

A rank inequality for the annular Khovanov homology of 2-periodic links

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For a 2-periodic link \tilde{L} in the thickened annulus and its quotient link L , we exhibit a spectral sequence with

$$E^1 \cong \text{AKh}(\tilde{L}) \otimes_{\mathbb{F}} \mathbb{F}[\theta, \theta^{-1}] \rightrightarrows E^\infty \cong \text{AKh}(L) \otimes_{\mathbb{F}} \mathbb{F}[\theta, \theta^{-1}].$$

This spectral sequence splits along quantum and \mathfrak{sl}_2 weight-space gradings, proving a rank inequality $\text{rkAKh}^{j,k}(L) \leq \text{rkAKh}^{2j-k,k}(\tilde{L})$ for every pair of quantum and \mathfrak{sl}_2 weight-space gradings (j, k) . We also present a few decategorified consequences and discuss partial results toward a similar statement for the Khovanov homology of 2-periodic links, as well as some frameworks for obstructing 2-periodicity in links.

57M25, 57M27; 57M60

1 Introduction

The world of topological objects is rich with spaces exhibiting interesting symmetries. These symmetries manifest themselves in the algebraic invariants we use to study the objects and can be manipulated to provide insights into the object itself. As fundamental, ubiquitous objects in low-dimensional topology, knots and links are some of the most natural objects to have algebraic invariants associated to them.

The rise of homology-type invariants categorifying polynomial link invariants have led to the study of equivariant homology theories by Chbili [7], Seidel and Smith [23], Hendricks [11; 12], Politarczyk [18; 19], Borozik and Politarczyk [6], and many others, building and improving upon earlier work of Murasugi [16; 17], Yokota [24], and Przytycki [20] relating the polynomial invariants of periodic links with those of their quotient links.

A link \tilde{L} in S^3 is p -periodic if there is a p -periodic automorphism τ of S^3 with fixed set an unknot B restricting to a diffeomorphism on \tilde{L} ; the *quotient link* L is the quotient of \tilde{L} under the action of τ . Since \tilde{L} can naturally be viewed in a thickened annulus $A \times I$ by deleting the fixed set B from S^3 , Asaeda, Przytycki, and Sikora's

annular Khovanov homology [2] seems particularly suitable for probing periodic links. We refer to a link in the thickened annulus as an *annular link*.

In this article, we construct a Tate-like bicomplex (Section 4) to show rank inequalities for the annular Khovanov homology of a 2-periodic annular link \tilde{L} and that of its quotient link L . For any annular link L , let $\text{AKh}^{j,k}(L)$ denote the annular Khovanov homology of L at quantum grading level j and \mathfrak{sl}_2 weight-space grading level k .

Theorem 1 *Let \tilde{L} be a 2-periodic link with quotient link L . For each pair of integers (j, k) , there is a spectral sequence with*

$$E^1 \cong \text{AKh}^{2j-k,k}(\tilde{L}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}] \rightrightarrows E^\infty \cong \text{AKh}^{j,k}(L) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}].$$

A rank inequality immediately follows:

Corollary 2 *For \tilde{L} and L as above,*

$$\text{rk}_{\mathbb{F}_2} \text{AKh}^{j,k}(L) \leq \text{rk}_{\mathbb{F}_2} \text{AKh}^{2j-k,k}(\tilde{L}).$$

This result generalizes [8], where Cornish proves a rank inequality between the next-to-top \mathfrak{sl}_2 weight-space gradings in the annular Khovanov homologies of a 2-periodic link and that of its quotient by employing the Lipshitz–Treumann spectral sequence [14] with Auroux, Grigsby, and Wehrli’s identification [3] of the next-to-top winding grading with the Hochschild homology of a Khovanov–Seidel bimodule. He remarks that by using Beliakova, Putyra and Wehrli’s generalization [5] of Auroux, Grigsby, and Wehrli’s work, one should be able to prove similar rank inequalities in other \mathfrak{sl}_2 weight-space gradings, but that this requires checking the Lipshitz–Treumann algebraic conditions for larger dg algebras.

While our result does not require the links to be braid closures, annular Khovanov homology is often used to study braids, as the set of conjugacy classes of a braid group B_n embed into the set of isotopy classes of annular links. In particular, B_n can be viewed as the mapping class group of a disk with n punctures, and Corollary 2 suggests that the rank of annular Khovanov homology could provide a measure of complexity in the mapping class group.

Following the notation in Roberts [21] (page 418, under Definition 2.2; also see Section 7 in this article), the decategorification of Theorem 1 can be written as follows:

Corollary 3 For all j and k ,

$$\langle q_{k, \tilde{L}}(-1, q), q^{2j-k} \rangle \equiv \langle q_{k, L}(-1, q), q^j \rangle \pmod{2},$$

where $\langle f, g \rangle$ denotes the coefficient of g in f .

However, the spectral sequence hints that the grading $j_1 := j - k$ is actually more pertinent to annular links (see Grigsby, Licata and Wehrli [10; 9] for more evidence). With this in mind, one can write down the decategorification as a statement similar to Murasugi’s theorem on the Jones polynomial of periodic links [17]:

Corollary 4 $V_{\tilde{L}}(1, q, q^{-1}) \equiv [V_L(1, q, q^{-1})]^2 \pmod{2}.$

The Jones polynomial of a periodic link \tilde{L} is related to the decategorification of the annular Khovanov homology of its quotient link L in the following way:

Corollary 5 $V_{\tilde{L}}(1, q, 1) \equiv V_L(1, q^2, q^{-1}) \pmod{2}.$

Moreover, we conjecture that a similar spectral sequence also relates the Khovanov homology [13] of \tilde{L} with the *annular* Khovanov homology of L (see Section 8):

Conjecture 6 Let \tilde{L} be a 2-periodic link in S^3 with quotient link L . There is a spectral sequence with

$$E^1 \cong \text{Kh}(\tilde{L}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}] \rightrightarrows E^\infty \cong \text{AKh}(L) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}].$$

This implies the cascade of rank inequalities

$$\text{rk}_{\mathbb{F}_2} \text{AKh}(\tilde{L}) \geq \text{rk}_{\mathbb{F}_2} \text{Kh}(\tilde{L}) \geq \text{rk}_{\mathbb{F}_2} \text{AKh}(L) \geq \text{rk}_{\mathbb{F}_2} \text{Kh}(L),$$

where the first and third inequalities are due to Roberts’ spectral sequence [21] coming from the \mathfrak{sl}_2 weight-space filtration on $\text{CKh}(\mathcal{D}(\tilde{L}))$ and $\text{CKh}(\mathcal{D}(L))$.

Note that this conjecture predicts much more than a rank inequality for Khovanov homology. In many contexts, spectral sequences from the homology of an associated graded object like AKh to the homology of the filtered complex tends to dramatically decrease rank, so it is surprising to find so much evidence for the middle inequality $\text{rk}_{\mathbb{F}_2} \text{Kh}(\tilde{L}) \geq \text{rk}_{\mathbb{F}_2} \text{AKh}(L)$. In fact, Corollary 5 would be the decategorification of Conjecture 6. We can show that the conjecture and consequent rank inequalities hold for the following family of annular links:

Theorem 7 *If the annular braid closure $L = \widehat{\beta}$ has a diagram with at most 1 positive crossing, then the spectral sequence in Conjecture 6 exists, and the cascade of rank inequalities holds.*

Furthermore, it follows from Theorem 7 that the cascade of rank inequalities also holds for positive braid closures (and more):

Corollary 8 *If the annular braid closure $L = \widehat{\beta}$ has a diagram with at most 1 negative crossing, then the cascade of rank inequalities holds.*

We conclude by discussing frameworks for obstructing 2–periodicity in links which arise from our results.

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2 Algebraic preliminaries

We begin by discussing the necessary homological algebra. The main theorem follows from computing two related spectral sequences arising from a particular bicomplex (constructed in Section 4). Spectral sequences from bicomplexes are a special case of spectral sequences from filtered complexes, which we discuss first.

The spectral sequences will be computed explicitly by making use of a well-known *cancellation lemma* (Lemma 10). Due to a special property of our complexes (see Definition 9), the algebraic computations can be described visually using dots and arrows; we describe this at the end of Section 2.1.

In Section 2.5, we give a brief overview of the ideas in Borel equivariant cohomology which motivate the constructions in this article.

2.1 Filtered chain complexes

Let \mathcal{C}_\bullet be a chain complex of \mathbb{F}_2 vector spaces, with differential ∂ .

A decreasing \mathbb{Z} -filtration of \mathcal{C}_\bullet is a sequence of subcomplexes indexed by decreasing integers:

$$\cdots \subset F_n \mathcal{C}_\bullet \subset F_{n-1} \mathcal{C}_\bullet \subset F_{n-2} \mathcal{C}_\bullet \subset \cdots \subset F_m \mathcal{C}_\bullet \subset \cdots \subset \mathcal{C}_\bullet.$$

For the chains in \mathcal{C}_\bullet the filtration $F_* \mathcal{C}_\bullet$ provides a filtration grading: $x \in \mathcal{C}_\bullet$ has filtration grading p if and only if $x \in F_p \mathcal{C}_\bullet$ and $x \notin F_{p+1} \mathcal{C}_\bullet$. The associated graded vector space $G_* \mathcal{C}_\bullet$ is given by

$$G_p \mathcal{C}_\bullet = F_p \mathcal{C}_\bullet / F_{p+1} \mathcal{C}_\bullet.$$

The existence of the filtration on \mathcal{C}_\bullet implies that every component of ∂ originating from $x \in \mathcal{C}_\bullet$ maps to targets y where the filtration grading of y is at least that of x ; we say that ∂ is nonnegative with respect to the filtration grading.

Let \mathcal{H}_\bullet denote the homology of the chain complex \mathcal{C}_\bullet . The filtration $F_* \mathcal{C}_\bullet$ induces a filtration $F_* \mathcal{H}_\bullet$ on \mathcal{H}_\bullet : the class $[x] \in \mathcal{H}_\bullet$ is in $F_p \mathcal{H}_\bullet$ if and only if it is represented by some chain $x \in [x]$ where $x \in F_p \mathcal{C}_\bullet$. The induced filtration gives rise to the associated graded vector space for homology, $G_* \mathcal{H}_\bullet$.

Definition 9 In our work, the underlying vector space of \mathcal{C}_\bullet is always freely generated by a finite collection of distinguished generators which come equipped with some \mathbb{Z} -grading gr , and the differential is nonnegative with respect to gr . In this case, gr induces a \mathbb{Z} -filtration and takes on the role of the filtration grading. We call the distinguished generators a *filtered graded basis*. A component ∂_s of the differential shifts the filtration grading by $s \in \mathbb{Z}$ or is of degree s with respect to the filtration grading if $\partial_s(x)$ is a linear combination of basis elements in grading $\text{gr}(x) + s$ for all filtered graded basis elements x .

Similarly, a bigraded underlying vector space induces a $\mathbb{Z} \oplus \mathbb{Z}$ -filtration if the differential is monotone with respect to each grading.

Let (\mathcal{C}, ∂) be such a complex. We can visualize (\mathcal{C}, ∂) by using dots to represent the distinguished generators and arrows to represent components of the differential, as shown in Figure 1. We write $\partial(x, y)$ for the coefficient of y in $\partial(x)$, so that $\partial(x)$ is the sum of $\partial(x, y) \cdot y$ as y runs over all the distinguished generators of \mathcal{C} . If $\partial(x, y) = 1$, we draw an arrow from x to y .

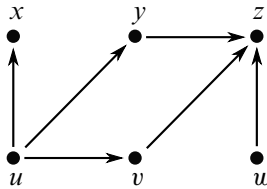


Figure 1

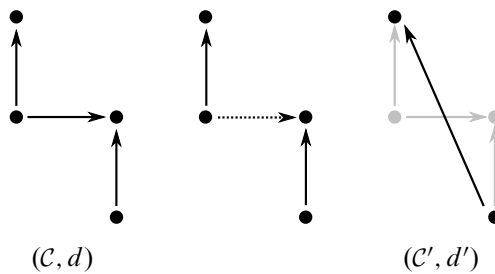


Figure 2: The cancellation process. On the left is the original complex. In the middle, the dotted arrow represents the arrow about to be canceled. On the right is the resulting complex, with fewer generators and arrows, but with the same homology as the original complex.

2.2 Spectral sequences from filtered complexes

Spectral sequences are used to gradually approximate the true homology of a complex by gradually simplifying the complex, whilst preserving the homology. We execute this by repeatedly applying the *cancellation* operation (Figure 2), which relies on the following well-known “cancellation lemma”.

Lemma 10 [4, Lemma 4.1] *Let (C, ∂) be a complex of \mathbb{F}_2 vector spaces freely generated by elements x_i . Let $\partial(x_i, x_j)$ be the coefficient of x_j in $\partial(x_i)$. If $\partial(x_k, x_l) = 1$, define a new complex (C', ∂') with generators $\{x_i \mid i \neq k, l\}$ and differential*

$$\partial'(x_i) = \partial(x_i) + \partial(x_i, x_l)\partial(x_k).$$

Then (C', ∂') is chain homotopy equivalent to (C, ∂) .

Given a decreasing \mathbb{Z} -filtered complex F_*C_\bullet with filtered differential ∂ , we can build a spectral sequence to compute the homology by starting with the associated graded complex and sequentially canceling components of the differential of larger and larger filtration degree.

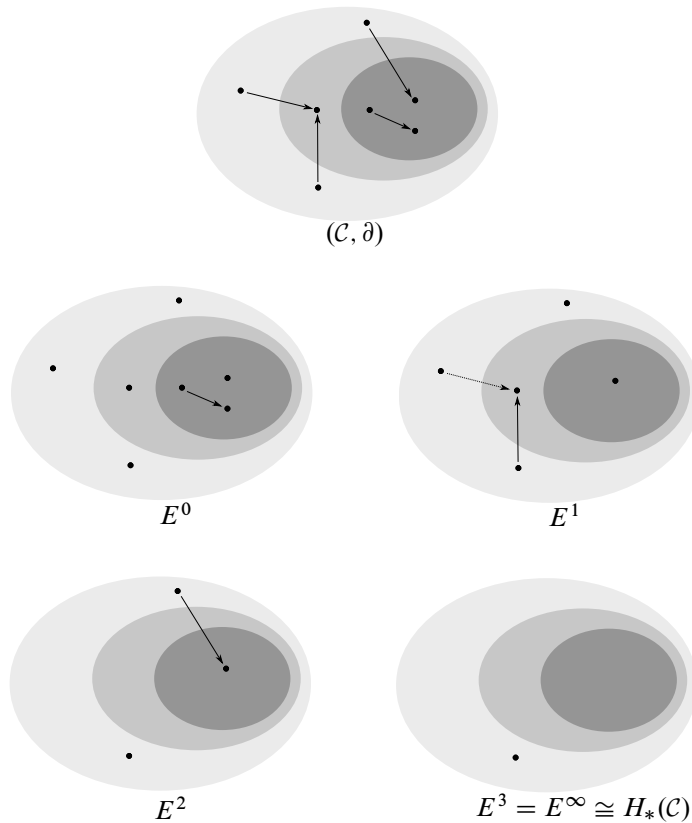


Figure 3: The spectral sequence which computes the homology of a filtered complex

The pages of our spectral sequence are denoted by (E^r, d^r) , where E_p^r is the vector space at filtration grading p surviving to page r , $E^r = \bigoplus_p E_p^r$, and d^r is the induced differential on E^r induced by cancellation on the previous pages. For example, $E_p^0 = G_p \mathcal{C} / G_{p-1} \mathcal{C}$, and d^0 is the sum of all the components of ∂ which preserve the filtration grading gr . The next page (E^1, d^1) is obtained by canceling all the components of d^0 , and is chain homotopy equivalent to (E^0, d^0) ; the underlying vector space \mathcal{C}' injects into \mathcal{C} and therefore has an induced filtration.

Iterating this process, we obtain pages (E^r, d^r) for all $r \geq 0$. In some situations, the spectral sequence eventually *collapses*, ie there is some N such that $d^r = 0$ for all $r \geq N$, and all pages are identical from then on. In such cases, the limit term $E^\infty := E^N = E^{N+1} = \dots$ is identified with the homology of the original complex (\mathcal{C}, ∂) . Figure 3 shows the cancellation process (the spectral sequence) used to compute the homology of a filtered complex.

