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Groups of homotopy classes of phantom maps

HIROSHI KIHARA

We introduce a new approach to phantom maps which largely extends the rationalization-completion approach developed by Meier and Zabrodsky. Our approach enables us to deal with the set $\mathrm{Ph}(X,Y)$ of homotopy classes of phantom maps and the subset $\mathrm{SPh}(X,Y)$ of homotopy classes of special phantom maps simultaneously. We give a sufficient condition for $\mathrm{Ph}(X,Y)$ and $\mathrm{SPh}(X,Y)$ to have natural group structures, which is much weaker than the conditions obtained by Meier and McGibbon. Previous calculations of $\mathrm{Ph}(X,Y)$ have generally assumed that $[X,\Omega\widehat{Y}]$ is trivial, in which case generalizations of Miller's theorem are directly applicable, and calculations of $\mathrm{SPh}(X,Y)$ have rarely been reported. Here, we calculate not only $\mathrm{Ph}(X,Y)$ but also $\mathrm{SPh}(X,Y)$ in many important cases of nontrivial $[X,\Omega\widehat{Y}]$.

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1 Introduction

Given two pointed CW-complexes X and Y, a map $f\colon X\to Y$ is called a *phantom map* if for any finite complex K and any map $h\colon K\to X$, the composite fh is null-homotopic. Let $\mathrm{Ph}(X,Y)$ denote the subset of $[X,Y]$ consisting of homotopy classes of phantom maps. Since $\mathrm{Ph}(X,Y)=0$ for any finite complex X, the phantom map concept is a key to understanding maps with infinite-dimensional sources, and has been an important topic in homotopy theory since its discovery; see McGibbon [14] and Roitberg [24].

We briefly review the main results of the theory of phantom maps, focusing on the two most important subjects: natural group structure on $\mathrm{Ph}(X,Y)$ and calculation of $\mathrm{Ph}(X,Y)$.

Recall that a space is called an H_0–space (resp. co-H_0–space) if its rationalization is homotopy equivalent to a product of Eilenberg–Mac Lane spaces (resp. coproduct of Moore spaces). Let $\widehat{\mathbb{Z}}$ denote the product $\prod_p \widehat{\mathbb{Z}}_p$ of the p–completions of $\mathbb{Z}$, in which $\mathbb{Z}$ is diagonally contained.

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Theorem A *Let X be a connected CW-complex and Y a nilpotent CW-complex of finite type with π_1 finite.*

1. *If X is a co-H_0–space or Y is an H_0–space,* $\mathrm{Ph}(X,Y)$ *has a natural, divisible, abelian group structure.*
2. *Suppose that X is a Postnikov space with π_1 locally finite, the classifying space of a compact Lie group, or an infinite loop space with π_1 torsion. Suppose that Y is a nilpotent finite complex or an iterated loop space of such a space. Then there is a noncanonical bijection*
$$\mathrm{Ph}(X,Y) \cong \prod_{i>0} H^i(X;\pi_{i+1}(Y)\otimes \widehat{\mathbb{Z}}/\mathbb{Z}).$$
Furthermore, if X is a co-H_0–space or Y is an H_0–space, this bijection can be taken as an isomorphism of groups.

Remark 1.1 (1) Zabrodsky stated [34, Theorem B(d)] that $\mathrm{Ph}(X,Y)$ is a divisible abelian group assuming only that X and Y are 1–connected with finite type. However, it is thought that this assertion is incorrect; see [14, page 1236]. Meier, McGibbon, Roitberg and coauthors have attempted to find a sufficient condition such that $\mathrm{Ph}(X,Y)$ has a natural group structure. Theorem A(1) was essentially proposed by Meier [19] and McGibbon [12, Theorem 4]; see also Roitberg and Touhey [25, page 302].

(2) $\mathrm{Ph}(X,Y)$ is described as the quotient of the set that is (noncanonically) isomorphic to the product $\prod_{i>0} H^i(X;\pi_{i+1}(Y)\otimes\widehat{\mathbb{Z}}/\mathbb{Z})$ by the group action of $[X,\Omega\widehat{Y}]$ (see the works of Meier [20], Zabrodsky [34], and Roitberg and Touhey [25]). Theorem A(2), which is a central result in the calculation of $\mathrm{Ph}(X,Y)$, follows immediately from this description since $[X,\Omega\widehat{Y}]$ is trivial by generalizations of Miller's theorem (see Proposition 6.3(1)). Though most experts on phantom maps have attempted to calculate $\mathrm{Ph}(X,Y)$, an effective method has not been found in cases where $[X,\Omega\widehat{Y}]$ is nontrivial. Thus, such a method is most desirable.

We are also interested in the subset $\mathrm{SPh}(X,Y)$ of $\mathrm{Ph}(X,Y)$ consisting of homotopy classes of *special phantom maps*, defined by the exact sequence of pointed sets

$$0\to \mathrm{SPh}(X,Y)\to \mathrm{Ph}(X,Y)\xrightarrow{e_{Y\sharp}} \mathrm{Ph}(X,\check{Y}), \tag{1-1}$$

where $e_Y\colon Y\to\check{Y}=\prod_p Y_{(p)}$ is a natural map called the *local expansion* (see [24, page 150]). The target Y is usually assumed to be nilpotent of finite type.

$\mathrm{SPh}(X,Y)$ is less clearly understood than $\mathrm{Ph}(X,Y)$, and although we know that a result similar to Theorem A(1) holds under the finite-type condition on X (see [12, Theorem 4]), calculations of $\mathrm{SPh}(X,Y)$ have rarely been reported.

In this paper, we develop a new approach to phantom maps which enables us to deal with $\mathrm{Ph}(X,Y)$ and $\mathrm{SPh}(X,Y)$ simultaneously and to address the following two problems:

Problem 1 Identify a weaker sufficient condition such that $\mathrm{Ph}(X,Y)$ and $\mathrm{SPh}(X,Y)$ have natural group structures.

Problem 2 Calculate $\mathrm{Ph}(X,Y)$ and $\mathrm{SPh}(X,Y)$, especially in the cases where $[X,\Omega\widehat{Y}]$ is nontrivial.

Our solution to Problem 1 is given in Theorem 2.3(1). Our solution to Problem 2 is given in Proposition 2.5 and Theorem 2.7. Proposition 2.5 directly generalizes Theorem A(2) (see Remark 2.6). Theorem 2.7 gives a method for calculating not only $\mathrm{Ph}(X,Y)$ but also $\mathrm{SPh}(X,Y)$ in various important cases of *highly nontrivial* $[X,\Omega\widehat{Y}]$; the power of Theorem 2.7 is illustrated by Corollaries 2.8–2.10.

2 Main results

2.1 Solution to Problem 1

Let $\mathcal{CW}$ denote the category of pointed connected CW-complexes and homotopy classes of maps and let $\mathcal{N}$ denote the full subcategory of $\mathcal{CW}$ consisting of nilpotent CW-complexes of finite type.

Definition 2.1 Let $\mathcal{Q}$ be the full subcategory of $\mathcal{CW}^{\mathrm{op}}\times\mathcal{N}$ consisting of (X,Y) satisfying the following condition:

(Q) For each pair $i,j>0$, the rational cup product

$$\cup\colon H^i(X;\mathbb{Q})\otimes H^j(X;\mathbb{Q})\to H^{i+j}(X;\mathbb{Q})$$

or the rational Whitehead product

$$[\,,]\colon (\pi_{i+1}(Y)\otimes\mathbb{Q})\otimes(\pi_{j+1}(Y)\otimes\mathbb{Q})\to\pi_{i+j+1}(Y)\otimes\mathbb{Q}$$

is trivial.

Remark 2.2 A pair $(X,Y)\in\mathcal{CW}^{\mathrm{op}}\times\mathcal{N}$ is in $\mathcal{Q}$ if X is a co-H_0–space or Y is an H_0–space. $\mathcal{Q}$ contains many other pairs (see Section 4.2).

We have the following fundamental theorem regarding natural group structures on $\mathrm{Ph}(X,Y)$ and $\mathrm{SPh}(X,Y)$.

Theorem 2.3 *Let (X,Y) be an object of $\mathcal{Q}$.*

(1) $\mathrm{Ph}(X,Y)$ *and* $\mathrm{SPh}(X,Y)$ *have natural divisible abelian group structures, for which* $\mathrm{SPh}(X,Y)$ *is a subgroup of* $\mathrm{Ph}(X,Y)$.

(2) *Let* $(f^{\mathrm{op}},g)\colon (K,L)\to(X,Y)$ *be a morphism of* $\mathcal{CW}^{\mathrm{op}}\times\mathcal{N}$. *Then the images* $\mathrm{Im}\,\mathrm{Ph}(f,g)$ *and* $\mathrm{Im}\,\mathrm{SPh}(f,g)$ *are divisible abelian subgroups of* $\mathrm{Ph}(X,Y)$ *and* $\mathrm{SPh}(X,Y)$*, respectively.*

(3) *If X is a co-H–space or Y is an H–space, the group structures on* $\mathrm{Ph}(X,Y)$ *and* $\mathrm{SPh}(X,Y)$ *are compatible with the multiplicative structures on* $[X,Y]$.

Remark 2.4 (1) By Remark 2.2, Theorem 2.3(1) generalizes Theorem A(1). Furthermore, we have a natural epimorphism

$$\prod_{i>0} H^i(X;\pi_{i+1}(Y)\otimes\widehat{\mathbb{Z}}/\mathbb{Z})\to \mathrm{Ph}(X,Y)$$

and a similar natural epimorphism onto $\mathrm{SPh}(X,Y)$ for $(X,Y)\in\mathcal{Q}$ (Propositions 5.10 and 5.12); such natural epimorphisms are obtained for the first time (see Theorem A(2) and Remark 1.1(2)).

(2) Part 2 of Theorem 2.3 describes a new feature of natural group structures on phantom maps and provides a means of calculating $\mathrm{Ph}(X,Y)$ and $\mathrm{SPh}(X,Y)$ (see Theorem 2.7).

2.2 Solution to Problem 2

We begin with a direct generalization of Part 2 of Theorem A.

A space whose i^{th} homotopy group is zero for $i\le n$ and locally finite for $i=n+1$ is said to be $n\frac{1}{2}$*–connected*. Let us define the classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}'$ and $\mathcal{B}'$ by:

$\mathcal{A}=$ the class of $\frac{1}{2}$–connected Postnikov spaces, the classifying spaces of compact Lie groups, $\frac{1}{2}$–connected infinite loop spaces and their iterated suspensions.

