Topological equivalences of
E-infinity differential graded algebras

HALDUN ÖZGÜR BAYINDIR

Two DGAs are said to be topologically equivalent when the corresponding Eilenberg–MacLane ring spectra are weakly equivalent as ring spectra. Quasi-isomorphic DGAs are topologically equivalent, but the converse is not necessarily true. As a counterexample, Dugger and Shipley showed that there are DGAs that are nontrivially topologically equivalent, ie topologically equivalent but not quasi-isomorphic.

In this work, we define E_∞ topological equivalences and utilize the obstruction theories developed by Goerss, Hopkins and Miller to construct first examples of nontrivially E_∞ topologically equivalent E_∞ DGAs. Also, we show using these obstruction theories that for coconnective E_∞ $\mathbb{F}_p$–DGAs, E_∞ topological equivalences and quasi-isomorphisms agree. For E_∞ $\mathbb{F}_p$–DGAs with trivial first homology, we show that an E_∞ topological equivalence induces an isomorphism in homology that preserves the Dyer–Lashof operations and therefore induces an H_∞ $\mathbb{F}_p$–equivalence.

1 Introduction

Dugger and Shipley [7] defined a new equivalence relation between associative differential graded algebras (which we call DGAs) that they call topological equivalences. To define topological equivalences, they use the Quillen equivalence between R–DGAs and HR–algebras, where R denotes a discrete commutative ring; see Shipley [24]. Two R–DGAs X and Y are said to be *topologically equivalent* if the corresponding HR–algebras HX and HY are weakly equivalent as $\mathbb{S}$–algebras, where $\mathbb{S}$ denotes the sphere spectrum. Using Quillen equivalences in [24], it is easy to see that topologically equivalent DGAs are Morita equivalent. Furthermore, topological equivalences appear in one of the equivalent definitions of Morita equivalences of DGAs; see Theorem 1.4 of [7].

By the Quillen equivalence between R–DGAs and HR–algebras, two R–DGAs are quasi isomorphic if and only if the corresponding HR–algebras are weakly equivalent as HR–algebras. Because the forgetful functor from HR–algebras to $\mathbb{S}$–algebras

preserves weak equivalences, it is clear that quasi-isomorphic DGAs are always topologically equivalent. One of the main results of [7] is that there are DGAs that are not quasi-isomorphic but are topologically equivalent. Such DGAs are called nontrivially topologically equivalent. On the other hand, another theorem in [7] states that there are no examples of nontrivial topological equivalences in $\mathbb{Q}$–DGAs; ie topologically equivalent $\mathbb{Q}$–DGAs are quasi-isomorphic. See Theorem 1.8 below.

Because there is also a Quillen equivalence between E_∞ R–DGAs and commutative HR–algebras (see Richter and Shipley [21]), topological equivalences for E_∞ DGAs can also be considered. Now we explain what we mean by topological equivalences for DGAs and E_∞ DGAs. For DGAs, we have the following definition of topological equivalence.

Definition 1.1 Two R–DGAs X and Y are *topologically equivalent* if the corresponding HR–algebras HX and HY are weakly equivalent as $\mathbb{S}$–algebras. This is the same as the definition of topological equivalence in [7].

The definition for topological equivalence of E_∞ DGAs is the following.

Definition 1.2 Two E_∞ R–DGAs X and Y are E_∞ *topologically equivalent* if the corresponding commutative HR–algebras HX and HY are weakly equivalent as commutative $\mathbb{S}$–algebras.

Our methods make use of obstruction theories for ring spectra. In [22], Robinson develops an obstruction theory for showing existence of ring structures on spectra. This obstruction theory is generalized for commutative ring spectra by Robinson in [23]. Based on the obstruction spectral sequence of Bousfield [4], Hopkins and Miller developed another obstruction theory; see Rezk [20]. Their obstruction theory provides an obstruction spectral sequence for calculating mapping spaces of ring spectra and also an obstruction theory for showing the existence of ring structures on spectra. This obstruction theory is generalized to commutative ring spectra by Goerss and Hopkins [9]. In this work, we use the T–algebra spectral sequence of Johnson and Noel [10], which is a generalization of the obstruction spectral sequence of Hopkins and Miller, to calculate mapping spaces of commutative ring spectra.

In this work, we construct the first examples of nontrivially E_∞ topologically equivalent E_∞ DGAs. One of these examples is in E_∞ $\mathbb{F}_p$–DGAs. This is particularly interesting because one of the open questions in [7] asks if there are any examples of

nontrivial topological equivalences of k–DGAs for a field k. Our example provides a positive answer to this question in E_∞ DGAs. Although there is an example of nontrivial E_∞ topological equivalences over $\mathbb{F}_p$, our nonexistence results for E_∞ topological equivalences hint that such examples are not common.

Before stating our nonexistence results, we note that topologically equivalent DGAs have isomorphic homology rings. This is because the Quillen equivalence between R–DGAs and HR–algebras gives an isomorphism between the homology ring of an R–DGA and the homotopy ring of the corresponding ring spectra. Therefore, if X and Y are topologically equivalent DGAs, then $H_*(X) \cong \pi_*(HX) \cong \pi_*(HY) \cong H_*(Y)$, where the isomorphisms are ring isomorphisms and the isomorphism in the middle is induced by the topological equivalence. The same is true for E_∞ topological equivalences, but as Example 5.1 indicates, the isomorphism of homology rings may not preserve Dyer–Lashof operations. However, by Theorem 1.6, if X and Y are E_∞ topologically equivalent E_∞ $\mathbb{F}_p$–DGAs with trivial first homology, then the isomorphism of homology rings induced by the E_∞ topological equivalence preserves Dyer–Lashof operations; ie it is an isomorphism of algebras over the Dyer–Lashof algebra.

For coconnective E_∞ $\mathbb{Z}/(m)$–DGAs, we prove that there are no nontrivial E_∞ topological equivalences where m is a nonunital integer (ie $m \in \mathbb{Z}$ with $m \neq \pm 1$). This in particular implies that there are no nontrivial E_∞ topological equivalences of coconnective E_∞ $\mathbb{Z}$–DGAs.

Theorem 1.3 *E_∞ topologically equivalent coconnective E_∞ $\mathbb{Z}/(m)$–DGAs are quasi-isomorphic as E_∞ $\mathbb{Z}/(m)$–DGAs. In other words, E_∞ topological equivalences and quasi-isomorphisms agree for coconnective E_∞ $\mathbb{Z}/(m)$–DGAs.*

There is an important class of examples for this theorem, namely the cochain complex of a topological space with coefficients in $\mathbb{Z}/(m)$; this is the function spectrum $F(\Sigma^\infty X_+, H\mathbb{Z}/(m))$ for a topological space X. Note that since we use homological grading, the cochain complex of a space is coconnective. Moreover, Mandell's result [13] states that finite-type nilpotent spaces are weakly equivalent if and only if their cochain complexes with integer coefficients are quasi-isomorphic as E_∞ $\mathbb{Z}$–DGAs. Combining Mandell's result with Theorem 1.3, we obtain the following corollary.

Corollary 1.4 *Finite-type nilpotent spaces are weakly equivalent if and only if their cochain complexes with integer coefficients are E_∞ topologically equivalent.*

Remark 1.5   It is also interesting to consider the following consequence of Theorem 1.3. For a coconnective commutative $\mathbb{S}$–algebra X, we use the E_∞ connective cover $H\pi_0 X \to X$ to obtain a map $H\mathbb{Z} \to X$ which gives X a commutative $H\mathbb{Z}$–algebra structure. This says that there is an E_∞ $\mathbb{Z}$–DGA corresponding to a coconnective commutative $\mathbb{S}$–algebra. By this and Theorem 1.3, we deduce that weak equivalence classes of coconnective commutative $\mathbb{S}$–algebras are uniquely determined by the quasi-isomorphism classes of the corresponding E_∞ $\mathbb{Z}$–DGAs.

In Example 5.1, we construct E_∞ $\mathbb{F}_p$–DGAs that are nontrivially E_∞ topologically equivalent. Therefore, it is not possible to generalize Theorem 1.3 to all E_∞ $\mathbb{F}_p$–DGAs. However, for E_∞ $\mathbb{F}_p$–DGAs with trivial first homology, we have the following result.

Theorem 1.6   *Let X and Y be E_∞ $\mathbb{F}_p$–DGAs with trivial first homology group. If X and Y are E_∞ topologically equivalent, then they are equivalent as H_∞ $\mathbb{F}_p$–algebras. Furthermore, an $\mathbb{S}$–algebra equivalence between HX and HY induces an isomorphism of the homology rings that preserves Dyer–Lashof operations.*

We actually prove a stronger result. Theorem 7.1 states that for H_∞ $H\mathbb{F}_p$–algebras with trivial first homotopy, H_∞ $\mathbb{S}$–algebra equivalence implies H_∞ $H\mathbb{F}_p$–algebra equivalence.

The condition of trivial first homology is due to the fact that the dual Steenrod algebra is generated by an element of degree 1 as a ring with Dyer–Lashof operations. Again by Example 5.1, this condition cannot be removed from this theorem.

Remark 1.7   In [11], Lawson produces examples of H_∞ $\mathbb{S}$–algebras whose H_∞ $\mathbb{S}$–algebra structures do not lift to commutative $\mathbb{S}$–algebra structures. One of the intermediate results of [11] states that Theorem 7.1 is still true without the restriction on the first homotopy, but Example 5.1 contradicts this. The examples of spectra constructed in [11] are coconnective. Therefore, Theorem 7.1 recovers the main result of [11]. We elaborate on this in Section 7.

The proof of the nonexistence theorem in [7] for $\mathbb{Q}$–DGAs also works for E_∞ $\mathbb{Q}$–DGAs. We obtain the following.

Theorem 1.8   *(E_∞) topologically equivalent (E_∞) $\mathbb{Q}$–DGAs are quasi-isomorphic. That is, (E_∞) topological equivalences and (E_∞) quasi-isomorphisms agree in (E_∞) $\mathbb{Q}$–DGAs.*

In Section 2, we explain the examples of nontrivial topological equivalences given in [7], and in the appendix, we make a correction to a mistake in the construction of these examples. Section 3 discusses the obstruction spectral sequences that we will use for calculating mapping spaces of ring spectra, and Section 4 describes the dual Steenrod algebra and the Dyer–Lashof operations on it. Section 5 is devoted to our examples of nontrivial E_∞ topological equivalences. Section 6 contains the proof of Theorem 1.3, and Section 7 contains the proof of Theorem 1.6.

Notation   As noted earlier, for a commutative ring R, when we say R–DGAs we mean associative R–DGAs. Similarly, for a commutative ring spectrum R, by R–algebras we mean associative R–algebras. A smash product without a subscript $\wedge$ denotes the smash product over the sphere spectrum. The category of spectra we use is symmetric spectra in topological spaces with the positive model structure as in Mandell, May, Schwede and Shipley [14].

2   Previous examples of topological equivalences

In this section, we discuss the examples of nontrivial topological equivalences in [7].

Example 2.1   There are exactly two non quasi-isomorphic DGAs whose homology is $\Lambda_{\mathbb{F}_p}(x_{2p-2})$, the exterior algebra over $\mathbb{F}_p$ with a single generator in degree $2p-2$, and these DGAs are topologically equivalent. Therefore, they are nontrivially topologically equivalent. For $p = 2$, one of these DGAs is the formal one, and the other one is given by

$$\mathbb{Z}[e_1; de_1 = 2]/(e_1^4), \quad \text{where } |e_1| = 1.$$

This example is constructed by classifying weak equivalence classes of Postnikov extensions which are obtained using topological Hochschild cohomology. However, this construction in [7] contains a gap. The weak equivalences classes calculated

in [7] are weak equivalence classes of Postnikov extensions. This in general may not correspond to weak equivalence classes that should be considered here, namely the weak equivalence classes of $H\mathbb{Z}$–algebras. In the appendix, we explain this in detail and correct the mistake in [7] by showing that these two equivalence classes agree for this particular example.

Example 2.2 The second example of Dugger and Shipley has a simpler construction. They start with $H\mathbb{Z} \wedge H\mathbb{F}_2$ and give this $\mathbb{S}$–algebra two $H\mathbb{Z}$–algebra structures using the maps $H\mathbb{Z} \cong H\mathbb{Z} \wedge \mathbb{S} \to H\mathbb{Z} \wedge H\mathbb{F}_2$ and $H\mathbb{Z} \cong \mathbb{S} \wedge H\mathbb{Z} \to H\mathbb{Z} \wedge H\mathbb{F}_2$. These two $H\mathbb{Z}$–algebras are not weakly equivalent but their underlying $\mathbb{S}$–algebras are the same. This means that we have two DGAs that are not quasi-isomorphic but are topologically equivalent. In Theorem 5.3, we provide a generalization of this example in E_∞ DGAs.