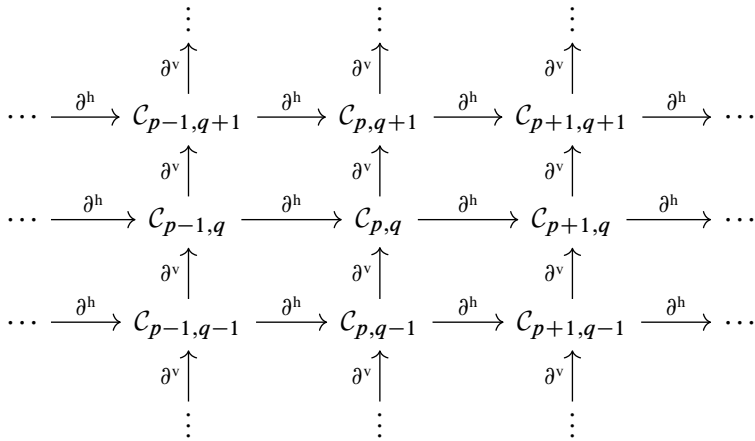


Figure 4

2.3 Bicomplexes

A bicomplex $C_{\bullet, \bullet}$ (Figure 4) is a bigraded \mathbb{F}_2 vector space with differentials

$$\partial_{p,q}^h: C_{p,q} \rightarrow C_{p+1,q}, \quad \partial_{p,q}^v: C_{p,q} \rightarrow C_{p,q+1}$$

such that $\partial^h \circ \partial^v + \partial^v \circ \partial^h = 0$. Here ∂^h is the sum of all $\partial_{p,q}^h$, the *horizontal differentials*, and ∂^v is the sum of all $\partial_{p,q}^v$, the *vertical differentials*.

The corresponding *total complex* $\text{Tot}(C)_{\bullet}$ is given by

$$\text{Tot}(C)_n = \bigoplus_{p+q=n} C_{p,q}$$

with differential $\partial^{\text{Tot}} = \partial^h + \partial^v$.

The total complex has two standard filtrations vF (the *rowwise filtration*) and hF (the *columnwise filtration*) induced by the rows and columns of the bicomplex, respectively, defined below, and depicted in Figure 5:

$$({}^vF_m C)_{p,q} = \begin{cases} C_{p,q} & \text{if } q \leq m, \\ 0 & \text{otherwise,} \end{cases} \quad ({}^hF_m C)_{p,q} = \begin{cases} C_{p,q} & \text{if } p \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

With respect to either of these filtrations, the total differential ∂^{Tot} is nonnegative.

2.4 Spectral sequences from bicomplexes

The standard filtrations vF and hF induce spectral sequences of filtered complexes ${}^{h^v}E$ and ${}^{v^h}E$, respectively.

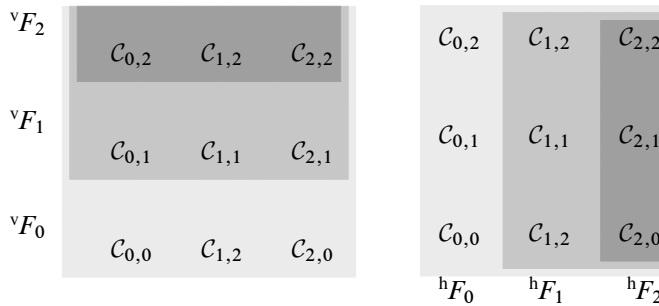


Figure 5

For example, consider the rowwise filtration vF shown on the left of Figure 5. The pages of the attached spectral sequence ${}^{hv}E$ approximate $H_*(\text{Tot}(\mathcal{C}))$ via the following process. On page ${}^{hv}E^0$, the differential d^0 consists of all the components of ∂^{Tot} which preserve the filtration grading; that is, $d^0 = \partial^h$. The induced differential on the next page $(\partial^{\text{Tot}})'$ now only has components which shift the filtration grading by 1 or more. The following proposition says that in certain situations, both spectral sequences eventually collapse to ${}^{hv}E^\infty \cong {}^{vh}E^\infty \cong H_*(\text{Tot}(\mathcal{C}))$, where the vector space generated by the surviving basis elements at the level set with filtration level $p + q = m$ is identified with $H_m(\text{Tot}(\mathcal{C}))$. A proof may be found in [15].

Proposition 11 *If for each n , the number of p such that $C_{p,n-p} \neq 0$ is finite, then*

$${}^{hv}E_{p,q}^\infty = {}^vF_q H_{p+q}(\text{Tot}(\mathcal{C})) / {}^vF_{q+1} H_{p+q}(\text{Tot}(\mathcal{C})),$$

$${}^{vh}E_{p,q}^\infty = {}^vF_p H_{p+q}(\text{Tot}(\mathcal{C})) / {}^vF_{p+1} H_{p+q}(\text{Tot}(\mathcal{C})).$$

Both spectral sequences converge to the total homology.

2.5 Equivariant homology and the Tate spectral sequence

Our bicomplex will be modeled on the bicomplex used to compute the Borel equivariant cohomology of a topological space X with an involution τ :

$$H_{\mathbb{Z}/2\mathbb{Z}}^*(X; \mathbb{F}_2) := \text{Ext}_{\mathbb{F}_2[\mathbb{Z}/2\mathbb{Z}]}(C_*(X), \mathbb{F}_2).$$

Here $C_*(X)$ is the singular chain complex for X with \mathbb{F}_2 coefficients. Since we are working over \mathbb{F}_2 coefficients, $(1 + \tau)^2 = 0$, so this is indeed a complex. Moreover, one can check that the differentials of $C_*(X)$ and the induced involution $\tau^\#$ commute, and therefore so do their duals. Thus we can build the double complex

$$0 \rightarrow C^*(X; \mathbb{F}_2) \xrightarrow{1+\tau^\#} C^*(X; \mathbb{F}_2) \xrightarrow{1+\tau^\#} C^*(X; \mathbb{F}_2) \xrightarrow{1+\tau^\#} \dots$$

which computes the Borel equivariant cohomology of X . Equipped with the rowwise and columnwise filtration, this is a bicomplex.

We can view the underlying vector space of the bicomplex as a module

$$C^*(X; \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta],$$

where θ shifts the columnwise filtration degree by 1 (to the right).

Localizing at θ gives the bi-infinite *Tate bicomplex*

$$C^*(X; \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}] = (\cdots \xrightarrow{1+\tau^\#} C^*(X; \mathbb{F}_2) \xrightarrow{1+\tau^\#} C^*(X; \mathbb{F}_2) \xrightarrow{1+\tau^\#} \cdots),$$

whose total homology, under some finiteness conditions, is isomorphic to

$$H^*(X^{\text{fix}}; \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}].$$

Here X^{fix} is the τ -invariant topological subspace of X . See Section 2 in [14] for more details on $\mathbb{Z}/2\mathbb{Z}$ -localization in Borel equivariant cohomology.

3 Topological preliminaries

The purpose of this section is to give an overview of the *annular Khovanov homology* of a link while establishing the notation to be used in the proof of the main result. In Section 3.1 we define annular links and relate them to braid and tangle closures. As annular Khovanov homology is computed from an annular link diagram, in Section 3.2 we set up the annular link diagram such that the topological involution on the link translates nicely to an involution on the diagram. Section 3.3 reviews the construction of the annular Khovanov chain complex via a *cube of resolutions* and defines the three gradings attached to the complex.

Throughout, we work in the smooth category with oriented links and \mathbb{F}_2 coefficients.

3.1 Annular links and tangles

An *annular link* is a finite disjoint union of embedded circles $\bigsqcup_1^n S^1$ in a thickened annulus $A \times I$. See Figure 6 for an example.

Annular links are *tangle closures*. A *tangle* \mathcal{T} is a properly embedded compact 1-manifold in $D^2 \times I$, with $\partial\mathcal{T} \subset D^2 \times \{0, 1\}$. If the number of boundary components of \mathcal{T} on the two disks agree, then after arranging for $\partial\mathcal{T} = \{x_1, \dots, x_n\} \otimes \{0, 1\}$, we

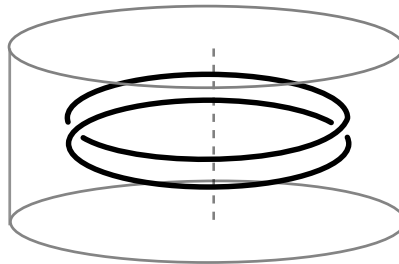


Figure 6: A Hopf link embedded in a thickened annulus

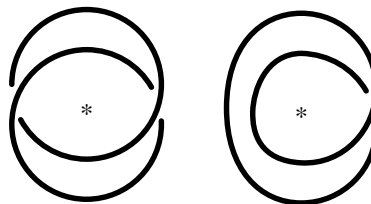


Figure 7: On the left is an annular diagram for the Hopf link from Figure 6. On the right is an annular diagram for the quotient knot, a positively stabilized annular unknot.

can glue $D^2 \times \{0\}$ to $D^2 \times \{1\}$ (via id_{D^2}) to obtain a link $\widehat{\mathcal{T}} = \mathcal{T}/(x_i, 0) \sim (x_i, 1)$ in a solid torus $D^2 \times I / (D^2 \times \{0\} \sim D^2 \times \{1\})$.

Braids constitute a well-studied case of tangles. Closures of n -strand braids form the braid group B_n . Isotopy classes of n -braid closures correspond to the conjugacy classes in B_n . With this important case in mind, we think of A as the xy -plane punctured at the origin $*$ and call $\{*\} \times I \subset \mathbb{R}^2 \times I \cong D^2 \times I$ the *braid axis*.

A generic projection of an annular link $L \subset A \times I$ to A is an immersion with at most double points, as in Figure 7. If we also remember the crossing information (which strand was over the other before the projection), then we call the image of this projection a *diagram* of L in A , denoted by $\mathcal{D}(L)$. By abuse of notation, sometimes we will think of $\mathcal{D}(\cdot)$ as the projection $A \times I \rightarrow A$, and refer to the image $\mathcal{D}(L) \subset A$ as the *annular link diagram*, even though the image doesn't actually capture the crossing information.

3.2 2-periodic links and their diagrams

Given a 2-periodic link $\tilde{L} \subset S^3$ with involution $\tau: S^3 \rightarrow S^3$, the fixed point set B (for “braid axis”) is an unknotted S^1 in S^3 .

With some choice of coordinates on S^3 minus a point $\{\infty\}$ on B , we can view B as the z -axis in \mathbb{R}^3 and τ as the rotation of 180° about the z -axis, and let A be the xy -plane minus the origin. Hence $\tilde{L} \hookrightarrow A \times \mathbb{R} \cong A \times I$ is an annular link.

Remark 12 We define a 2-periodic link $\tilde{L} \subset S^3$ as one that comes with a chosen involution of S^3 which fixes an unknot in S^3 . It is important to note that a link \tilde{L} in S^3 may come with multiple involutions τ_i in the sense that the link $\tilde{L} \cup B_1$ is not isotopic to $\tilde{L} \cup B_2$ (where B_i is the fixed-point set of involution τ_i). In this case \tilde{L} with involution τ_1 defines a different annular link from \tilde{L} with involution τ_2 .

Maintaining symmetry under τ , isotope \tilde{L} so that the projection $\mathcal{D}: A \times I \rightarrow A$ takes \tilde{L} to a link diagram $\mathcal{D}(\tilde{L})$ in A . At the double points, we keep the crossing information. Employing a small isotopy, we may also assume the double points occur away from the y -axis, so that there is a clear notion of the “right side” of $A \times I$ or A (where $x \geq 0$), and the “left side” of $A \times I$ or A (where $x \leq 0$).

The preimage of the right side of $\mathcal{D}(\tilde{L})$ is a tangle \mathcal{T} , which by symmetry has the same number of loose strands on both ends. The quotient link of \tilde{L} with respect to the involution τ is the tangle closure $L := \hat{\mathcal{T}}$. A diagram for L in A is given by $\mathcal{D}(\mathcal{T})$ on the right side of A together with an identity braid on the left side of A . See Figure 7 for example.

3.3 Annular Khovanov homology

Annular Khovanov homology (with \mathbb{F}_2 coefficients) is a TQFT from properly embedded 1-manifolds in $A \times I$ and cobordisms between them to \mathbb{F}_2 vector spaces and linear maps between them. Asaeda, Przytycki, and Sikora define this annular link invariant in [2], where they construct a *cube of resolutions* from the diagram $\mathcal{D}(L)$ and associate to it a triply graded chain complex $\text{CKh}(\mathcal{D}(L))$ which computes their annular link invariant $\text{AKh}(L)$.

We define two types of cubes, related by the TQFT. See Figure 8 and Figure 9 for examples.

Definition 13 (cube of resolutions) We will use the notation $\text{Cube}(\mathcal{D})$ to denote the *cube of resolutions* of the diagram \mathcal{D} , where

- vertices are resolutions of \mathcal{D} , which are closed 1-manifolds embedded in A ,
- edges are 2-dimensional saddle cobordisms between two vertices.

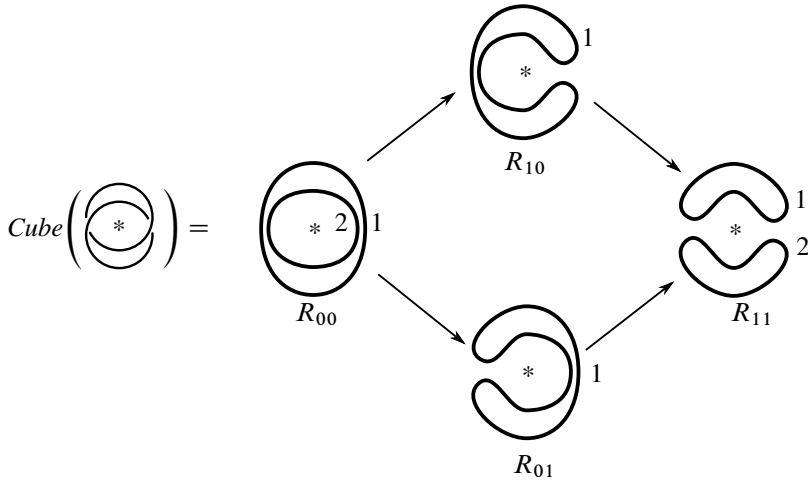


Figure 8: The cube of resolutions for the annular Hopf link in Figure 7

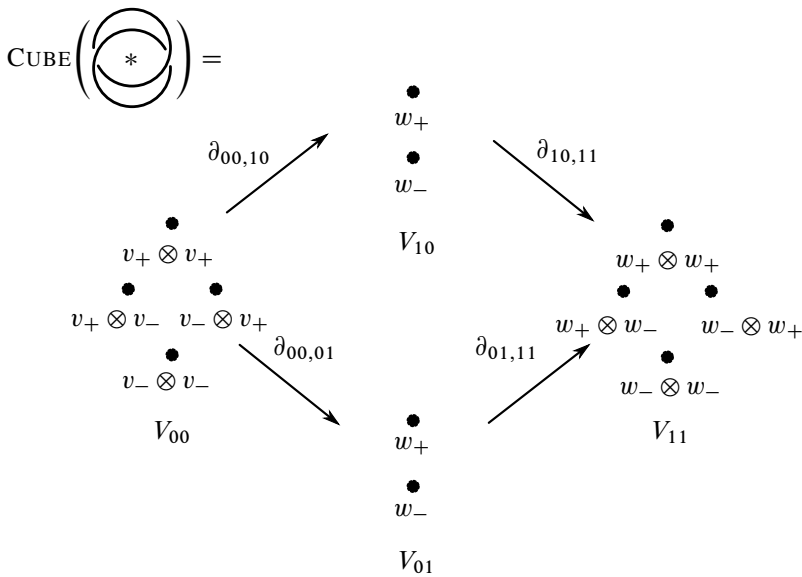


Figure 9: The cube of chains for the annular Hopf link in Figure 7

Definition 14 (cube of chains) The annular Khovanov TQFT takes $Cube(\mathcal{D})$ to $CUBE(\mathcal{D})$, the *cube of chains*, where

- vertices are \mathbb{F}_2 vector spaces,
- edges are linear maps between the vertices.

