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Quasi-invariant measures for some amenable groups acting on the line

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We show that if G is a solvable group acting on the line and if there is $T \in G$ having no fixed points, then there is a Radon measure μ on the line quasi-invariant under G. In fact, our method allows for the same conclusion for G inside a class of groups that is closed under extensions and contains all solvable groups and all groups of subexponential growth.

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1 Introduction

Let G be a group acting by homeomorphism of the line. We say that a Borel measure on the line μ is quasi-invariant[1] or quasipreserved under the action of G if, for every $g \in G$, there is $\lambda_g \in \mathbb{R}$ such that $g_*\mu = \lambda_g \mu$, where $g_*\mu\colon B \mapsto \mu(g^{-1}(B))$. We say that μ is preserved by G if $g_*\mu = \mu$ for all $g \in G$. We will also say that a measure ν on the line is *proper* if $\nu([0,x)) \to \infty$ when $x \to \infty$ and $\nu([x,0)) \to -\infty$ when $x \to -\infty$.

Aiming to decide the (non)amenability of Thompson's group F (see [7] for an introduction to this group, and Juschenko [12] for an introduction to amenability) a very interesting criterion was proposed by L Beklaryan [4, Theorem B].

Criterion *If an amenable group acts by order-preserving homeomorphism of the line with an element acting freely, then there is a Radon*[2] *measure on the line quasi-invariant under the group action.*

Since for the natural — piecewise affine — action of F on $(0,1)$ there is no quasi-invariant measure, the criterion implies the nonamenability of F. However, the claim of validity of the criterion was withdrawn in Beklaryan [5], apparently by the appearance

[1] Notice that in the literature the term quasi-invariant measure is also used to denote a measure for which the given action preserves the 0–measure sets. Thus, the notion we work with is stronger.

[2] A Borel measure μ is said to be a Radon measure if it gives finite mass to compact sets.

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of the preprint of Akhmedov [1], where it is claimed that the criterion fails already for the class of solvable groups.

This note is intended to clarify the discussion around the validity of the criterion. We became interested in this problem after we discovered a flaw in (the first version of) [1]. In fact, we will prove that the criterion is valid in a class of groups that is closed under extensions and includes all solvable groups and all groups of subexponential growth (see de la Harpe [11] for the definition of group growth). We were, however, unable to decide whether Beklaryan criterion holds in the class of all amenable groups.

We will say that a group G *locally* has subexponential growth if any of its finitely generated subgroups has subexponential growth. Let $\mathcal{S}$ denote the class of groups G for which there is a finite normal filtration

$$\{\mathrm{id}\} = G^{d+1} \lhd G^d \lhd \cdots \lhd G^1 \lhd G^0 = G$$

with the property that G_{i-1}/G_i locally has subexponential growth for $i = 1, \dots, d$. Observe that G_{i-1}/G_i may not be finitely generated. In this note, the *degree* of a group G in $\mathcal{S}$ is the length of the shortest filtration in which each successive quotient has locally subexponential growth. So for instance a group of subexponential growth has degree 1.

Clearly, any solvable group is in $\mathcal{S}$, and any group in $\mathcal{S}$ is amenable (see for instance Juschenko [12]). We will show:

Theorem A *Let G be a group in $\mathcal{S}$ that is acting on the line by order-preserving homeomorphisms. Assume that there is $T \in G$ having no fixed points. Then there is a proper Radon measure μ on the line which is quasipreserved by G.*

Remark 1 Plante [15] considers a class of groups $\mathcal{S}_0$ that contains all polycyclic groups,[3] and all finitely generated groups of subexponential growth. He proves that *any* action on the line of a group in $\mathcal{S}_0$ quasipreserves a Radon measure.

The class $\mathcal{S}_0$ however does not contain either all solvable groups or all groups that have locally subexponential growth. In fact, counterexamples of Plante's theorem among finitely generated (infinite-rank) solvable group are easy to find: there are actions of some solvable groups not allowing for a quasi-invariant measure, for instance some actions of $\mathbb{Z} \wr \mathbb{Z}$ — see Plante [15] and Rivas and Tessera [17] — or some even more

[3] A solvable group is polycyclic if and only if it admits a filtration such that each successive quotient is cyclic.

