Z}/8$ *is nontrivial but not the entire group* $\mathbb{Z}/8$*. It is either* $\mathbb{Z}/2$ *or* $\mathbb{Z}/4$ *as a subgroup of* $\mathbb{Z}/8$*.*

3.2 Computations in higher dimensions

We follow up the computations in Section 3.1 by demonstrating that there are many examples where the inclusion of the bottom cell in $D(\mathbb{C}P^{8n+2})$ carries a nontrivial kernel in π_0. The methods here are easier, involving computations of Steenrod operations and the existence of certain stable homotopy classes.

The next example after 18 is 26. Recall from [25] that the group $\pi_{26}^s \cong \mathbb{Z}/2\{\mu_{26}\} \oplus \mathbb{Z}/2\{h_2^2 g\} \oplus \mathbb{Z}/3\{\beta_2\}$. We know that the class μ_{26} survives to $\mathbb{C}P^{13}$ from [8]. We have the following result:

Theorem 3.6 *The class $h_2^2 g$ maps to 0 in $\pi_0 D(\mathbb{C}P^{13}/\mathbb{C}P^{11})$. It follows that the kernel of the map $\pi_{26}^s \to \pi_0(D\mathbb{C}P^{13})$ is at least $\mathbb{Z}/2$.*

Proof As in Section 3.1, we have the filtration

$$S^{-26} = D(\mathbb{C}P^{13}/\mathbb{C}P^{12}) \to D(\mathbb{C}P^{13}/\mathbb{C}P^{11}) \to \cdots \to D(\mathbb{C}P^{13}/\mathbb{C}P^{0}) = D(\mathbb{C}P^{13})$$

and the map $S^{-26} \to D(\mathbb{C}P^{13})$ is the inclusion of the bottom cell. We show that $h_2^2 g$ is in the kernel. It suffices to work 2–locally. We have the cofibre

$$S^{-25} \to S^{-26} \to D(\mathbb{C}P^{13}/\mathbb{C}P^{11}).$$

The map $S^{-25} \to S^{-26}$ is an element of $\pi_1^s = \mathbb{Z}/2\{h_1\}$ and, since $\mathrm{Sq}^2(x^{12}) = 0$ in the cohomology of $\mathbb{C}P^n$, the map is trivial. It follows that $D(\mathbb{C}P^{13}/\mathbb{C}P^{11}) \simeq S^{-26} \vee S^{-24}$. Next we have the cofibre

$$S^{-23} \to D(\mathbb{C}P^{13}/\mathbb{C}P^{11}) \to D(\mathbb{C}P^{13}/\mathbb{C}P^{10}).$$

The map $S^{-23} \to D(\mathbb{C}P^{13}/\mathbb{C}P^{11})$ is given by a pair of maps to S^{-26} and S^{-24}. We note the Steenrod squares $\mathrm{Sq}^2(x^{11}) = x^{12}$ and $\mathrm{Sq}^4(x^{11}) = x^{13}$ in the cohomology of $\mathbb{C}P^n$. Therefore the map is given by (h_1, h_2). On π_0 we have the sequence

$$\pi_{23}^s \xrightarrow{(h_1,h_2)} \pi_{24}^s \oplus \pi_{26}^s \to \pi_0 D(\mathbb{C}P^{13}/\mathbb{C}P^{10}).$$

Now note that $h_2 g$ represents a nontrivial element in π_{23}^s and $h_1 h_2 g = 0$. Thus $h_2^2 g$ is in the image of the left-hand map and hence maps to 0 in $\pi_0 D(\mathbb{C}P^{13}/\mathbb{C}P^{10})$. Therefore, $h_2^2 g$ maps to 0 in $\pi_0 D\mathbb{C}P^{13}$, proving the theorem. □

We have the corresponding result for the Problem 2.8.