We outline the construction of $Cube(\mathcal{D})$ and $CUBE(\mathcal{D})$ in the rest of this subsection. Let n be the number of crossings in $\mathcal{D}(L)$, and label the crossings 1 through n in any order. Let n_+ denote the number of positive crossings, and n_- the number of negative crossings.

Definition 15 (bitstring notation) We call $\alpha \in \{0, 1\}^n$ a *bitstring of length n* . We make the following definitions:

- $\alpha[i]$ denotes the bit at the i^{th} position, where indexing begins at $i = 0$.
- $|\alpha| = \sum_{i=0}^{n-1} \alpha[i]$ is the *Hamming weight* of α .
- Let $\alpha' \in \{0, 1\}^n$. Suppose there is an index k such that $\alpha[k] = 0$, $\alpha'[k] = 1$, and $\alpha[j] = \alpha'[j]$ for all $j \neq k$. In this case, we say that α' is a *bit increment from α* .
- Let $\beta \in \{0, 1\}^n$. Bitstrings of length n form a poset: we write $\alpha < \beta$ if $\alpha[i] \leq \beta[i]$ for all i .
- $\alpha\beta$ represents the concatenation of α and β , in that order. That is, $\alpha\beta$ is the length- $2n$ bitstring where

$$\alpha\beta[i] = \begin{cases} \alpha[i] & \text{for } i < n, \\ \beta[i - n] & \text{for } i \geq n. \end{cases}$$

- For $\alpha\beta$ as above, there is an involution τ defined by $\tau(\alpha\beta) = \beta\alpha$.

Remark 16 We will also use the symbol τ to denote some topological involutions, but the distinction from this bitstring operation should be clear from context. See Section 4.1 for more on the notation for various involutions.

Vertices Let $\alpha \in \{0, 1\}^n$ be a length- n bitstring. Figure 10 shows two ways of resolving a local picture of a crossing. Associate to α a resolution R_α of $\mathcal{D}(L)$, where at crossing i we replace a local picture with a 0-resolution if $\alpha[i] = 0$, or a 1-resolution if $\alpha[i] = 1$. The resulting resolution R_α is a closed 1-manifold embedded in the annulus A .

Let $|R_\alpha|$ be the number of components of R_α , and label the components $C_1, \dots, C_{|R_\alpha|}$. Let $\mathbb{V} = \langle v_-, v_+ \rangle$ be the 2-dimensional \mathbb{F}_2 vector space generated by the symbols v_- and v_+ . We associate to the resolution R_α a vector space $V_\alpha = \mathbb{V}^{\otimes |R_\alpha|}$, generated by the pure tensors of the form $v = v_{p_1} \otimes v_{p_2} \otimes \dots \otimes v_{p_{|R_\alpha|}} \in \{v_-, v_+\}^{\otimes |R_\alpha|}$. We

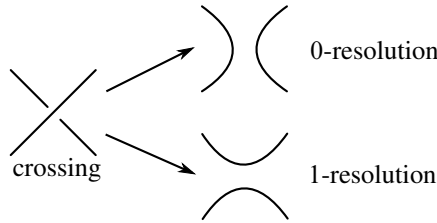


Figure 10

endow $\text{CKh}(\mathcal{D}(L))$ with the following three gradings by assigning gradings to these distinguished generators. This notation is slightly modified from that in [9]:

- The (homological) i -grading of v is

$$\text{gr}_i(v) = |\alpha| - n_-.$$

- Define $p(v) = \#\{p_i : p_i = +\} - \#\{p_i : p_i = -\}$. The (quantum) j -grading of v is

$$\text{gr}_j(v) = p(v) + |\alpha| + n_+ - 2n_-.$$

- The components of R_α come in two flavors with respect to the basepoint $*$:
 - The *nontrivial circles* have nonzero winding number with respect to $*$ (with respect to either orientation).
 - The *trivial circles* have zero \mathfrak{sl}_2 weight space with respect to $*$ (with respect to either orientation).

The (\mathfrak{sl}_2 weight-space) k -grading of v is given by

$$\text{gr}_k(v) = \#\{p_i : p_i = +, C_i \text{ nontrivial}\} - \#\{p_i : p_i = -, C_i \text{ nontrivial}\}.$$

We think of the i^{th} tensor factor in $\mathbb{V}^{\otimes |R_\alpha|}$ as corresponding to a labeling of the i^{th} component C_i with a $-$ or $+$ symbol. As such, for a trivial circle C_i we will henceforth write the corresponding factor as $\mathbb{W} = \langle w_-, w_+ \rangle$, reserving the notation $\mathbb{V} = \langle v_-, v_+ \rangle$ for nontrivial circles. For example, if R_α consists of one nontrivial circle and two trivial circles, labeled in that order, one of the generators of V_α is $v_- \otimes w_+ \otimes w_-$.

Remark 17 In [10], Grigsby, Licata, and Wehrli show that annular Khovanov homology over \mathbb{C} enjoys a Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ -action, which is why we refer to the k -grading as the “ \mathfrak{sl}_2 weight-space grading”.

Edges Let $\alpha, \alpha' \in \{0, 1\}^n$. If α' is a bit increment from α , we say that the resolution R'_{α} is a *successor* to R_{α} . In our cube of resolutions, we draw a directed edge $R_{\alpha} \rightarrow R'_{\alpha}$ corresponding to a linear map $\partial_{\alpha, \alpha'}: V_{\alpha} \rightarrow V_{\alpha'}$. This map depends on a few characteristics of the resolutions R_{α} and $R_{\alpha'}$:

- Since R_{α} and $R_{\alpha'}$ differ at the site of exactly one crossing, all but one or two components remain unchanged between R_{α} and $R_{\alpha'}$. The linear map $\partial_{\alpha, \alpha'}$ may be a *merge map* or a *split map*. If the site of the crossing touches only one component C_i of R_{α} , then C_i has split into two components in $R_{\alpha'}$. If the site of the crossing touches two components C_i and C_j of R_{α} , then C_i and C_j merge into one component in $R_{\alpha'}$.
- The participating components may be trivial or nontrivial circles. The linear map $\partial_{\alpha, \alpha'}$ is constructed so that gr_j and gr_k are preserved, and gr_i is increased by 1. Roberts shows in [21] that there are exactly six possible types of maps corresponding to an edge in the cube resolutions. We can visually describe these maps by drawing dots representing the distinguished generators of V_{α} and $V_{\alpha'}$, aligning them in rows of constant gr_j , and drawing arrows between the dots to represent components of the differential. Figure 11 shows the six maps defining all possible $\partial_{\alpha, \alpha'}$ using this visual representation.

One can check that $\partial_{\text{AKh}} = \sum \partial_{\alpha, \alpha'}$ is a differential by verifying that each face of the cube of chains commutes. The triply graded homology of this complex $\text{CKh}(\mathcal{D}(L))$ is independent of the choice of diagram $\mathcal{D}(L)$ [2].

4 The annular Khovanov–Tate bicomplex

We are now prepared to construct the bicomplex which defines the spectral sequence in Theorem 1. The motivation for this construction is discussed in Section 2.5.

4.1 An involution on CKh

The first task is to define an involution on $\text{CKh}(\mathcal{D}(\tilde{L}))$ induced by the topological involution τ .

Definition 18 We will need to discuss many related involutions. By abuse of notation, we utilize only three symbols; the distinctions within each group will be clear from context. After presenting the notation, we discuss the involutions and their relationships with each other in more detail.

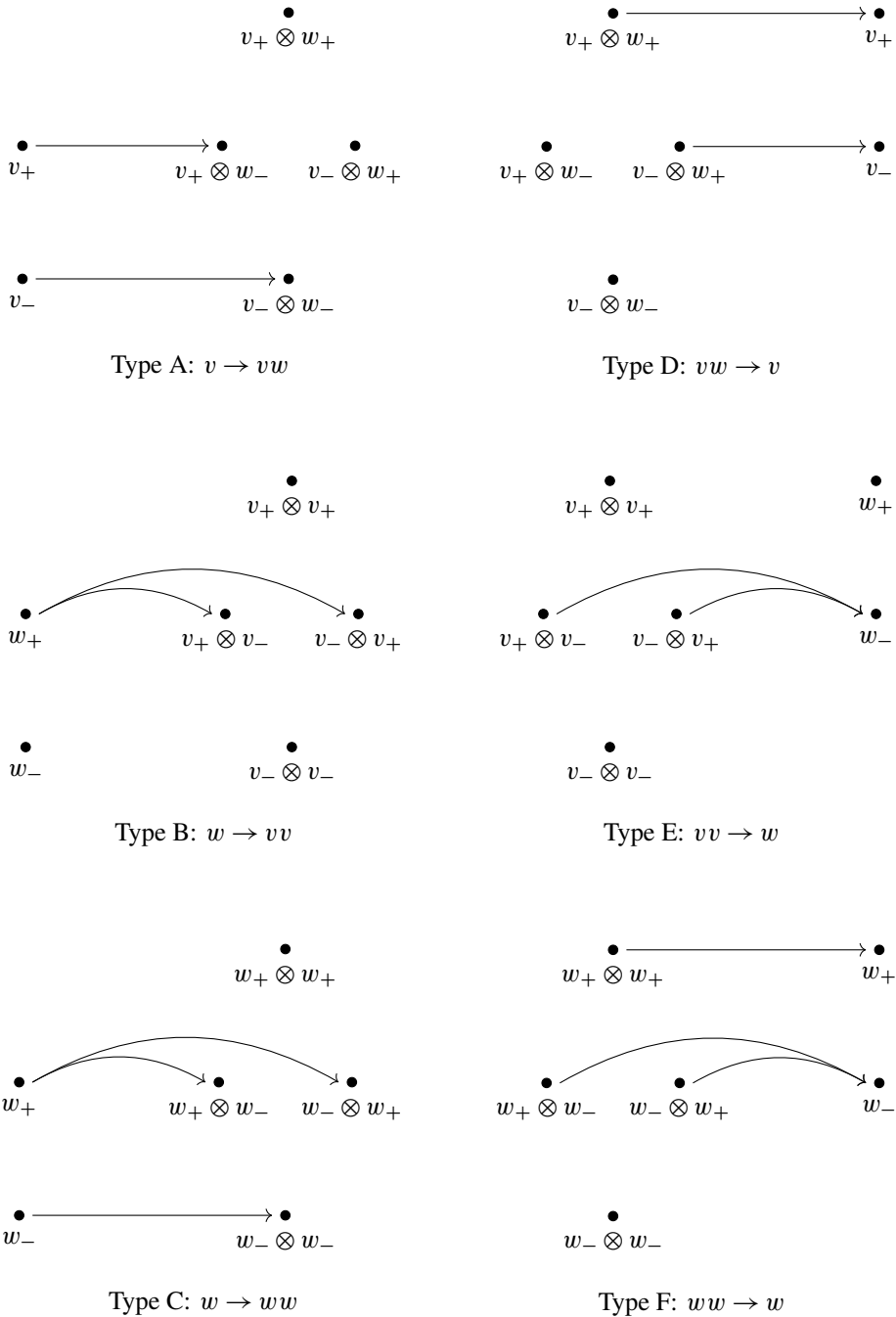


Figure 11: The AKh differentials. Within each diagram, the rows have constant j -grading.

In the following contexts, we use the symbol τ :

- the bitstring involution defined in Definition 15,
- the topological involution on S^3 with fixed point set B ,
- the restriction of τ to the 2-periodic link $\tilde{L} \subset S^3$,
- the involution induced on $A \times I$ and its restriction to $\tilde{L} \hookrightarrow A \times I$.

We use the symbol τ_A to denote the following:

- the involution induced on A from τ on $A \times I$,
- the restriction of τ_A to $\mathcal{D}(\tilde{L}) \subset A$,
- the involution induced on an equivariant resolution $R_{\alpha\alpha} \subset A$.

Finally, $\tau^\#$ is used to denote the following:

- the involution induced on the chains of $\text{CKh}(\mathcal{D}(\tilde{L}))$,
- the involutions induced by $\tau^\#$ on the pages of the spectral sequences.

We continue to use the setting described in Section 3.2. Let n be the number of crossings in the quotient link diagram $\mathcal{D}(L)$. Label the $2n$ crossings in $\mathcal{D}(\tilde{L})$ by first labeling crossings of $\mathcal{D}(\tilde{L})$ on the right side of A ($x > 0$), then labeling the crossings on the left side ($x < 0$) so that τ_A takes the i^{th} crossing on the right to the $(i + n)^{\text{th}}$ crossing on the left.

With this assignment, each $2n$ -bitstring $\alpha = \alpha_1\alpha_2$ is the concatenation of two n -bitstrings α_1 and α_2 , where α_1 represents a sequence of resolution choices for the crossings on the right and α_2 represents a sequence of resolution choices for the crossings on the left. Thus τ_A takes $R_\alpha = R_{\alpha_1\alpha_2}$ to $R_{\tau(\alpha)} = R_{\alpha_2\alpha_1}$.

Thinking of the distinguished generators of the cube of chains $\text{CUBE}(\mathcal{D}(\tilde{L}))$ as marked resolutions, τ_A induces an involution $\tau^\#$ on the chains of $\text{CKh}(\mathcal{D}(\tilde{L}))$.

Definition 19 A resolution R_α is *equivariant* if $\tau_A(R_\alpha) = R_\alpha$. A generator x of the cube of chains $\text{CUBE}(\mathcal{D}(\tilde{L}))$ is *equivariant* if $\tau^\#(x) = x$. If a resolution or generator is not equivariant, it is *nonequivariant*.

In fact, $\tau^\#$ is an involution on the complex $(\text{CKh}(\mathcal{D}(\tilde{L})), \partial_{\text{AKh}})$:

Lemma 20 *The maps ∂_{AKh} and $\tau^\#$ commute.*

Proof It suffices to show that every edge belongs to a well-defined pair of edges $\{\partial_{\alpha,\alpha'}, \partial_{\tau(\alpha),\tau(\alpha')}\}$ of $\text{CUBE}(\mathcal{D}(\tilde{L}))$, where α' is a bit increment from α , so that the following diagram commutes:

$$\begin{array}{ccc} V_{\alpha'} & \xrightarrow{\tau^\#} & \tau^\#(V_{\alpha'}) \\ \partial_{\alpha,\alpha'} \uparrow & & \uparrow \partial_{\tau(\alpha),\tau(\alpha')} \\ V_\alpha & \xrightarrow{\tau^\#} & \tau^\#(V_\alpha) \end{array}$$

Since ∂_{AKh} is the sum of all such pairs of edges, ∂_{AKh} commutes with $\tau^\#$.

The edge $\partial_{\alpha,\alpha'}$ may correspond to a change of resolution on the right or left side of $\mathcal{D}(\tilde{L})$. Without loss of generality, consider the case where the resolution change is on the right:

$$\begin{array}{ccc} V_{\alpha'_1\alpha_2} & \xrightarrow{\tau^\#} & V_{\alpha_2\alpha'_1} \\ \partial_{\alpha,\alpha'} \uparrow & & \uparrow \partial_{\tau(\alpha),\tau(\alpha')} \\ V_{\alpha_1\alpha_2} & \xrightarrow{\tau^\#} & V_{\alpha_2\alpha_1} \end{array}$$

The components of $R_{\alpha_1\alpha_2}$ and $R_{\alpha_2\alpha_1}$ are identified in pairs by the diffeomorphism τ_A ; the same holds for the pair $R_{\alpha'_1\alpha_2}$ and $R_{\alpha_2\alpha'_1}$. In this way, a merge (resp. split) map $V_{\alpha_1\alpha_2} \rightarrow V_{\alpha'_1\alpha_2}$ corresponds to the isomorphic merge (resp. split) map $V_{\alpha_2\alpha_1} \rightarrow V_{\alpha_2\alpha'_1}$.