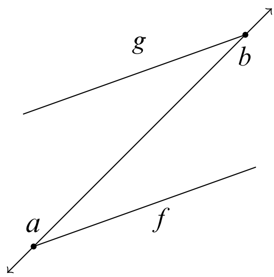

Figure 1: A crossing

exotic, as in Botto Mura and Rhemtulla [6, Section 6.2]. In these actions, though there are no global fixed points, each element of the group has at least one fixed point.

It is therefore natural to impose a priori in the criterion the condition that G has an element acting without fixed points.

Besides the groundwork provided by Plante, our main tool is the notion of *crossed elements*. This notion was introduced in Beklaryan [3], but has been extensively studied/exploited in its connection with total orderings on groups (see Deroin, Navas and Rivas [8], Navas [13] and Rivas [16]).

Definition 1 We say f and g, two order-preserving homeomorphisms of the line, are crossed if there are $a, b \in \mathbb{R}$ such that $a < b$, $f(a) = a$, $g(b) = b$, and $f(x) < x$ and $g(x) > x$ for every $x \in (a, b)$. See Figure 1, where the graphs of f and g are depicted.

For us, the main importance of crossed elements is that they entail exponential growth. Indeed, if f and g are crossed elements, then there is $n \in \mathbb{N}$ such that f^n and g^n generate a free semigroup (see for instance Navas [14]). In particular the group generated by f and g has exponential growth.

1.1 Quasi-invariant measures and semiconjugacy to affine actions

Definition 2 Two representations $\rho_1, \rho_2\colon G \to \mathrm{Homeo}_+(\mathbb{R})$ are *semiconjugated* if there is a monotone map (ie nondecreasing) $c\colon \mathbb{R} \to \mathbb{R}$ which is proper (ie c^{-1} sends compact sets to bounded sets, or, equivalently since c is monotone, $c(\mathbb{R})$ is unbounded in both directions) and such that, for all $g \in G$,

$$c \circ \rho_1(g) = \rho_2(g) \circ c.$$

Remark 2 The above definition is the analog for actions of the line of the definition of semiconjugacy for groups acting on the circle from Ghys [9]. Though, sometimes one also insists in the continuity of c above (for instance in Ghys [10] and Navas [14]), without the continuity assumption semiconjugacy becomes an equivalence relation (see [9] for the case of the circle and Alonso, Brum and Rivas [2] for the case of the line). In this note, we do not assume continuity.

Observe that for $G \subseteq \mathrm{Homeo}_+(\mathbb{R})$ acting without global fixed points, the presence of a Radon measure μ quasi-invariant by G provides us, for every $g \in G$, the affine map of the line

$$A_g(x) = \frac{1}{\lambda_g} x + \mu\big([0, g(0))\big), \tag{1}$$

where, by convention, if $g(0) < 0$ then $\mu\big([0, g(0))\big)$ means $-\mu\big([g(0), 0)\big)$. This convention is used throughout this note.

This association is in fact a representation of G into the affine group:

$$\begin{aligned} A_{fg}(x) &= \frac{1}{\lambda_f}\Big(\frac{1}{\lambda_g} x + \mu\big([f^{-1}(0), g(0))\big)\Big) \\ &= \frac{1}{\lambda_f}\Big(\frac{1}{\lambda_g} x + \mu\big([0, g(0))\big) + \mu\big([f^{-1}(0), 0)\big)\Big) \\ &= \frac{1}{\lambda_f}\Big(\frac{1}{\lambda_g} x + \mu\big([0, g(0))\big)\Big) + \mu\big([0, f(0))\big) = A_f \circ A_g(x). \end{aligned}$$

Further, if the quasi-invariant measure μ is proper, then its cumulative distribution function $F(x) = \mu([0, x))$ is also proper, so the G–action is semiconjugated to the affine action above. Indeed,

$$\begin{aligned} F(g(x)) &= \mu\big([0, g(x))\big) \\ &= \mu\big([g(0), g(x))\big) + \mu\big([0, g(0))\big) \\ &= g_*^{-1}\mu([0, x)) + \mu\big([0, g(0))\big) \\ &= \frac{1}{\lambda_g} F(x) + \mu\big([0, g(0))\big) = A_g(F(x)). \end{aligned}$$