Corollary 3.7 *The kernel $\mathrm{Ker}(f^*_{\mathbb{C}P^{13}}) \subset \pi_{26}^s = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$ is nontrivial and contains a summand $\mathbb{Z}/2$.*

Next we show that techniques as in Theorem 3.6 exist in many high dimensions. More precisely, we demonstrate examples in higher dimensions where the map $S^{-8n-2} \to D(\mathbb{C}P^{8n+2})$ has a nontrivial kernel on π_0 using some p–local computations. We use a result from [20]: For $p \geq 7$ the classes $\alpha_1\beta_1^r\gamma_t$ are nontrivial in the stable homotopy groups of S^0 for $2 \leq t \leq p-1$ and $r \leq p-2$ (in dimension $n(t,p,r) = [2(tp^3-t-p^2)+2r(p^2-1-p)-2]$). With these assumptions, $\beta_1^r\gamma_t$ is also nontrivial in dimension $n(t,p,r)-(2p-3)$.

We note:

Proposition 3.8 *In $H^*(\mathbb{C}P^n;\mathbb{Z}/p)$, $P^1(x^k) \neq 0$ if and only if p does not divide k.*

In the next theorem, note that if $(p-1)(t-r)+r \equiv 3 \pmod 4$ then $n(t,p,r) \equiv 2 \pmod 8$.

Theorem 3.9 *Suppose that p is a prime ≥ 7, $2 \leq t \leq p-1$ and $r \leq p-2$. Assume that p does not divide $t+r$. Then the map $\pi^s_{n(t,p,r)} \to \pi_0 D(\mathbb{C}P^{n(t,p,r)/2})$ has nontrivial p–torsion in the kernel.*

Proof Note that the condition p does not divide $t+r$ implies that $n(t,p,r)-2(p-1)$ is not divisible by p. We work p–locally. The first nontrivial element in $\pi^s_* \otimes \mathbb{Z}_{(p)}$ in positive dimension is α_1 in dimension $2p-3$, and this is detected by the Steenrod operation P^1.

Constructing $D(\mathbb{C}P^{n(t,p,r)/2})$ cell by cell as above in the p–local category, the first possible nontrivial attaching map is the map

$$S^{-n(t,p,r)+2p-3} \to D(\mathbb{C}P^{n(t,p,r)/2}/\mathbb{C}P^{(n(t,p,r)-2(p-1))/2}).$$

The assumption that p does not divide $t+r$ implies that, in $\mathbb{C}P^N$ for $N \gg 0$,

$$P^1(x^{(n(t,p,r)-2(p-1))/2}) = x^{n(t,p,r)/2}.$$

Therefore the map $S^{-n(t,p,r)+2p-3} \to D(\mathbb{C}P^{n(t,p,r)/2}/\mathbb{C}P^{(n(t,p,r)-2(p-1))/2})$ is given by α_1 on the bottom cell. Now, in the long exact sequence of homotopy groups

$$\cdots \to \pi^s_{n(t,p,r)-(2p-3)} \to \pi_0 D(\mathbb{C}P^{n(t,p,r)/2}/\mathbb{C}P^{(n(t,p,r)-2(p-1))/2}) \\ \to \pi_0 D(\mathbb{C}P^{n(t,p,r)/2}/\mathbb{C}P^{(n(t,p,r)-2(p-1))/2}) \to \cdots,$$

the nontrivial element $\beta_1^r\gamma_t$ maps to the nontrivial element $\alpha_1\beta_1^r\gamma_t$. It follows that the nontrivial element $\alpha_1\beta_1^r\gamma_t \in \pi^s_{n(t,p,r)}$ goes to 0 in $\pi_0 D(\mathbb{C}P^{n(t,p,r)/2})$. This completes the proof. □

Remark 3.10 The above conditions are easily satisfied. For example, if $t=3,\ r=1$ and $p\geq 7$, we have $n(t,p,r)=6p^3-2p-10$, which is $\equiv 2 \pmod 8$.

The following result is an immediate consequence of stringing together Theorems 2.4 and 2.5, Corollaries 3.5 and 3.7, and Theorem 3.9.

Theorem 3.11 (i) *$I(\mathbb{C}P^9)=\mathbb{Z}_2$ or $\mathbb{Z}_4$.*

(ii) *$I(\mathbb{C}P^{13})\supseteq \mathbb{Z}/2$.*

(iii) *Suppose that p is a prime $\geq 7,\ 2\leq t\leq p-1$ and $r\leq p-2$. Assume that p does not divide $t+r$ and $(p-1)(t-r)+r\equiv 3 \pmod 4$. Then $I(\mathbb{C}P^{n(t,p,r)/2})\supseteq \mathbb{Z}/p$.*

4 Geometric structures of inequivalent smooth structures

In this section we explore some consequences of the computations in the previous section. It is known that inequivalent smoothings need not share certain basic geometric properties with the standard smooth structure. Hitchin [10] has shown that certain homotopy spheres do not admit a Riemannian metric of positive scalar curvature, while the round metric on the standard S^m has positive sectional curvature. In this section, we also show that there exist two smooth structures on $\mathbb{C}P^9$ such that one admits a metric of nonnegative scalar curvature and the other does not.