$\mathcal{B}=$ the class of nilpotent finite complexes, the classifying spaces of compact Lie groups and their iterated loop spaces.

$\mathcal{A}'=$ the class of $1\frac{1}{2}$–connected Postnikov spaces of finite type and their iterated suspensions.

$\mathcal{B}'=$ the class of BU, BO, $B\,\mathrm{Sp}$, $B\,\mathrm{SO}$, U/Sp, Sp/U, SO/U, U/SO, and their iterated loop spaces.

A pair $(X,Y)\in\mathcal{A}\times\mathcal{B}$ is in $\mathcal{Q}$ if Y is an iterated loop space of a nilpotent finite complex or if Y is the classifying space of a connected Lie group. Any pair $(X,Y)\in\mathcal{A}'\times\mathcal{B}'$ is in $\mathcal{Q}$.

By showing that $[X,\Omega\widehat{Y}]=0$ if (X,Y) is in $\mathcal{A}\times\mathcal{B}$ or $\mathcal{A}'\times\mathcal{B}'$ (Corollary 6.4), we generalize Theorem A(2). Let $\check{\mathbb{Z}}$ denote the product $\prod_p \mathbb{Z}_{(p)}$ of the p–localizations of $\mathbb{Z}$, in which $\mathbb{Z}$ is diagonally contained.

Proposition 2.5 *Let (X,Y) be in $\mathcal{A}\times\mathcal{B}$ or $\mathcal{A}'\times\mathcal{B}'$. Then there exist bijections*

$$\mathrm{Ph}(X,Y)\cong\prod_{i>0}H^i(X;\pi_{i+1}(Y)\otimes\widehat{\mathbb{Z}}/\mathbb{Z}),$$
$$\mathrm{SPh}(X,Y)\cong\prod_{i>0}H^i(X;\pi_{i+1}(Y)\otimes\check{\mathbb{Z}}/\mathbb{Z}).$$

If (X,Y) is in $\mathcal{Q}$, then these bijections can be taken to be natural isomorphisms of abelian groups.

Remark 2.6 The following notes relate to Proposition 2.5:

(1) We should note the following points:

- Besides Miller's theorem, the theorem of Anderson and Hodgkin can be generalized and used to calculate $\mathrm{Ph}(X,Y)$ (see Proposition 6.3 and Corollary 6.4).
- In the generalized Miller's theorem for calculating $\mathrm{Ph}(X,Y)$, the target spaces can be the classifying spaces of compact Lie groups.

These points have been largely overlooked in the literature; an exception is Meier [20], who proved Proposition 6.3(2) for $A=K(\mathbb{Z},n)$ with $n\geq 3$ and $B=BU$, and hence calculated the group of homotopy classes of phantom maps.

(2) Recall that calculations of $\mathrm{SPh}(X,Y)$ have rarely been reported. We should also note the following points:

- The vanishing of $[X,\Omega\widehat{Y}]$ enables us to calculate not only $\mathrm{Ph}(X,Y)$, but also $\mathrm{SPh}(X,Y)$ (see Remark 6.2(2)).
- The last assertion on the group structures on $\mathrm{Ph}(X,Y)$ and $\mathrm{SPh}(X,Y)$ is novel.

Next, we present a new method for calculating the groups $\mathrm{Ph}(X,Y)$ and $\mathrm{SPh}(X,Y)$ for $(X,Y)\in\mathcal{Q}$ with $[X,\Omega\widehat{Y}]\neq 0$. Note that in the following theorem, $p^\sharp\,\mathrm{Ph}(K,Y)$ and $p^\sharp\,\mathrm{SPh}(K,Y)$ are the subgroups of $\mathrm{Ph}(X,Y)$ and $\mathrm{SPh}(X,Y)$ (Theorem 2.3(2)).

Theorem 2.7 *Let (X,Y) be in $\mathcal{Q}$. Let $X' \xrightarrow{i} X \xrightarrow{p} K$ be a cofibration sequence with $[X', \Omega\widehat{Y}] = 0$, or a fibration sequence with weakly contractible $\mathrm{map}_*(X', \Omega\widehat{Y})$. Then there exist natural split exact sequences of abelian groups given by*

$$0 \to p^\sharp\,\mathrm{Ph}(K,Y) \to \mathrm{Ph}(X,Y) \to \prod_{i>0} H^i(X;\pi_{i+1}(Y)\otimes\widehat{\mathbb{Z}}/\mathbb{Z})/p^*H^i(K;\pi_{i+1}(Y)\otimes\widehat{\mathbb{Z}}/\mathbb{Z}) \to 0,$$

$$0 \to p^\sharp\,\mathrm{SPh}(K,Y) \to \mathrm{SPh}(X,Y) \to \prod_{i>0} H^i(X;\pi_{i+1}(Y)\otimes\check{\mathbb{Z}}/\mathbb{Z})/p^*H^i(K;\pi_{i+1}(Y)\otimes\check{\mathbb{Z}}/\mathbb{Z}) \to 0.$$

Using Theorem 2.7, we can produce infinitely many computational examples. We illustrate its power through the following computational results.

Corollary 2.8 *Let X be a connected infinite loop space with $\pi_1(X)/\mathrm{torsion} \cong \mathbb{Z}^m$ for some $m \geq 0$, and let Y be in $\mathcal{B}$. Suppose that (X,Y) is in $\mathcal{Q}$. Then we have the natural isomorphisms of groups*

$$\mathrm{Ph}(X,Y) \cong \prod_{i>0} H^i(X;\pi_{i+1}(Y)\otimes\widehat{\mathbb{Z}}/\mathbb{Z})/p^*H^i(T;\pi_{i+1}(Y)\otimes\widehat{\mathbb{Z}}/\mathbb{Z}),$$
$$\mathrm{SPh}(X,Y) \cong \prod_{i>0} H^i(X;\pi_{i+1}(Y)\otimes\check{\mathbb{Z}}/\mathbb{Z})/p^*H^i(T;\pi_{i+1}(Y)\otimes\check{\mathbb{Z}}/\mathbb{Z}),$$

where the map p from X to the m–dimensional torus T is defined to be the composite $X \to K(\pi_1(X),1) \to K(\pi_1(X)/\mathrm{torsion},1) = T$ of the canonical maps.

Let $K\langle n\rangle$ denote the n–connected cover of K.

Corollary 2.9 *Suppose that K is a $1\frac{1}{2}$–connected finite complex and Y is in $\mathcal{B}$, or that K is a $2\frac{1}{2}$–connected finite complex and Y is in $\mathcal{B}'$. If $(K\langle n\rangle, Y)$ is in $\mathcal{Q}$, there exist natural isomorphisms of groups*

$$\mathrm{Ph}(K\langle n\rangle,Y) \cong \prod_{i>0} H^i(K\langle n\rangle;\pi_{i+1}(Y)\otimes\widehat{\mathbb{Z}}/\mathbb{Z})/p^*H^i(K;\pi_{i+1}(Y)\otimes\widehat{\mathbb{Z}}/\mathbb{Z}),$$
$$\mathrm{SPh}(K\langle n\rangle,Y) \cong \prod_{i>0} H^i(K\langle n\rangle;\pi_{i+1}(Y)\otimes\check{\mathbb{Z}}/\mathbb{Z})/p^*H^i(K;\pi_{i+1}(Y)\otimes\check{\mathbb{Z}}/\mathbb{Z}),$$

where $p\colon K\langle n\rangle \to K$ is the canonical map.

Corollary 2.10 *Let (X, Y) be in $\mathcal{Q}$. Suppose that X is the product of connected CW-complexes X' and K (ie $X' \times K$). Suppose that (X', Y) is in $\mathcal{A} \times \mathcal{B}$ or $\mathcal{A}' \times \mathcal{B}'$. Then there exist natural isomorphisms of groups*

$$\mathrm{Ph}(X, Y) \cong \mathrm{Ph}(K, Y) \oplus \mathrm{Hom}(\widetilde{H}_*(X'; \mathbb{Q}) \otimes H_*(K; \mathbb{Q}), \pi_{*+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}),$$
$$\mathrm{SPh}(X, Y) \cong \mathrm{SPh}(K, Y) \oplus \mathrm{Hom}(\widetilde{H}_*(X'; \mathbb{Q}) \otimes H_*(K; \mathbb{Q}), \pi_{*+1}(Y) \otimes \check{\mathbb{Z}}/\mathbb{Z}),$$

where Hom *denotes the module of homomorphisms of graded $\mathbb{Q}$–modules.*

See Example 6.6 for another computational result on $\mathrm{Ph}(X, Y)$ and $\mathrm{SPh}(X, Y)$.

In the forthcoming paper, we use our approach to study the group of weak identities [23; 16]. We demonstrate that our approach is also useful for investigating the Gray index of phantom maps [5; 6; 7; 18] in the succeeding article.

The paper is organized as follows. For a given map $\varphi\colon Y \to Y'$, the subset $P^{\varphi}(X, Y)$ of $[X, Y]$ is defined as

$$P^{\varphi}(X, Y) = \{[f] \in [X, Y] \mid \varphi_{\sharp}([f]) = 0\}.$$

In Section 3, we develop a general theory leading to the double coset formula for $P^{\varphi}(X, Y)$ (Proposition 3.6) that enables us to treat $\mathrm{Ph}(X, Y)$, $\mathrm{SPh}(X, Y)$ and $\mathrm{Ph}(X, \check{Y})$ simultaneously. This formula encourages the study of the conditions under which the group $[X, \Omega Y_{(0)}] \cong [\Sigma X, Y_{(0)}]$ is abelian, to which Section 4 is devoted. In Section 5, the results of Sections 3 and 4 are applied to phantom maps, and Theorem 2.3 is established. In Section 6, we prove Proposition 2.5, Theorem 2.7, and Corollaries 2.8–2.10.

3 Double coset formula

In this section, we develop a general theory that enables simultaneous treatment of $\mathrm{Ph}(X, Y), \mathrm{SPh}(X, Y)$ and $\mathrm{Ph}(X, \check{Y})$; see the exact sequence (1-1) of pointed sets.