Notice that $\partial_{\alpha,\alpha'} = \partial_{\tau(\alpha)}$ if and only if they have the same source and target. This is impossible as one of $|\alpha|$ and $|\alpha'|$ is odd, and hence corresponds to a nonequivariant resolution. □

4.2 Construction of the AKh^{Tate} bicomplex

This allows us to define a variant of the Tate bicomplex (see Section 2.5), constructed as follows:

- Each column is a copy of the complex $\text{CKh}(\mathcal{D}(\tilde{L}))$, where the vertical filtration is induced by gr_i .
- The underlying vector space is the $\mathbb{F}_2[\theta]$ -module $\text{CKh}(\mathcal{D}(\tilde{L})) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta]$, where θ acts on the columns by shifting right by one column.
- The horizontal differential is $1 + \tau^\#$.

Observe that this complex is quadruply graded by $\text{gr}_i, \text{gr}_j, \text{gr}_k$, and a column grading gr_t , where t is the exponent of θ .

Definition 21 We call this bicomplex the *annular Khovanov–Tate complex* for the 2–periodic link \tilde{L} with involution τ , abbreviated $\text{AKh}^{\text{Tate}}(\tilde{L})$. We denote the total complex of this bicomplex by $\text{Tot}(\text{AKh}^{\text{Tate}}(\tilde{L}))$.

Remark 22 This notation requires some justification. We will show that

$$H^*(\text{Tot}(\text{AKh}^{\text{Tate}}(\tilde{L}))) \cong \text{AKh}(L) \otimes \mathbb{F}_2[\theta, \theta^{-1}],$$

where the latter is independent of the choice of diagram by [2].

5 Proof of Theorem 1

We are now ready to prove the main result. Recall that $\text{AKh}^{j,k}(L)$ denotes the part of $\text{AKh}(L)$ at quantum grading j and \mathfrak{sl}_2 weight-space grading k .

Theorem 1 *Let \tilde{L} be a 2–periodic link with quotient link L . For each pair of integers (j, k) , there is a spectral sequence with*

$$E^1 \cong \text{AKh}^{2j-k,k}(\tilde{L}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}] \rightrightarrows E^\infty \cong \text{AKh}^{j,k}(L) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}].$$

Before diving into the details, we first give a sketch of the main ideas behind the proof. The reader may also wish to refer to the example in Section 6.

Since the complex $\text{CKh}(\mathcal{D}(\tilde{L}))$ is finite-dimensional, our bicomplex $\text{AKh}^{\text{Tate}}(\mathcal{D}(\tilde{L}))$ has finite-dimensional columns, so by Proposition 11, the ${}^{\text{hv}}E$ and ${}^{\text{vh}}E$ spectral sequences both converge to $H_*(\text{Tot}(\text{AKh}^{\text{Tate}}(\mathcal{D}(\tilde{L}))))$. We will compute the ${}^{\text{hv}}E$ spectral sequence to find that ${}^{\text{hv}}E^\infty \cong \text{AKh}(L) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$. Then we will show that ${}^{\text{vh}}E^1 = \text{AKh}(\tilde{L}) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$. By Proposition 11, ${}^{\text{vh}}E^\infty \cong {}^{\text{hv}}E^\infty$, with the isomorphism respecting the $(i+t)$ –grading. Observing that each diagonal level set in ${}^{\text{vh}}E^\infty$ can be identified with $\bigoplus_i \text{AKh}_i(L)$ (where $\text{AKh}_i(L)$ refers to the piece of $\text{AKh}(L)$ at $\text{gr}_i = i$), we see that ${}^{\text{vh}}E$ is a spectral sequence from the $(i+t)$ –filtered $\text{AKh}(\tilde{L}) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$ to $\text{AKh}(L) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$. The total differential preserves j – and k –gradings, so ${}^{\text{vh}}E$ splits along j – and k –gradings.

5.1 The first pages

The first pages of either spectral sequence can be understood readily and have topological interpretations.

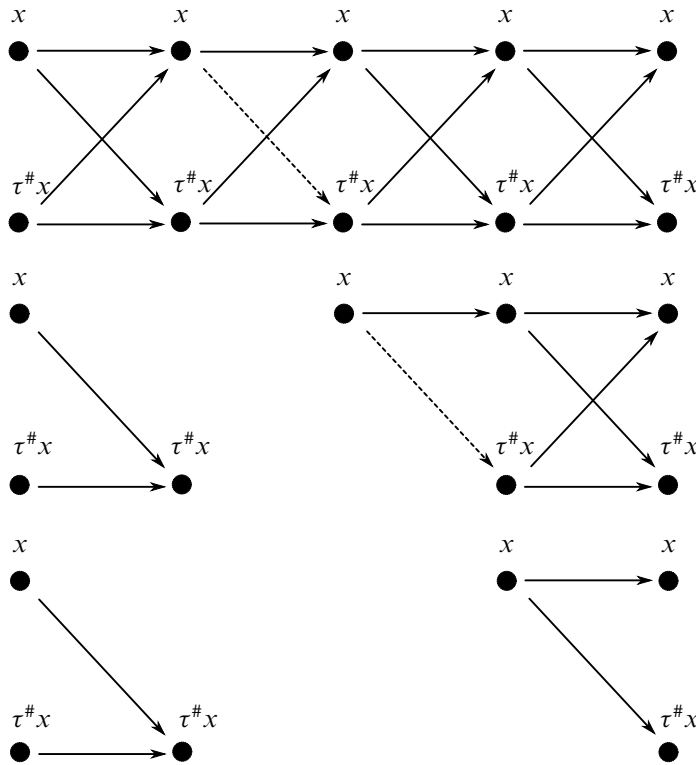


Figure 12: Cancellation of the $\tau^\#$ arrows eliminates all the nonequivariant generators.

Lemma 23 ${}^{\text{vh}}E^1 \cong \text{AKh}(\tilde{L}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}]$.

Proof In each column, the cancellation of all the vertical arrows computes the homology $\text{AKh}(\mathcal{D}(\tilde{L}))$. The fact that cancellation actually occurs in the greater context of the bicomplex is irrelevant because any induced maps are no longer vertical. \square

Lemma 24 ${}^{\text{hv}}E^1$ is generated by the equivariant generators of $\text{AKh}^{\text{Tate}}(\mathcal{D}(\tilde{L}))$.

Proof Since $1 + \tau^\#$ vanishes on all equivariant generators, it suffices to show that all the nonequivariant generators vanish upon cancellation of all the horizontal arrows in the bicomplex. In fact, cancellation of all the $\tau^\#$ arrows suffices, as shown in Figure 12. \square

Definition 25 A staircase of arrows (Figure 13) is a finite collection of arrows in a bicomplex (generated by a distinguished filtered graded basis) which form a path with the following properties:

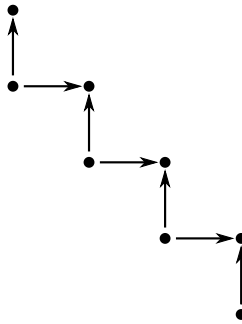


Figure 13: A staircase of arrows. In this example, the staircase has *height* 4, so it induces an arrow of *length* 4.

- The path alternates between vertical and horizontal arrows.
- Both arrows at a corner either both point away from or both point toward the corner.
- If a vertical arrow and a horizontal arrow meet at a corner of the path, they either have the same source or the same target.

The *height* of a staircase of arrows with respect to a given filtration is the change in filtration degree between the generators at the ends of the path.

With respect to the rowwise filtration, if both ends of the staircase begin with a vertical arrow, then cancellation of the horizontal arrows in the staircase (in any order) induces a single arrow from one end of the path to the other. The *length* of this induced arrow is the height of the staircase.

Lemma 26 *If r is odd, then ${}^{\text{hv}}d^r = 0$.*

Proof By Lemma 24, only equivariant generators survive to ${}^{\text{hv}}E^1$. Since these belong to vector spaces corresponding to equivariant resolutions of $\mathcal{D}(\tilde{L})$, their Hamming weights must be even. The i -grading is a shift of Hamming weight, so every other row vanishes on ${}^{\text{hv}}E^1$. Components of ${}^{\text{hv}}d_r$ for odd r are represented by arrows of length r , which must map to or from a vanishing row, so ${}^{\text{hv}}d_r = 0$. \square

Therefore the next interesting differential in the spectral sequence is ${}^{\text{hv}}d^2$.

Lemma 27 *Every component of ${}^{\text{hv}}d^2$ is induced by the cancellation of a $\tau^\#$ arrow at a row corresponding to an odd Hamming weight.*

Proof Since we begin with only (rowwise) filtration degree 0 and 1 maps, the only way to induce a length-2 differential is by canceling the horizontal arrow on a staircase

of height 2. The source and target of a length-2 differential are equivariant generators, so the canceled horizontal arrow must lie in a row corresponding to an odd Hamming weight. \square

Remark 28 Later on it will become useful to view the computation of the induced differentials in the spectral sequence as a sequence of maps (by ∂_{AKh}) and lifts (by $1 + \tau^\#$), traveling up the staircase in Figure 13. The maps constructed this way are the same as the maps induced by cancellation of the $\tau^\#$ arrows in the staircase. This is a more intuitive reason why cancellation of only the $\tau^\#$ arrows is sufficient for eliminating the nonequivariant generators.

5.2 Relationship between $({}^{\text{hv}}E^2, {}^{\text{hv}}d^2)$ and the quotient link

The crux of the theorem comes from the observation that the generators and differentials on page 2 of the ${}^{\text{hv}}E$ spectral sequence correspond exactly with distinguished generators in the cube of chains for the diagram of the quotient link. The following two propositions verify the details of this correspondence.

Proposition 29 *There is a one-to-one correspondence*

$$\{\text{generators of } {}^{\text{hv}}E^2\} \longleftrightarrow \{\text{generators of } \text{CKh}(L) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}]\}, \quad \tilde{v} \longleftrightarrow v$$

induced by τ such that

- $\text{gr}_i(\tilde{v}) = 2 \text{gr}_i(v)$,
- $\text{gr}_j(\tilde{v}) = 2 \text{gr}_j(v) - \text{gr}_k(v)$,
- $\text{gr}_k(\tilde{v}) = \text{gr}_k(v)$,
- $\text{gr}_t(\tilde{v}) = \text{gr}_t(v)$.

Proof By Lemma 24, we have that ${}^{\text{hv}}E^1$ is generated by the equivariant generators of $\text{AKh}^{\text{Tate}}(\mathcal{D}(\tilde{L}))$, and Lemma 26 shows ${}^{\text{hv}}d^1 = 0$, so ${}^{\text{hv}}E^1 = {}^{\text{hv}}E^2$.

Bijection of generators First of all, the correspondence is given by the identity on the $\mathbb{F}_2[\theta, \theta^{-1}]$ factor.

Let \tilde{v} denote a generator of ${}^{\text{hv}}E^2$, coming from the equivariant resolution $R_{\alpha\alpha}$. Thinking of \tilde{v} as a τ_A -equivariant assignment of pluses and minuses to the components of $R_{\alpha\alpha}$, let v be the quotient of \tilde{v} by τ_A .

In the other direction, for v a generator of $\text{CKh}(L) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}]$ coming from a resolution R_α , view v as an assignment of pluses and minuses to the components

of R_α . Pick any path connecting the two components of ∂A and take a double cover of R_α . The lift of v is \tilde{v} .

Grading relationships The t -grading relationship follows by definition. For the other three relationships, suppose v consists of

- a nontrivial circles labeled “+”,
- b nontrivial circles labeled “-”,
- c trivial circles labeled “+”,
- d trivial circles labeled “-”.

In this case, write $v = v_+^a v_-^b w_+^c w_-^d$ to denote the distinguished generator, suppressing the tensor signs in favor of compact notation. Then $\tilde{v} = v_+^a v_-^b w_+^{2c} w_-^{2d}$.

Let n_+ and n_- denote the number of positive and negative crossings in $\mathcal{D}(L)$, respectively. For v ,

- $\text{gr}_i(v) = |\alpha| - n_-$,
- $\text{gr}_j(v) = |\alpha| + (a - b + c - d) + n_+ - 2n_-$,
- $\text{gr}_k(v) = a - b$.

Since the diagram $\mathcal{D}(\tilde{L})$ obtained by taking the double cover of $\mathcal{D}(L)$ has twice as many positive and negative crossings as $\mathcal{D}(L)$, respectively, we compute that

- $\text{gr}_i(\tilde{v}) = 2|\alpha| - 2n_- = 2 \text{gr}_i(v)$,
- $\text{gr}_j(\tilde{v}) = 2|\alpha| + (a - b + 2c - 2d) + 2n_+ - 4n_- = 2 \text{gr}_j(v) - (a - b) = 2 \text{gr}_j(v) - \text{gr}_k(v)$,
- $\text{gr}_k(\tilde{v}) = a - b = \text{gr}_k(v)$. □

Proposition 30 *Under the correspondence above, each line with -2 slope in the bicomplex $({}^{\text{hv}}E^2, {}^{\text{hv}}d^2)$ is isomorphic to the complex $(\text{CKh}(L), \partial_{\text{AKh}})$.*

Proof Let α' be a bit increment from α . Lemma 27 implies that any component of ${}^{\text{hv}}d^2$ lies along a line with -2 slope.

Upstairs, consider the components of ${}^{\text{hv}}d^2$ from $V_{\alpha\alpha} \otimes \theta^t$ to $V_{\alpha'\alpha'} \otimes \theta^{t-1}$; see Figure 14. The two resolutions $R_{\alpha\alpha'}$ and $R_{\alpha'\alpha}$ connecting $R_{\alpha\alpha}$ and $R_{\alpha'\alpha'}$ correspond to changing the resolutions of some pair of crossings c_i and c_{i+n} from their 0-resolution in $R_{\alpha\alpha}$ to their 1-resolution in $R_{\alpha'\alpha'}$.

Downstairs, consider the edge of $\text{CUBE}(\mathcal{D}(L))$ going from V_α to $V_{\alpha'}$. This edge falls into one of the six types of AKh differentials, as discussed in Section 3.3.

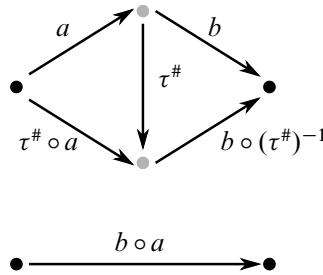


Figure 14: Cancellation of $\tau^\#$ arrows induce ${}^{\text{hv}}d^2$ arrows. For each diamond of the form above, we obtain a length-2 differential.