3   Obstruction theories for ring spectra

For a commutative $\mathbb{S}$–algebra X, a commutative $H\mathbb{Z}$–algebra structure on X is given by a map $H\mathbb{Z} \to X$ of commutative $\mathbb{S}$–algebras. In other words, the category of commutative $H\mathbb{Z}$–algebras is the category commutative $\mathbb{S}$–algebras under $H\mathbb{Z}$. Therefore, it is natural to consider the maps from $H\mathbb{Z}$ to a commutative $\mathbb{S}$–algebra X for the purpose of studying E_∞ topological equivalences. For this, we employ an obstruction spectral sequence to calculate homotopy class of maps in commutative ring spectra.

The obstruction spectral sequence we use relies on Bousfield's obstruction spectral sequence [4]. The first application of Bousfield's obstruction theory to ring spectra was in the Hopkins–Miller theorem; see [20]. The obstruction theory of Hopkins and Miller is for associative ring spectra. It is used for showing existence of ring structures on spectra and for calculating mapping spaces of ring spectra. Hopkins and Miller use this obstruction theory to show that the Morava stabilizer group acts on the Morava E–theory spectrum E_n. Later, Goerss and Hopkins generalized this theory to commutative ring spectra [9].

Johnson and Noel generalized the obstruction theory of the Hopkins–Miller theorem to calculate mapping spaces of algebras over a general monad in a model category in [10]. This is called the T–algebra spectral sequence.

Generalizing the obstruction theory of the Hopkins–Miller theorem to commutative ring spectra is not trivial because of the following problem. For a spectrum X and a

homology theory E_* corresponding to another spectrum E, if $E_* X$ is flat over E_*, then $E_* T(X)$ is the free associative E_*–algebra over $E_* X$, where $T(X)$ is the free associative ring spectrum over X. For commutative ring spectra, one uses the free commutative ring spectra functor $\mathbb{P}_{\mathbb{S}}$, but $E_* \mathbb{P}_{\mathbb{S}}(X)$ may not have a nice description, even under the above flatness assumption. However, for calculating mapping spaces of commutative $H\mathbb{F}_p$–algebras one uses the fact that $\mathbb{P}_{H\mathbb{F}_p}(X)_*$ is the free unstable algebra over the Dyer–Lashof algebra generated by X_*. Noel uses this with the results of [10] and constructs a spectral sequence that calculates mapping spaces of commutative $H\mathbb{F}_p$–algebras; see Proposition 2.2 of [18]. More generally, his spectral sequence calculates mapping spaces of commutative Hk–algebras for any field k. Using the adjunction between commutative $\mathbb{S}$–algebras and commutative $H\mathbb{F}_p$–algebras, we obtain the following spectral sequence from Noel's spectral sequence.

Theorem 3.1 [18, Proposition 2.2] *Let X be a commutative $\mathbb{S}$–algebra and let Y be a commutative $H\mathbb{F}_p$–algebra. Given a map $\phi \colon X \to Y$ of commutative $\mathbb{S}$–algebras, there is a spectral sequence abutting to $\pi_{t-s} \, \mathrm{map}_{\mathbb{S}\text{–cAlg}}(X, Y)$, where $\mathbb{S}$–cAlg denotes commutative $\mathbb{S}$–algebras. The E_2 term of this spectral sequence is given by*

$$E_2^{0,0} = \mathrm{Hom}_{\mathcal{R}\text{–alg}}(H\mathbb{F}_{p*}X, Y_*),$$

and for $t > 0$,

$$E_2^{s,t} = \mathrm{Der}^s_{\mathcal{R}\text{–alg}}(H\mathbb{F}_{p*}X, Y_*^{S^t}),$$

where $\mathrm{Der}^s_{\mathcal{R}\text{–alg}}(-,-)$ denotes the sth André–Quillen cohomology for unstable algebras with Dyer–Lashof operations [19], Y^{S^t} denotes the mapping spectrum from the t–sphere to Y and $\mathrm{Hom}_{\mathcal{R}\text{–alg}}(H\mathbb{F}_{p*}X, Y_*)$ denotes morphisms preserving Dyer–Lashof operations.

Obstructions to lifting a morphism in $E_2^{0,0}$ to a morphism of commutative $\mathbb{S}$–algebras lie in $\mathrm{Der}^t_{\mathcal{R}\text{–alg}}(H\mathbb{F}_{p*}X, Y_*^{S^{t-1}})$ for $t \geq 2$.

Obstructions to up-to-homotopy uniqueness of a lift lie in $\mathrm{Der}^t_{\mathcal{R}\text{–alg}}(H\mathbb{F}_{p*}X, Y_*^{S^t})$ for $t \geq 1$.

Proof The adjunction between commutative $\mathbb{S}$–algebras and commutative $H\mathbb{F}_p$–algebras gives

$$\mathrm{map}_{\mathbb{S}\text{–cAlg}}(X, Y) \cong \mathrm{map}_{H\mathbb{F}_p\text{–cAlg}}(H\mathbb{F}_p \wedge X, Y).$$

Therefore, the setting of Noel's spectral sequence that calculates the homotopy groups of $\mathrm{map}_{H\mathbb{F}_p\text{–cAlg}}(H\mathbb{F}_p \wedge X, Y)$ provides us the spectral sequence above.

Noel's spectral sequence is a special case of the T–algebra spectral sequence of [10]. Therefore, Theorem 4.5 of [10] gives us the obstruction theoretical results. □

4 Dyer–Lashof operations and the dual Steenrod algebra

For a commutative ring spectrum R, we denote the free commutative algebra functor from R–modules to commutative R–algebras by $\mathbb{P}_R$. This functor is homotopically well behaved and induces a monad on $\mathrm{Ho}(R\text{–mod})$, and the algebras over this monad are called H_∞ R–algebras. Therefore, an E_∞ algebra is an H_∞ algebra. The converse to this is shown to be false by counterexamples in [17] and [11].

Dyer–Lashof operations are power operations, just like the Steenrod operations, that are constructed in a way to act on the homotopy ring of H_∞ $H\mathbb{F}_p$–algebras in [5]. Equivalently, they act on the homology ring of H_∞ $\mathbb{F}_p$–DGAs. Indeed, the category of H_∞ $H\mathbb{F}_p$–algebras is equivalent to the category of graded commutative rings over $\mathbb{F}_p$ with Dyer–Lashof operations satisfying the allowability and p^th power conditions, which are called unstable algebras over the Dyer–Lashof algebra; see the discussion in Section 3 of [11].

For each integer s, there is a Dyer–Lashof operation denoted by Q^s. These operations are preserved under H_∞ $H\mathbb{F}_p$–algebra morphisms and hence E_∞ $H\mathbb{F}_p$–algebra morphisms. The operation Q^s increases the degree by $2s(p-1)$ for odd primes and by s for $p=2$. For an element x in the homotopy ring of a commutative $H\mathbb{F}_p$–algebra, the unstable Dyer–Lashof operations satisfy the following properties (the properties for $p=2$ are given in parentheses):

$$Q^s x = 0 \quad \text{for } 2s < |x| \ \text{(for } s < |x|),$$
$$Q^s x = x^p \quad \text{for } 2s = |x| \ \text{(for } s = |x|),$$
$$Q^s 1 = 0 \quad \text{for } s \neq 0.$$

Also, these operations satisfy the Cartan formula and the Adem relations as in Chapter III, Theorem 1.1 of [5].

As mentioned earlier, for X a commutative $H\mathbb{F}_p$–algebra, $\mathbb{P}_{H\mathbb{F}_p}(X)_*$ is the free unstable algebra over the Dyer–Lashof algebra generated by X_*.

Theorem 4.1 [1] $\mathbb{P}_{H\mathbb{F}_p}(X)_*$ *is the free commutative graded $\mathbb{F}_p$–algebra generated by $Q^I x_j$, where the x_j form a basis for X_* and $I = (\varepsilon_1, i_1, \varepsilon_2, \ldots, \varepsilon_n, i_n)$ is admissible and satisfies* $\mathrm{excess}(I) + \varepsilon_1 > |x_j|$.

The definition of admissibility and excess can be found in [15].

Dual Steenrod algebra   Now we discuss the dual Steenrod algebra and the Dyer–Lashof operations on it. The dual Steenrod algebra is first described by Milnor in [16] and the Dyer–Lashof operations on it are first studied in Chapter III of [5]. We also recommend [2]. The dual Steenrod algebra $\mathcal{A}_* \cong H\mathbb{F}_{p*}H\mathbb{F}_p$ is a free graded commutative $\mathbb{F}_p$–algebra. For $p = 2$, it is given by two different standard sets of generators

$$\mathcal{A}_* = \mathbb{F}_2[\xi_r \mid r \geq 1] = \mathbb{F}_2[\zeta_r \mid r \geq 1],$$

where $|\xi_r| = |\zeta_r| = 2^r - 1$. The transpose map of the smash product applied to $H\mathbb{F}_p \wedge H\mathbb{F}_p$ induces an automorphism of the dual Steenrod algebra denoted by χ. The reason we have two different set of generators above is to keep track of the action of χ. We have $\chi(\xi_r) = \zeta_r$. The generating series $\xi(t) = t + \sum_{r \leq 1} \xi_r t^{2^r}$ and $\zeta(t) = t + \sum_{r \leq 1} \zeta_r t^{2^r}$ are composition inverses in the sense that $\zeta(\xi(t)) = t = \xi(\zeta(t))$. This in particular shows that $\xi_1 = \zeta_1$.

Since commutative $H\mathbb{F}_p$–algebras form the category of commutative $\mathbb{S}$–algebras under $H\mathbb{F}_p$, we can give $H\mathbb{F}_p \wedge H\mathbb{F}_p$ two different commutative $H\mathbb{F}_p$–algebra structures using the maps

$$H\mathbb{F}_p = H\mathbb{F}_p \wedge \mathbb{S} \to H\mathbb{F}_p \wedge H\mathbb{F}_p \quad \text{and} \quad H\mathbb{F}_p = \mathbb{S} \wedge H\mathbb{F}_p \to H\mathbb{F}_p \wedge H\mathbb{F}_p.$$

We call these maps g_1 and g_2 respectively. We denote the Dyer–Lashof operations induced on $\mathcal{A}_*$ from the first structure map by Q^s and the second structure map by $\widetilde{Q}^s$.

Since the transpose map induces an isomorphism of commutative $H\mathbb{F}_p$–algebras, χ preserves the corresponding Dyer–Lashof operations; ie $\chi(Q^s x) = \widetilde{Q}^s \chi(x)$. For $p = 2$, $\mathcal{A}_*$ is generated as an algebra over the Dyer–Lashof algebra by ξ_1, and we have

$$Q^{2^s - 2}\xi_1 = \zeta_s \quad \text{for } s \geq 1.$$

By using the fact that χ preserves Dyer–Lashof operations, one obtains

$$\widetilde{Q}^{2^s - 2}\xi_1 = \xi_s \quad \text{for } s \geq 1.$$

For an odd prime p, the dual Steenrod algebra is given by

$$\mathcal{A}_* = \mathbb{F}_p[\xi_r \mid r \geq 1] \otimes \Lambda(\tau_s \mid s \geq 0) = \mathbb{F}_p[\zeta_r \mid r \geq 1] \otimes \Lambda(\overline{\tau}_s \mid s \geq 0),$$

where $|\xi_r| = |\zeta_r| = 2(p^r - 1)$ and $|\tau_s| = |\overline{\tau}_s| = 2p^s - 1$. The action of the antipode map is given by $\chi(\xi_r) = \zeta_r$ and $\chi(\tau_r) = \overline{\tau}_r$. We use the following formula to relate

the two set of generators for the dual Steenrod algebra; see Section 7 of [16] and the proof of Lemma 4.7 in [2]:

$$(1) \qquad \overline{\tau}_s + \overline{\tau}_{s-1}\xi_1^{p^{s-1}} + \overline{\tau}_{s-2}\xi_2^{p^{s-2}} + \cdots + \overline{\tau}_0\xi_s + \tau_s = 0.$$

The dual Steenrod algebra is generated by τ_0 as an algebra over the Dyer–Lashof algebra. We have the following formulae for the Dyer–Lashof operations for $s \geq 1$:

$$Q^{(p^s-1)/(p-1)}\tau_0 = (-1)^s\overline{\tau}_s,$$
$$\beta Q^{(p^s-1)/(p-1)}\tau_0 = (-1)^s\zeta_s.$$

More can be found on the Dyer–Lashof operations on the dual Steenrod algebra in [2] and in Chapter III of [5].

5   Examples of nontrivial E_∞ topological equivalences

5A   Examples in E_∞ $\mathbb{F}_p$–DGAs

Here, we discuss the first examples of E_∞ DGAs that are not quasi-isomorphic but are E_∞ topologically equivalent, ie nontrivially E_∞ topologically equivalent.

The first example we construct is in E_∞ $\mathbb{F}_p$–DGAs. By the equivalence of E_∞ $\mathbb{F}_p$–DGAs and commutative $H\mathbb{F}_p$–algebras, constructing nontrivially topologically equivalent E_∞ $\mathbb{F}_p$–DGAs is the same as constructing commutative $H\mathbb{F}_p$–algebras that are not weakly equivalent as commutative $H\mathbb{F}_p$–algebras but are weakly equivalent as commutative $\mathbb{S}$–algebras.