Let $\mathrm{Ker}(d_{\mathbb{R}})$ denote the kernel of the Adams d–invariant $d_{\mathbb{R}}\colon \pi^s_{8n+2}\to \mathbb{Z}/2$; see [1]. Under the isomorphism $\Theta_{8n+2}\cong \pi^s_{8n+2}$, the Adams d–invariant $d_{\mathbb{R}}\colon \pi^s_{8n+2}\to\mathbb{Z}/2$ may be identified with the α–invariant homomorphism $\alpha\colon \Theta_{8n+2}\to\mathbb{Z}/2$; see [10]. Therefore $\mathrm{Ker}(d_{\mathbb{R}})$ consists of homotopy $8n+2$–spheres which bound spin manifolds.

In [12], we studied the Adams d–invariant and asked the following question to determine the inertia group $I(\mathbb{C}P^{4n+1})$:

Question 4.1 *Let $f\colon \mathbb{C}P^{4n+1}\to S^{8n+2}$ be any degree 1 map with $n\geq 1$.*

Does there exist an element $\eta\in \mathrm{Ker}(d_{\mathbb{R}})\subset \pi^s_{8n+2}=\Theta_{8n+2}$ such that the following is true:

($\star$) *If any map $h\colon S^{q+8n+2}\to S^q$ represents η, then*

$$h\circ\Sigma^q f\colon \Sigma^q\mathbb{C}P^{4n+1}\to S^q$$

is not null-homotopic.

The following theorem shows that the answer to the above question is yes for $n=2$, where the Adams d–invariant $d_{\mathbb{R}}\colon \pi_{18}^s=\mathbb{Z}/2\{h_1P^2h_1\}\oplus\mathbb{Z}/8\{h_2h_4\}\to\mathbb{Z}/2$ is such that $\mathrm{Ker}(d_{\mathbb{R}})=\mathbb{Z}_8\{h_2h_4\}$; see [1]. From Theorem 3.11 we readily deduce:

Proposition 4.2 *$I(\mathbb{C}P^9)$ is properly contained in* $\mathrm{Ker}(d_{\mathbb{R}})=\mathbb{Z}/8$.

Picking up an element in $\mathrm{Ker}(d_{\mathbb{R}})$ which is not in $I(\mathbb{C}P^9)$, we have a class $\{S^{18},S^0\}$ which maps nontrivially in $\{\mathbb{C}P^9,S^0\}$. Thus, for every representative $S^{q+18}\to S^q$, the corresponding map $\Sigma^q\mathbb{C}P^9\to S^q$ is not null-homotopic. This answers Question 4.1 in the case $n=2$.

The existence of classes outside the inertia group for $\mathbb{C}P^9$ also has the following consequence:

Theorem 4.3 *There exist three homotopy 18–spheres Σ_1^{18}, Σ_2^{18} and Σ_3^{18} such that the following is true:*

(i) *The manifolds $\mathbb{C}P^9$, $\mathbb{C}P^9\#\Sigma_1^{18}$, $\mathbb{C}P^9\#\Sigma_2^{18}$ and $\mathbb{C}P^9\#\Sigma_3^{18}$ are pairwise nondiffeomorphic.*

(ii) *The manifolds $\mathbb{C}P^9\#\Sigma_2^{18}$ and $\mathbb{C}P^9\#\Sigma_3^{18}$ do not admit a metric of nonnegative scalar curvature but $\mathbb{C}P^9\#\Sigma_1^{18}$ does.*