For a given map $\varphi\colon Y \to Y'$, the subset $P^{\varphi}(X, Y)$ of $[X, Y]$ is defined as

$$P^{\varphi}(X, Y) = \{[f] \in [X, Y] \mid \varphi_{\sharp}([f]) = 0\}.$$

Under certain conditions on φ, we determine the rational homotopy structure of the double looping of the fibration sequence

$$F \to Y \xrightarrow{\varphi} Y'$$

(Proposition 3.1) and establish the double coset formula for $P^{\varphi}(X, Y)$ (Proposition 3.6). By applying these results to the profinite completion $c_Y\colon Y \to \widehat{Y}$, the local expansion $e_Y\colon Y \to \check{Y}$, and the natural map $d_Y\colon \check{Y} \to \widehat{Y}$, we can investigate $\mathrm{Ph}(X, Y)$, $\mathrm{SPh}(X, Y)$ and $\mathrm{Ph}(X, \check{Y})$ simultaneously (see Corollary 5.3, Propositions 5.4 and 5.7).

Notation Here and throughout, the subscript (0) denotes the rationalization of a nilpotent space or a nilpotent group.

Proposition 3.1 *Let Y, Y' be 1–connected CW-complexes. Suppose that $\varphi\colon Y \to Y'$ induces monomorphisms on the rational homotopy groups. Then:*

(1) *The homotopy fiber F of φ is nilpotent and the principal fibration*

$$(\Omega Y)_{(0)} \to (\Omega Y')_{(0)} \to F_{(0)}$$

is trivial.

(2) *F is rationally equivalent to $\prod_{i>0} K(\pi_{i+1}(Y')_{(0)}/\pi_{i+1}(Y)_{(0)}, i)$.*

To prove Proposition 3.1, we require the following lemma.

Lemma 3.2 *Let $Z \xrightarrow{\psi} Z' \xrightarrow{p} B$ be a fibration sequence such that Z, Z', and B have the homotopy type of a CW-complex. Suppose that Z and Z' are simple rational spaces with trivial Postnikov invariants and that $\psi\colon Z \to Z'$ induces monomorphisms on the homotopy groups. Then the fibration $p\colon Z' \to B$ is trivial and the base B is a simple rational space with trivial Postnikov invariants.*

Proof First, we construct a homotopy left inverse σ of $\psi\colon Z \to Z'$. Fix homotopy equivalences $Z \simeq \prod K(\pi_i(Z), i)$ and $Z' \simeq \prod K(\pi_i(Z'), i)$ (see Remark 3.3). Choose left inverses s_i of the monomorphisms $\pi_i(\psi)\colon \pi_i(Z) \to \pi_i(Z')$ of $\mathbb{Q}$–modules and define the homotopy equivalence $h\colon Z \to Z$ to be the composite

$$Z \xrightarrow{\psi} Z' \simeq \prod K(\pi_i(Z'), i) \xrightarrow{\prod K(s_i, i)} \prod K(\pi_i(Z), i) \simeq Z.$$

Then, the desired homotopy left inverse σ of $\psi\colon Z \to Z'$ is defined to be the composite

$$Z' \simeq \prod K(\pi_i(Z'), i) \xrightarrow{\prod K(s_i, i)} \prod K(\pi_i(Z), i) \simeq Z \xrightarrow{g} Z,$$

where g is a homotopy inverse of h.

Next, consider the morphism of fibrations

$$\begin{array}{ccc} Z' & \xrightarrow{(p,\sigma)} & B \times Z \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{proj}} & \\ & B & \end{array}$$

We can then easily see that this morphism is a trivialization of the fibration of interest. Since $(p, \sigma)\colon Z' \to B \times Z$ is a homotopy equivalence, B admits an H–space structure, and hence all the Postnikov invariants of B vanish. □

Proof of Proposition 3.1 Since F is nilpotent by [11, Proposition 4.4.1], we have the fibration

$$(\Omega Y)_{(0)} \xrightarrow{(\Omega\varphi)_{(0)}} (\Omega Y')_{(0)} \xrightarrow{\partial_{(0)}} F_{(0)}.$$

We can apply Lemma 3.2 to complete the proof. □

Remark 3.3 Since the product of infinitely many Eilenberg–Mac Lane complexes need not have the homotopy type of a CW-complex, the product in Proposition 3.1 should be interpreted as the weak product (see [32]). We can now operate in the category $\mathcal{CW}$. This interpretation is used throughout the paper.

Remark 3.4 In Proposition 3.1, the assumption that Y and Y' are 1–connected can be replaced by the assumption that $\varphi_\sharp\colon \pi_1(Y) \to \pi_1(Y')$ is a monomorphism. Then, by the universal covering argument, we observe that $\pi_0(F) \cong \pi_1(Y')/\pi_1(Y)$ and that each component of F is rationally equivalent to $\prod_{i>0} K(\pi_{i+1}(Y')_{(0)}/\pi_{i+1}(Y)_{(0)}, i)$.

Recall that for a connected space X and a map $\varphi\colon Y \to Y'$, the subset $P^\varphi(X, Y)$ of $[X, Y]$ is defined by

$$P^\varphi(X, Y) = \{[f] \in [X, Y] \mid \varphi_\sharp([f]) = 0\}.$$

Consider the principal $\Omega Y'$–fibration $\Omega Y' \to F \to Y$, where F is the homotopy fiber of φ. This fibration generates an exact sequence of pointed sets

$$[X, \Omega Y'] \to [X, F] \to P^\varphi(X, Y) \to 0$$

and the group $[X, \Omega Y']$ acts on $[X, F]$ in an obvious way.

Lemma 3.5 (orbit formula) *$P^\varphi(X,Y)$ is isomorphic to the orbit space $[X,F]/[X,\Omega Y']$.*

Proof See [11, Lemma 1.4.7]. □

A more useful formula for $P^{\varphi}(X,Y)$ can be established under certain conditions.

Proposition 3.6 (double coset formula) *Suppose that the map $\varphi\colon Y \to Y'$ is compatible with Proposition 3.1 and that the homotopy fiber F of φ is rational. Then we have the isomorphisms*

$$\begin{aligned} P^{\varphi}(X,Y) &\cong (\Omega\varphi_{(0)})_{\sharp}[X,\Omega Y_{(0)}]\backslash[X,\Omega Y'_{(0)}]/(\Omega r')_{\sharp}[X,\Omega Y'] \\ &\cong \varphi_{(0)\sharp}[\Sigma X, Y_{(0)}]\backslash[\Sigma X, Y'_{(0)}]/r'_{\sharp}[\Sigma X, Y'], \end{aligned}$$

where $r'\colon Y' \to Y'_{(0)}$ denotes the rationalization of Y'.

Proof Rationalizing φ yields the following commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\;r\;} & Y_{(0)} \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi_{(0)}} \\ Y' & \xrightarrow[\;r'\;]{} & Y'_{(0)} \end{array}$$

Since F is rational, this diagram is a homotopy pullback. Thus the natural map

$$(r_{\sharp},\varphi_{\sharp})\colon [X,Y] \to [X,Y_{(0)}] \times_{[X,Y'_{(0)}]} [X,Y']$$

has a kernel that is isomorphic to

$$(\Omega\varphi_{(0)})_{\sharp}[X,\Omega Y_{(0)}]\backslash[X,\Omega Y'_{(0)}]/(\Omega r')_{\sharp}[X,\Omega Y']$$

(see [11, Proposition 2.2.2]). Thus, we merely require that if $\varphi_{\sharp}([f]) = 0$, then $r_{\sharp}([f]) = 0$. This is easily seen from Proposition 3.1(1). □

4 The functor $[\,\cdot\,, \Omega\ \cdot_{(0)}]$

By Proposition 3.6, if $[X,\Omega Y'_{(0)}] \cong [\Sigma X, Y'_{(0)}]$ is abelian, $P^{\varphi}(X,Y)$ has an abelian group structure. Motivated by this inference, we investigate the functor $[\,\cdot\,,\Omega\,\cdot_{(0)}] \cong [\Sigma\,\cdot\,,\ \cdot_{(0)}]$ and use the results of Scheerer [26] to find a necessary and sufficient condition under which the group $[X,\Omega Y_{(0)}] \cong [\Sigma X, Y_{(0)}]$ is abelian. We also study spaces with trivial rational cup and Whitehead products.

4.1 Necessary and sufficient condition that $[X, \Omega Y_{(0)}]$ is abelian

First we review the relevant background material (see [2, Chapter VI, Section 1]).

Let L be a nilpotent Lie algebra over $\mathbb{Q}$. Then L can be endowed with a group structure via the Baker–Campbell–Hausdorff formula

$$x \cdot y = x + y + \tfrac{1}{2}[x, y] + \tfrac{1}{12}[[x, y], y] + \tfrac{1}{12}[[y, x], x] + \cdots .$$

This group $(L, \cdot)$, denoted by $\exp L$, is a nilpotent rational group [8, Chapter 10].

Let (C, Δ) be a connected cocommutative graded coalgebra over $\mathbb{Q}$, and $(\pi, [\ ,\ ])$ a reduced graded Lie algebra over $\mathbb{Q}$. The $\mathbb{Q}$–module $\mathrm{Hom}_{\mathbb{Q}}(C, \pi)$ of homomorphisms of graded $\mathbb{Q}$–modules is a (nongraded) Lie algebra over $\mathbb{Q}$ with the Lie bracket defined by

$$[f, g]\colon C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} \pi \otimes \pi \xrightarrow{[\ ,\ ]} \pi.$$

If $C_n = 0$ for sufficiently large n, $\mathrm{Hom}_{\mathbb{Q}}(C, \pi)$ is a nilpotent Lie algebra. Thus we define the group

$$\exp \mathrm{Hom}_{\mathbb{Q}}(C, \pi) = \varprojlim_{n} \exp \mathrm{Hom}_{\mathbb{Q}}(C_{\leq n}, \pi)$$

as the inverse limit of nilpotent groups, where $C_{\leq n}$ is the subcoalgebra of C of all elements of degree $\leq n$.

Recall that the rational homology $H_*(X; \mathbb{Q})$ of an arcwise connected space X is a connected cocommutative graded coalgebra over $\mathbb{Q}$ whose coproduct is induced by the diagonal map of X. Recall also that for a 1–connected space Y, the rational homotopy group $\pi_*(\Omega Y) \otimes \mathbb{Q}$ equipped with the Samelson product is a reduced graded Lie algebra over $\mathbb{Q}$, and that the Samelson product of ΩY is the Whitehead product of Y.

Let 1-$\mathcal{CW}$ denote the full subcategory of $\mathcal{CW}$ consisting of 1–connected CW-complexes, and let Gr denote the category of groups.