As we noted earlier, commutative $H\mathbb{F}_p$–algebras form the category of commutative $\mathbb{S}$–algebras under $H\mathbb{F}_p$. There is a model structure induced on the undercategory, where the weak equivalences, cofibrations and fibrations are precisely the same as for commutative $\mathbb{S}$–algebras.

In our example, we start with a commutative $\mathbb{S}$–algebra X and induce two different commutative $H\mathbb{F}_p$–algebra structures on this object by providing two different commutative $\mathbb{S}$–algebra maps from $H\mathbb{F}_p$ to X. Clearly these two commutative $H\mathbb{F}_p$–algebras are weakly equivalent (even isomorphic) as commutative $\mathbb{S}$–algebras. We show that these two commutative $H\mathbb{F}_p$–algebras are not weakly equivalent as commutative $H\mathbb{F}_p$–algebras by showing that their homotopy rings are not isomorphic as algebras over the Dyer–Lashof algebra. By the discussion of Section 4, this shows that these

nontrivially E_∞ topologically equivalent E_∞ $\mathbb{F}_p$–DGAs are furthermore not equivalent as H_∞ $\mathbb{F}_p$–DGAs.

Example 5.1 For an odd prime p, the E_∞ $\mathbb{F}_p$–DGAs we produce have the same homology ring, given by $\Lambda_{\mathbb{F}_p}[\tau_0, \xi_1, \tau_1]/(\tau_0\tau_1, \tau_1\xi_1, \tau_0\xi_1 - \tau_1)$, where the degrees of the generators are those of the dual Steenrod algebra, ie $|\tau_0| = 1$, $|\xi_1| = 2(p-1)$ and $|\tau_1| = 2p - 1$. However, the homology groups of these E_∞ $\mathbb{F}_p$–DGAs are not isomorphic as algebras over the Dyer–Lashof algebra. In one of them, $Q^1(\tau_0) = \tau_1$, and in the other, $Q^1(\tau_0) = 0$. Therefore, these two E_∞ $\mathbb{F}_p$–DGAs are not equivalent as H_∞ $\mathbb{F}_p$–DGAs, and therefore they are not quasi-isomorphic.

For $p = 2$, the homology ring of the two E_∞ topologically equivalent E_∞ $\mathbb{F}_2$–DGAs is $\mathbb{F}_2[\xi_1]/(\xi_1^4)$, where $|\xi_1| = 1$ as in the dual Steenrod algebra. In the homology of the first E_∞ $\mathbb{F}_2$–DGA, $Q^2(\xi_1) = \xi_1^3$, and in the other one, $Q^2(\xi_1) = 0$. Again, these two E_∞ $\mathbb{F}_2$–DGAs are not quasi-isomorphic because their homology rings are not isomorphic as algebras over the Dyer–Lashof algebra, and therefore they are not equivalent as H_∞ $\mathbb{F}_2$–DGAs.

First, we discuss our example for $p = 2$. For this, we are going to use Postnikov sections for commutative ring spectra. These were first introduced in Section 8 of [3] for studying Postnikov towers of commutative ring spectra. The nth Postnikov section of a connected commutative ring spectrum Z is a map $Z \to P_n Z$ that induces an isomorphism on $\pi_i(Z) \to \pi_i(P_n Z)$ for $i \leq n$ and for which $\pi_i P_n Z = 0$ for $i > n$. Let $H\mathbb{F}_2 \wedge H\mathbb{F}_2 \to P_3(H\mathbb{F}_2 \wedge H\mathbb{F}_2)$ be the third Postnikov section of $H\mathbb{F}_2 \wedge H\mathbb{F}_2$ as a commutative $\mathbb{S}$–algebra. We have $\pi_*(P_3(H\mathbb{F}_2 \wedge H\mathbb{F}_2)) = \mathbb{F}_2[\xi_1, \xi_2]/(\xi_1^4, \xi_2^2, \xi_1\xi_2)$ with $|\xi_1| = 1$ and $|\xi_2| = 3$, ie the dual Steenrod algebra quotiented out by the ideal of elements of degree 4 and higher. By using Lemma 5.2, we kill the element $\xi_1^3 + \xi_2$ in $\pi_*(P_3(H\mathbb{F}_2 \wedge H\mathbb{F}_2))$, and then taking the third Postnikov section, we obtain another commutative $\mathbb{S}$–algebra X with $\pi_*(X) = \mathbb{F}_2[\xi_1, \xi_2]/(\xi_1^4, \xi_2^2, \xi_1\xi_2, \xi_1^3 + \xi_2) = \mathbb{F}_2[\xi_1]/(\xi_1^4)$. The reason we kill $\xi_1^3 + \xi_2$ is because it is equal to ζ_2 in the dual Steenrod algebra; this follows from the generating series we discuss in Section 4. Note that for Lemma 5.2, one can use the commutative $H\mathbb{F}_2$–algebra structure on $P_3(H\mathbb{F}_2 \wedge H\mathbb{F}_2)$ induced by the map $H\mathbb{F}_2 \wedge \mathbb{S} \to H\mathbb{F}_2 \wedge H\mathbb{F}_2 \to P_3(H\mathbb{F}_2 \wedge H\mathbb{F}_2)$.

Furthermore, we have a map $P_3(H\mathbb{F}_2 \wedge H\mathbb{F}_2) \to X$ with the induced map on the homotopy rings being the canonical one. By precomposing this map with the map into the Postnikov section $H\mathbb{F}_2 \wedge H\mathbb{F}_2 \to P_3(H\mathbb{F}_2 \wedge H\mathbb{F}_2)$, we obtain a map of

commutative $\mathbb{S}$–algebras $f\colon H\mathbb{F}_2 \wedge H\mathbb{F}_2 \to X$. We construct two commutative $\mathbb{S}$–algebra maps from $H\mathbb{F}_2$ to X as shown in the diagram below:

(2)
$$
\begin{array}{ccc}
H\mathbb{F}_2 \cong H\mathbb{F}_2 \wedge \mathbb{S} & & \\
& \xrightarrow{\;g_1\;} & \\
& H\mathbb{F}_2 \wedge H\mathbb{F}_2 & \xrightarrow{\;f\;} X \\
& \xrightarrow{\;g_2\;} & \\
H\mathbb{F}_2 \cong \mathbb{S} \wedge H\mathbb{F}_2 & &
\end{array}
$$

The maps g_1 and g_2 induce two commutative $H\mathbb{F}_2$–algebra structures on X. The commutative $H\mathbb{F}_2$–algebra with unit $f \circ g_1$ is denoted by X_1 and with unit $f \circ g_2$ by X_2. As we discuss in Section 4, $H\mathbb{F}_2 \wedge H\mathbb{F}_2$ can be given two commutative $H\mathbb{F}_2$–algebra structures through the maps g_1 and g_2, and we call the two associated commutative $H\mathbb{F}_2$–algebras Y_1 and Y_2. The Dyer–Lashof operations on $\pi_*(Y_1)$ are denoted by Q^s and the Dyer–Lashof operations on $\pi_*(Y_2)$ are denoted by $\widetilde{Q}^s$ as in Section 4.

Because morphisms in commutative $H\mathbb{F}_2$–algebras are morphisms of commutative $\mathbb{S}$–algebras under $H\mathbb{F}_2$, from the map f alone we obtain two $H\mathbb{F}_2$–algebra maps $g\colon Y_1 \to X_1$ and $h\colon Y_2 \to X_2$. These maps induce maps that preserve Dyer–Lashof operations in the homotopy rings, and we use this to understand the Dyer–Lashof operations on $\pi_*(X_1)$ and $\pi_*(X_2)$. On $\pi_*(X_1)$,

$$Q^2(\xi_1) = Q^2(g_*(\xi_1)) = g_*(Q^2(\xi_1)) = g_*(\zeta_2) = \zeta_2 = \xi_1^3 + \xi_2 = 0.$$

On $\pi_*(X_2)$,

$$Q^2(\zeta_1) = Q^2(h_*(\zeta_1)) = h_*(\widetilde{Q}^2(\zeta_1)) = h_*(\xi_2) = \xi_2 \neq 0.$$

Therefore, $\pi_*(X_1)$ and $\pi_*(X_2)$ are not isomorphic as algebras over the Dyer–Lashof algebra as desired.

For odd primes p, the construction of an example is similar. By (1) in Section 4, $\overline{\tau}_1 = \tau_0\xi_1 - \tau_1$. Therefore, one can use Lemma 5.2 to kill $\tau_0\xi - \tau_1$, which kills $Q^1\tau_0$. The rest of the arguments follow similarly.

Lemma 5.2 *Let X be a connective commutative $H\mathbb{F}_p$–algebra with $\pi_0(X) = \mathbb{F}_p$ and $\pi_i(X) = 0$ for $i > n$. Given $x \in \pi_n(X)$, there is a commutative $\mathbb{S}$–algebra Y and a map of commutative $\mathbb{S}$–algebras $X \to Y$ which induces the morphism $X_* \to X_*/(x)$ on the level of homotopy groups.*

Proof Let the $H\mathbb{F}_p$–module map $\Sigma^n H\mathbb{F}_p \to X$ represent x. By adjunction and by applying the nth Postnikov section functor, we obtain the map $P_n(\mathbb{P}_{H\mathbb{F}_p}(\Sigma^n H\mathbb{F}_p)) \to P_n(X) \simeq X$. The homotopy ring $\pi_*(\mathbb{P}_{H\mathbb{F}_p}(\Sigma^n H\mathbb{F}_p))$ is the free unstable algebra

over the Dyer–Lashof algebra generated by an element of degree n. Therefore, we obtain that $\pi_*(P_n(\mathbb{P}_{H\mathbb{F}_p}(\Sigma^n H\mathbb{F}_p))) = \Lambda_{\mathbb{F}_p}[x_n]$, where $|x_n| = n$. Let Z denote $P_n(\mathbb{P}_{H\mathbb{F}_p}(\Sigma^n H\mathbb{F}_p))$. The required Y is $P_n(H\mathbb{F}_p \wedge_Z X)$, where $H\mathbb{F}_p$ is a commutative Z–algebra by the map $Z \to P_0 Z = H\mathbb{F}_p$.

The homotopy groups of $H\mathbb{F}_p \wedge_Z X$ can be calculated using the Künneth spectral sequence whose E^2 page is $\mathrm{Tor}_{*,*}^{\Lambda_{\mathbb{F}_p}[x_n]}(\mathbb{F}_p, X_*)$. We want to show that $\pi_*(H\mathbb{F}_p \wedge_Z X)$ and $\mathbb{F}_p \otimes_{\Lambda_{\mathbb{F}_p}[x_n]} X_*$ agree for degree less than $n + 1$.

There is a standard resolution of $\mathbb{F}_p$ over $\Lambda_{\mathbb{F}_p}[x_n]$ which is $\Sigma^{kn}\Lambda_{\mathbb{F}_p}[x_n]$ at homological degree k. Therefore, we have $\mathrm{Tor}_{k,l}^{\Lambda_{\mathbb{F}_p}[x_n]}(\mathbb{F}_p, X_*) = 0$ for $k > 0$ and $l < n$, and hence the only terms that contribute to $\pi_i(H\mathbb{F}_p \wedge_Z X)$ for $i \leq n$ are in $E_{0,*}^2$. Since the differentials on these terms hit the second quadrant, they are zero. We obtain $\pi_i(H\mathbb{F}_p \wedge_Z X) \cong \mathrm{Tor}_{0,i}^{\Lambda_{\mathbb{F}_p}[x_n]}(\mathbb{F}_p, X_*) \cong (\mathbb{F}_p \otimes_{\Lambda_{\mathbb{F}_p}} X_*)_i$ for $i \leq n$. Because $\mathbb{F}_p \otimes_{\Lambda_{\mathbb{F}_p}[x_n]} X_*$ is concentrated in degrees between 0 and n, we have $\pi_* Y \cong \pi_*(P_n(H\mathbb{F}_p \wedge_Z X)) \cong \mathbb{F}_p \otimes_{\Lambda_{\mathbb{F}_p}} X_*$. Since the image of x_n in $\pi_*(X)$ is x, we have $\pi_* Y \cong \mathbb{F}_p \otimes_{\Lambda_{\mathbb{F}_p}} X_* \cong \pi_*(X)/(x)$, and $X \to Y$ induces the desired morphism $\pi_* X \to \pi_* X/(x)$ on the homotopy ring. For this lemma, we forget the $H\mathbb{F}_p$ structure and use the corresponding commutative $\mathbb{S}$–algebra map $X \to Y$. $\square$

5B Examples in E_∞ $\mathbb{Z}$–DGAs

Now we discuss examples of nontrivial E_∞ topological equivalences in E_∞ $\mathbb{Z}$–DGAs. Theorem 5.4 below gives a general scenario where examples of nontrivially E_∞ topologically equivalent E_∞ $\mathbb{Z}$–DGAs occur. We start with Theorem 5.3, which provides examples that have a simple construction. This theorem uses the construction of the example of Dugger and Shipley that we discuss in Example 2.2 above. For E_∞ DGAs, this construction is generalized to odd primes.