Observe that μ may have atoms, for instance when the G–action admits a discrete invariant set. We also have a converse:

Proposition 1.1 *If an action of G without global fixed points is semiconjugated to an affine action, then there is a proper measure quasipreserved by G.*

Proof Suppose there is a semiconjugacy

$$F \circ g = A(g) \circ F$$

for some affine representation $A\colon G \to \mathrm{Aff}(\mathbb{R})$. Since G has no global fixed points and the increasing map F is proper, we also have that the affine action has no global fixed points (for instance, the orbit of $F(0)$ under $A(G)$ is unbounded in both directions). Now, for $I = [a,b)$ let $\mu([a,b)) := \mathrm{Leb}\big([F(a), F(b))\big)$. By Carathéodory's extension theorem (see Royden and Fitzpatrick [18]) there is a unique Borel measure extending μ. Thus, to check that μ is quasipreserved by G, we only need to observe that

$$\begin{aligned} g_*\mu(I) &= \mu\big([g^{-1}(a), g^{-1}(b))\big) \\ &= \mathrm{Leb}\big([F \circ g^{-1}(a), F \circ g^{-1}(b))\big) \\ &= \mathrm{Leb}\big(A(g^{-1})\big([F(a), F(b))\big)\big) \\ &= \lambda_g \mathrm{Leb}\big([F(a), F(b))\big) = \lambda_g \mu(I), \end{aligned}$$

where λ_g is precisely the dilation factor of the affine map $A(g^{-1})$. Finally, we also have that μ is a proper measure since F is a proper function. □

2 Proof of Theorem A

We begin with the next proposition, which is the first step in an induction argument. The proposition is known, but we provide a full proof since the arguments in it will be used in the proof of Theorem A.

Proposition 2.1 *Let G be a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ locally of subexponential growth. Assume that there is $T \in G$ having no fixed points. Then there is a proper Radon measure μ on the line which is preserved by G.*

Proof We first observe that the presence of the fixed-point-free element $T \in G$ implies that there is a nonempty minimal invariant set (that is, a closed set invariant under G, and having no closed proper subset invariant under G). Indeed, let x_0 be any point of the line and let $I = [x_0, T(x_0)]$ when $T(x_0) > x_0$, or $I = [T(x_0), x_0]$ otherwise. Let $\mathcal{F}$ be the family of nonempty closed sets which are G–invariant. Since T has no fixed points, any G–orbit intersects I; therefore, $\Lambda \cap I \neq \varnothing$ for any $\Lambda \in \mathcal{F}$.

Let us consider an order relation $\preceq$ in $\mathcal{F}$, defined as $\Lambda_1 \succeq \Lambda_2$ if $\Lambda_1 \cap I \subseteq \Lambda_2 \cap I$ (note that by G–invariance this implies that $\Lambda_1 \subseteq \Lambda_2$). Let $(\Lambda_i)_{i \in I}$ be a totally ordered family of elements of $\mathcal{F}$. Since I is compact, and the Λ_i are nested, the set $\bigcap_i \Lambda_i$ is G–invariant and nonempty. Thus, by Zorn's lemma, there exists a maximal element for $\preceq$, ie a closed *minimal* (with respect to inclusion) G–invariant set, which we denote by Λ. There are three possibilities.

(1) **Λ' (the accumulation points of Λ) is empty** In this case, Λ is discrete and $\mu = \sum_{m \in \Lambda} \delta_m$ is a proper Radon G–invariant measure.

(2) **$\partial\Lambda$ (the boundary of Λ) is empty** In this case, $\Lambda = \mathbb{R}$, so the action of G is minimal (that is, every orbit is dense). We claim that this action is also free. Indeed, if the action is not free, then there exists $f \in G \setminus \{\mathrm{id}\}$ having at least one fixed point. We let $I = [a, a')$, where $a \in \mathbb{R}$ and $a' \in \mathbb{R} \cup \{\infty\}$, be a nonempty component of $\mathbb{R} \setminus \mathrm{Fix}(f)$. By possibly changing f by its inverse, we can assume that $f(x) < x$ for all $x \in I$. Since the action is minimal, there is $h \in G$ such that $h(a) \in I$. Now consider the element $g = hf^n$. Since, for any $x \in I$, $f^n(x) \to a$ as n tends to ∞, we have that $g(x) > x$ for every $x \in [a, h(a)]$, but, if n is large enough, we have that $g(f^{-1}h(a)) = hf^n(f^{-1}h(a)) < f^{-1}h(a)$. Thus g has a fixed point that is greater than $h(a)$. Let b be the infimum of these fixed points. Then f and g are crossed elements exactly as in Figure 1. This contradicts the fact that G locally has subexponential growth.