Proposition 4.1 *The functor* $[\,\cdot\,, \Omega \,\cdot_{(0)}]\colon \mathcal{CW}^{\mathrm{op}} \times 1\text{-}\mathcal{CW} \to \mathrm{Gr}$ *is naturally isomorphic to the functor* $\exp\ \mathrm{Hom}_{\mathbb{Q}}(H_*(\,\cdot\,; \mathbb{Q}),\ \pi_*(\Omega\,\cdot) \otimes \mathbb{Q})$.

Proof The proof follows from Proposition 1 in [26, Section 0.1], the remark on [26, page 71] and the natural isomorphism $H_*(\Omega\,\cdot\,; \mathbb{Q}) \cong U\pi_*(\Omega \cdot_{(0)})$ of graded Hopf algebras, where U denotes the universal enveloping algebra functor [22, Appendix]. □

We introduce the following full subcategory of $\mathcal{CW}^{\mathrm{op}} \times 1\text{-}\mathcal{CW}$.

Definition 4.2 Let $\mathcal{Q}^{\sim}$ be the full subcategory of $\mathcal{CW}^{\mathrm{op}} \times 1\text{-}\mathcal{CW}$ whose members (X, Y) satisfy the condition (Q) (see Definition 2.1).

Theorem 4.3 (1) *Let (X, Y) be an object of $\mathcal{CW}^{\mathrm{op}} \times 1\text{-}\mathcal{CW}$. Then (X, Y) is in $\mathcal{Q}^{\sim}$ if and only if $[X, \Omega Y_{(0)}]$ is abelian.*

(2) *The functor $[\,\cdot\,, \Omega \,\cdot_{(0)}] = [\Sigma \,\cdot\,, \,\cdot_{(0)}]\colon \mathcal{Q}^{\sim} \to \mathrm{Gr}$ is naturally isomorphic to the functor $\prod_{i>0} H^i(\,\cdot\,; \pi_{i+1}(\,\cdot\,)_{(0)})$. In particular, $[\,\cdot\,, \Omega \,\cdot_{(0)}] = [\Sigma \,\cdot\,, \,\cdot_{(0)}]$ on $\mathcal{Q}^{\sim}$ is a $\mathbb{Q}$–module-valued functor.*

Proof (1) By definition, we have

$$\exp \mathrm{Hom}_{\mathbb{Q}}(H_*(X;\mathbb{Q}), \pi_*(\Omega Y) \otimes \mathbb{Q}) = \varprojlim_n \exp \mathrm{Hom}_{\mathbb{Q}}(H_{\leq n}(X;\mathbb{Q}), \pi_*(\Omega Y) \otimes \mathbb{Q}).$$

Thus, Proposition 4.1 and the Mal'cev correspondence [8, page 118] imply that the group $[X, \Omega Y_{(0)}]$ is abelian if and only if the Lie algebra $\mathrm{Hom}_{\mathbb{Q}}(H_*(X;\mathbb{Q}), \pi_*(\Omega Y) \otimes \mathbb{Q})$ is abelian. Hence, we show that (X, Y) is in $\mathcal{Q}^{\sim}$ if and only if the Lie algebra $\mathrm{Hom}_{\mathbb{Q}}(H_*(X;\mathbb{Q}), \pi_*(\Omega Y) \otimes \mathbb{Q})$ is abelian.

$(\Rightarrow)$ We observe that the bracket $[\,,]$ of $\mathrm{Hom}_{\mathbb{Q}}(H_*(X;\mathbb{Q}), \pi_*(\Omega Y) \otimes \mathbb{Q})$ is trivial. It is sufficient to observe that the composite

$$\begin{aligned}[f, g]_{p,q}\colon H_n(X;\mathbb{Q}) &\xrightarrow{\Delta_{p,q}} H_p(X;\mathbb{Q}) \otimes H_q(X;\mathbb{Q}) \\ &\xrightarrow{f_p \otimes g_q} (\pi_p(\Omega Y) \otimes \mathbb{Q}) \otimes (\pi_q(\Omega Y) \otimes \mathbb{Q}) \xrightarrow{[\,,]} \pi_n(\Omega Y) \otimes \mathbb{Q}\end{aligned}$$

is zero for any f, g and any p, q, n with $p + q = n$, where $\Delta_{p,q}$ is the composition of Δ and the projection onto $H_p(X;\mathbb{Q}) \otimes H_q(X;\mathbb{Q})$. It is easily verified that $\cup\colon H^p(X;\mathbb{Q}) \otimes H^q(X;\mathbb{Q}) \to H^n(X;\mathbb{Q})$ is zero if and only if $\Delta_{p,q}$ is zero. Thus we have $[f, g]_{p,q} = 0$.

$(\Leftarrow)$ Suppose that

$$\cup\colon H^i(X;\mathbb{Q}) \otimes H^j(X;\mathbb{Q}) \to H^{i+j}(X;\mathbb{Q})$$

and

$$[\,,]\colon (\pi_{i+1}(Y) \otimes \mathbb{Q}) \otimes (\pi_{j+1}(Y) \otimes \mathbb{Q}) \to \pi_{i+j+1}(Y) \otimes \mathbb{Q}$$

are nonzero for some i and j. Then it is not difficult to find

$$f, g \in \mathrm{Hom}_{\mathbb{Q}}(H_*(X;\mathbb{Q}), \pi_*(\Omega Y) \otimes \mathbb{Q})$$

with $[f, g] \neq 0$.

(2) As is shown in the proof of part (1), the Lie algebra

$$\mathrm{Hom}_{\mathbb{Q}}(H_*(X;\mathbb{Q}), \pi_*(\Omega Y) \otimes \mathbb{Q})$$

is abelian for $(X,Y)\in\mathcal{Q}^{\sim}$. Thus we have

$$\begin{aligned}[X,\Omega Y_{(0)}] &\cong \exp \mathrm{Hom}_{\mathbb{Q}}(H_*(X;\mathbb{Q}),\pi_*(\Omega Y)\otimes\mathbb{Q}) \\ &= \mathrm{Hom}_{\mathbb{Q}}(H_*(X;\mathbb{Q}),\pi_*(\Omega Y)\otimes\mathbb{Q}) \\ &= \prod_{i>0} H^i(X;\pi_{i+1}(Y)_{(0)}). \qquad \square\end{aligned}$$

4.2 C_0–spaces and W_0–spaces

The spaces defined below are important in the context of Theorem 4.3.

Definition 4.4 (1) X is called a C_0*–space* if the cup product on the reduced cohomology $\widetilde{H}^*(X;\mathbb{Q})$ is trivial.

(2) Y is called a W_0*–space* if the Whitehead product on the higher rational homotopy groups $\bigoplus_{i>1}\pi_i(Y)\otimes\mathbb{Q}$ is trivial.

See [9, Proposition 5.5] and [10] for properties characterizing C_0–spaces. Spaces with trivial Whitehead product have been investigated by Siegel [27], who referred to them as W–spaces.

Example 4.5 The classes of C_0– and W_0–spaces properly contain co-H_0–spaces and H_0–spaces, respectively. Many finite Grassmannians (eg $\mathbb{C}P^n$ with $n>1$) are W_0–spaces but not H_0–spaces.

A pair $(X,Y)\in\mathcal{CW}^{\mathrm{op}}\times 1\text{-}\mathcal{CW}$ is in $\mathcal{Q}^{\sim}$ if X is a C_0–space or Y is a W_0–space. Let us see that $\mathcal{Q}^{\sim}$ contains many pairs (X,Y) in which X is not a C_0–space and Y is not a W_0–space.

Given a pair (X,Y) of connected spaces, define $D_{X,Y}$ by

$$D_{X,Y}=\{i>0 \mid H^i(X;\mathbb{Q})\neq 0 \text{ and } \pi_{i+1}(Y)\otimes\mathbb{Q}\neq 0\}.$$

Then the condition (Q) is equivalent to the same vanishing condition for any $i,j\in D_{X,Y}$ with $i+j\in D_{X,Y}$. Note that if Y is a loop space, a classifying space, or a homogeneous space of a Lie group, then $D_{X,Y}$ is finite.

Example 4.6 (1) (X,S^n) is in $\mathcal{Q}^{\sim}$ if and only if either n is odd or the rational cup product

$$\cup\colon H^{n-1}(X;\mathbb{Q})\otimes H^{n-1}(X;\mathbb{Q})\to H^{2n-2}(X;\mathbb{Q})$$

is trivial.

(2) Let X be a connected CW-complex with $H_*(X;\mathbb{Q}) = H_{\mathrm{ev}}(X;\mathbb{Q})$. Then (X,Y) is in $\mathcal{Q}^\sim$ if the Whitehead product on $\pi_{\mathrm{odd}}(Y)\otimes\mathbb{Q}$ is trivial. In particular, if Y is a homogeneous space of a Lie group, then (X,Y) is in $\mathcal{Q}^\sim$.

Proof (1) Recall that

$$\pi_*(S^n)\otimes\mathbb{Q} = \begin{cases} \mathbb{Q}\cdot 1_n & \text{if } n \text{ is odd,} \\ \mathbb{Q}\cdot 1_n \oplus \mathbb{Q}\cdot[1_n,1_n] & \text{if } n \text{ is even,} \end{cases}$$

where 1_n denotes the homotopy class of the identity map of S^n. Then the proof follows immediately from the definition of $\mathcal{Q}^\sim$.

(2) The first clause is obvious. To prove the latter, we set $Y = G/H$ and consider the fiber sequence $H \xrightarrow{i} G \xrightarrow{p} G/H$. Since $\pi_*(H)\otimes\mathbb{Q} = \pi_{\mathrm{odd}}(H)\otimes\mathbb{Q}$, the map $p_\sharp\colon \pi_{\mathrm{odd}}(G)\otimes\mathbb{Q}\to\pi_{\mathrm{odd}}(G/H)\otimes\mathbb{Q}$ is surjective. Since the Whitehead product on $\pi_{\mathrm{odd}}(G)\otimes\mathbb{Q}$ is trivial, the Whitehead product on $\pi_{\mathrm{odd}}(G/H)\otimes\mathbb{Q}$ is also trivial. □

5 Applications to phantom maps

In this section, we apply the results of Sections 3 and 4 to phantom maps, and also prove Theorem 2.3.