Theorem 5.3 *Let X and Y denote the commutative $H\mathbb{Z}$–algebras whose underlying commutative $\mathbb{S}$–algebras are $H\mathbb{Z} \wedge H\mathbb{F}_p$ and whose commutative $H\mathbb{Z}$–algebra structures are respectively given by the maps*

$$H\mathbb{Z} \cong H\mathbb{Z} \wedge \mathbb{S} \to H\mathbb{Z} \wedge H\mathbb{F}_p \quad and \quad H\mathbb{Z} \cong \mathbb{S} \wedge H\mathbb{Z} \to H\mathbb{Z} \wedge H\mathbb{F}_p.$$

These commutative $H\mathbb{Z}$–algebras X and Y are not weakly equivalent. Since the underlying commutative $\mathbb{S}$–algebras of X and Y are the same, we deduce that the E_∞ $\mathbb{Z}$–DGAs corresponding to X and Y are not quasi-isomorphic but are E_∞ topologically equivalent.

Proof Assume that X and Y are weakly equivalent as commutative $H\mathbb{Z}$–algebras. Taking cofibrant and fibrant replacements, we assume that there is a weak equivalence $\psi\colon X \xrightarrow{\sim} Y$ of commutative $H\mathbb{Z}$–algebras. This means that there is the following diagram in commutative $\mathbb{S}$–algebras:

(3)
$$\begin{array}{ccc} & H\mathbb{Z} & \\ {\scriptstyle \varphi_X}\swarrow & & \searrow{\scriptstyle \varphi_Y} \\ U(X) & \xrightarrow[\simeq]{\psi} & U(Y) \end{array}$$

Here, U denotes the forgetful functor to commutative $\mathbb{S}$–algebras and φ_X and φ_Y denote the commutative $H\mathbb{Z}$–algebra structure maps of X and Y, respectively. Note that by the Künneth spectral sequence,

$$H\mathbb{F}_{p*}U(X) \cong H\mathbb{F}_{p*}(H\mathbb{Z} \wedge H\mathbb{F}_p) \cong H\mathbb{F}_{p*}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{p*}H\mathbb{F}_p.$$

Taking the $H\mathbb{F}_p$ homology of the diagram above, we obtain the following:

(4)
$$\begin{array}{ccc} & H\mathbb{F}_{p*}H\mathbb{Z} & \\ {\scriptstyle H\mathbb{F}_{p*}\varphi_X}\swarrow & & \searrow{\scriptstyle H\mathbb{F}_{p*}\varphi_Y} \\ H\mathbb{F}_{p*}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{p*}H\mathbb{F}_p & \xrightarrow[\simeq]{H\mathbb{F}_{p*}\psi} & H\mathbb{F}_{p*}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{p*}H\mathbb{F}_p \end{array}$$

With this identification, $H\mathbb{F}_{p*}\varphi_X(x) = x \otimes 1$. Similarly, we have $H\mathbb{F}_{p*}U(Y) \cong H\mathbb{F}_{p*}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{p*}H\mathbb{F}_p$ and $H\mathbb{F}_{p*}\varphi_Y(x) = 1 \otimes x$. As noted earlier, the canonical map $H\mathbb{F}_{p*}H\mathbb{Z} \to H\mathbb{F}_{p*}H\mathbb{F}_p$ is an inclusion, and $H\mathbb{F}_{p*}H\mathbb{Z}$ is a free commutative ring generated by the same generators as $H\mathbb{F}_{p*}H\mathbb{F}_p$ except τ_0. The only degree-1 element in $H\mathbb{F}_{p*}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{p*}H\mathbb{F}_p$ is $1 \otimes \tau_0$, and since $H\mathbb{F}_{p*}\psi$ is an isomorphism, this is mapped by $H\mathbb{F}_{p*}\psi$ to $1 \otimes \tau_0$. Since $Q^1(1 \otimes \tau_0) = 1 \otimes \overline{\tau}_1$, we have $H\mathbb{F}_{p*}\psi(1 \otimes \overline{\tau}_1) = 1 \otimes \overline{\tau}_1$. However, the commutativity of the triangle forces $H\mathbb{F}_{p*}\psi(\overline{\tau}_1 \otimes 1) = 1 \otimes \overline{\tau}_1$, which contradicts the injectivity of $H\mathbb{F}_{p*}\psi$. Thus, X and Y are not equivalent as commutative $H\mathbb{Z}$–algebras. The argument for $p = 2$ is similar. □

Theorem 5.4 *Let X be either an E_∞ $\mathbb{F}_p$–DGA for an odd prime p that satisfies*

(1) *$H_i X = 0$ for $i = 1$ and $i > 2p^2 - 4$,*

(2) *$H_i X \neq 0$ for either $i = 2p - 1$ or $i = 2p - 2$,*

or an E_∞ $\mathbb{F}_2$–DGA that satisfies

(1) *$H_i X = 0$ for $i = 1$ and $i > 4$,*

(2) *$H_i X \neq 0$ for $i = 2$.*

For such an X, there exists an E_∞ $\mathbb{Z}$–DGA that is E_∞ topologically equivalent to X but not quasi-isomorphic to X.

Indeed, we show that the E_∞ $\mathbb{Z}$–DGA we construct in the proof of the above theorem is not quasi-isomorphic to any E_∞ $\mathbb{F}_p$–DGA.

It is clear that E_∞ $\mathbb{F}_p$–DGAs that satisfy the conditions of this theorem exist. One can start with a graded commutative ring that satisfies the above conditions and use the corresponding formal commutative $\mathbb{F}_p$–DGA.

Remark 5.5 A special case of the above theorem gives the example of Dugger and Shipley that we discuss in Example 2.1. For this, one uses the formal commutative $\mathbb{F}_2$–DGA with homology the exterior algebra with a generator in degree 2 for X. Moreover, our theorem provides a generalization of this example. We start with an X as described in the $p = 2$ case above and let Y be the E_∞ $\mathbb{Z}$–DGA we produce in the proof. Then X and Y are not quasi-isomorphic as associative $\mathbb{Z}$–DGAs although they are topologically equivalent; ie X and Y are nontrivially topologically equivalent as associative DGAs.

Proof To produce our E_∞ $\mathbb{Z}$–DGA, we start with an X as above and construct a commutative $\mathbb{S}$–algebra map $\varphi_Y \colon H\mathbb{Z} \to U(X)$ using obstruction theory, where U is the forgetful functor to commutative $\mathbb{S}$–algebras. This gives us a new commutative $H\mathbb{Z}$–algebra Y whose underlying commutative $\mathbb{S}$–algebra is $U(X)$. Obstruction theory gives us control over the map induced by φ_Y on the $H\mathbb{F}_p$ homology. Using this and Dyer–Lashof operations, we show that Y is not weakly equivalent as a commutative $H\mathbb{Z}$–algebra to any commutative $H\mathbb{F}_p$–algebra.

We describe our example when p is an odd prime and $H_{2p-1}X \neq 0$. The case $H_{2p-2}X \neq 0$ is similar. We explain the $p = 2$ example at the end.

For the obstruction spectral sequence of Theorem 3.1, we use the composite map $H\mathbb{Z} \to H\mathbb{F}_p \xrightarrow{\varphi_X} U(X)$ as a basepoint, where the map φ_X is the $H\mathbb{F}_p$ structure morphism of X. Using this basepoint and by setting up the obstruction spectral sequence to calculate the commutative $\mathbb{S}$–algebra maps from $H\mathbb{Z}$ to X, we obtain that obstructions to lifting a morphism of unstable algebras over the Dyer–Lashof algebra in

$$E_2^{0,0} = \mathrm{Hom}_{\mathcal{R}\text{–alg}}(H\mathbb{F}_{p\,*}H\mathbb{Z}, X_*)$$

to a commutative $\mathbb{S}$–algebra map from $H\mathbb{Z}$ to $U(X)$ lie in the cohomology groups

$$\mathrm{Der}_{\mathcal{R}\text{–alg}}^t(H\mathbb{F}_{p\,*}H\mathbb{Z}, X_*^{S^{t-1}}) \quad \text{for } t \geq 2.$$

Here, $X^{S^{t-1}}$ denotes the mapping spectrum from the $(t-1)$–sphere S^{t-1} to X. In Lemma 5.6 below, we show that these groups that contain the obstructions are trivial. Therefore, every map in $\mathrm{Hom}_{\mathcal{R}\text{–alg}}(H\mathbb{F}_{p*}H\mathbb{Z}, X_*)$ lifts to a commutative $\mathbb{S}$–algebra map.

The canonical morphism $H\mathbb{F}_{p*}H\mathbb{Z} \to H\mathbb{F}_{p*}H\mathbb{F}_p$ is an injection and the image of this morphism is the free commutative algebra generated by ζ_i and $\bar{\tau}_i$ for $i \geq 1$ when p is odd. For $p = 2$, the image is generated by ξ_1^2 and ζ_i for $i \geq 1$. Since the above inclusion comes from a map of commutative $H\mathbb{F}_p$–algebras, it preserves Dyer–Lashof operations. This says that the Dyer–Lashof operations on $H\mathbb{F}_{p*}H\mathbb{Z}$ are those of $H\mathbb{F}_{p*}H\mathbb{F}_p$.

To construct the commutative $\mathbb{S}$–algebra map $\varphi_Y\colon H\mathbb{Z} \to U(X)$ that defines Y as desired, we start with any morphism f in $E_2^{0,0} = \mathrm{Hom}_{\mathcal{R}\text{–alg}}(H\mathbb{F}_{p*}H\mathbb{Z}, X_*)$ that maps τ_1 to a nonzero element in $H_{2p-1}X$ and use the lift of this map to a commutative $\mathbb{S}$–algebra map. Here, φ_Y being a lift of f means that the map

$$H\mathbb{F}_p \wedge H\mathbb{Z} \xrightarrow{\mathrm{id}\wedge\varphi_Y} H\mathbb{F}_p \wedge U(X) \to U(X)$$

induces f in homotopy, where the second map is given by the $H\mathbb{F}_p$–module structure map of X.

At this point, we need to show that the E_∞ $\mathbb{Z}$–DGAs corresponding to X and Y are not quasi-isomorphic; ie X and Y are not weakly equivalent as commutative $H\mathbb{Z}$–algebras. Assume that they are weakly equivalent over $H\mathbb{Z}$. We start with an X that is cofibrant and fibrant as a commutative $H\mathbb{F}_p$–algebra. Therefore, Y is also fibrant as a commutative $H\mathbb{Z}$–algebra because the underlying commutative $\mathbb{S}$–algebra of Y is the underlying commutative $\mathbb{S}$–algebra of X. Our X is also cofibrant as a commutative $H\mathbb{Z}$–algebra since we use a cofibrant $H\mathbb{F}_p$ so that the initial map $H\mathbb{Z} \hookrightarrow H\mathbb{F}_p \hookrightarrow X$ is a composition of two cofibrations. Recall that cofibrations of commutative $H\mathbb{F}_p$–algebras are those of commutative $H\mathbb{Z}$–algebras. Therefore, we have a weak equivalence of commutative $H\mathbb{Z}$–algebras $\psi\colon X \xrightarrow{\sim} Y$. That is, we have the following commuting diagram in commutative $\mathbb{S}$–algebras:

(5)

$$
\begin{array}{ccc}
& H\mathbb{Z} & \\
{\scriptstyle \varphi_Y}\downarrow & \searrow & \\
& & H\mathbb{F}_p \\
& & \quad\searrow {\scriptstyle \varphi_X} \\
U(X) = U(Y) & \xleftarrow[\simeq]{\psi} & U(X)
\end{array}
$$

From the above diagram, by applying the $H\mathbb{F}_p$ homology functor, we obtain the following diagram:

$$(6)$$

$$
\begin{array}{c}
H\mathbb{F}_{p*}H\mathbb{Z} \\
\Big\downarrow{\scriptstyle H\mathbb{F}_{p*}\varphi_Y} \qquad\qquad\searrow \\
\qquad\qquad H\mathbb{F}_{p*}H\mathbb{F}_p \\
\qquad\qquad\qquad\qquad\searrow{\scriptstyle H\mathbb{F}_{p*}\varphi_X} \\
H\mathbb{F}_{p*}X \xleftarrow[\simeq]{H\mathbb{F}_{p*}\psi} H\mathbb{F}_{p*}X \\
\Big\downarrow \\
X_*
\end{array}
$$

Therefore, all the morphisms in this triangle preserve Dyer–Lashof operations. The bottom left arrow is induced by the $H\mathbb{F}_p$–module structure map of X. This is a morphism of commutative $H\mathbb{F}_p$–algebras, therefore the bottom left vertical arrow also preserves Dyer–Lashof operations operations. In conclusion, all the arrows in this diagram preserve Dyer–Lashof operations.