Since the action of G is free and minimal, Hölder's theorem states that G is topologically conjugate to a group of translations (see [14]). We can pull back the Lebesgue measure by this conjugacy to obtain a proper invariant Radon measure for the G–action.

(3) **$\Lambda' = \partial\Lambda = \Lambda$** In this case Λ is "locally" a Cantor set, namely its intersection with any closed interval is either empty, a point or a Cantor set. So, one can collapse each interval in the complement of Λ to a point to obtain another (topological) line endowed with a G–action which is semiconjugated to the original one.

More precisely, there is a surjective, nondecreasing (hence proper) and continuous map $c\colon \mathbb{R} \to \mathbb{R}$ that is constant in the complement of Λ. Since Λ — and hence its complement — is G–invariant, we can define $\psi\colon G \to \mathrm{Homeo}_+(\mathbb{R})$ satisfying

$$\psi(g) \circ c = c \circ g \quad \text{for all } g \in G. \tag{2}$$

In this way, since $c(\Lambda) = \mathbb{R}$, the subgroup $\psi(G) \subset \mathrm{Homeo}_+(\mathbb{R})$ acts minimally on the line. As in the previous case, the group $\psi(G)$ also acts freely, thus, by Hölder's theorem, $\psi(G)$ is conjugated to a group of translations. In particular, by conjugating $\psi(G)$ if necessary, we can assume that $\psi(G)$ preserves the Lebesgue measure Leb.

We can now pull back Leb by the semiconjugacy to obtain a G–invariant Radon measure. More precisely, for A a Borel subset of Λ, we define $\widetilde{\mu}(A) = \mathrm{Leb}(c(A))$. This is indeed a measure on Λ since there is a countable subset M of Λ such that $c\colon \Lambda \setminus M \to \mathbb{R}$ is injective (in fact M is the subset of Λ made of endpoints of complementary intervals). The G invariance and properness of $\widetilde{\mu}$ follows easily from (2). Finally, $\mu(A) := \widetilde{\mu}(A \cap \Lambda)$ is the required measure. □

Proof of Theorem A We argue by contradiction. Suppose there is a group G in $\mathcal{S}$ contradicting the conclusion of Theorem A. We choose G with the least possible degree. From Proposition 2.1 the degree of G is greater than 1; say it has degree $d+1$ and the filtration witnessing this degree is $\{\mathrm{id}\} = G^{d+1} \lhd G^d \lhd \cdots \lhd G^1 \lhd G^0 = G$. We fix the action of G on the line having no proper, Radon, quasi-invariant measure, and also fix an element $T \in G$ having no fixed points. By possibly changing T to its inverse, we can (and will) assume that $T(x) > x$ for all $x \in \mathbb{R}$. As in the proof of Proposition 2.1, we have that G has a nonempty, closed minimal invariant set Λ.

If Λ is discrete, then $\mu = \sum_{m \in \Lambda} \delta_m$ is a proper, Radon, G–invariant measure, contradicting our assumption. So Λ can not be discrete. We make two reductions.

Step 1 We first argue that we can reduce to the case where G^d has no (global) fixed points.

Suppose G^d has at least one fixed point. Since G^d is normal in G, we have that $X := \mathrm{Fix}(G^d)$, the set of points fixed by G^d, is an infinite, closed, G–invariant set unbounded in both directions of the line. Since Λ is nondiscrete, the closure of every G–orbit contains Λ. Indeed, this is obvious in the minimal case, and if Λ is "locally" a Cantor set, then, by the minimality of the G action on Λ, the orbit of any component I of the complement of Λ visits any neighborhood of any point in Λ (see [14, Section 2.1] for more details). In particular, $\Lambda \subseteq X$.