The source is considered to be a connected CW-complex and the target is a nilpotent CW-complex of finite type over $\mathbb{Z}_l$, where $\mathbb{Z}_l$ is the localization of $\mathbb{Z}$ at a nonempty set l of primes. (A nilpotent space of finite type over $\mathbb{Z}_l$ is defined in [11, page 79], referred to as an $f\mathbb{Z}_l$–nilpotent space.) A nilpotent space of finite type over $\mathbb{Z}$ defines a nilpotent space of finite type in the usual sense.

For a nilpotent space Y of finite type over $\mathbb{Z}_l$, the profinite l–completion $\widehat{Y}$ and the l–local expansion $\check{Y}$ are defined by $\widehat{Y} = \prod_{p\in l}\widehat{Y}_p$ and $\check{Y} = \prod_{p\in l} Y_{(p)}$, respectively, where $\widehat{Y}_p$ and $Y_{(p)}$ are the p–profinite completion and the p–localization of Y respectively [30]. Thus, we can establish a commutative diagram of natural transformations

$$\begin{array}{ccc} & Y & \\ {\scriptstyle e_Y}\swarrow & & \searrow{\scriptstyle c_Y} \\ \check{Y} & \xrightarrow[d_Y]{} & \widehat{Y} \end{array}$$

Similarly, for a finitely generated $\mathbb{Z}_l$–module M, the profinite l–completion $\widehat{M}$ and the l–local expansion $\check{M}$ are defined by $\widehat{M} = \prod_{p\in l}\widehat{M}_p$ and $\check{M} = \prod_{p\in l} M_{(p)}$, respectively. Then there exist natural isomorphisms

$$\widehat{M} \cong M \otimes \widehat{\mathbb{Z}}_l \quad \text{and} \quad \check{M} \cong M \otimes \check{\mathbb{Z}}_l,$$

and the commutative diagram of natural transformations is given as

$$\begin{array}{ccc} & M & \\ {\scriptstyle e_M} \swarrow & & \searrow {\scriptstyle c_M} \\ \check{M} & \xrightarrow[d_M]{} & \widehat{M} \end{array}$$

The following lemma generalizes Meier and Zabrodsky's result [34, page 137] (see also [14, Theorem 5.1] and [25, Section 1]).

Proposition 5.1 *Let X be a connected CW-complex and Y a nilpotent CW-complex of finite type over $\mathbb{Z}_l$. Then a map $f\colon X \to Y$ is a phantom map if and only if the composite $X \xrightarrow{f} Y \xrightarrow{c_Y} \widehat{Y}$ is null-homotopic.*

Proof ($\Rightarrow$) The proof is outlined in [34, page 137].

($\Leftarrow$) Let p be a prime in l. Under the assumption on Y, we have $\pi_i(\widehat{Y}_p) \cong \pi_i(Y)\widehat{}_p$ (see the proof of [31, Theorem 3.9]). Thus the p–profinite completion of Y coincides with the completion of Y at p in the context of [11, Definition 10.2.3] (see [11, Theorems 11.1.1 and 11.1.2]). Hence, the reverse implication immediately follows from [11, Theorem 13.1.1]. □

Remark 5.2 (1) Recall that we have assumed that $l \neq \varnothing$ (ie $\mathbb{Z}_l \subsetneqq \mathbb{Q}$). In the case where $l = \varnothing$, $\widehat{Y}$ is a singleton, and hence Proposition 5.1 does not hold. But we know that any phantom map to a nilpotent space of finite type over $\mathbb{Q}$ is trivial (see [20, Theorem 1] or [29, Theorem 3.3(b)]).

(2) As implemented by Meier and Zabrodsky [34] and Roitberg and Touhey [25], our approach to phantom maps is based on Proposition 5.1. Though another characterization of phantom maps, Theorem B(b) in [34], is also easily generalizable to an arbitrary nilpotent source, it is not relevant to our approach. See [14] for the other approaches to phantom maps.

(3) Hereafter, for simplicity, we assume that the target Y is nilpotent of finite type. However, all of the results remain valid under appropriate modification, even if the target is regarded as nilpotent of finite type over $\mathbb{Z}_l$ rather than nilpotent of finite type.

Recall the definition of $P^{\varphi}(X,Y)$ in Section 3.

Corollary 5.3 *Let X be a connected CW-complex and Y a nilpotent CW-complex of finite type. Then*

$$\begin{aligned}\mathrm{Ph}(X,Y) &= P^{c_Y}(X,Y),\\ \mathrm{SPh}(X,Y) &= P^{e_Y}(X,Y),\\ \mathrm{Ph}(X,\check{Y}) &= P^{d_Y}(X,\check{Y}).\end{aligned}$$

Proof The first equality follows immediately from Proposition 5.1. Since $d_Y = \prod_p c_{Y_{(p)}}$, the third equality also follows from Proposition 5.1. Futhermore, since $d_Y e_Y = c_Y$, the second equality follows from the definition of $\mathrm{SPh}(X,Y)$ and the first equality. □

Using Corollary 5.3, we apply the results of Section 3 to the study of phantom maps. Let F_Y, F'_Y and F''_Y denote the homotopy fibers of c_Y, e_Y and d_Y, respectively.

Proposition 5.4 *Let Y be a 1–connected CW-complex of finite type.*

(1) *F_Y, F'_Y and F''_Y are homotopically equivalent, respectively, to the products*

$$\prod_{i>0} K(\pi_{i+1}(Y)\otimes\hat{\mathbb{Z}}/\mathbb{Z}, i),$$
$$\prod_{i>0} K(\pi_{i+1}(Y)\otimes\check{\mathbb{Z}}/\mathbb{Z}, i),$$
$$\prod_{i>0} K(\pi_{i+1}(Y)\otimes\hat{\mathbb{Z}}/\check{\mathbb{Z}}, i).$$

(2) *The natural sequence*

$$F'_Y \xrightarrow{i_Y} F_Y \xrightarrow{p_Y} F''_Y$$

is a trivial fibration sequence.

Proof (1) Note that

$$\begin{aligned}\pi_i(\hat{Y}) &= \pi_i(Y)\otimes\hat{\mathbb{Z}},\\ \pi_i(\check{Y}) &= \pi_i(Y)\otimes\check{\mathbb{Z}}\end{aligned}$$

and that $\hat{\mathbb{Z}}/\mathbb{Z}$, $\check{\mathbb{Z}}/\mathbb{Z}$ and $\hat{\mathbb{Z}}/\check{\mathbb{Z}}$ are $\mathbb{Q}$–modules. Then the homotopy types of F_Y, F'_Y and F''_Y are determined by Proposition 3.1.

(2) Since $c_Y = d_Y e_Y$, we have the natural fibration sequence

$$F'_Y \to F_Y \to F''_Y.$$

This sequence is trivial as per Lemma 3.2. □

The triviality of the fibration sequence $F'_Y \to F_Y \to F''_Y$ is exploited in the calculation of $\mathrm{SPh}(X,Y)$ in Section 6.

Remark 5.5 For a nilpotent CW-complex Y of finite type, the homotopy fibers F_Y, F'_Y, and F''_Y are determined by Proposition 5.4 and Remark 3.4. In the case of finite $\pi_1(Y)$, the homotopy type of F_Y is determined by the method described by Roitberg and Touhey [25]. Their proof is based on the results of Meier [20] and Steiner [29], whose works are mentioned in Remark 5.2(1).

Remark 5.6 From the definition of a phantom map, it is easily seen that $\mathrm{Ph}(X,Y) = \mathrm{Ph}(X,\widetilde{Y})$ holds (without the finite-type assumption imposed on Y), where $\widetilde{Y}$ denotes the universal cover of Y. Consequently, our approach is valid under the weaker assumption that the target has finitely generated higher homotopy groups. In [17, page 367], such a space is called a *finite-type target*. The classifying space of any compact Lie group is a finite-type target.

The following proposition describes a special case of Proposition 3.6.

Proposition 5.7 (double coset formula) *Let X be a connected CW-complex, and Y a nilpotent CW-complex of finite type. Then there exist natural isomorphisms*

$$\begin{aligned}\mathrm{Ph}(X,Y) &\cong (\Omega c_{(0)})_\sharp[X,\Omega Y_{(0)}]\backslash[X,\Omega\widehat{Y}_{(0)}]/(\Omega\hat{r})_\sharp[X,\Omega\widehat{Y}]\\ &\cong c_{(0)\sharp}[\Sigma X, Y_{(0)}]\backslash[\Sigma X,\widehat{Y}_{(0)}]/\hat{r}_\sharp[\Sigma X,\widehat{Y}],\\ \mathrm{SPh}(X,Y) &\cong (\Omega e_{(0)})_\sharp[X,\Omega Y_{(0)}]\backslash[X,\Omega\check{Y}_{(0)}]/(\Omega\check{r})_\sharp[X,\Omega\check{Y}]\\ &\cong e_{(0)\sharp}[\Sigma X, Y_{(0)}]\backslash[\Sigma X,\check{Y}_{(0)}]/\check{r}_\sharp[\Sigma X,\check{Y}],\\ \mathrm{Ph}(X,\check{Y}) &\cong (\Omega d_{(0)})_\sharp[X,\Omega \check{Y}_{(0)}]\backslash[X,\Omega\widehat{Y}_{(0)}]/(\Omega\hat{r})_\sharp[X,\Omega\widehat{Y}]\\ &\cong d_{(0)\sharp}[\Sigma X, \check{Y}_{(0)}]\backslash[\Sigma X,\widehat{Y}_{(0)}]/\hat{r}_\sharp[\Sigma X,\widehat{Y}],\end{aligned}$$

where $\hat{r}$ and $\check{r}$ denote the rationalization of $\widehat{Y}$ and $\check{Y}$ respectively.

Proof By Remark 5.6, it is sufficient to prove that Proposition 5.7 holds when Y is 1–connected. This follows from Proposition 3.6, Corollary 5.3 and Proposition 5.4. □

Remark 5.8 Proposition 5.7 is analogous to the double coset formulas describing local and adelic genera of a group and a space; see [11, pages 147, 169, 242 and 262] and [13].

The following corollary generalizes McGibbon's result [12, page 266].