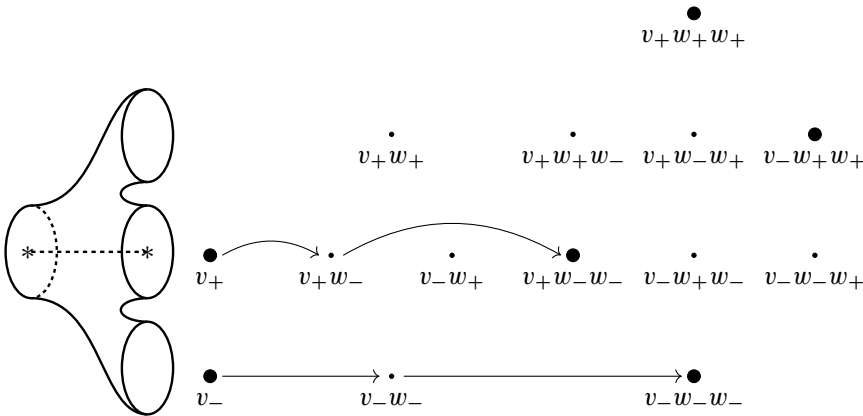


Figure 15: Type \mathcal{A} : $\tilde{v} \rightarrow \tilde{v}\tilde{w}$

We compare the ${}^{\text{hv}}d^2$ components induced by the cancellation of the $\tau^\#$ arrows going between $V_{\alpha\alpha'}$ and $V_{\alpha'\alpha}$ with the components of ∂_{AKh} , in the six different cases. Figures 15 through 20 show the computation of ${}^{\text{hv}}d^2$ in all six cases. Compare these figures with Figure 11. Note that in contrast with the notation used in Figure 11, the tensor signs are now suppressed for the sake of compactness. By inspection, under the correspondence from Proposition 29, the induced length-2 arrows upstairs align exactly with the differential downstairs. \square

5.3 Visualizing higher differentials

Just as all components of ${}^{\text{hv}}d^2$ are induced by cancellation of some $\tau^\#$ arrow, longer differentials are induced by the cancellation of multiple $\tau^\#$ arrows. This can be seen by inducting on the applications of the cancellation operation that result in a differential of length $2r$. At each step, the slope of the arrows representing the induced differentials

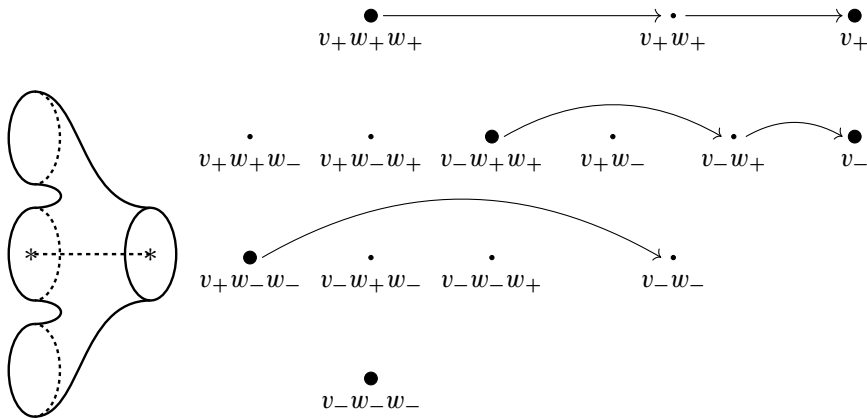


Figure 16: Type \mathcal{D} : $\tilde{v}\tilde{w} \rightarrow \tilde{v}$

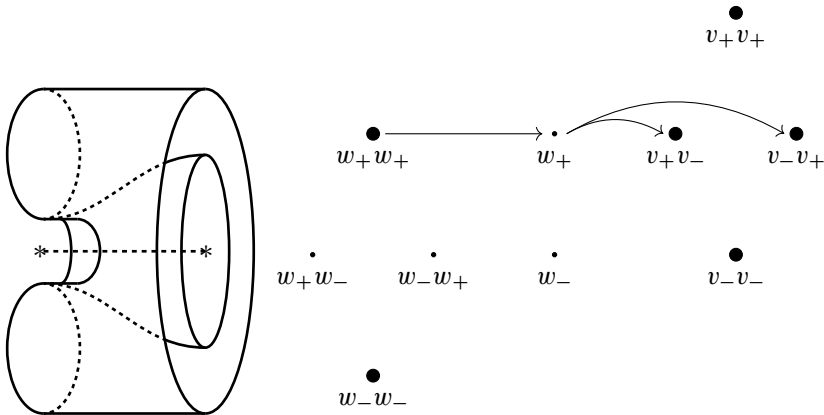


Figure 17: Type \mathcal{B} : $\tilde{w} \rightarrow \tilde{v}\tilde{v}$

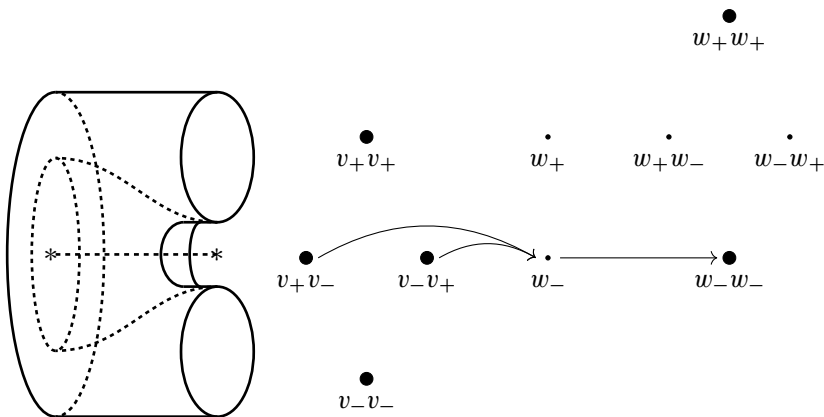


Figure 18: Type \mathcal{E} : $\tilde{v}\tilde{v} \rightarrow \tilde{w}$

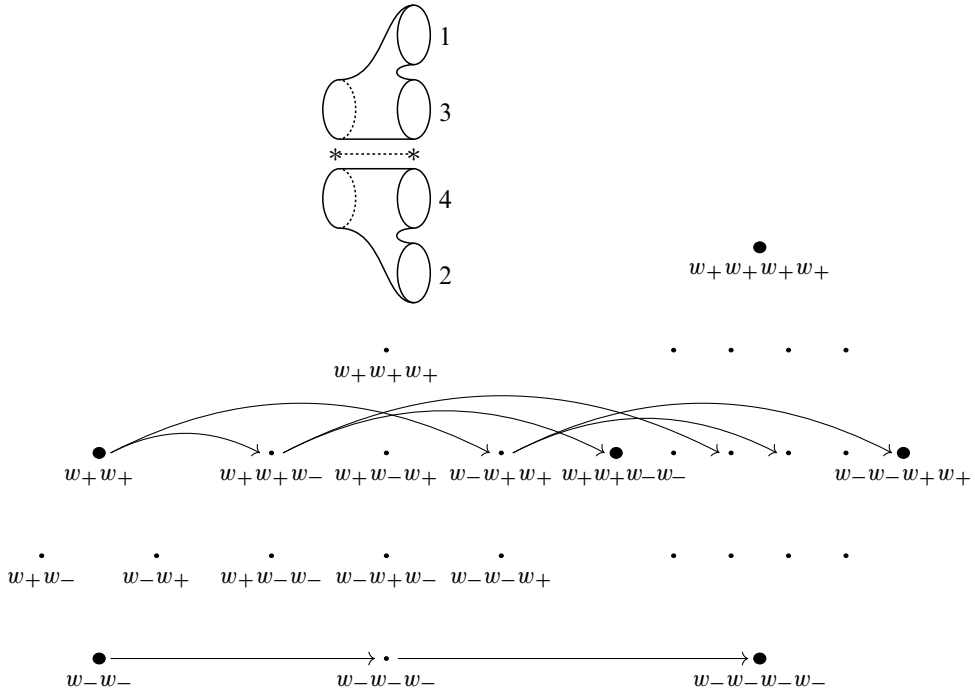


Figure 19: Type \mathcal{E} : $\tilde{w} \rightarrow \tilde{w}\tilde{w}$. We begin with $C_1 \cup C_2$. In the first step, C_1 splits into $C_1 \cup C_3$. In the second step, C_2 splits into $C_2 \cup C_4$.

are each of the form $-k/(k - 1)$. The generators at the ends of the staircase must be equivariant, and all the intermediate generators (ie the corners of the staircase) must be nonequivariant because they are adjacent to $\tau^\#$ arrows.

To prove that there are no higher differentials in the ${}^{hv}E$ spectral sequence, we focus on individual staircases. This directs us to restrict our focus to a local piece of the complex surrounding the staircase which is large enough to determine the behavior of the map induced by the staircase.

Definition 31 Let \mathcal{D} be an annular link diagram with n crossings. Let α and β be two n -bitstrings where $\alpha < \beta$. The *subcube of resolutions* from R_α to R_β , denoted by $Cube(R_\alpha, R_\beta)$, is the subgraph of $Cube(\mathcal{D})$ such that

- the vertex set consists of resolutions R_γ , where γ is an n -bitstring with $\gamma[i] = \alpha[i] = \beta[i]$ whenever $\alpha[i] = \beta[i]$,
- the edge set consists of all edges of $Cube(\mathcal{D})$ whose endpoints are both in the vertex set.

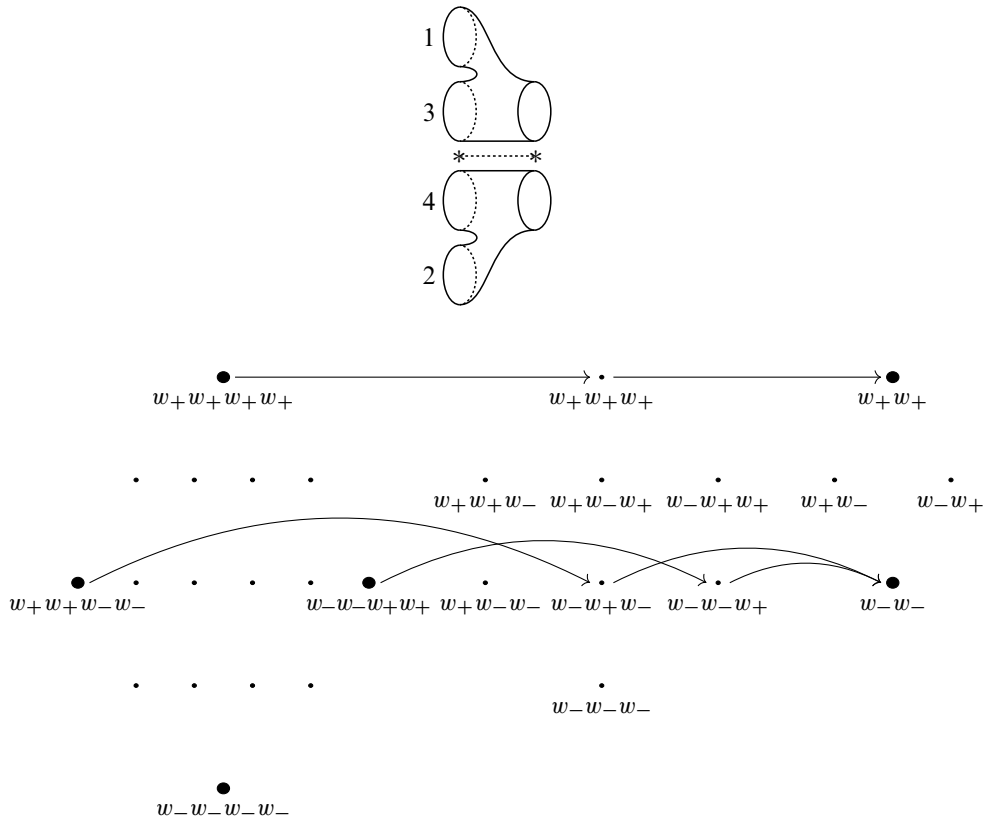


Figure 20: Type \mathcal{F} : $\tilde{w}\tilde{w} \rightarrow \tilde{w}$. We start with $C_1 \cup C_2 \cup C_3 \cup C_4$, where $C_1 \cup C_2$ and $C_3 \cup C_4$ are each equivariant pairs of circles. In the first step, $C_1 \cup C_3$ merges to C_1 and C_2 stays put. In the second step, $C_2 \cup C_4$ merges to C_2 and C_1 stays put.

In $\text{CUBE}(\mathcal{D})$, the *subcube of chains* from V_α to V_β , denoted by $\text{CUBE}(V_\alpha, V_\beta)$, is defined similarly, with

- underlying chain groups consisting of the vector spaces corresponding to the vertex set of $\text{Cube}(R_\alpha, R_\beta)$,
- differentials corresponding to the edge set of $\text{Cube}(R_\alpha, R_\beta)$.

In this case the crossings c_i in the diagram \mathcal{D} for which $\alpha[i] \neq \beta[i]$ are called *participating crossings*, and all other crossings are called *nonparticipating crossings*. Let $\mathcal{D}(\alpha, \beta)$ denote the link diagram obtained from $\mathcal{D}(L)$ by smoothing only the nonparticipating crossings based on the common values of α and β at these bits. Then $\text{Cube}(\mathcal{D}(\alpha, \beta)) \cong \text{Cube}(R_\alpha, R_\beta)$ and $\text{CUBE}(\mathcal{D}(\alpha, \beta)) \cong \text{CUBE}(V_\alpha, V_\beta)$.

Similarly, the components of R_α fall into two categories: the *participating circles*, which are adjacent to the sites of participating crossings, and the *nonparticipating circles*, which are the components of R_α that do not participate in a merge or split along any edge in $Cube(R_\alpha, R_\beta)$.

Definition 32 If R_α and R_β are equivariant resolutions of an equivariant annular link diagram \mathcal{D} , and $\alpha < \beta$, we say $Cube(R_\alpha, R_\beta)$ is *equivariantly split* if the set of participating circles can be divided into two or more equivalence classes under the equivalence relation

$$C_i \sim C_j \text{ if there is a (site of a) crossing adjacent to both } C_i \text{ and } C_j.$$

Proposition 33 For an equivariantly split 2-periodic diagram \mathcal{D} with $2r$ crossings, there are no induced differentials of length $2r$.

Proof If \mathcal{D} were equivariantly split, it could be written as a split diagram $\mathcal{D}_1 \sqcup \mathcal{D}_2$, and hence

$$CUBE(\mathcal{D}) \cong CUBE(\mathcal{D}_1) \otimes CUBE(\mathcal{D}_2).$$

Let $\emptyset \neq \{C_j\}_{j \in J_1}$ denote the set of circles in R_α corresponding to the all-zeros resolution of \mathcal{D}_1 , and $\emptyset \neq \{C_j\}_{j \in J_2}$ the set of circles in R_α corresponding to the all-zeros resolution of \mathcal{D}_2 .

The marked resolution x corresponds to a choice of markings $a \in \{v_-, v_+\}^{|J_1|}$ for $\{C_j\}_{j \in J_1}$ plus a choice of markings $\{v_-, v_+\}^{|J_2|}$ for $\{C_j\}_{j \in J_2}$, so for computational reasons we write $x = a \otimes b$, even though in reality we might have ordered the tensor components differently. (We are implicitly using the isomorphism $CUBE(\mathcal{D}) \cong CUBE(\mathcal{D}_1) \otimes CUBE(\mathcal{D}_2)$.) Since $x = a \otimes b$ is an equivariant generator, a and b must be equivariant generators for $CUBE(\mathcal{D}_1)$ and $CUBE(\mathcal{D}_2)$, respectively.

As the differential respects the tensor factors, we may use the product rule to compute $d^{2r}(x)$:

$$\begin{aligned} x &= a \otimes b, \\ \partial x &= \partial a \otimes b + a \otimes \partial b, \\ (1 + \tau^\#)^{-1} \partial x &= (1 + \tau^\#)^{-1} \partial a \otimes b + a \otimes (1 + \tau^\#)^{-1} \partial b, \\ \partial(1 + \tau^\#)^{-1} \partial x &= \partial(1 + \tau^\#)^{-1} \partial a \otimes b + (1 + \tau^\#)^{-1} \partial a \otimes \partial b \\ &\quad + \partial a \otimes (1 + \tau^\#)^{-1} \partial b + a \otimes \partial(1 + \tau^\#)^{-1} \partial b, \\ &\quad \vdots \end{aligned}$$

Here we abuse notation and write ∂ to mean the differential in $\text{CUBE}(\mathcal{D})$, $\text{CUBE}(\mathcal{D}_1)$, or $\text{CUBE}(\mathcal{D}_2)$, depending on context. Writing $\tilde{\partial} := (1 + \tau^\#)^{-1}\partial$, where $(1 + \tau^\#)^{-1}$ represents “lifting by $1 + \tau^\#$ ” (see Remark 28), the above computation becomes

$$\begin{aligned} d^{2r}x &= \partial\tilde{\partial}^{2r-1}x \\ &= \partial \sum_{i=0}^{2r-1} \binom{2r-1}{i} \tilde{\partial}^i a \otimes \tilde{\partial}^{2r-1-i} b \\ &= \sum_{i=0}^{2r-1} \binom{2r-1}{i} \partial\tilde{\partial}^i a \otimes \tilde{\partial}^{2r-1-i} b + \tilde{\partial}^i a \otimes \partial\tilde{\partial}^{2r-1-i} b. \end{aligned}$$

We want to show that each summand must be 0 on E^{2r} , where all the survivors are necessarily equivariant. Note that in order for $a' \otimes b' \in \text{CUBE}(\mathcal{D}_1) \otimes \text{CUBE}(\mathcal{D}_2)$ to be equivariant, a' and b' must be simultaneously equivariant, and hence must fall in a column corresponding to even Hamming weight in their respective CUBES.