The composition of the vertical arrows on the left gives the map f as chosen above. Therefore, $\overline{\tau}_1$ in $H\mathbb{F}_{p*}H\mathbb{Z}$ is mapped to a nonzero element in X_* by the composition of the vertical arrows. Because the triangle above commutes, if we travel $\overline{\tau}_1$ through the diagonal arrow to $H\mathbb{F}_{p*}H\mathbb{F}_p$ and then to X_*, we see that $\overline{\tau}_1$ in $H\mathbb{F}_{p*}H\mathbb{F}_p$ must also be mapped to a nonzero element in X_*. Because $\pi_1 X$ is trivial, τ_0 in $H\mathbb{F}_{p*}H\mathbb{F}_p$ is mapped to zero in X_*. However this, and the fact that $\overline{\tau}_1 = -\beta Q^1 \tau_0$ imply that $\overline{\tau}_1$ in $H\mathbb{F}_{p*}H\mathbb{F}_p$ is mapped to zero in X_*. This contradicts $f(\overline{\tau}_1) \neq 0$. Therefore, Y and X are not weakly equivalent as commutative $H\mathbb{Z}$–algebras.

For the case $H_{2p-2}X \neq 0$, we use an f in $\mathrm{Hom}_{\mathcal{R}-\mathrm{alg}}(H\mathbb{F}_{p*}H\mathbb{Z}, X_*)$ that maps ζ_1 to a nonzero element in $\pi_{2p-2}X$. Since $Q^1\tau_0 = -\zeta_1$, the rest of the argument follows similarly.

For $p=2$, we start with an f in $\mathrm{Hom}_{\mathcal{R}-\mathrm{alg}}(H\mathbb{F}_{2*}H\mathbb{Z}, X_*)$ that maps ξ_1^2 in $H\mathbb{F}_{2*}H\mathbb{Z}$ to a nonzero element in $\pi_2 X$. Note that ξ_1^2 in $H\mathbb{F}_{2*}H\mathbb{Z}$ is a free algebra generator and $\xi_1 \notin H\mathbb{F}_{2*}H\mathbb{Z}$. The arguments are similar but we do not use Dyer–Lashof operations in this case. Again considering diagram (6), ξ_1^2 in $H\mathbb{F}_{2*}H\mathbb{F}_2$ is mapped to a nonzero element in $\pi_2 X$ but since $\pi_1 X = 0$, we have that ξ_1 is mapped to zero, and this is a contradiction. Since we haven't used Dyer–Lashof operations, we may consider the underlying associative $H\mathbb{Z}$–algebras of X and Y and the above arguments still work;

ie the associative $\mathbb{Z}$–DGAs corresponding to X and Y are nontrivially topologically equivalent.

What is left to prove is the following lemma which says that the obstructions in the above setting for lifting f to a map of commutative $\mathbb{S}$–algebras are zero.   □

Lemma 5.6   *In the setting of Theorem 5.4,*

$$\mathrm{Der}^t_{\mathcal{R}-\mathrm{alg}}(H\mathbb{F}_{p_*}H\mathbb{Z}, X^{S^{t-1}}_*) = 0 \quad \text{for } t \geq 2.$$

Proof   We describe the odd prime case; the proof is similar for $p = 2$. Let $F_{\mathcal{R}}(\zeta_1, \overline{\tau}_1)$ denote the free unstable algebra over the Dyer–Lashof algebra generated by two elements whose degrees are the degrees of the corresponding generators in the dual Steenrod algebra. The free unstable algebra over the Dyer–Lashof algebra is described in Theorem 4.1. The lowest-degree generator of $F_{\mathcal{R}}(\zeta_1, \overline{\tau}_1)$ after ζ_1 and $\overline{\tau}_1$ is $\beta Q^p \zeta_1$, with degree $2p^2 - 3$. Note that $H\mathbb{F}_{p_*}H\mathbb{Z}$ has no free algebra generators in this degree, showing that $H\mathbb{F}_{p_*}H\mathbb{Z}$ cannot be the free unstable algebra over the Dyer–Lashof algebra generated by ζ_1 and $\overline{\tau}_1$. However, also note that $H\mathbb{F}_{p_*}H\mathbb{Z}$ agrees with $F_{\mathcal{R}}(\zeta_1, \overline{\tau}_1)$ up to degree $2p^2 - 4$. Therefore, the morphism $F_{\mathcal{R}}(\zeta_1, \overline{\tau}_1) \to H\mathbb{F}_{p_*}H\mathbb{Z}$ which preserves Dyer–Lashof operations maps ζ_1 to ζ_1 and $\overline{\tau}_1$ to $\overline{\tau}_1$ is an isomorphism up to degree $2p^2 - 4$.

Furthermore, considering the free simplicial resolution of these objects as unstable algebras over the Dyer–Lashof algebra, we get a morphism

$$F^{\bullet+1}_{\mathcal{R}}(F_{\mathcal{R}}(\zeta_1, \overline{\tau}_1)) \to F^{\bullet+1}_{\mathcal{R}}(H\mathbb{F}_{p_*}H\mathbb{Z})$$

of simplicial unstable algebras over the Dyer–Lashof algebra. Up to degree $n > 0$, $F_{\mathcal{R}}(M)$ only depends on the part of the vector space M up to degree n. Therefore, the morphism above is an isomorphism up to degree $2p^2 - 4$ at each simplicial degree.

Let the functor

$$\mathrm{Der}_{\mathcal{R}-\mathrm{alg}}(-, X^{S^{t-1}}_*) = \mathrm{map}_{\mathcal{R}-\mathrm{alg}\downarrow X_*}(-, X^{S^{t-1}}_*)$$

denote the degree-preserving derivations that preserve Dyer–Lashof operations. Since $X^{S^{t-1}}_*$ is concentrated in degree $2p^2 - 4$ and below, this functor depends only on the input up to degree $2p^2 - 4$. Therefore, the morphism of simplicial sets above induces an isomorphism

$$\mathrm{Der}_{\mathcal{R}-\mathrm{alg}}(F^{\bullet+1}_{\mathcal{R}}(H\mathbb{F}_{p_*}H\mathbb{Z}), X^{S^{t-1}}_*) \cong \mathrm{Der}_{\mathcal{R}-\mathrm{alg}}(F^{\bullet+1}_{\mathcal{R}}(F_{\mathcal{R}}(\zeta_1, \overline{\tau}_1)), X^{S^{t-1}}_*).$$

In cohomology, this induces the isomorphism

$$\operatorname{Der}^t_{\mathcal{R}-\mathrm{alg}}(H\mathbb{F}_{p*}H\mathbb{Z}, X^{S^{t-1}}_*) \cong \operatorname{Der}^t_{\mathcal{R}-\mathrm{alg}}(F_{\mathcal{R}}(\zeta_1, \overline{\tau}_1), X^{S^{t-1}}_*) = 0 \quad \text{for } t > 0.$$

The last equality follows because André–Quillen cohomology of a free object is trivial above degree 0. □

6 Proof of Theorem 1.3

To prove Theorem 1.3, we need to show that two coconnective commutative $H\mathbb{Z}/(m)$–algebras that are weakly equivalent as commutative $\mathbb{S}$–algebras are also weakly equivalent as commutative $H\mathbb{Z}/(m)$–algebras for any $m \in \mathbb{Z}$ with $m \neq \pm 1$.

Since the category of commutative $H\mathbb{Z}/(m)$–algebras is the category of commutative $\mathbb{S}$–algebras under $H\mathbb{Z}/(m)$, for our purpose, it is natural to consider the homotopy class of commutative $\mathbb{S}$–algebra maps from $H\mathbb{Z}/(m)$ to a coconnective $H\mathbb{Z}/(m)$–algebra X. We omit the forgetful functor to commutative $\mathbb{S}$–algebras and denote this by $\pi_0 \operatorname{map}_{\mathbb{S}-\mathrm{cAlg}}(H\mathbb{Z}/(m), X)$. We show in Proposition 6.1 that there is a unique homotopy class of maps in $\operatorname{map}_{\mathbb{S}-\mathrm{cAlg}}(H\mathbb{Z}/(m), X)$. The proof of Theorem 1.3 is based on this fact.

Proof of Theorem 1.3 Let X and Y be coconnective commutative $H\mathbb{Z}/(m)$–algebras that are weakly equivalent as commutative $\mathbb{S}$–algebras. We assume X and Y are cofibrant and fibrant as commutative $H\mathbb{Z}/(m)$–algebras. Recall that cofibrations, fibrations and weak equivalences of commutative $H\mathbb{Z}/(m)$–algebras are precisely those of commutative $\mathbb{S}$–algebras. Therefore, X and Y are also fibrant as commutative $\mathbb{S}$–algebras. Furthermore, they are also cofibrant commutative $\mathbb{S}$–algebras because the initial map $\mathbb{S} \to U(X)$ factors as a composition of two commutative $\mathbb{S}$–algebra cofibrations as $\mathbb{S} \hookrightarrow H\mathbb{Z}/(m) \hookrightarrow U(X)$, where U denotes the forgetful functor to commutative $\mathbb{S}$–algebras.

Since X and Y are weakly equivalent as commutative $\mathbb{S}$–algebras and they are cofibrant and fibrant as commutative $\mathbb{S}$–algebras, there is a weak equivalence of commutative $\mathbb{S}$–algebras $\psi \colon U(X) \xrightarrow{\sim} U(Y)$. Let $\varphi_X \colon H\mathbb{Z}/(m) \to U(X)$ and $\varphi_Y \colon H\mathbb{Z}/(m) \to U(Y)$ denote the commutative $\mathbb{S}$–algebra maps that are the $H\mathbb{Z}/(m)$ structure maps of X and Y, respectively. Since ψ is only a commutative $\mathbb{S}$–algebra map, it may not preserve the $H\mathbb{Z}/(m)$ structure; ie $\psi \circ \varphi_X$ is not necessarily equal to φ_Y.

Let Y' be the commutative $H\mathbb{Z}/(m)$–algebra whose underlying commutative $\mathbb{S}$–algebra is $U(Y)$ and whose $H\mathbb{Z}/(m)$ structure map is $\psi \circ \varphi_X$. With this $H\mathbb{Z}/(m)$ structure of Y', the map ψ becomes a weak equivalence of commutative $H\mathbb{Z}/(m)$–algebras from X to Y'. Therefore, it is sufficient to show that Y' and Y are weakly equivalent as $H\mathbb{Z}/(m)$–algebras.

By Proposition 6.1, $\pi_0 \operatorname{map}_{\mathbb{S}-\mathrm{cAlg}}(H\mathbb{Z}/(m), Y) = \{*\}$. Therefore, φ_Y and $\psi \circ \varphi_X$, the structure maps of Y and Y' respectively, are homotopic. A homotopy between φ_Y and $\psi \circ \varphi_X$ is given by the following diagram:

(7)
$$
\begin{array}{c}
\xymatrix{
 & & & U(Y) \\
H\mathbb{Z}/(m) \ar[r]^{f} & U(Y)^I \ar[ur]^{p_1}_{\simeq} \ar[dr]_{p_2}^{\simeq} & \\
 & & & U(Y) = U(Y')
}
\end{array}
$$

Here, $U(Y)^I$ denotes a path object of $U(Y)$. This is a diagram in commutative $\mathbb{S}$–algebras. However, if we give $U(Y)^I$ a commutative $H\mathbb{Z}/(m)$–algebra structure using f and call this commutative $H\mathbb{Z}/(m)$–algebra Z, then p_1 becomes a weak equivalence of commutative $H\mathbb{Z}/(m)$–algebras from Z to Y, and p_2 becomes a weak equivalence of commutative $H\mathbb{Z}/(m)$–algebras from Z to Y'. Therefore, Y and Y' are weakly equivalent commutative $H\mathbb{Z}/(m)$–algebras and so are Y and X.   □

What is left to prove is the following proposition.

Proposition 6.1 *For a coconnective commutative $H\mathbb{Z}/(m)$–algebra X, the mapping space* $\operatorname{map}_{\mathbb{S}-\mathrm{cAlg}}(H\mathbb{Z}/(m), X)$ *is contractible.*

We use the obstruction spectral sequence to show that all the homotopy groups of this mapping space are trivial. However, since we work over a general $\mathbb{Z}/(m)$ where m may not be a prime, we do not have a description of the E_2 page of the spectral sequence as in Theorem 3.1. It turns out that it is sufficient to consider the E_1 page only. For this purpose, we use the spectral sequence in Theorem A of [10], which we can do by Corollary 4.13 of [10]. Because X is a commutative $H\mathbb{Z}/(m)$–algebra, there is a map of commutative $\mathbb{S}$–algebras $H\mathbb{Z}/(m) \to U(X)$ that serves as a basepoint for this spectral sequence.