Corollary 5.9 *Let X be a connected CW-complex and Y a nilpotent CW-complex of finite type. Then the following natural exact sequence of pointed sets exists:*

$$0 \to \mathrm{SPh}(X, Y) \to \mathrm{Ph}(X, Y) \xrightarrow{e_\sharp} \mathrm{Ph}(X, \check{Y}) \to 0.$$

Proof According to Proposition 5.7, both $\mathrm{Ph}(X, Y)$ and $\mathrm{Ph}(X, \check{Y})$ are quotient sets of $[X, (\Omega\widehat{Y})_{(0)}]$, implying that $e_\sharp$ is surjective. The remainder of the proof follows from the definition of $\mathrm{SPh}(X, Y)$. □

We now closely study natural group structures on phantom maps.

Let $0 \to \check{\mathbb{Z}}/\mathbb{Z} \xrightarrow{\iota} \widehat{\mathbb{Z}}/\mathbb{Z} \xrightarrow{\varpi} \widehat{\mathbb{Z}}/\check{\mathbb{Z}} \to 0$ denote the canonical exact sequence of $\mathbb{Q}$–modules.

Proposition 5.10 *Let (X, Y) be an object of $\mathcal{Q}$. Then there are natural isomorphisms*

$$[X, F_Y] \cong \prod_{i>0} H^i(X; \pi_{i+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}),$$
$$[X, F'_Y] \cong \prod_{i>0} H^i(X; \pi_{i+1}(Y) \otimes \check{\mathbb{Z}}/\mathbb{Z}),$$
$$[X, F''_Y] \cong \prod_{i>0} H^i(X; \pi_{i+1}(Y) \otimes \widehat{\mathbb{Z}}/\check{\mathbb{Z}}).$$

Under these isomorphisms, $i_{Y\sharp}\colon [X, F'_Y] \to [X, F_Y]$ and $p_{Y\sharp}\colon [X, F_Y] \to [X, F''_Y]$ are identified with $\prod_{i>0} H^i(X; \pi_{i+1}(Y) \otimes \iota)$ and $\prod_{i>0} H^i(X; \pi_{i+1}(Y) \otimes \varpi)$, respectively.

Proof By Remarks 5.5 and 5.6, we may assume that Y is 1–connected. For a simultaneous treatment of $c_Y\colon Y \to \widehat{Y}$, $e_Y\colon Y \to \check{Y}$ and $d_Y\colon \check{Y} \to \widehat{Y}$, we denote one mapping by $\varphi = \varphi_Y\colon \alpha Y \to \beta Y$ and let F^φ_Y be its homotopy fiber. Then $P^\varphi(X, \alpha Y)$ coincides with $\mathrm{Ph}(X, Y)$, $\mathrm{SPh}(X, Y)$ and $\mathrm{Ph}(X, \check{Y})$ in the respective cases. Let α, β and φ also denote the corresponding functors and natural transformations for finitely generated $\mathbb{Z}$–modules.

Since the homotopy fiber F_Y^{φ} is rational (Proposition 5.4), Proposition 3.1 implies that the principal fibration

$$(\Omega\alpha Y)_{(0)} \to (\Omega\beta Y)_{(0)} \to F_Y^{\varphi}$$

is trivial; hence $[X, F_Y^{\varphi}]$ is isomorphic to the orbit space $[X, (\Omega\alpha Y)_{(0)}]\backslash[X, (\Omega\beta Y)_{(0)}]$ by [11, Lemma 1.4.7]. Note that $1\times\alpha$ and $1\times\beta$ preserve condition (Q). Then, from Theorem 4.3,

$$(\Omega\varphi_Y)_{(0)\sharp}\colon [X, (\Omega\alpha Y)_{(0)}] \to [X, (\Omega\beta Y)_{(0)}]$$

is identified with

$$\prod H^i(X;\pi_{i+1}(Y)\otimes\varphi_{\mathbb{Z}(0)})\colon \prod H^i(X;\pi_{i+1}(Y)\otimes(\alpha\mathbb{Z})_{(0)}) \to \prod H^i(X;\pi_{i+1}(Y)\otimes(\beta\mathbb{Z})_{(0)}).$$

Note that

$$(\beta\mathbb{Z})_{(0)}/(\alpha\mathbb{Z})_{(0)} = (\beta\mathbb{Z}/\alpha\mathbb{Z})_{(0)} = \beta\mathbb{Z}/\alpha\mathbb{Z}.$$

Then we obtain the natural isomorphism

$$[X, F_Y^{\varphi}] \cong \prod_{i>0} H^i(X;\pi_{i+1}(Y)\otimes\beta\mathbb{Z}/\alpha\mathbb{Z}).$$

Theorem 4.3 also implies that $i_{Y\sharp}$ and $p_{Y\sharp}$ are identified. □

Proof of Theorem 2.3 By Remarks 5.5 and 5.6, we assume that Y is 1–connected, and use the notations introduced in the proof of Proposition 5.10; note that $\alpha = 1$ since $\varphi_Y = c_Y$ or e_Y.

(1) Since, by Theorem 4.3, $[X, (\Omega\beta Y)_{(0)}]$ is a $\mathbb{Q}$–module, we have

$$\begin{aligned} P^{\varphi}(X,Y) &\cong (\Omega\varphi)_{(0)\sharp}[X,(\Omega Y)_{(0)}]\backslash[X,(\Omega\beta Y)_{(0)}]/(\Omega r')_{\sharp}[X,\Omega\beta Y] \\ &\cong [X,(\Omega\beta Y)_{(0)}]/(\Omega\varphi)_{(0)\sharp}[X,(\Omega Y)_{(0)}] + (\Omega r')_{\sharp}[X,\Omega\beta Y] \\ &\cong [X, F_Y^{\varphi}]/\text{ the image of } [X,\Omega\beta Y] \end{aligned}$$

by the proof of Proposition 5.10, where r' denotes the rationalization of βY. Thus, a natural, divisible, abelian group structure exists on $P^{\varphi}(X,Y)$.

We can easily see that the inclusion $\mathrm{SPh}(X,Y) \hookrightarrow \mathrm{Ph}(X,Y)$ is a homomorphism of abelian groups (Proposition 5.10).

(2) We treat only the case of $\operatorname{Im}\mathrm{Ph}(f,g)$; the same argument applies to $\operatorname{Im}\mathrm{SPh}(f,g)$.

By Proposition 5.7, we have the morphism of exact sequences of pointed sets

$$\begin{array}{ccccc} [K,\Omega\widehat{L}_{(0)}] & \longrightarrow & \mathrm{Ph}(K,L) & \longrightarrow & 0 \\ \big\downarrow{\scriptstyle [f,\Omega\widehat{g}_{(0)}]} & & \big\downarrow{\scriptstyle \mathrm{Ph}(f,g)} & & \\ [X,\Omega\widehat{Y}_{(0)}] & \longrightarrow & \mathrm{Ph}(X,Y) & \longrightarrow & 0 \end{array}$$

Recall that the map $[X,\Omega\widehat{Y}_{(0)}]\to\mathrm{Ph}(X,Y)$ is an epimorphism of abelian groups which is the composite of the natural epimorphisms $[X,\Omega\widehat{Y}_{(0)}]\to[X,F_Y]$ and $[X,F_Y]\to\mathrm{Ph}(X,Y)$ (see the proof of part (1) above). By Proposition 4.1, Theorem 4.3, and the proof of Proposition 5.10, the image of the composite

$$[K,\Omega\widehat{L}_{(0)}]\xrightarrow{[f,\Omega\widehat{g}_{(0)}]}[X,\Omega\widehat{Y}_{(0)}]\to[X,F_Y]$$

is identified with $\prod_{i>0}\mathrm{Im}\,H^i(f;\pi_{i+1}(g)\otimes\widehat{\mathbb{Z}}/\mathbb{Z})$, which shows that $\mathrm{Im}\,\mathrm{Ph}(f,g)$ is a divisible abelian subgroup of $\mathrm{Ph}(X,Y)$.

(3) We treat first the case of $\mathrm{Ph}(X,Y)$.

Define the maps ι and ∂ by the long fibration sequence

$$\cdots\to\Omega\widehat{Y}\xrightarrow{\partial}F_Y\xrightarrow{\iota}Y\xrightarrow{c_Y}\widehat{Y}.$$

Consider the commutative diagram

$$\begin{array}{ccc} [X,\Omega\widehat{Y}_{(0)}] & & \\ \big\downarrow{\scriptstyle \partial_{(0)\sharp}} & & \\ [X,F_Y] & \xrightarrow{\iota_\sharp} & [X,Y] \\ \big\downarrow{\scriptstyle \pi} & \nearrow & \\ \mathrm{Ph}(X,Y) & & \end{array}$$

where π is the identification map by the action of the group $[X,\Omega\widehat{Y}]$. Since the group structure on $\mathrm{Ph}(X,Y)$ is characterized by the property that the composite $\pi\partial_{(0)\sharp}$ is an epimorphism of groups (see the proof of part (1) above), it is sufficient to see that the composite

$$[X,\Omega\widehat{Y}_{(0)}]\xrightarrow{\partial_{(0)\sharp}}[X,F_Y]\xrightarrow{\iota_\sharp}[X,Y] \tag{5-1}$$

preserves multiplication.

First, suppose that X is a co-H–space. Then the result follows from [32, Theorem 5.21 in Chapter III].

Second, suppose that Y is an H–space. Then, F_Y admits an H–structure such that $\partial\colon \Omega\widehat{Y} \to F_Y$ and $\iota\colon F_Y \to Y$ are H–maps (see the proof of [28, Theorem 9.1] and [33, page 15]). Thus, we see that the composite

$$\Omega\widehat{Y}_{(0)} \xrightarrow{\partial_{(0)}} F_Y \xrightarrow{\iota} Y$$

is an H–map, and hence that the composite (5-1) preserves multiplication.

The result for $\mathrm{SPh}(X,Y)$ follows from that for $\mathrm{Ph}(X,Y)$ since $\mathrm{SPh}(X,Y)$ is a subgroup of $\mathrm{Ph}(X,Y)$ for $(X,Y) \in \mathcal{Q}$. □

Remark 5.11 From the proof of Theorem 2.3, it is obvious that a result similar to Theorem 2.3 holds for $\mathrm{Ph}(X,\check{Y})$.