The action of G on X factors throughout an action of G/G^d. Moreover, the G/G^d–action on X can be extended to an action on the whole real line, for instance by taking linear interpolation on the open components of the complement of X (see for instance the proof of Theorem 6.8 in [10]). Denote this new action by $\psi\colon G \to \mathrm{Homeo}_+(\mathbb{R})$. Observe that in this construction, $\psi(T)$ has no fixed points, G^d acts trivially on the line and $g(x) = \psi(g)(x)$ for every $x \in \Lambda$ and every $g \in G$. It follows that Λ is also the minimal invariant set for $\psi(G)$.

From the minimality of the degree of G, we have that the action ψ of G/G^d on the line admits a proper quasi-invariant Radon measure μ. In particular, $\mathrm{supp}(\mu)$, the support of μ, is closed and G/G^d–invariant, and hence $\Lambda \subseteq \mathrm{supp}(\mu)$. We claim that $\Lambda = \mathrm{supp}(\mu)$. To see this, first note that, as in Section 1.1, the presence of the proper quasi-invariant Radon measure implies that $\psi(G)$ is semiconjugated to an affine group. Precisely, if we let A_g be as in (1) and $F(x) := \mu([0,x))$, then F is a proper function and

$$A_g \circ F = F \circ \psi(g) \quad \text{for all } g \in G.$$

This affine action has no global fixed points since $\psi(T)$ has no fixed points and F is a proper function so, in particular, A_T has no fixed points. Moreover, the orbits in this affine action are not discrete since $\Lambda \subseteq \mathrm{supp}(\mu)$ is not discrete and $\psi(G)$ acts minimally on Λ.

Since affine actions never allow a minimal invariant set which is locally a Cantor set, it follows that the affine action is minimal: the closure of every orbit is the whole real line. In particular F is continuous and hence $\mathbb{R} = F(\mathrm{supp}(\mu)) = F(\Lambda)$. As a consequence we have that components of the complement of Λ are mapped to points by F. If we observe that two points x and y are identified under F if and only if $\mu([x,y)) = 0$, we obtain that components of the complement of Λ are also components of the complement of $\mathrm{supp}(\mu)$. Therefore, $\Lambda = \mathrm{supp}(\mu)$, as claimed.

The preceding claim implies, in particular, that $\Lambda = \mathrm{supp}(\mu) \subseteq X$. Thus G^d, in the original action of G, fixes every point in $\mathrm{supp}(\mu)$. Therefore, μ is G^d–invariant, and hence μ is a proper Radon measure quasipreserved by G. This contradicts our choice of G. Hence, we conclude that G^d has no global fixed points.

Step 2 If there is an element $T \in G^d$ acting freely, then Theorem A is ensured by Plante [15] and Proposition 2.1.

Indeed, Proposition 2.1 ensures the existence of a proper G^d–invariant Radon measure μ on the line. Since T has no fixed points, the translation number homomorphism $\tau_\mu\colon g \mapsto \mu[0, g(0))$ defined on G^d is nontrivial. Theorem A then follows from Lemmas 4.1 and 4.2 in [15]; alternatively, the argument below also works.

So, we are left with the case where G^d has no global fixed point but there is no element of G^d acting freely. We claim that, in the presence of the freely acting element $T \in G$, this case is also not possible.

Indeed, since G^d has no global fixed points, there is $f \in G^d$ such that $T(0) < f(0)$. Since f has at least one fixed point, there is an f–invariant interval, containing 0, of the form $I = (a,b)$, where at least one of the endpoints is in $\mathbb{R}$ (and the other may be $\pm\infty$) and such that f has no fixed point in its interior (hence $f(x) > x$ for all $x \in I$). For concreteness we assume that a is a point in the real line, the other case being analogous.

Let $h = T^{-1} f T$. Since G^d is normal in G, we have that $h \in G^d$. Observe that $h(a) \in I$. Then, proceeding in the same way that in item (2) of the proof of Proposition 2.1, we can build $g \in G^d$ so that f and g are crossed. This contradicts that G^d has locally subexponential growth. This last contradiction finishes the proof of Theorem A. □

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