In this setting, the E_1 page of this spectral sequence is given by

$$E_1^{s,t} = \pi_t \, \mathrm{map}_{\mathbb{S}-\mathrm{cAlg}}(\mathbb{P}_{\mathbb{S}}^{s+1}(H\mathbb{Z}/(m)), X),$$

where the homotopy groups are calculated at the given basepoint; see the proof of Theorem A of [10]. This spectral sequence abuts to

$$\pi_{t-s} \, \mathrm{map}_{\mathbb{S}-\mathrm{cAlg}}(H\mathbb{Z}/(m), X).$$

The free commutative $\mathbb{S}$–algebra functor induces a monad in the homotopy category of $\mathbb{S}$–modules and let $h\mathbb{P}_{\mathbb{S}}$ denote this monad. We use that the $E_2^{0,0}$ term of this spectral sequence is given by

$$E_2^{0,0} = \mathrm{Hom}_{h\mathbb{P}_{\mathbb{S}}-\mathrm{alg}}(H\mathbb{Z}/(m), X),$$

which denotes morphisms of algebras over $h\mathbb{P}_{\mathbb{S}}$ in $\mathbb{S}$–modules. Note that $E_2^{0,0}$ term of this spectral sequence is just a set.

Proof We will show that in the above spectral sequence, $E_1^{s,t} = 0$ for $t > 0$ and $E_2^{0,0} = \mathrm{pt}$. This is sufficient to show that the homotopy groups of the mapping space are trivial.

We start by showing that $E_2^{0,0} = 0$. We have the following isomorphisms for $E_1^{0,0}$ that we explain below:

$$\begin{aligned}
E_1^{0,0} &= \pi_0 \, \mathrm{map}_{\mathbb{S}-\mathrm{cAlg}}(\mathbb{P}_{\mathbb{S}}(H\mathbb{Z}/(m)), X) \\
&\cong \pi_0 \, \mathrm{map}_{\mathbb{S}-\mathrm{mod}}(H\mathbb{Z}/(m), X) \\
&\cong \mathrm{Hom}_{\mathbb{Z}-\mathrm{mod}}(H\mathbb{Z}_* H\mathbb{Z}/(m), X_*) \\
&\cong \mathrm{Hom}_{\mathbb{Z}-\mathrm{mod}}(\mathbb{Z}/(m), X_0).
\end{aligned}$$

The first isomorphism follows by adjunction. For the second isomorphism, we use the universal coefficient spectral sequence of Theorem 4.5 in Chapter IV of [8] with respect to the homology theory $H\mathbb{Z}_*$. This works because X is an $H\mathbb{Z}$–module by forgetting the $H\mathbb{Z}/(m)$–module structure through the map $H\mathbb{Z} \to H\mathbb{Z}/(m)$. For this spectral sequence, we have

$$E_2^{p,q} = \mathrm{Ext}_{\mathbb{Z}}^{p,-q}(H\mathbb{Z}_* H\mathbb{Z}/(m), X_*), \quad d_{p,q}^r \colon E_2^{p,q} \to E_2^{p+r,q-r+1},$$

where p denotes the cohomological degree and $-q$ denotes the internal degree; particularly, it denotes the Ext groups calculated by considering the maps that increase the

degree by $-q$. This spectral sequence abuts to

$$\pi_{-(p+q)} \, \mathrm{maps}_{\mathbb{S}-\mathrm{mod}}(H\mathbb{Z}/(m), X).$$

Since $\mathbb{Z}$ has global dimension one, $E_2^{p,q} = 0$ for $p > 1$ and therefore all differentials are zero and the only terms that contribute to π_0 are $E_2^{0,0}$ and $E_2^{1,-1}$. Note that by Proposition 1.2 in Chapter IV of [8], $H\mathbb{Z}_* H\mathbb{Z}/(m)$ is connective. Therefore, there is a projective resolution of $H\mathbb{Z}_* H\mathbb{Z}/(m)$ that is connective in each resolution degree. From such a resolution, there are no maps of degree 1 to the coconnective object X_*. Hence, $E_2^{1,-1} = 0$. This proves the second isomorphism. By the Tor spectral sequence of Theorem 4.1 in Chapter IV of [8], $H\mathbb{Z}_0 H\mathbb{Z}/(m) \cong \mathbb{Z}/(m)$. The third isomorphism follows from this because X is coconnective.

By the description of the $E_2^{0,0}$ above, a morphism in $E_2^{0,0}$ that lifts to $E_2^{0,0}$ should preserve the multiplicative identity. Since $\mathbb{Z}/(m)$ is generated as an abelian group by 1, there is only one such morphism in $E_1^{0,0} \cong \mathrm{Hom}_{\mathbb{Z}-\mathrm{mod}}(\mathbb{Z}/(m), X_0)$. Furthermore, we know that this morphism lifts to the $E_2^{0,0}$ term because it is represented by a morphism which is an actual commutative $\mathbb{S}$–algebra map $H\mathbb{Z}/(m) \to X$ which is our basepoint. In conclusion, $E_2^{0,0} = \mathrm{pt}$.

Now we will show that $E_1^{s,t} = 0$ for $t > 0$. Again by adjunction, we have

$$E_1^{s,t} = \pi_t \, \mathrm{maps}_{\mathbb{S}-\mathrm{cAlg}}(\mathbb{P}_{\mathbb{S}}^{s+1}(H\mathbb{Z}/(m)), X)$$
$$\cong \pi_t \, \mathrm{maps}_{\mathbb{S}-\mathrm{mod}}(\mathbb{P}_{\mathbb{S}}^s(H\mathbb{Z}/(m)), X).$$

There is a spectral sequence for calculating homotopy groups of the homotopy orbit of a spectrum with an action of a group G:

$$H_p(G, \pi_q Y) \implies \pi_{p+q} Y_{hG}.$$

Using this spectral sequence, it is clear that the homotopy orbit spectrum of a connective spectrum is connective. Therefore, $\mathbb{P}_{\mathbb{S}}(Y)$ is connective when Y is because $\mathbb{P}_{\mathbb{S}}(Y)$ is wedges of homotopy orbits of $Y^{\wedge n}$ with respect to the action of the symmetric group. This means that $\mathbb{P}_{\mathbb{S}}^s(H\mathbb{Z}/(m))$ is connective. Since X is coconnective by hypothesis, we have

$$\pi_t \, \mathrm{maps}_{\mathbb{S}-\mathrm{mod}}(\mathbb{P}_{\mathbb{S}}^s(H\mathbb{Z}/(m)), X) \cong 0 \quad \text{for } t > 0$$

by Proposition 1.4 in Chapter IV of [8].   □

7 Proof of Theorem 1.6

In this section, we prove the following theorem.

Theorem 7.1 Let X and Y be H_∞ $H\mathbb{F}_p$–algebras with trivial first homotopy groups. If X and Y are equivalent as H_∞ $\mathbb{S}$–algebras, then they are equivalent as H_∞ $H\mathbb{F}_p$–algebras.

This is a slightly stronger result than Theorem 1.6. If two E_∞ DGAs are E_∞ topologically equivalent, then the corresponding ring spectra are commutative $\mathbb{S}$–algebra equivalent and therefore H_∞ $\mathbb{S}$–algebra equivalent. Therefore, Theorem 1.6 is a corollary of Theorem 7.1.

Remark 7.2 As mentioned in Remark 1.7, one of the intermediate results of [11], Proposition 5, states that Theorem 7.1 is still true for H_∞ $H\mathbb{F}_p$–algebras with nontrivial first homology and this contradicts Example 5.1. The proof of Proposition 5 of [11] ends by stating that the canonical map

$$\text{(8)} \qquad\qquad [\mathbb{P}_{H\mathbb{F}_p}(M), M]_{H\mathbb{F}_p-\text{mod}} \to [\mathbb{P}_{\mathbb{S}}(M), M]_{\mathbb{S}-\text{mod}}$$

between homotopy classes of maps in $H\mathbb{F}_p$–modules to $\mathbb{S}$–modules is injective, where M is an $H\mathbb{F}_p$–module. This says that H_∞ $H\mathbb{F}_p$–algebra structure maps forget injectively to H_∞ $\mathbb{S}$–algebra structure maps. However, this does not imply the desired result since one needs to consider H_∞ $H\mathbb{F}_p$–equivalences and H_∞ $\mathbb{S}$–equivalences between different H_∞ $H\mathbb{F}_p$–algebra and H_∞ $\mathbb{S}$–algebra structures on M.

In the proof of Theorem 7.1, we use the following facts about H_∞ algebras which can be derived using the results of [5]. In the items below, X denotes an H_∞ $H\mathbb{F}_p$–algebra.

(1) A morphism of H_∞ $\mathbb{S}$–algebras induces a map of rings in the homotopy groups.

(2) The structure map $\mu_X\colon H\mathbb{F}_p \wedge X \to X$ induced by the $H\mathbb{F}_p$–module structure on X is a map of H_∞ $H\mathbb{F}_p$–algebras. Therefore, this map preserves Dyer–Lashof operations on the homotopy ring.

(3) There is an equivalence $H\mathbb{F}_p \wedge X \cong (H\mathbb{F}_p \wedge H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} X$. Using this, we obtain the identification $\pi_*(H\mathbb{F}_p \wedge X) \cong \mathcal{A}_* \otimes_{\mathbb{F}_p} X_*$. Note that the Dyer–Lashof operations on $\mathcal{A}_* \otimes_{\mathbb{F}_p} X_*$ are not those of the tensor product because the $H\mathbb{F}_p$ structure on $H\mathbb{F}_p \wedge X$ is given by multiplication with the $H\mathbb{F}_p$ factor on the left. With this identification, μ_{X*} is given by $\mu_{X*}(a \otimes x) = ax$ if $a \in \mathcal{A}_0 = \mathbb{F}_p$ and $\mu_{X*}(a \otimes x) = 0$ if $a \in \mathcal{A}_i$ for $i > 0$.

(4) The unit map $\eta_X \colon \mathbb{S} \wedge X \to H\mathbb{F}_p \wedge X$ satisfies $\mu_X \circ \eta_X = \mathrm{id}$. However, η_X is only a map of H_∞ $\mathbb{S}$–algebras and it may not preserve Dyer–Lashof operations in the homotopy ring. By the identification of $\pi_*(H\mathbb{F}_p \wedge X)$ above, the morphism induced by η_X on the homotopy ring is given by $\eta_{X*}(x) = 1 \otimes x$.

Proof of Theorem 7.1 Let $\varphi \colon X \to Y$ be an equivalence of H_∞ $\mathbb{S}$–algebras. This implies that φ_* is an isomorphism of rings. We will show that φ induces an equivalence of H_∞ $H\mathbb{F}_p$–algebras by showing that φ_* preserves Dyer–Lashof operations. This is sufficient because an H_∞ $H\mathbb{F}_p$–algebra equivalence type is determined by the isomorphism class of its homotopy ring as an algebra over the Dyer–Lashof algebra; see Theorem 4 in [11] and the discussion after it.

We have the following diagram:

(9)
$$
\begin{array}{ccc}
X & \xrightarrow{\ \varphi\ } & Y \\
\downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} \\
H\mathbb{F}_p \wedge X & \xrightarrow{\ \mathrm{id} \wedge \varphi\ } & H\mathbb{F}_p \wedge Y \\
\downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle \mu_Y} \\
X & & Y
\end{array}
$$

Applying the homotopy functor to this diagram produces the following:

(10)
$$
\begin{array}{ccc}
X_* & \xrightarrow{\ \varphi_*\ } & Y_* \\
\downarrow{\scriptstyle \eta_{X*}} & & \downarrow{\scriptstyle \eta_{Y*}} \\
\mathcal{A}_* \otimes_{\mathbb{F}_p} X_* & \xrightarrow{\ \psi\ } & \mathcal{A}_* \otimes_{\mathbb{F}_p} Y_* \\
\downarrow{\scriptstyle \mu_{X*}} & & \downarrow{\scriptstyle \mu_{Y*}} \\
X_* & \xrightarrow{\ \varphi_*\ } & Y_*
\end{array}
$$

The middle horizontal morphism ψ is the morphism induced on the homotopy groups by $\mathrm{id} \wedge \varphi$. Because we do not assume φ to be a map of H_∞ $H\mathbb{F}_p$–algebras, ψ may not be induced by two morphisms on the tensor factors. However, ψ preserves Dyer–Lashof operations because it is the morphism in $H\mathbb{F}_p$ homology induced by φ.

The top square in diagram (10) commutes because it is induced by the commutative square in diagram (9). Although the bottom square is not induced by a commuting square, we show that it also commutes. For this purpose, we need to know more about the Dyer–Lashof operations on $\mathcal{A}_* \otimes_{\mathbb{F}_p} X_*$.

We have the map

$$ H\mathbb{F}_p \wedge H\mathbb{F}_p \cong (H\mathbb{F}_p \wedge H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} H\mathbb{F}_p \to (H\mathbb{F}_p \wedge H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} X \cong H\mathbb{F}_p \wedge X $$

induced by the map of H_∞ $H\mathbb{F}_p$–algebras $H\mathbb{F}_p \to X$. This is a map of H_∞ $H\mathbb{F}_p$–algebras when the $H\mathbb{F}_p$ multiplication on $H\mathbb{F}_p \wedge H\mathbb{F}_p$ is given by that of the $H\mathbb{F}_p$ factor on the left. Therefore, the morphism $\mathcal{A}_* \to \mathcal{A}_* \otimes_{\mathbb{F}_p} X_*$ induced on the homotopy groups preserves Dyer–Lashof operations. This says that on $\mathcal{A}_* \otimes_{\mathbb{F}_p} \{1\} \subseteq \mathcal{A}_* \otimes_{\mathbb{F}_p} X_*$, Dyer–Lashof operations are given by the ones on the dual Steenrod algebra, that is, $Q^s(a \otimes 1) = (Q^s a) \otimes 1$.