Proposition 5.12 *Let (X,Y) be an object of $\mathcal{Q}$. Then there exists a natural commutative diagram of groups*

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
[X,\Omega\check{Y}] & \longrightarrow & [X,F'_Y] & \longrightarrow & \mathrm{SPh}(X,Y) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle i_{Y\sharp}} & & \downarrow & & \\
[X,\Omega\widehat{Y}] & \longrightarrow & [X,F_Y] & \longrightarrow & \mathrm{Ph}(X,Y) & \longrightarrow & 0 \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle p_{Y\sharp}} & & \downarrow{\scriptstyle e_{Y\sharp}} & & \\
[X,\Omega\widehat{Y}] & \longrightarrow & [X,F''_Y] & \longrightarrow & \mathrm{Ph}(X,\check{Y}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

where all rows and all columns except the left column are exact; $[X,F_Y]$, $[X,F'_Y]$ and $[X,F''_Y]$ are regarded as groups by the natural isomorphisms in Proposition 5.10.

Proof The exactness of the rows follows immediately from the proof of Theorem 2.3(1) and Remark 5.11. The exactness of the right and the middle columns follows from Corollary 5.9 and Proposition 5.10 respectively. The commutativity is obvious. □

6 Calculations of $\mathrm{Ph}(X,Y)$ and $\mathrm{SPh}(X,Y)$

In this section, we prove Proposition 2.5, Theorem 2.7 and Corollaries 2.8–2.10.

6.1 $\mathrm{Ph}(X,Y)$ and $\mathrm{SPh}(X,Y)$ in the case of trivial $[X,\Omega\widehat{Y}]$

In this subsection, we prove Proposition 2.5, generalizing Theorem A(2) not only for $\mathrm{Ph}(X,Y)$, but also for $\mathrm{SPh}(X,Y)$. Though analogous results in all cases can be found for $\mathrm{Ph}(X,\check{Y})$, we omit their statements and proofs, since they follow from the results for $\mathrm{Ph}(X,Y)$, Remark 5.2(3), and the isomorphism $\mathrm{Ph}(X,\check{Y})\cong\prod_p \mathrm{Ph}(X,Y_{(p)})$.

First, let us consider pairs (X,Y) with $[X,\Omega\widehat{Y}]$ locally finite (and, in particular, with $[X,\Omega\widehat{Y}]$ trivial).

Proposition 6.1 *Let X be a connected CW-complex and Y a nilpotent CW-complex of finite type. If the group $[X,\Omega\widehat{Y}]$ is locally finite, we have the isomorphisms*

$$\mathrm{Ph}(X,Y)\cong\prod_{i>0}H^i(X;\pi_{i+1}(Y)\otimes\widehat{\mathbb{Z}}/\mathbb{Z}),$$
$$\mathrm{SPh}(X,Y)\cong\prod_{i>0}H^i(X;\pi_{i+1}(Y)\otimes\check{\mathbb{Z}}/\mathbb{Z}).$$

Further, if (X,Y) is in $\mathcal{Q}$, these isomorphisms can be taken as natural isomorphisms of abelian groups.

Proof We may assume that Y is 1–connected by Remark 5.6.

The case of $\mathrm{Ph}(X,Y)$ By Proposition 5.7, we have

$$\mathrm{Ph}(X,Y)\cong(\Omega c_{(0)})_\sharp[X,\Omega Y_{(0)}]\backslash[X,\Omega\widehat{Y}_{(0)}]/(\Omega\hat{r})_\sharp[X,\Omega\widehat{Y}].$$

Define the map

$$\psi_k\colon \Omega\widehat{Y}_{(0)}\to\Omega\widehat{Y}_{(0)},\quad \psi_k(\alpha)=\alpha^k\qquad\text{for } k\geq 1.$$

Since ψ_k is a homotopy equivalence, $(\Omega\hat{r})_\#[X,\Omega\widehat{Y}]=0$, and $\mathrm{Ph}(X,Y)$ is given by

$$\mathrm{Ph}(X,Y)\cong(\Omega c)_{(0)\sharp}[X,\Omega Y_{(0)}]\backslash[X,\Omega\widehat{Y}_{(0)}]\cong[X,F_Y]$$

by Proposition 3.1. The desired isomorphism is implied by Proposition 5.4. If (X,Y) is in $\mathcal{Q}$, the isomorphism can be taken as a natural isomorphism of abelian groups by Proposition 5.10 and the proof of Theorem 2.3(1).

The case of SPh(X, Y) By Corollary 5.3 and Lemma 3.5, we have the commutative diagram

$$\begin{array}{ccc} [X, F'_Y] & \longrightarrow & [X, F_Y] \\ \downarrow & & \downarrow \\ \mathrm{SPh}(X,Y) & \longrightarrow & \mathrm{Ph}(X,Y) \end{array}$$

whose vertical arrows are quotient maps. As previously determined, the right vertical arrow is bijective. Since the upper arrow is injective by Proposition 5.4(2), the left vertical arrow must be bijective. The above argument completes the proof. □

Remark 6.2 The following notes relate to Proposition 6.1.

(1) Meier [20] and Zabrodsky [34] identified the set $\mathrm{Ph}(X, Y)$ when investigating the case of $[X, \Omega\widehat{Y}] = 0$ (see [14, Section 5]). In [25, Theorem 1.5], $\mathrm{Ph}(X, Y)$ is identified in the more general case of finite $[X, \Omega\widehat{Y}]$.

(2) Naturally, the local finiteness of $[X, \Omega\check{Y}]$ implies the same results for $\mathrm{SPh}(X, Y)$. However, the above assertion regarding $\mathrm{SPh}(X, Y)$ is more useful since important vanishing results appear in the form $[X, \Omega\widehat{Y}] = 0$, as demonstrated in the next proposition.

Proposition 6.3 (1) (generalization of Miller [21, Theorem A]) *Let A be in $\mathcal{A}$ and let B be a nilpotent finite complex. Then* $\mathrm{map}_*(A, \widehat{B})$ *is weakly contractible.*

(2) (generalization of Anderson and Hodgkin [1, Theorem I]) *Let A be in $\mathcal{A}'$ and let B be (the identity component of) a space in $\mathcal{B}'$. Then* $\mathrm{map}_*(A, \widehat{B})$ *is weakly contractible.*

Proof (1) This follows immediately from the generalizations of Miller's theorem [21] developed by Zabrodsky [34], Friedlander and Mislin [4] and McGibbon [15]; the finite-type condition on the Postnikov space was removed in [15, page 3244].

(2) Here, we may assume that A is a $1\frac{1}{2}$–connected Postnikov space of finite type. We prove the weak contractibility of $\mathrm{map}_*(A, \widehat{B})$ in four steps.

Step 1 We prove the result for $B = BU$; similar proofs can be found for BO and $B\,\mathrm{Sp}$.

Suppose that A is an Eilenberg–Mac Lane space $K(\pi, n)$ with π finite and $n = 2$ or π finitely generated, with $n \geq 3$. The rationalization $\pi \to \pi_{(0)}$ induces the following morphism of fibration sequences:

$$\begin{array}{ccc} \mathrm{map}_*(K(\pi,n), F_{BU}) & \xleftarrow{\varphi'} & \mathrm{map}_*(K(\pi_{(0)},n), F_{BU}) \\ \downarrow & & \downarrow \\ \mathrm{map}_*(K(\pi,n), BU) & \xleftarrow{\varphi} & \mathrm{map}_*(K(\pi_{(0)},n), BU) \\ \downarrow & & \downarrow \\ \mathrm{map}_*(K(\pi,n), \widehat{BU}) & \xleftarrow{\varphi''} & \mathrm{map}_*(K(\pi_{(0)},n), \widehat{BU}) \end{array}$$

By [1, Corollary 4.8], φ is a weak homotopy equivalence. Since F_{BU} is rational (Proposition 5.4), φ' is a weak homotopy equivalence [11, Section 19.5]. Since $K(\pi_{(0)},n) \to *$ is an $H\mathbb{Z}/p$–equivalence for any prime p, $\mathrm{map}_*(K(\pi_{(0)},n), \widehat{BU})$ is weakly contractible [11, Sections 19.2 and 19.5], and hence the null component of $\mathrm{map}_*(K(\pi,n), \widehat{BU})$ is weakly contractible. On the other hand, we have

$$\begin{aligned}[K(\pi,n), \widehat{BU}] &\cong [K(\pi,n), \Omega^2\widehat{BU}] \\ &\cong \pi_2(\mathrm{map}_*(K(\pi,n), \widehat{BU})) = 0,\end{aligned}$$

which implies that $\mathrm{map}_*(K(\pi,n), \widehat{BU})$ is weakly contractible.

For general A, the result is proved by induction using Dwyer's version of the Zabrodsky lemma [3, Proposition 3.4].

Step 2 The result for $B = B\,\mathrm{SO}$ is obtained from the result for BO using the fibration sequence

$$\{\pm 1\} \to B\,\mathrm{SO} \to BO.$$

Step 3 The result for $B = U/\mathrm{Sp}$ is obtained from the results for BU and $B\,\mathrm{Sp}$ using the fibration sequence

$$U/\mathrm{Sp} \to B\,\mathrm{Sp} \to BU.$$

Applying similar arguments, we can obtain results for $B = \mathrm{Sp}/U$, SO/U and U/SO.

Step 4 We express the identity component of ΩB as $\Omega_0 B$. Observe that if $\mathrm{map}_*(A, \widehat{B})$ is weakly contractible, then

$$\mathrm{map}_*(A, \Omega^k \widehat{B}) \cong \mathrm{map}_*(A, \widehat{\Omega_0^k B})$$

is also weakly contractible. This statement completes the proof. □

Corollary 6.4 *Let (X,Y) be in $\mathcal{A}\times\mathcal{B}$ or $\mathcal{A}'\times\mathcal{B}'$. Then the mapping space* $\mathrm{map}_*(X,\Omega\widehat{Y})$ *is weakly contractible.*

Proof This immediately follows from Proposition 6.3. □

Remark 6.5 Proposition 6.3(2) and its use in proving the results apply equally to the representing spaces of conjugate K–theory KC_* [1].