Case 1 For i odd, $\partial\tilde{\partial}^i a$ and $\tilde{\partial}^{2r-1-i} b$ have even Hamming weight, while $\tilde{\partial}^i a$ and $\partial\tilde{\partial}^{2r-1-i} b$ have odd Hamming weight. If any component of $\partial\tilde{\partial}^i a \otimes \tilde{\partial}^{2r-1-i} b$ is equivariant, then $\tilde{\partial}^{2r-1-i} b$ is equivariant, contradicting the fact that $(1 + \tau^\#)\tilde{\partial}^{2r-1-i} b = \partial\tilde{\partial}^{2r-2-i} b \neq 0$.

Case 2 For i even, $\partial\tilde{\partial}^i a$ and $\tilde{\partial}^{2r-1-i} b$ have odd Hamming weight, while $\tilde{\partial}^i a$ and $\partial\tilde{\partial}^{2r-1-i} b$ have even Hamming weight. If any component of $\tilde{\partial}^i a \otimes \partial\tilde{\partial}^{2r-1-i} b$ is equivariant, then $\tilde{\partial}^i a$ is equivariant, which contradicts the fact that $(1 + \tau^\#)\tilde{\partial}^i a = \partial\tilde{\partial}^{i-1} a \neq 0$. □

5.4 Grading shifts of higher differentials

Again let α, β be equivariant $2r$ -bitstrings with $\alpha < \beta$. By Proposition 33, we may now assume that $\mathcal{D}(\alpha, \beta)$ is not equivariantly split. This will allow us to perform computations by describing parts of the complex using connected graphs.

A first obstruction to the existence of a ${}^{\text{hv}}d^{2r}$ map between distinguished generators x and y is that x and y must have the same j -grading. For this reason, we compare the j -span of V_α and V_β , defined as follows.

Definition 34 The j -span of the vector space V_ω associated with the binary string ω is the range of j -gradings of the distinguished generators of V_ω .

Split maps shift the upper bound of the j -span up by 2, and merge maps shift the lower bound of the j -span up by 2.

Definition 35 A path through a cube of resolutions is a sequence of vertices R_{α_i} such that α_{i+1} is a bit increment from α_i , together with the edges between successive resolutions.

Since any component of $h^v d^{2r}$ will come from repeatedly mapping via ∂ and lifting by $1 + \tau^\#$ along a (directed) path from V_α to V_β in $\text{CUBE}(\mathcal{D}(\alpha, \beta))$, and $1 + \tau^\#$ affects neither of the j - or k -gradings, we focus on paths through $\text{CUBE}(\mathcal{D}(\alpha, \beta))$ and which combinations of merges and splits they involve.

Lemma 36 Any two paths through a cube $\text{CUBE}(\mathcal{D})$ from the all-zeros resolution to the all-ones resolution have the same number of merges, and also have the same number of splits.

Proof Let $\#m$ and $\#\Delta$ denote the number of merge and split maps along a path from V_α to V_β . These quantities satisfy the linear equations

$$\#m + \#\Delta = 2r, \quad |R_\beta| - |R_\alpha| = \#\Delta - \#m. \quad \square$$

So, in order to count merge and splits, we may without loss of generality choose any path we'd like. Choose γ to be a path through the cube for which every other vertex is equivariant. In other words, after every two steps along this path, we land on an equivariant resolution. Each such set of two-steps corresponds to one step in a path in $\text{CUBE}(\mathcal{D}(\alpha, \beta)/\tau_A)$, since each equivariant pair of crossings corresponds to a crossing downstairs. We can thus classify the six different two-step paths in γ by looking at the classification of edges in the cube downstairs.

Definition 37 We can describe each two-step using modular pieces of a graph, which we call *atoms* (Figure 21). These pieces are oriented in the sense that the lines on the left of the circle represent *input circles*, or circles in the source resolution, and the lines on the right of the circle represent the *output circles*, or circles in the target resolution of the two-step. A double bond indicates an equivariant twin pair of W circles, and a single bond indicates a single equivariant V circle. While attaching these atoms together, we can only attach V circles to V circles, and equivariant pairs of W circles to equivariant pairs of W circles. A connected graph obtained by attaching these atoms together is called a *molecule*.

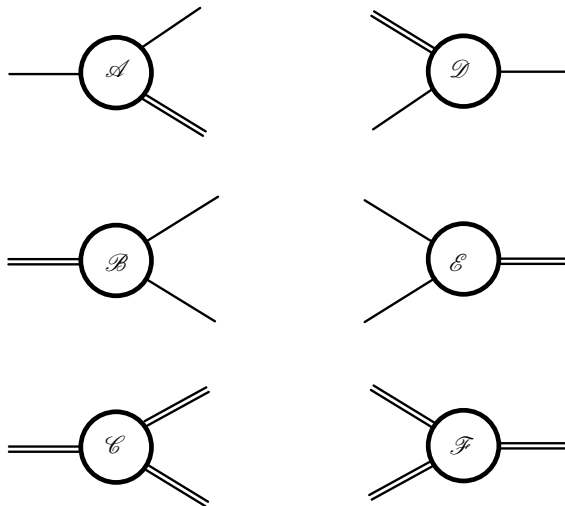


Figure 21: The six *atoms* representing the six possible equivariant two-steps

Since the $\mathcal{D}(\alpha, \beta)/\tau_A$ diagram downstairs is nonsplit, when the pieces of γ are attached at their equivariant ends, we obtain a single molecule Γ . Let A, B, C, D, E and F denote the number of modular pieces of type $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ and \mathcal{F} , respectively. In particular, this means that $A + B + C + D + E + F = r$.

Proposition 38 *Given an equivariant path γ through an equivariantly nonsplit cube, the size of the j -span overlap between the vector spaces corresponding to the all-zeros resolution V_α and the all-ones resolution V_β is*

$$2(A + B + 2C + D + E + 2F - N_b),$$

where

- A, B, C, D, E and F are the number of atoms of type $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ and \mathcal{F} , respectively, in the molecule Γ corresponding to γ ,
- N_b is the total number of bonds (counted with multiplicity) in Γ .

Proof Let j_0 denote the minimal j -grading of a distinguished generator of V_α . The number of components of R_α is equal to the number of input circles of the atoms separately, minus the number of input circles which are then identified with output circles of other atoms, which is equal to the number of total bonds, counted with multiplicity. Let N_b denote the total number of bonds. The source j -span is therefore

$$[j_0, j_0 + 2(A + 2B + 2C + 3D + 2E + 4F - N_b)].$$

Traveling through the path γ , we ultimately shift the bottom of the j -span by (two times) the number of merges, which is $2(B + 2D + E + 2F)$. Hence the possible j -span overlap between V_α and V_β is

$$[j_0 + 2(B + 2D + E + 2F), j_0 + 2(A + 2B + 2C + 3D + 2E + 4F - N_b)],$$

which is a difference of $2(A + B + 2C + D + E + 2F - N_b)$. □

Thus the largest j -span overlap using the atoms in Γ is achieved when Γ has a minimal total number of bonds.

Lemma 39 *The minimal total number of bonds achievable by a connected molecule Γ comprised of a fixed set of atoms is achieved when Γ is composed to two trees, one containing all the \mathcal{A} , \mathcal{B} , \mathcal{D} and \mathcal{E} atoms, and the other containing all the \mathcal{C} and \mathcal{F} atoms, with the two trees connected by a unique double bond.*

Proof The structure of a minimally bonded Γ must be a tree: if Γ has a cycle, we can decrease the number of bonds by cutting any side of the cycle.

Furthermore, when attaching atoms of type \mathcal{A} , \mathcal{B} , \mathcal{D} and \mathcal{E} , the number of bonds is minimized by connecting to the molecule via a single rather than a double bond. When these atoms are arranged as a subtree of Γ , the number of single bonds is minimized. The \mathcal{C} and \mathcal{F} atoms will invariably attach to the tree via a double bond, so the two-tree molecule described has the same number of double bonds as any minimally bonded Γ . □

5.5 Higher differentials vanish

The final major step in the proof of Theorem 1 uses the j - and k -gradings to show that the ${}^{\text{hv}}E$ spectral sequence collapses on ${}^{\text{hv}}E^3$.

Proposition 40 *For all $r \geq 2$, we have ${}^{\text{hv}}d^{2r} = 0$.*

Proof By Proposition 33, we may assume that $\text{Cube}(R_\alpha, R_\beta)$ is equivariantly nonsplit. Pick an equivariant path γ through $\text{Cube}(R_\alpha, R_\beta)$, and let Γ denote the associated molecule. Let's first assume that at least one of the atoms of Γ is *not* of type \mathcal{C} or \mathcal{F} . In the worst case, the total number of bonds is the sum of $A + B + D + E - 1$ from the $\mathcal{A}, \mathcal{B}, \mathcal{D}, \mathcal{E}$ tree, $2C + 2F - 2$ from the \mathcal{C}, \mathcal{F} tree, and 2 from connecting the two

trees. Hence $N_b = A + B + 2C + D + E + 2F - 1$. The size of the overlap is then

$$2(A + 2B + 2C + 3D + 2E + 4F - (A + B + 2C + D + E + 2F - 1)) - 2(B + 2D + E + 2F) = 2.$$

Therefore x must have either the maximum or next-to-maximum j -grading of the distinguished generators of V_α . Since x is equivariant, any equivariant pair of trivial circles must have the same marking, so since x can have at most one component marked with a v_- , all trivial circles in x are marked w_+ . Moreover, from Figures 15 through 20 we see that if any equivariant paths out of V_α are not of type \mathcal{E} , then x is not in $\ker(\text{hv}d^2)$. So all equivariant paths out of V_α must be of type \mathcal{E} . This indicates that x is a marked resolution consisting solely of v_+ -marked concentric circles. But this means x is unique in its k -grading in the entire $\text{CUBE}(V_\alpha, V_\beta)$. Therefore there cannot be a $\text{hv}d^{2r}$ component from x to y in $\text{CUBE}(V_\alpha, V_\beta)$.

In the case where *all* the atoms are of type \mathcal{C} or \mathcal{F} , the worst case Γ has $2C + 2F - 2$ bonds, producing a j -span overlap of

$$2(2C + 4F - (2C + 2F - 2)) - 2(2F) = 4.$$

Therefore x must have at most two circles marked with w_- . Again, we see in Figures 19 and 20 that x is not in $\ker(\text{hv}d^2)$.

Thus there cannot be a $\text{hv}d^{2r}$ component from (a θ -power translate of) x to (a θ -power translate of) y . □

As $\text{AKh}^{\text{Tate}}(\mathcal{D}(\tilde{L}))$ is 1-periodic in the horizontal direction, its homology is 1-periodic as well, so the total rank along each diagonal corresponding to the induced filtration gradings on H^{Tot} is the same as the total rank down along a vertical column. Therefore $\text{AKh}(L) \otimes_{\mathbb{F}_2} \mathbb{F}[\theta, \theta^{-1}] \cong \text{hv}E^3 = \text{hv}E^\infty \cong \text{vh}E^\infty$, so the $\text{vh}E$ spectral sequence has $\text{vh}E^1 \cong \text{AKh}(\tilde{L}) \otimes_{\mathbb{F}_2} \mathbb{F}[\theta, \theta^{-1}]$ and $\text{vh}E^\infty \cong \text{AKh}(L) \otimes_{\mathbb{F}_2} \mathbb{F}[\theta, \theta^{-1}]$.

Since all the maps in the double complex preserve j - and k -grading, the spectral sequence splits along j - and k -gradings. This concludes the proof of Theorem 1.

Finally, since each page of a spectral sequence is a subquotient of the previous page, Corollary 2 follows:

Corollary 2 *For any annular tangle closure L , quantum grading j and \mathfrak{sl}_2 weight-space grading k ,*

$$\text{rk}_{\mathbb{F}_2} \text{AKh}^{j,k}(L) \leq \text{rk}_{\mathbb{F}_2} \text{AKh}^{2j-k,k}(\tilde{L}).$$

6 Example: Hopf link and stabilized unknot

To help convey the construction to the reader, we describe the simplest nontrivial example, a two-crossing annular Hopf link as the 2-braid closure $\tilde{L} := \hat{\sigma}_1^2$. The quotient link is a positively stabilized unknot $L = \hat{\sigma}_1$.

Annular diagrams for both \tilde{L} and L are shown in Figure 7. We label the crossings of $\mathcal{D}(\tilde{L})$ in accordance to our convention (Section 4.1); the first crossing is on the right, and the second crossing is on the left. The cube of resolutions $Cube(\mathcal{D}(\tilde{L}))$ is shown in Figure 8. In order to simplify the notation in this small example, we give the distinguished generators unique, concise names in the cube of chains shown in Figure 22. For example, “ a_{+-} ” is the generator $v_+ \otimes v_-$ in V_{00} , corresponding to the marked resolution shown in Figure 23.

Putting this all together, the bicomplex $Akh^{Tate}(\mathcal{D}(\tilde{L}))$ locally looks like Figure 24.

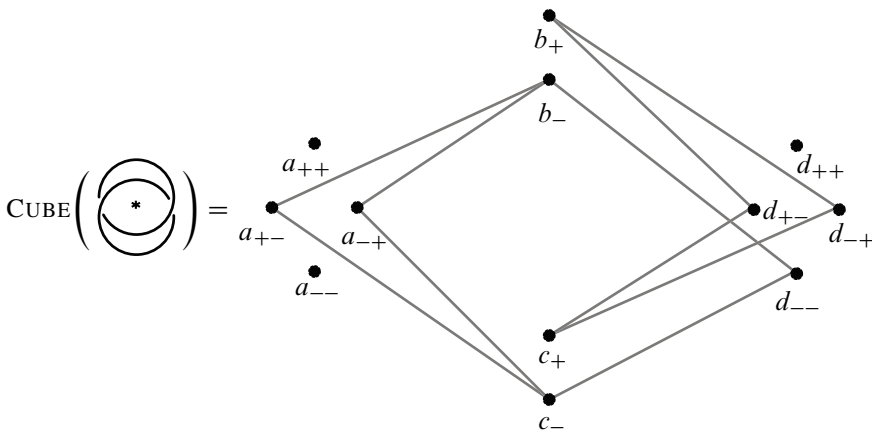


Figure 22: The cube of chains for the annular Hopf link from Figure 7, with AKh differentials drawn as solid lines and $\tau^\#$ drawn as dotted lines. The arrow heads have been dropped; gr_i increases to the right. This is a finer version of Figure 9.

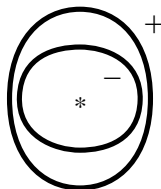


Figure 23: The marked resolution corresponding to the generator a_{+-}

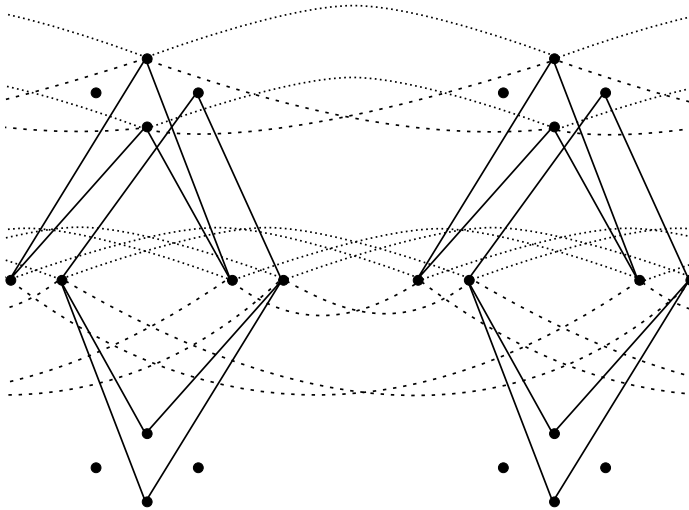


Figure 24: A local picture of the $\text{AKh}^{\text{Tate}}(\mathcal{D}(\tilde{L}))$ bicomplex for the Hopf link example. There are two copies of the cube of chains in this figure, rotated 90° clockwise from Figure 9. The dotted lines which bend up represent the “identity” map which increases the t -grading, and the dashed lines which bend down represent the $\tau^\#$ map. This local picture is repeated horizontally to form $\text{AKh}^{\text{Tate}}(\mathcal{D}(\tilde{L}))$.