Now we show that the bottom square in diagram (10) commutes. We first show this for elements of the form $a \otimes x \in \mathcal{A}_* \otimes_{\mathbb{F}_p} X_*$ with $|a| > 0$. By the description of μ_{X_*} in the paragraph before this proof, we have $\mu_{X_*}(a \otimes x) = 0$ and therefore $\varphi_* \circ \mu_{X_*}(a \otimes x) = 0$. Therefore, our goal is to show that $\mu_{Y_*} \circ \psi(a \otimes x) = 0$. Let τ_0 denote the degree-1 element in $\mathcal{A}_*$ that generates it as an algebra over the Dyer–Lashof algebra (this element is called ξ_1 for $p = 2$). Because $\pi_1(Y) = 0$, we have $\mu_{Y_*} \circ \psi(\tau_0 \otimes 1) = 0$. Since $\mu_{Y_*} \circ \psi$ is a morphism of rings that preserves Dyer–Lashof operations, $\mu_{Y_*} \circ \psi(a \otimes 1) = 0$ whenever $|a| > 0$. Therefore, when $|a| > 0$,

$$\mu_{Y_*} \circ \psi(a \otimes x) = (\mu_{Y_*} \circ \psi(a \otimes 1)) \cdot (\mu_{Y_*} \circ \psi(1 \otimes x)) = 0.$$

After this, we just need to show that the bottom square in diagram (10) commutes for elements in $\mathcal{A}_* \otimes_{\mathbb{F}_p} X_*$ of the form $a \otimes x$, where $a \in \mathcal{A}_0 = \mathbb{F}_p$. Clearly, it is sufficient to work only with the elements of the form $1 \otimes x$. By the description of μ_{X_*} in the paragraph before this proof, we have $\varphi_* \circ \mu_{X_*}(1 \otimes x) = \varphi_*(x)$. Therefore, our goal is to show that $\mu_{Y_*} \circ \psi(1 \otimes x) = \varphi_*(x)$. Because the top square in diagram (10) commutes, we deduce that

$$(11) \qquad \psi(1 \otimes x) = \psi(\eta_{X_*}(x)) = \eta_{Y_*}(\varphi_*(x)) = 1 \otimes \varphi_*(x).$$

Using this, we obtain what we wanted to show:

$$\mu_{Y_*} \circ \psi(1 \otimes x) = \mu_{Y_*}(1 \otimes \varphi_*(x)) = \varphi_*(x).$$

At this point, we know that the bottom square in diagram (10) commutes and we are ready to show that φ_* preserves Dyer–Lashof operations. Given $x \in X_*$, we have

$$\varphi_*(Q^s x) = \varphi_*(Q^s \mu_{X_*}(1 \otimes x)) = \varphi_* \circ \mu_{X_*}(Q^s(1 \otimes x)) = \mu_{Y_*} \circ \psi(Q^s(1 \otimes x)).$$

Therefore, we need to show that $\mu_{Y_*} \circ \psi(Q^s(1 \otimes x)) = Q^s \varphi_*(x)$. This is given by the following chain of equalities

$$\mu_{Y_*} \circ \psi(Q^s(1 \otimes x)) = Q^s \mu_{Y_*}(\psi(1 \otimes x)) = Q^s \mu_{Y_*}(1 \otimes \varphi_*(x)) = Q^s \varphi_*(x).$$

The first equality follows because both ψ and μ_{Y*} preserve Dyer–Lashof operations and the second equality follows by (11). Since φ_* preserves Dyer–Lashof operations, it induces an isomorphism between X_* and Y_* as algebras over the Dyer–Lashof algebra and therefore X and Y are equivalent as $H_\infty \ H\mathbb{F}_p$–algebras. $\square$

Appendix

In Section 2, we discuss the first class of examples of nontrivial topological equivalences provided in [7]. These examples rely on the classification of Postnikov extensions of ring spectra developed in [6]. In this appendix, we point out a mistake in the construction of these examples and provide a correction which recovers the classification of quasi-isomorphism classes of $\mathbb{Z}$–DGAs with homology ring $\Lambda_{\mathbb{F}_p}(x_n)$ for $n > 0$, ie the exterior algebra with a single generator in a positive degree.

Let R be a connective commutative ring spectrum. We first explain the classification of Postnikov extensions of connective (trivial negative homotopy groups) R–algebras developed in [6]. For a connective R–algebra X, the nth Postnikov section of X is a map of R–algebras $X \to P_n X$ which induces an isomorphism on $\pi_i(X) \to \pi_i(P_n X)$ for $i \leq n$ and with $\pi_i(P_n X) = 0$ for $i > n$. Given a connective R–algebra Y with $P_{n-1}Y \simeq Y$ and a $\pi_0(Y)$–bimodule M, a Postnikov extension of Y of type (M, n) is a map of R–algebras $X \to Y$ which satisfies the following properties:

(1) $\pi_i(X) = 0$ for $i > n$.

(2) $\pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i < n$.

(3)   There is an isomorphism of $\pi_0(X)$–bimodules $\pi_n(X) \cong M$, where $\pi_0(X)$–bimodule structure of M is obtained by the map $\pi_0(X) \to \pi_0(Y)$.

The moduli space of Postnikov extensions of Y of type (M, n), which we denote by $\mathcal{M}_R(Y + M, n)$, is defined to be the category whose objects are Postnikov extensions of Y of type (M, n), and a morphism between two extensions $X_1 \to Y$ to $X_2 \to Y$ is a weak equivalence $X_1 \xrightarrow{\simeq} X_2$ for which the following triangle commutes:

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\ \simeq\ } & X_2 \\
 & \searrow \quad \swarrow & \\
 & Y &
\end{array}
$$

The main result of [6] is a classification of these Postnikov extensions in terms of topological Hochschild cohomology.

Theorem A.1 [6] *Assuming X is cofibrant as an R–module, we have a bijection*

$$\pi_0 \mathcal{M}_R(X + M, n) \cong \mathrm{THH}_R^{n+2}(X, M)/\mathrm{Aut}(M).$$

This result is used in [7] to classify weak equivalence classes of $\mathbb{Z}$–DGAs with homology ring $\Lambda_{\mathbb{F}_p}(x_n)$, the exterior algebra over $\mathbb{F}_p$ with a single generator in degree n for $n > 0$. Any such DGA is a Postnikov extension of $\mathbb{F}_p$ of type $(\mathbb{F}_p, n)$. By the Quillen equivalence of $\mathbb{Z}$–DGAs and $H\mathbb{Z}$–algebras, this is the same as classifying $H\mathbb{Z}$–algebras with homotopy ring $\Lambda_{\mathbb{F}_p}(x_n)$, and such $H\mathbb{Z}$–algebras are Postnikov extensions of $H\mathbb{F}_p$ of type $(H\mathbb{F}_p, n)$ in $H\mathbb{Z}$–algebras.

At this point, we note the piece of explanation that is missing in [7] about this classification. In [7], it is claimed that weak equivalence classes of $H\mathbb{Z}$–algebras whose homotopy ring are $\Lambda_{\mathbb{F}_p}(x_n)$ are classified by $\pi_0 \mathcal{M}_{H\mathbb{Z}}(H\mathbb{F}_p + H\mathbb{F}_p, n)$. However in $\mathcal{M}_{H\mathbb{Z}}(H\mathbb{F}_p + H\mathbb{F}_p, n)$, the morphisms are weak equivalences of Postnikov extensions; ie for two Postnikov extensions of $H\mathbb{F}_p$ of type $(H\mathbb{F}_p, n)$, namely $X_1 \to H\mathbb{F}_p$ and $X_2 \to H\mathbb{F}_p$, a morphism in $\mathcal{M}_{H\mathbb{Z}}(H\mathbb{F}_p + H\mathbb{F}_p, n)$ is a weak equivalence of $H\mathbb{Z}$–algebras $X_1 \xrightarrow{\sim} X_2$ for which the triangle above commutes. In the classification we are concerned with here, a morphism of two Postnikov extensions is just a weak equivalence of $H\mathbb{Z}$–algebras $X_1 \xrightarrow{\sim} X_2$. In general, one should not expect these two classifications to be the same. However, in this case we will show that they are actually the same. Unfortunately we cannot use simple point set arguments to prove this, even if we work with $\mathbb{Z}$–DGAs instead of $H\mathbb{Z}$–algebras, because one needs to use a cofibrant and fibrant $H\mathbb{Z}$–algebra model of $H\mathbb{F}_p$, and also because we need the same result over $\mathbb{S}$–algebras, not only for $H\mathbb{Z}$–algebras.

We prove that these two classifications are the same by first showing that there is a unique homotopy class of $H\mathbb{Z}$–algebra maps from X to $H\mathbb{F}_p$. We prove this in Lemma A.3 by using the obstruction theory of the Hopkins–Miller theorem. Using this fact, we prove that $\pi_0 \mathcal{M}_{H\mathbb{Z}}(H\mathbb{F}_p + H\mathbb{F}_p, n)$ actually classifies weak equivalence classes of $H\mathbb{Z}$–algebras with homotopy ring $\Lambda_{\mathbb{F}_p}(x_n)$.

Proposition A.2 *The set $\pi_0 \mathcal{M}_{H\mathbb{Z}}(H\mathbb{F}_p + H\mathbb{F}_p, n)$ is in bijective correspondence with weak equivalence classes of $H\mathbb{Z}$–algebras with homotopy ring $\Lambda_{\mathbb{F}_p}(x_n)$ for $n > 0$. This statement holds for $\mathbb{S}$–algebras too. Namely, the set $\pi_0 \mathcal{M}_{\mathbb{S}}(H\mathbb{F}_p + H\mathbb{F}_p, n)$ is in bijective correspondence with weak equivalence classes of $\mathbb{S}$–algebras with homotopy ring $\Lambda_{\mathbb{F}_p}(x_n)$ for $n > 0$.*

Proof We will only prove the statement for $H\mathbb{Z}$–algebras. The proof is similar for $\mathbb{S}$–algebras; the only important difference is that one uses the second part of Lemma A.3 instead of the first part.

Since up to weak equivalence there is a unique $H\mathbb{Z}$–algebra with homotopy $\mathbb{F}_p$ concentrated at degree 0, every $H\mathbb{Z}$–algebra with homotopy ring $\Lambda_{\mathbb{F}_p}(x_n)$ is a Postnikov extension of $H\mathbb{F}_p$ of type $(H\mathbb{F}_p, n)$. Given two such Postnikov extensions $\varphi_1\colon X_1 \to H\mathbb{F}_p$ and $\varphi_2\colon X_2 \to H\mathbb{F}_p$, if these extensions are weakly equivalent in $\mathcal{M}_{H\mathbb{Z}}(H\mathbb{F}_p + H\mathbb{F}_p, n)$ then they are clearly weakly equivalent as $H\mathbb{Z}$–algebras. We need to show that when these Postnikov extensions are weakly equivalent as $H\mathbb{Z}$–algebras, they are also weakly equivalent in $\mathcal{M}_{H\mathbb{Z}}(H\mathbb{F}_p + H\mathbb{F}_p, n)$.

Let X_1 and X_2 be weakly equivalent as $H\mathbb{Z}$–algebras, we show that φ_1 and φ_2 are weakly equivalent in $\mathcal{M}_{H\mathbb{Z}}(H\mathbb{F}_p + H\mathbb{F}_p, n)$. We assume that X_1 and X_2 are both fibrant and cofibrant and $H\mathbb{F}_p$ is fibrant as $H\mathbb{Z}$–algebras.