Proof of Proposition 2.5 The result follows from Proposition 6.1 and Corollary 6.4. □

6.2 Ph(X, Y) and SPh(X, Y) in the case of nontrivial $[X, \Omega\widehat{Y}]$

In this subsection we prove Theorem 2.7 and Corollaries 2.8–2.10

Proof of Theorem 2.7 **The case of Ph(X, Y)** By Corollary 5.3 and the comment before Lemma 3.5, the morphism of exact sequences of pointed sets is given by

$$\begin{array}{ccccccc}
[X,\Omega\widehat{Y}] & \longrightarrow & [X,F_Y] & \longrightarrow & \mathrm{Ph}(X,Y) & \longrightarrow & 0 \\
{\scriptstyle p^\sharp}\uparrow & & {\scriptstyle p^\sharp}\uparrow & & {\scriptstyle p^\sharp}\uparrow & & \\
[K,\Omega\widehat{Y}] & \longrightarrow & [K,F_Y] & \longrightarrow & \mathrm{Ph}(K,Y) & \longrightarrow & 0
\end{array}$$

We observe that $p^\sharp\colon [K,\Omega\widehat{Y}]\to[X,\Omega\widehat{Y}]$ is surjective. In the cofibration case, the surjectivity follows immediately from the assumption; in the fibration case, the surjectivity is inferred from the assumption via Dwyer's version of the Zabrodsky lemma [3, Proposition 3.4]. Next, consider the induced morphism

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{[X,\Omega\widehat{Y}]} & \longrightarrow & [X,F_Y] & \longrightarrow & \mathrm{Ph}(X,Y) & \longrightarrow & 0 \\
 & & {\scriptstyle \psi}\uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Ker}\,\phi & \longrightarrow & p^\sharp[K,F_Y] & \xrightarrow{\ \phi\ } & p^\sharp\,\mathrm{Ph}(K,Y) & \longrightarrow & 0
\end{array}$$

of exact sequences of pointed sets, where $\overline{[X,\Omega\widehat{Y}]}$ denotes the image of $[X,\Omega\widehat{Y}]$ and ϕ denotes the map induced by the natural quotient map $[K,F_Y]\to\mathrm{Ph}(K,Y)$. By Theorem 2.3 and its proof, this is a morphism of exact sequences of abelian groups. Since $p^\sharp\colon[K,\Omega\widehat{Y}]\to[X,\Omega\widehat{Y}]$ is surjective, ψ is also surjective. We regard this morphism of exact sequences as a short exact sequence of chain complexes, and take its homology exact sequence. Thus we find that

$$\mathrm{Ph}(X,Y)/p^\sharp\,\mathrm{Ph}(K,Y)\cong[X,F_Y]/p^\sharp[K,F_Y],$$

which gives the desired exact sequence (see Proposition 5.10 and its proof, Proposition 4.1 and Theorem 4.3). Since $p^\sharp \operatorname{Ph}(K,Y)$ is a divisible abelian group (Theorem 2.3(2)), the short exact sequence splits.

The case of $\mathbf{SPh}(X,Y)$ As mentioned in the introduction of Section 6.1, $\operatorname{Ph}(X,Y)$ and $\operatorname{Ph}(X,\check{Y})$ generate analogous results. Thus there exists a morphism of exact sequences of abelian groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & p^\sharp \operatorname{Ph}(K,Y) & \longrightarrow & \operatorname{Ph}(X,Y) & \longrightarrow & \displaystyle\prod_{i>0} \frac{H^i(X;\pi_{i+1}(Y)\otimes \widehat{\mathbb{Z}}/\mathbb{Z})}{p^* H^i(K;\pi_{i+1}(Y)\otimes \widehat{\mathbb{Z}}/\mathbb{Z})} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \epsilon} & & \Big\downarrow{\scriptstyle e_{Y\sharp}} & & \Big\downarrow & & \\
0 & \longrightarrow & p^\sharp \operatorname{Ph}(K,\check{Y}) & \longrightarrow & \operatorname{Ph}(X,\check{Y}) & \longrightarrow & \displaystyle\prod_{i>0} \frac{H^i(X;\pi_{i+1}(Y)\otimes \widehat{\mathbb{Z}}/\check{\mathbb{Z}})}{p^* H^i(K;\pi_{i+1}(Y)\otimes \widehat{\mathbb{Z}}/\check{\mathbb{Z}})} & \longrightarrow & 0
\end{array}$$

Regarding this morphism as a short exact sequence of chain complexes, we take its homology exact sequence. Thus we obtain the exact sequence

$$0 \to \operatorname{Ker}\epsilon \to \operatorname{SPh}(X,Y) \to \prod_{i>0} \frac{H^i(X;\pi_{i+1}(Y)\otimes \check{\mathbb{Z}}/\mathbb{Z})}{p^* H^i(K;\pi_{i+1}(Y)\otimes \check{\mathbb{Z}}/\mathbb{Z})} \to 0$$

by Corollary 5.9. Define the abelian group $\widetilde{\operatorname{Ker}\epsilon}$ by the pullback diagram

$$\begin{array}{ccc}
\widetilde{\operatorname{Ker}\epsilon} & \hookrightarrow & \displaystyle\prod_{i>0} H^i(X;\pi_{i+1}(Y)\otimes \check{\mathbb{Z}}/\mathbb{Z}) \\
\Big\downarrow & & \Big\downarrow \\
\operatorname{Ker}\epsilon & \hookrightarrow & \operatorname{SPh}(X,Y)
\end{array}$$

in the category of abelian groups, where the right vertical arrow is the natural quotient map (see Proposition 5.10 and the proof of Theorem 2.3(1)). Since the right vertical arrow is an epimorphism, this diagram is also a pushout diagram, implying that the two horizontal arrows share a single cokernel. Thus it is seen that

$$\widetilde{\operatorname{Ker}\epsilon} = \prod_{i>0} p^* H^i(K;\pi_{i+1}(Y)\otimes \check{\mathbb{Z}}/\mathbb{Z}),$$

which implies that $\operatorname{Ker}\epsilon = p^\sharp \operatorname{SPh}(K,Y)$ by the proof of Theorem 2.3(2), and the desired exact sequence is obtained. The short exact sequence splits by the divisibility of $p^\sharp \operatorname{SPh}(K,Y)$. □

We use Theorem 2.7 to prove Corollaries 2.8–2.10.

Proof of Corollary 2.8 Let X' be the fiber of $p\colon X \to T$. Note that X' is a $\frac{1}{2}$–connected infinite loop space. Since $\mathrm{map}_*(X', \Omega\widehat{Y})$ is weakly contractible by Corollary 6.4 and $\mathrm{Ph}(T, Y)$ vanishes, we obtain the result from Theorem 2.7. □

Proof of Corollary 2.9 Consider the fibration sequence

$$\Omega K^{(n)} \to K\langle n\rangle \xrightarrow{p} K,$$

where $K^{(n)}$ is the Postnikov n–stage of K. Since $\mathrm{map}_*(\Omega K^{(n)}, \Omega\widehat{Y})$ is weakly contractible by Corollary 6.4 and $\mathrm{Ph}(K, Y)$ vanishes, we obtain the result from Theorem 2.7. □

Proof of Corollary 2.10 Consider the fibration sequence

$$X' \xrightarrow{i} X \xrightarrow{p} K,$$

where i is the map $(1,0)\colon X' \to X' \times K = X$ and p is the canonical projection onto K. By Corollary 6.4, we can apply Theorem 2.7 to (X, Y). The canonical section $(0,1)\colon K \to X' \times K = X$ of the fibration p shows that $p^\sharp\colon \mathrm{Ph}(K, Y) \to \mathrm{Ph}(X, Y)$ is injective and splits the exact sequences of Theorem 2.7. The products of quotients of cohomology groups are easily identified with the Hom-modules in the statement. □

Last, we apply Theorem 2.7 to pairs (X, Y) in $\mathcal{Q}$ admitting a cofibration sequence $X' \xrightarrow{i} X \xrightarrow{p} K$ with $[X', \Omega\widehat{Y}] = 0$.

The Grassmannians $G_n(\mathbb{F})$ and $G_\infty(\mathbb{F})$ are defined by $G_n(\mathbb{F}) = \varinjlim_m G_{n,m}(\mathbb{F})$ and $G_\infty(\mathbb{F}) = \varinjlim_n G_n(\mathbb{F})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, where the finite Grassmannian $G_{n,m}(\mathbb{F})$ is the space of n–dimensional subspaces in $\mathbb{F}^{n+m}$.

Example 6.6 Let $G_n(\mathbb{F})/G_{n',m'}(\mathbb{F})$ be the quotient complex of $G_n(\mathbb{F})$ by $G_{n',m'}(\mathbb{F})$ $(n', m' < \infty,\ n' \le n \le \infty)$. Let Y be a space in $\mathcal{B}$ such that $(G_n(\mathbb{F})/G_{n',m'}(\mathbb{F}), Y)$ is in $\mathcal{Q}$. Then there exist natural isomorphisms of groups

$$\mathrm{Ph}(G_n(\mathbb{F})/G_{n',m'}(\mathbb{F}), Y) \cong \prod_{i>0} \frac{H^i\big(G_n(\mathbb{F})/G_{n',m'}(\mathbb{F}); \pi_{i+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}\big)}{p^* H^i\big(\Sigma G_{n',m'}(\mathbb{F}); \pi_{i+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}\big)},$$
$$\mathrm{SPh}(G_n(\mathbb{F})/G_{n',m'}(\mathbb{F}), Y) \cong \prod_{i>0} \frac{H^i\big(G_n(\mathbb{F})/G_{n',m'}(\mathbb{F}); \pi_{i+1}(Y) \otimes \check{\mathbb{Z}}/\mathbb{Z}\big)}{p^* H^i\big(\Sigma G_{n',m'}(\mathbb{F}); \pi_{i+1}(Y) \otimes \check{\mathbb{Z}}/\mathbb{Z}\big)},$$

where p is the canonical map $G_n(\mathbb{F})/G_{n',m'}(\mathbb{F}) \to \Sigma G_{n',m'}(\mathbb{F})$. Furthermore, if $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$, then

$$p^* H^i\big(\Sigma G_{n',m'}(\mathbb{F}); \pi_{i+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}\big) = p^* H^i\big(\Sigma G_{n',m'}(\mathbb{F}); \pi_{i+1}(Y) \otimes \check{\mathbb{Z}}/\mathbb{Z}\big) = 0.$$

Proof Consider the cofibration sequence

$$G_n(\mathbb{F}) \to G_n(\mathbb{F})/G_{n',m'}(\mathbb{F}) \xrightarrow{p} \Sigma G_{n',m'}(\mathbb{F}).$$

Since $[G_n(\mathbb{F}), \Omega \widehat{Y}] = 0$ by Corollary 6.4 and $\mathrm{Ph}(\Sigma G_{n',m'}(\mathbb{F}), Y) = 0$, we obtain the result from Theorem 2.7. □

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Center for Mathematical Sciences, University of Aizu
Aizuwakamatsu, Fukushima, Japan

kihara@u-aizu.ac.jp

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