After canceling all the $\tau^\#$ arrows, the only remaining interesting page to compute the homology of is ${}^{\text{hv}}E^2$, shown in Figure 25. As expected, each line with slope -2 in ${}^{\text{hv}}E^2$ corresponds to the cube of chains for the quotient link L , under the correspondence described by Propositions 29 and 30.

In this case, the fact that there are no higher differentials is clear from the fact that $\text{CKh}(\mathcal{D}(\tilde{L}))$ is only supported on three adjacent i -gradings. For comparison, the annular Khovanov homologies for \tilde{L} and L are listed in Table 1.

| (j, k) | $\text{rk}_{\mathbb{F}_2}(\text{AKh}^{j,k}(L))$ | $(2j - k, k)$ | $\text{rk}_{\mathbb{F}_2}(\text{AKh}_{2j-k,k}(\tilde{L}))$ |
|------------|---|---------------|--|
| $(3, 2)$ | 1 | $(4, 2)$ | 1 |
| $(1, 0)$ | 1 | $(2, 0)$ | 1 |
| $(-1, -2)$ | 1 | $(0, -2)$ | 1 |
| $(3, 0)$ | 1 | $(6, 0)$ | 1 |
| | | $(4, 0)$ | 2 |

Table 1: The annular Khovanov homologies for \tilde{L} and L

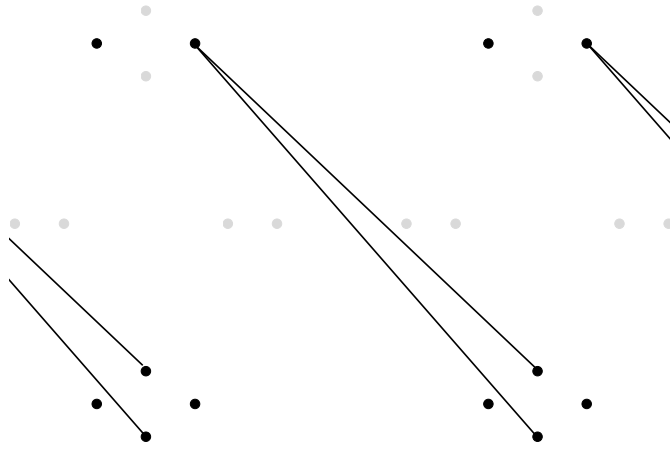


Figure 25: A local picture of the page ${}^{hv}E^2$ in our Hopf link example

7 Decategorification

Annular Khovanov homology was originally described as a categorification of the Kauffman bracket skein module in the trivial I -bundle over the annulus. In our case, we work with \mathbb{F}_2 coefficients and follow the conventions in [21].

Consider the $\mathbb{F}_2[q^{\pm 1}]$ -module consisting of all links in $A \times I$ modulo the usual Kauffman bracket skein relations. The set $\mathcal{B} = \{\hat{\mathbb{1}}_s : s = 0, 1, 2, \dots\}$ of all trivial braid closures forms a canonical basis for the skein module.

We use the variables t , q and x to record the i -, j -, and k -gradings, respectively. Setting $q = A^{-2}$ and introducing the k -grading to the skein relation, we have a variant of the Kauffman bracket skein relations. Choose a particular crossing c and let L_0 and L_1 denote the links obtained from L by smoothing c by the 0-resolution and 1-resolution, respectively. Then our relations are

$$\begin{aligned} \langle L \rangle &= \langle L_0 \rangle + tq \langle L_1 \rangle, \\ \langle L \cup U \rangle &= (q + q^{-1}) \langle L \rangle, \end{aligned}$$

where U is an annular unknot (a trivial circle).

Normalizing by setting $\hat{\mathbb{1}}_1 \mapsto qx + q^{-1}x^{-1}$, we obtain a map

$$\phi: \mathbb{F}_2[t, q^{\pm 1}, \mathcal{B}] \rightarrow \mathbb{F}_2[t, q^{\pm 1}, x^{\pm 1}]$$

by viewing \mathcal{B} as a monoid under disjoint union. Finally, we add in the proper grading shifts to obtain a variant of the Jones polynomial over \mathbb{F}_2 by setting $V(t, q, x) = t^{n_-} q^{n_+ - 2n_-} \phi(L)$. (To be clear, one needs to first orient L to obtain n_- and n_+ , and we are viewing the link L as an element of the skein module.)

Of interest to us is the annular link invariant $V_L(-1, q, x)$ which is the graded Euler characteristic of $\text{AKh}(L)$. (See Theorem 8.1 in [2] for details.) In other words,

$$V_L(t, q, x) = \sum_{i,j,k} t^i q^j x^k \text{rk}_{\mathbb{F}_2}(\text{AKh}^{i,j,k}(L)),$$

so the graded Euler characteristic is

$$V_L(-1, q, x) = \sum_{i,j,k} (-1)^i q^j x^k \text{rk}_{\mathbb{F}_2}(\text{AKh}^{i,j,k}(L)).$$

As in [2] and [21], we view the Kauffman bracket as a set of polynomials by defining $q_{k,L}(t, q)$ to be the coefficient of x^k in $V(t, q, x)$, so that

$$q_{k,L}(-1, q) = \sum_{i,j} (-1)^i q^j \text{rk}_{\mathbb{F}_2} \text{AKh}^{i,j,k}(L)$$

is the coefficient of x^k in the decategorification of $\text{AKh}(L)$.

The decategorification of Theorem 1 is the following:

Corollary 3 For all j and k ,

$$\langle q_{k,\tilde{L}}(-1, q), q^{2j-k} \rangle \equiv \langle q_{k,L}(-1, q), q^j \rangle \pmod{2},$$

where $\langle f, g \rangle$ denotes the coefficient of g in f .

Proof This follows from Theorem 1 and the fact that cancellation reduces rank by a multiple of 2 at each step (represented by erasing the dots at the head and tail of the arrow). □

In [16], Murasugi proved a relationship between the Alexander polynomials of a periodic link and that of its quotient; in [17], he proved an analogous formula for the Jones polynomial. By considering Theorem 1 with respect to the $j_1 = j - k$ grading rather than the j - and k -gradings individually, we obtain an annular analogue to Murasugi’s formulas:

Corollary 4 $V_{\tilde{L}}(1, q, q^{-1}) \equiv [V_L(1, q, q^{-1})]^2 \pmod{2}.$

If on the other hand one is more interested in the usual Jones polynomial of the 2-periodic link, we may forget the k -grading information by setting $x = 1$, obtaining the following relationship:

Corollary 5 $V_{\tilde{L}}(1, q, 1) \equiv V_L(1, q^2, q^{-1}) \pmod{2}.$

It is then natural to ask if there is a categorification of this relationship, coming from a spectral sequence from the Khovanov homology of \tilde{L} to the annular Khovanov homology of L (with some grading information sacrificed). In the next section, we show that this spectral sequence indeed exists for some families of annular links, and is likely to exist in general by way of a bicomplex very similar to the AKh^{Tate} bicomplex.

8 The Khovanov–Tate bicomplex

We can similarly define the *Khovanov–Tate bicomplex* $\text{Kh}^{\text{Tate}}(\mathcal{D}(\tilde{L}))$ by replacing the vertical differentials ∂_{AKh} in $\text{AKh}^{\text{Tate}}(\mathcal{D}(\tilde{L}))$ with the Khovanov differential ∂_{Kh} .

Conjecture 6 *Let \tilde{L} be a 2-periodic link in S^3 with quotient link L . There is a spectral sequence with*

$$E^1 \cong \text{Kh}(\tilde{L}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}] \rightrightarrows E^\infty \cong \text{AKh}(L) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}].$$

This would in turn imply the following cascade of rank inequalities:

$$\text{rk}_{\mathbb{F}_2} \text{AKh}(\tilde{L}) \geq \text{rk}_{\mathbb{F}_2} \text{Kh}(\tilde{L}) \geq \text{rk}_{\mathbb{F}_2} \text{AKh}(L) \geq \text{rk}_{\mathbb{F}_2} \text{Kh}(L),$$

where the first and third inequalities are given by the k -grading filtration on $\text{CKh}(\mathcal{D}(\tilde{L}))$ and $\text{CKh}(\mathcal{D}(L))$.

In Section 8.1 we give abbreviated proofs of the Kh analogues of the results from the AKh case which support Conjecture 6. But first, we discuss holistic reasons why we would expect this conjecture to hold.

Remark 41 Aside from the proofs of theorems given in Section 8.1, this conjecture would not be hard to believe. $H_*(\text{Tot}(\text{Kh}^{\text{Tate}}))$ is supposed to capture information about both the periodic link and the involution witnessing the symmetry, so the presence of the braid axis in S^3 is inherently captured in the data of the Kh^{Tate} bicomplex. Hence $H_*(\text{Tot}(\text{Kh}^{\text{Tate}}))$ should take on the form of an annular version of Khovanov homology.

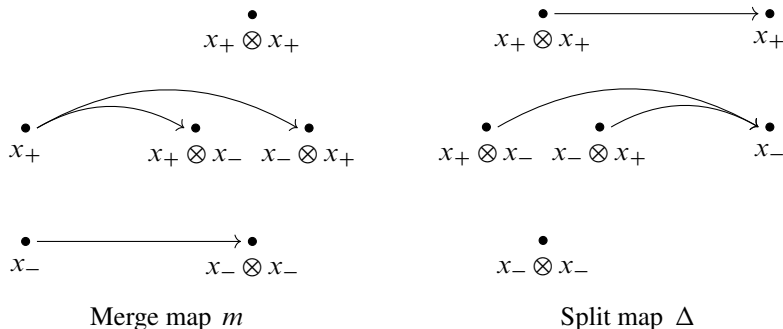


Figure 26: The Khovanov differentials. These do not depend on the base-point $*$, so x can stand for v or w .

Furthermore, Seidel and Smith proved the rank inequality for symplectic Khovanov homology [23], which is known to be equivalent to Khovanov homology over ground rings of characteristic 0 [1]:

$$\text{rk}_{\mathbb{F}_2} \text{Kh}_{\text{symp}}(\tilde{L}) \geq \text{rk}_{\mathbb{F}_2} \text{Kh}_{\text{symp}}(L).$$

They conjecture that symplectic Khovanov homology is also isomorphic to Khovanov homology in characteristic 2 [22], so some of the rank inequalities implied by this conjecture agree with their result.

8.1 Analogous statements in the Kh^{Tate} case

Khovanov homology can be viewed as a deformation of annular Khovanov homology, defined for knots in S^3 . This is why we have been denoting the underlying vector space for the annular Khovanov complex of a link diagram by CKh ; this is the same underlying vector space as in the Khovanov complex. Since there is no basepoint in the link diagram, the Khovanov distinguished generators are only doubly graded, by the homological i -grading and the quantum j -grading. There are only two types of edge maps: merge and split. Their definitions are shown in Figure 26 using the dot and arrow notation we used to describe the AKh differentials in back in Section 3.3.

Let ${}^{\text{h}v}E$ and ${}^{\text{v}h}E$ denote the spectral sequences induced by the rowwise and columnwise filtrations, respectively, of the bicomplex $\text{Kh}^{\text{Tate}}(\mathcal{D}(\tilde{L})) = \text{CKh}(\mathcal{D}(\tilde{L})) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}]$, where $\partial^{\text{v}} = \partial_{\text{Kh}}$ and $\partial^{\text{h}} = 1 + \tau^{\#}$.

Lemma 42 *The maps ∂_{Kh} and $\tau^{\#}$ commute.*

Proof We need to check that the two types of edge maps (merge and split) commute with $\tau^\#$. This is easily checked, and clear from thinking about the involution acting on the saddle cobordism representing the merge or split map, just as in the AKh cases. \square

Lemma 43 ${}^{\text{vh}}E^1 \cong \text{Kh}(\tilde{L}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}]$.

Proof This follows from the same argument as in the proof of Lemma 23. \square

Lemma 44 ${}^{\text{hv}}E^1$ is generated by the equivariant generators of $\text{Kh}^{\text{Tate}}(\mathcal{D}(\tilde{L}))$. Hence if r is odd, ${}^{\text{hv}}d^r = 0$.

Proof Since the rows of the $\text{Kh}^{\text{Tate}}(\mathcal{D}(\tilde{L}))$ bicomplex are identical to those of the $\text{AKh}^{\text{Tate}}(\mathcal{D}(\tilde{L}))$ complex, this is equivalent to Lemma 24. The second statement follows from the first just as in the proof of Lemma 26. \square

Lemma 45 Every component of ${}^{\text{hv}}d^2$ is induced by the cancellation of a $\tau^\#$ arrow at a row corresponding to an odd Hamming weight.

Proof The proof of Lemma 27 also holds in this case. \square

Proposition 46 There is a one-to-one correspondence

$$\{\text{generators of } {}^{\text{hv}}E^2\} \longleftrightarrow \{\text{generators of } \text{CKh}(L) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}]\}, \quad \tilde{v} \longleftrightarrow v$$

induced by τ .

Proof By Lemma 44, we have already identified the generators of ${}^{\text{hv}}E^2$ with the equivariant marked resolutions, which the proof of Proposition 29 shows to be in bijection with the generators of $\text{CKh}(\mathcal{D}(L)) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}]$ (since this is the underlying vector space in both the $\text{Kh}^{\text{Tate}}(\mathcal{D}(L))$ and $\text{AKh}^{\text{Tate}}(\mathcal{D}(L))$ bicomplexes). \square

Proposition 47 Under the correspondence above, each line with -2 slope in the bicomplex $({}^{\text{hv}}E^2, {}^{\text{hv}}d^2)$ is isomorphic to the complex $(\text{CKh}(\mathcal{D}(L)), \partial_{\text{AKh}})$.

Proof Again, we just need to compute the induced length-2 differentials. Since there are six types of AKh differentials, we can verify them case-by-case. The computations are shown in Figures 27 through 32. \square

Since the proof that higher differentials vanish relies on the k -grading, we cannot use the same methods to determine whether higher differentials vanish in the Kh^{Tate} case. However, the k -grading is only needed when considering atoms of type \mathcal{E} (see Section 5.4), so one can still consider special cases where this obstruction is not needed for the conjecture to hold.