Because X_1 and X_2 are weakly equivalent as $H\mathbb{Z}$–algebras and because X_1 is cofibrant and X_2 is fibrant, there is a weak equivalence of $H\mathbb{Z}$–algebras $\psi\colon X_1 \xrightarrow{\simeq} X_2$. Using this weak equivalence, we define another Postnikov extension of $H\mathbb{F}_p$ of type $(H\mathbb{F}_p, n)$ which is the composite $\varphi_2 \circ \psi\colon X_1 \to H\mathbb{F}_p$. This Postnikov extension, $\varphi_2 \circ \psi$, is weakly equivalent to φ_2 in $\mathcal{M}_{H\mathbb{Z}}(H\mathbb{F}_p + H\mathbb{F}_p, n)$ through ψ. Therefore, it is sufficient to show that $\varphi_2 \circ \psi$ is weakly equivalent to φ_1 in $\mathcal{M}_{H\mathbb{Z}}(H\mathbb{F}_p + H\mathbb{F}_p, n)$. By Lemma A.3, there is a unique homotopy class of maps from X_1 to $H\mathbb{F}_p$. Therefore, φ_1 and $\varphi_2 \circ \psi$ are homotopic. We have the following diagram in $H\mathbb{Z}$–algebras which corresponds to a homotopy between these maps, where $X_1 \wedge I$ is a path object of X_1:
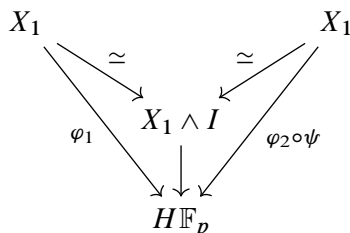
The map $X_1 \wedge I \to H\mathbb{F}_p$ in the above diagram is also a Postnikov extension of type $(H\mathbb{F}_p, n)$ of $H\mathbb{F}_p$ because it factors φ_1 by a weak equivalence. Therefore, the above diagram gives a zig-zag of weak equivalences between φ_1 and $\varphi_2 \circ \psi$ in $\mathcal{M}_{H\mathbb{Z}}(H\mathbb{F}_p + H\mathbb{F}_p, n)$. This shows that φ_1 and φ_2 are weakly equivalent in $\mathcal{M}_{H\mathbb{Z}}(H\mathbb{F}_p + H\mathbb{F}_p, n)$. $\square$

At this point, we are ready to provide the classification of weak equivalence classes of $H\mathbb{Z}$–algebras with homotopy ring $\Lambda_{\mathbb{F}_p}(x_n)$ for $n > 0$ and hence quasi-isomorphism classes of $\mathbb{Z}$–DGAs with homology ring $\Lambda_{\mathbb{F}_p}(x_n)$ for $n > 0$. By Proposition A.2 and Theorem A.1, this is given by $\mathrm{THH}_{H\mathbb{Z}}^{n+2}(H\mathbb{F}_p, H\mathbb{F}_p)/\mathrm{Aut}(\mathbb{F}_p)$. As described in Example 3.15 of [7], $\mathrm{THH}_{H\mathbb{Z}}^*(H\mathbb{F}_p, H\mathbb{F}_p) \cong \mathbb{F}_p[\sigma_2]$, a polynomial algebra with a generator in degree 2 (with cohomological grading). Calculating the quotient of $\mathbb{F}_p[\sigma_2]$ by the multiplicative action of $\mathbb{F}_p$, one obtains the following classification: for odd $n > 0$, there is a unique $H\mathbb{Z}$–algebra with homotopy ring $\Lambda_{\mathbb{F}_p}(x_n)$, and for even $n > 0$, there are exactly two $H\mathbb{Z}$–algebras that are not weakly equivalent whose homotopy ring is $\Lambda_{\mathbb{F}_p}(x_n)$. For $n = 0$, this classification says that there are two Postnikov extensions of $H\mathbb{F}_p$ of type $(H\mathbb{F}_p, 0)$ and these are $H\mathbb{Z}/p^2$ and $H\Lambda_{\mathbb{F}_p}(x_0)$.

Similarly, weak equivalence classes of $\mathbb{S}$–algebras with homotopy ring $\Lambda_{\mathbb{F}_p}(x_n)$ are given by $\mathrm{THH}_{\mathbb{S}}^{n+2}(H\mathbb{F}_p, H\mathbb{F}_p)/\mathrm{Aut}(\mathbb{F}_p)$. In this case, $\mathrm{THH}_{\mathbb{S}}^*(H\mathbb{F}_p, H\mathbb{F}_p) \cong \Gamma[\alpha_2]$ as rings, where $\Gamma[\alpha_2]$ is the divided polynomial algebra on a generator of degree 2 which is isomorphic to $\mathbb{F}_p[\sigma_2]$ as an $\mathbb{F}_p$–module; see Theorem 13.4.15 in [12]. Therefore, we get a similar classification result: there are exactly two $\mathbb{S}$–algebras that are not weakly equivalent with homotopy ring $\Lambda_{\mathbb{F}_p}(x_n)$ for odd $n > 0$ and there is only one for even $n > 0$.

What we are really interested in here is deciding which of these non-weakly equivalent $H\mathbb{Z}$–algebras are weakly equivalent as $\mathbb{S}$–algebras. For this, one considers the map

$$\mathrm{THH}_{H\mathbb{Z}}^{n+2}(H\mathbb{F}_p, H\mathbb{F}_p) \to \mathrm{THH}_{\mathbb{S}}^{n+2}(H\mathbb{F}_p, H\mathbb{F}_p)$$

induced by the forgetful functor from $H\mathbb{Z}$–algebras to $\mathbb{S}$–algebras. This corresponds to a morphism of rings $\varphi \colon \mathbb{F}_p[\sigma_2] \to \Gamma[\alpha_2]$, where φ maps σ_2 to α_2 because $H\mathbb{Z}/p^2$ and $H\Lambda_{\mathbb{F}_p}(x_0)$ are not weakly equivalent as $\mathbb{S}$–algebras. Since $\alpha_2^p = 0$ in $\Gamma[\alpha_2]$, we have $\varphi(\sigma_2^p) = 0$. This implies that the two non-weakly equivalent $H\mathbb{Z}$–algebras corresponding to σ_2^p and 0 are weakly equivalent as $\mathbb{S}$–algebras. These are the first example of nontrivial topological equivalences from [7]. That is, there are two $\mathbb{Z}$–DGAs that are not weakly equivalent with homology ring $\Lambda_{\mathbb{F}_p}(x_{2p-2})$ which are topologically equivalent.

To complete the proof of Proposition A.2, we need to prove the following lemma. Let X denote an $H\mathbb{Z}$–algebra or an $\mathbb{S}$–algebra with homotopy ring $\pi_*(X) \cong \Lambda_{\mathbb{F}_p}(x_n)$ for $|x_n| > 0$.

Lemma A.3 *If X is an $H\mathbb{Z}$–algebra as above, we have*

$$\pi_0 \operatorname{map}_{H\mathbb{Z}-\mathrm{alg}}(X, H\mathbb{F}_p) = \mathrm{pt}.$$

For an $\mathbb{S}$–algebra X as above, we have

$$\pi_0 \operatorname{map}_{\mathbb{S}-\mathrm{alg}}(X, H\mathbb{F}_p) = \mathrm{pt}.$$

Proof Hopkins–Miller obstruction theory states that obstructions to lifting a morphism in $E_2^{0,0} = \operatorname{Hom}_{\mathbb{F}_p-\mathrm{alg}}(\pi_* H\mathbb{F}_p \wedge_{H\mathbb{Z}} X, \mathbb{F}_p)$ to a map of $H\mathbb{Z}$–algebras lie in the André–Quillen cohomology for associative algebras,

$$E_2^{t,t-1} = \operatorname{Der}^t(\pi_* H\mathbb{F}_p \wedge_{H\mathbb{Z}} X, \Omega^{t-1}\mathbb{F}_p) \quad \text{for } t \geq 2,$$

and obstructions to homotopy uniqueness of the lift lie in

$$E_2^{t,t} = \operatorname{Der}^t(\pi_* H\mathbb{F}_p \wedge_{H\mathbb{Z}} X, \Omega^t \mathbb{F}_p) \quad \text{for } t \geq 1.$$

The desuspension functor Ω^s is defined by $\Omega^s M_* = M_{*+s}$ for a graded module M_*. By the Künneth spectral sequence, $\pi_0 H\mathbb{F}_p \wedge_{H\mathbb{Z}} X = \mathbb{F}_p$. Therefore, there is only a single map in $\operatorname{Hom}_{\mathbb{F}_p-\mathrm{alg}}(\pi_* H\mathbb{F}_p \wedge_{H\mathbb{Z}} X, \mathbb{F}_p)$ because these morphisms preserve the identity and the grading. To show that the obstructions to existence and uniqueness are zero, first note that by the Künneth spectral sequence, $\pi_* H\mathbb{F}_p \wedge_{H\mathbb{Z}} X$ is connected and $\Omega^t \mathbb{F}_p$ is in negative degrees for $t > 0$. Thus $\operatorname{Der}(F^{\bullet+1}(\pi_* H\mathbb{F}_p \wedge_{H\mathbb{Z}} X), \Omega^t \mathbb{F}_p) = 0$ for $t > 0$, where F denotes the free associative algebra functor. Therefore, the cohomology of this cosimplicial abelian group is also zero. This proves the desired result. The argument for the homotopy class of maps in $\mathbb{S}$–algebras is similar, the only difference is that one uses $\pi_* H\mathbb{F}_p \wedge_{\mathbb{S}} X$ instead of $\pi_* H\mathbb{F}_p \wedge_{H\mathbb{Z}} X$. $\square$

References

[1] **A Baker**, *Calculating with topological André–Quillen theory, I: Homotopical properties of universal derivations and free commutative S–algebras*, preprint (2012) arXiv

[2] **A Baker**, *Power operations and coactions in highly commutative homology theories*, Publ. Res. Inst. Math. Sci. 51 (2015) 237–272 MR

[3] **M Basterra**, *André–Quillen cohomology of commutative S–algebras*, J. Pure Appl. Algebra 144 (1999) 111–143 MR

[4] **A K Bousfield**, *Homotopy spectral sequences and obstructions*, Israel J. Math. 66 (1989) 54–104 MR

[5] **R R Bruner**, **J P May**, **J E McClure**, **M Steinberger**, H_∞ *ring spectra and their applications*, Lecture Notes in Math. 1176, Springer (1986) MR

[6] **D Dugger**, **B Shipley**, *Postnikov extensions of ring spectra*, Algebr. Geom. Topol. 6 (2006) 1785–1829 MR

[7] **D Dugger**, **B Shipley**, *Topological equivalences for differential graded algebras*, Adv. Math. 212 (2007) 37–61 MR

[8] **A D Elmendorf**, **I Kriz**, **M A Mandell**, **J P May**, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs 47, Amer. Math. Soc., Providence, RI (1997) MR

[9] **P G Goerss**, **M J Hopkins**, *Moduli spaces of commutative ring spectra*, from "Structured ring spectra" (A Baker, B Richter, editors), London Math. Soc. Lecture Note Ser. 315, Cambridge Univ. Press (2004) 151–200 MR

[10] **N Johnson**, **J Noel**, *Lifting homotopy T–algebra maps to strict maps*, Adv. Math. 264 (2014) 593–645 MR

[11] **T Lawson**, *A note on H_∞ structures*, Proc. Amer. Math. Soc. 143 (2015) 3177–3181 MR

[12] **J-L Loday**, *Cyclic homology*, 2nd edition, Grundl. Math. Wissen. 301, Springer (1998) MR Chapter 13 by the author in collaboration with Teimuraz Pirashvili

[13] **M A Mandell**, *Cochains and homotopy type*, Publ. Math. Inst. Hautes Études Sci. 103 (2006) 213–246 MR

[14] **M A Mandell**, **J P May**, **S Schwede**, **B Shipley**, *Model categories of diagram spectra*, Proc. London Math. Soc. 82 (2001) 441–512 MR

[15] **J P May**, *Homology operations on infinite loop spaces*, from "Algebraic topology" (R J Milgram, editor), Proc. Sympos. Pure Math. XXII, Amer. Math. Soc., Providence, RI (1971) 171–185 MR

[16] **J Milnor**, *The Steenrod algebra and its dual*, Ann. of Math. 67 (1958) 150–171 MR

[17] **J Noel**, $H_\infty \neq E_\infty$, from "An alpine expedition through algebraic topology" (C Ausoni, K Hess, B Johnson, W Lück, J Scherer, editors), Contemp. Math. 617, Amer. Math. Soc., Providence, RI (2014) 237–240 MR

[18] **J Noel**, *The T–algebra spectral sequence: comparisons and applications*, Algebr. Geom. Topol. 14 (2014) 3395–3417 MR

[19] **D Quillen**, *On the (co-) homology of commutative rings*, from "Applications of Categorical Algebra" (A Heller, editor), Proc. Sympos. Pure Math. XVII, Amer. Math. Soc., Providence, RI (1970) 65–87 MR

[20] **C Rezk**, *Notes on the Hopkins–Miller theorem*, from "Homotopy theory via algebraic geometry and group representations" (M Mahowald, S Priddy, editors), Contemp. Math. 220, Amer. Math. Soc., Providence, RI (1998) 313–366 MR

[21] **B Richter**, **B Shipley**, *An algebraic model for commutative $H\mathbb{Z}$–algebras*, Algebr. Geom. Topol. 17 (2017) 2013–2038 MR

[22] **A Robinson**, *Obstruction theory and the strict associativity of Morava K–theories*, from "Advances in homotopy theory" (S M Salamon, B Steer, W A Sutherland, editors), London Math. Soc. Lecture Note Ser. 139, Cambridge Univ. Press (1989) 143–152 MR

[23] **A Robinson**, *Gamma homology, Lie representations and E_∞ multiplications*, Invent. Math. 152 (2003) 331–348 MR

[24] **B Shipley**, *$H\mathbb{Z}$–algebra spectra are differential graded algebras*, Amer. J. Math. 129 (2007) 351–379 MR

Department of Mathematics, University of Illinois
Chicago, IL, United States

`hbayin2@uic.edu`