Proof We start with the first statement. Let Σ_1^{18}, Σ_2^{18} and Σ_3^{18} be homotopy spheres in $\Theta_{18}\cong\pi_{18}^s=\mathbb{Z}_2\{h_1P^2h_1\}\oplus\mathbb{Z}_8\{h_2h_4\}$ represented by the classes h_2h_4, $h_1P^2h_1$ and $h_1P^2h_1+h_2h_4$, respectively. Note that $\Sigma_i^{18}\notin I(\mathbb{C}P^9)$ by Theorem 3.4, where $i=1$, 2 or 3, and $\Sigma_1^{18}\in\mathrm{Ker}(d_{\mathbb{R}})$ but Σ_2^{18}, $\Sigma_3^{18}\notin\mathrm{Ker}(d_{\mathbb{R}})$. If $\mathbb{C}P^9\#\Sigma_i^{18}$ is diffeomorphic to $\mathbb{C}P^9\#\Sigma_j^{18}$, then $\Sigma_i^{18}\#(\Sigma_j^{18})^{-1}\in I(\mathbb{C}P^9)$. But, $\Sigma_i^{18}\#(\Sigma_j^{18})^{-1}$ equals Σ_k^{18} or $(\Sigma_k^{18})^{-1}$, where $k\neq i$, j. This implies that the manifolds $\mathbb{C}P^9$, $\mathbb{C}P^9\#\Sigma_1^{18}$, $\mathbb{C}P^9\#\Sigma_2^{18}$ and $\mathbb{C}P^9\#\Sigma_3^{18}$ are pairwise nondiffeomorphic. This proves (i).

Now we turn to (ii). Since $\Sigma_i^{18}\notin\mathrm{Ker}(d_{\mathbb{R}})$, where $i=2$ or 3, and $\mathbb{C}P^9$ is a spin manifold equipped with its natural metric (the Fubini study metric) of positive scalar curvature, the α–invariant satisfies $\alpha(\mathbb{C}P^9)=0$ and $\alpha(\Sigma_i^{18})\neq 0$; see [10]. Therefore,

$$\alpha(\mathbb{C}P^9\#\Sigma_i^{18})=\alpha(\mathbb{C}P^9)+\alpha(\Sigma_i^{18})\neq 0.$$

We now proceed by contradiction. Suppose there exists a nonnegative scalar curvature Riemannian metric g on $\mathbb{C}P^9\#\Sigma_i^{18}$. The nonvanishing of the α–invariant and the well-known deformation properties of scalar curvature [15] now imply that the Riemannian manifold $(\mathbb{C}P^9 \# \Sigma_i^{18}, g)$ must be scalar-flat and $\mathbb{C}P^9 \# \Sigma_i^{18}$ has a nontrivial parallel spinor. Therefore the Riemannian manifold $(\mathbb{C}P^9 \# \Sigma_i^{18}, g)$ has special holonomy [10]. This is a contradiction, since $\mathbb{C}P^9 \# \Sigma_i^{18}$ has generic holonomy. Hence $\mathbb{C}P^9 \# \Sigma_2^{18}$ and $\mathbb{C}P^9 \# \Sigma_3^{18}$ do not admit a metric of nonnegative scalar curvature. Now consider the connected sum $\mathbb{C}P^9 \# \Sigma_1^{18}$ and, by using the fact $\alpha(\Sigma_1^{18}) = 0$, it follows that $\mathbb{C}P^9 \# \Sigma_1^{18}$ admits a metric of positive scalar curvature [9]. This proves (ii). □

Another application is the following result, which is an immediate consequence of Theorem 4.3 and [12, Theorem 1.4]:

Theorem 4.4 *Let Σ_1^{18}, Σ_2^{18} and Σ_3^{18} be the specific homotopy 18–spheres posited in Theorem 4.3. Given a positive real number ϵ, there exists a closed complex hyperbolic manifold M^{18} of complex dimension 9 such that the following is true:*

(i) *The manifolds M^{18}, $M^{18} \# \Sigma_1^{18}$, $M^{18} \# \Sigma_2^{18}$ and $M^{18} \# \Sigma_3^{18}$ are pairwise nondiffeomorphic.*

(ii) *Each of the manifolds $M^{18} \# \Sigma_1^{18}$, $M^{18} \# \Sigma_2^{18}$ and $M^{18} \# \Sigma_3^{18}$ supports a negatively curved Riemannian metric whose sectional curvatures all lie in the closed interval $[-4-\epsilon, -1+\epsilon]$.*

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