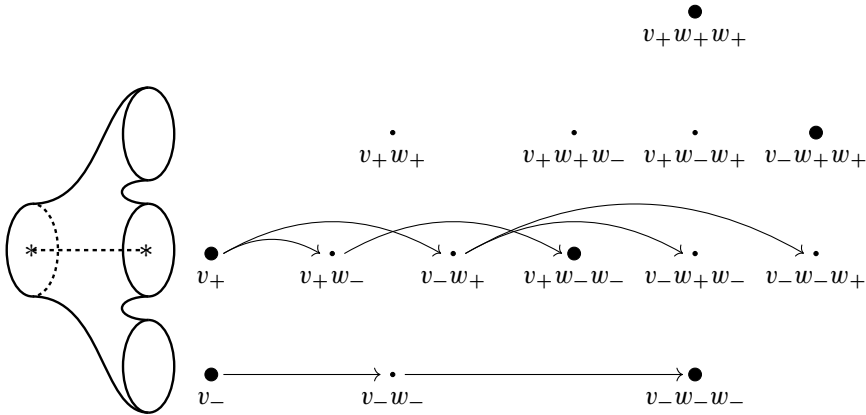


Figure 27: Type \mathcal{A} : $\tilde{v} \rightarrow \tilde{v}\tilde{w}$ for Khovanov differentials

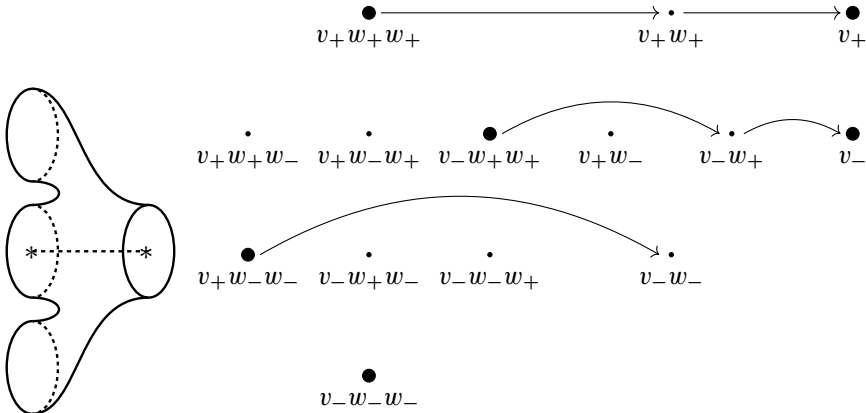


Figure 28: Type \mathcal{B} : $\tilde{v}\tilde{w} \rightarrow \tilde{v}$ for Khovanov differentials

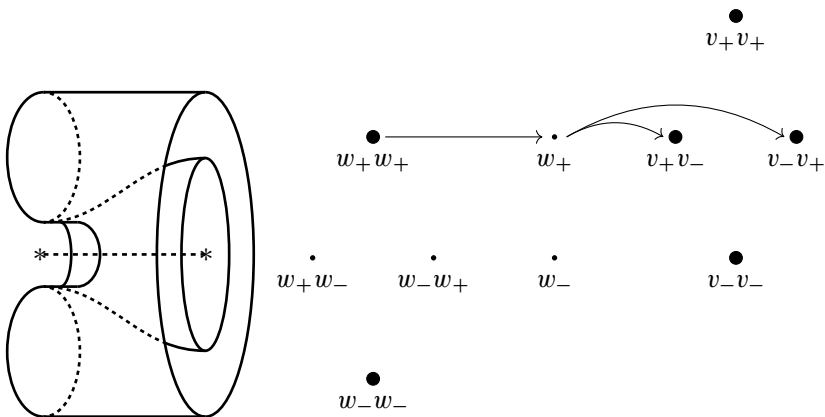


Figure 29: Type \mathcal{B} : $\tilde{w} \rightarrow \tilde{v}\tilde{v}$ for Khovanov differentials

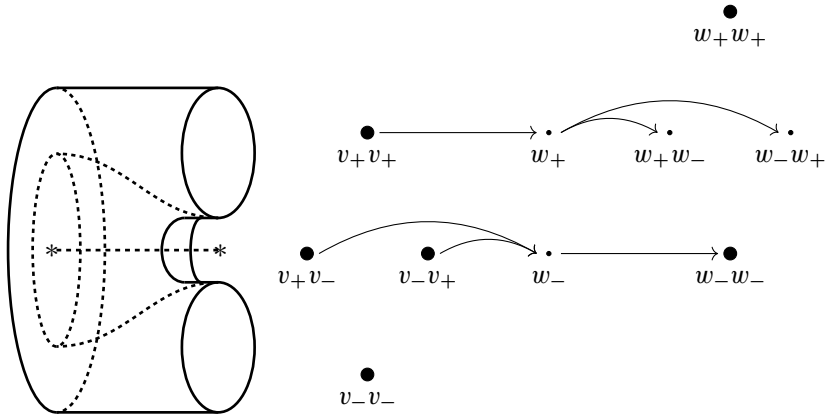


Figure 30: Type \mathcal{E} : $\tilde{v}\tilde{v} \rightarrow \tilde{w}$ for Khovanov differentials

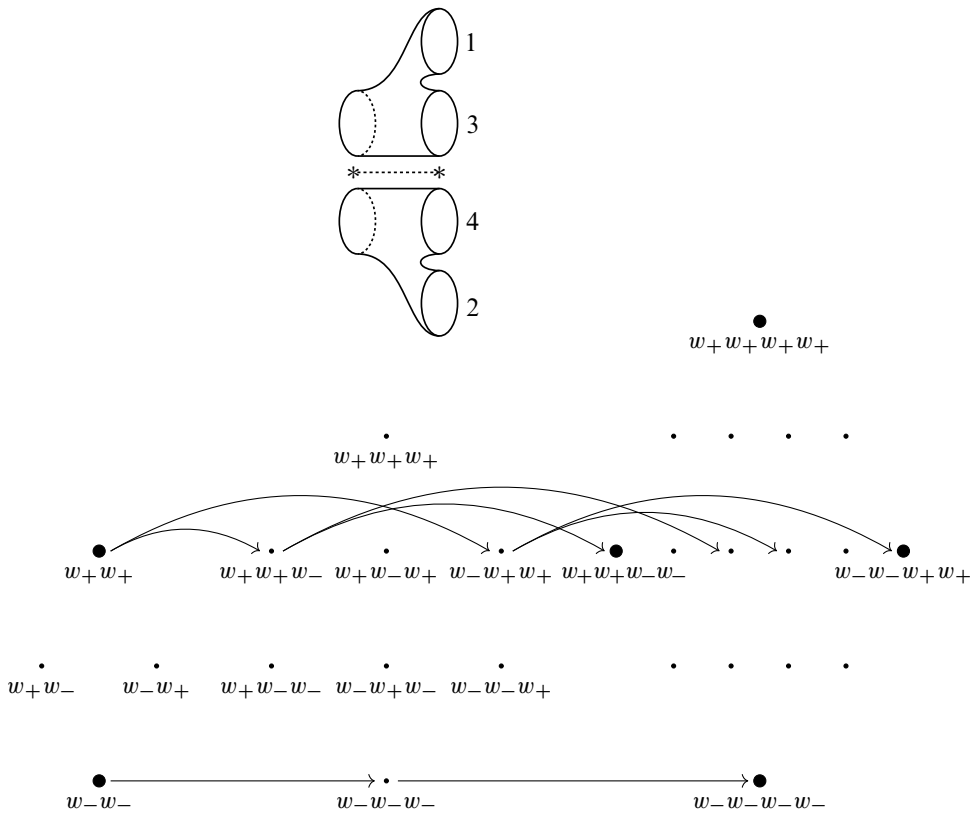


Figure 31: Type \mathcal{C} : $\tilde{w} \rightarrow \tilde{w}\tilde{w}$ for Khovanov differentials

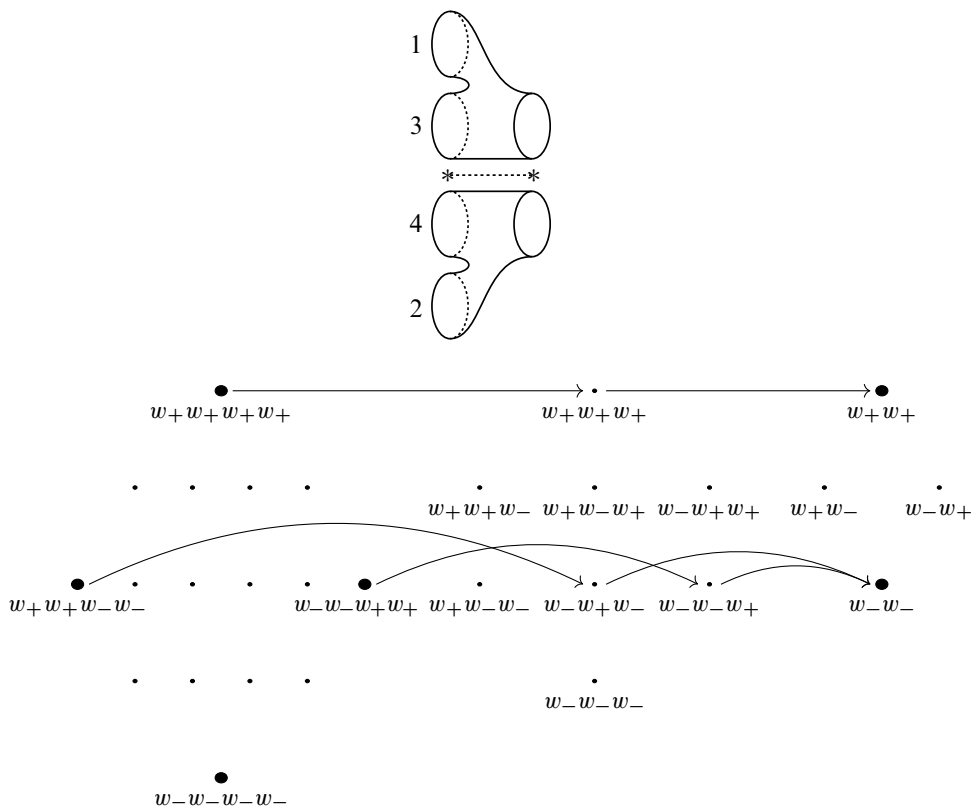


Figure 32: Type F: $\tilde{w}\tilde{w} \rightarrow \tilde{w}$ for Khovanov differentials

8.2 Positive and negative braid closures

Recall that the k -grading is used in the proof of Theorem 1 only when the j -grading was insufficient, namely in the situation where all the first equivariant steps through the subcube are \mathcal{E} atoms. From this we deduce that Conjecture 6 holds for “mostly negative” links.

Theorem 7 *If the annular braid closure $L = \hat{\beta}$ has a diagram with at most 1 positive crossing, then the spectral sequence in Conjecture 6 exists and the cascade of rank inequalities holds.*

Proof The proof of Theorem 1 relies mostly on the j -grading to prove that the ${}^{hv}E$ spectral sequence collapses on page ${}^{hv}E^3$, except for the case of the generator v_+v_+ in the starting resolution of type \mathcal{E} atoms. In the Kh^{Tate} case, again the j -grading is sufficient for all cases except for type \mathcal{E} , so for cases where at most one atom of

type \mathcal{E} appears, longer differentials cannot exist. This corresponds to the case where $Cube(\mathcal{D}(L))$ has at most one edge corresponding to a change of resolution from two nontrivial circles to two trivial circles. This means that the all-zeros resolution of $\mathcal{D}(L)$ corresponds to at most one braid-like resolution; that is, the braid β has at most one positive crossing. \square

On the other hand, we cannot use the same method to prove the existence of the spectral sequence for “mostly positive” braid closures. However, we can use duality in Khovanov and annular Khovanov homology to show that the cascade of rank inequalities still holds in this case.

Corollary 8 *If the annular braid closure $L = \widehat{\beta}$ has a diagram with at most 1 negative crossing, then the cascade of rank inequalities holds.*

Proof Let L be a link in S^3 and \mathcal{D} a diagram of L . Let $L^!$ denote the mirror of L , and $\mathcal{D}^!$ the dual diagram to \mathcal{D} (switch the signs of all crossings). Observe that the complex $(CKh(\mathcal{D}), \partial^{Kh})$ is dual to $(CKh(\mathcal{D}^!), \partial^{Kh})$. The cohomology of $CKh(\mathcal{D})$ is isomorphic to the homology of $CKh(\mathcal{D}^!)$, which in turn is isomorphic to the cohomology of $CKh(\mathcal{D}^!)$, as we are working over field coefficients. (It is now important to note that what we have been calling “Khovanov homology” is actually a cohomology theory.) Therefore $rk_{\mathbb{F}_2} Kh(\mathcal{D}) = rk_{\mathbb{F}_2} Kh(\mathcal{D}^!)$. For an annular link $L \subset A \times I$, the above paragraph holds analogously for annular Khovanov (co)homology. Therefore it follows from Theorem 7 that the cascade of rank inequalities also holds for mostly positive braid closures. \square

Aside from proving the conjecture, many related questions remain.

Question 48 Are there other families of links for which the conjecture holds? Do certain steps in the proof for the AKh^{Tate} spectral sequence point to other obstructions to longer differentials for the Kh^{Tate} spectral sequence? What are some other algebraic conditions on a braid closure which guarantee that the conjecture holds?

Question 49 While Khovanov-thin links have been studied for some time, at present I am unaware of any explorations of “annular Khovanov-thin links”. One could define an annular link L to be AKh -thin if for each pair (j, k) of quantum and \mathfrak{sl}_2 weight-space gradings, $AKh^{j,k}$ is supported on two adjacent i -gradings. Then AKh -thinness of the quotient link L suffices for the spectral sequence in Conjecture 6, since all differentials after page ${}^{hv}E^3$ are too long in i -degree to be nontrivial. Which classes of knots or links are AKh -thin?

8.3 Obstructing 2–periodicity

One motivation for pursuing Conjecture 6 arises from the study of obstructions to periodicity in links. The present method may overcome the following difficulties encountered by prior work.

First, obstructing p –periodicity for small p is particularly difficult. In particular, Murasugi’s Jones and Alexander polynomial congruences [16; 17] are weaker for $p = 2$ as mod 2 congruences only detect parity differences in coefficients. Comparing ranks in categorified invariants can potentially detect more differences. Borodzik and Politarczyk choose to work with the Khovanov polynomial rather than a graded Euler characteristic for this reason (see page 2 of [6]); however, their obstruction is only valid for $p > 3$.

Second, sometimes obstructions to knot periodicity do not extend to links with multiple components. For example, Hendricks’ rank inequality for $\widehat{\text{HFK}}$ pertains only to knots (see Section 3.2 in [12]). The present methods do not depend on the number of link components.

In this section, we formulate some frameworks for obstructing 2–periodicity as consequences of Theorem 1 and of various portions of Conjecture 6, including the special cases of Theorem 7 and Corollary 8.

Notation 50 Throughout this section, we fix the following notation. The symbols L and \tilde{L} indicate annular links (in $A \times I$), with $[L]$ and $[\tilde{L}]$ denoting their isotopy classes as links in S^3 . Since our frameworks consider a potentially periodic link $\tilde{L} = \hat{T}$ or $[\tilde{L}]$ and compare it with a potential quotient link L or $[L]$, a priori we do not know whether $\tilde{L} = \hat{T}^2$.

Corollary 2 can be framed as an obstruction as follows.

Corollary 51 *Let $L = \hat{T}$ and \tilde{L} be annular links. Suppose there is some (j, k) bigrading for which $\text{rk}_{\mathbb{F}_2} \text{AKh}^{2j-k, k}(\tilde{L}) \leq \text{rk}_{\mathbb{F}_2} \text{AKh}^{j, k}(L)$. Then $\tilde{L} \neq \hat{T}^2$.*

Now suppose a weaker version of Conjecture 6 were true; that is, suppose that the rank inequality holds for Khovanov homology. (Recall that Seidel and Smith proved this rank inequality holds for symplectic Khovanov homology over \mathbb{F}_2 [23], which is to date still only conjectured to be equivalent to Khovanov homology over \mathbb{F}_2 .)

Conjecture 52 *Let $[L]$ and $[\tilde{L}]$ be two links in S^3 . If $\text{rk}_{\mathbb{F}_2} \text{Kh}([\tilde{L}]) \leq \text{rk}_{\mathbb{F}_2} \text{Kh}([L])$, then $[\tilde{L}]$ is not 2–periodic with quotient link $[L]$. In other words, there are no representatives $\tilde{L} \in [\tilde{L}]$ and $L \in [L]$ such that $L = \hat{T}$ and $\tilde{L} = \hat{T}^2$.*

Remark 53 By comparing Propositions 29 and 46, it is clear that the present method will not yield a rank inequality for Khovanov homology splitting along any gradings, because the correspondence between equivariant generators of $\text{CKh}(\mathcal{D}(\tilde{L}))$ and the generators of $\text{CKh}(\mathcal{D}(L))$ indicates a dependence on the position of the basepoint via the k -grading. So, whenever we use rank inequalities involving Khovanov homology, we must use the rank of the entire homology.

For example, we deduce the following obstruction from Theorem 7 (and Corollary 8):

Corollary 54 Let $L = \hat{\beta}$ be an annular link which admits a **braided** annular diagram $\mathcal{D}(\hat{\beta})$ with at most one negative (or at most one positive) crossing.

- If $\text{rk}_{\mathbb{F}_2} \text{Kh}([\tilde{L}]) \leq \text{rk}_{\mathbb{F}_2} \text{Kh}([L])$, then $[\tilde{L}]$ is not 2-periodic with quotient link $[L]$.
- If $\text{rk}_{\mathbb{F}_2} \text{Kh}([\tilde{L}]) \leq \text{rk}_{\mathbb{F}_2} \text{AKh}(L)$, then $[\tilde{L}] \neq [\hat{\beta}^2]$.

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